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Design of elastic metamaterials

Yu-Chi Su
Purdue University

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By Yu-Chi Su

Entitled
Design of Elastic Metamaterials

For the degree of Doctor of Philosophy

Is approved by the final examining committee:

Chin-Teh Sun
Chair

J. Stuart Bolton

Weinong Chen

Vikas Tomar

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Approved by Major Professor(s): Chin-Teh Sun

Approved by: Weinong Chen 12/4/2015

Head of the Departmental Graduate Program Date
DESIGN OF ELASTIC METAMATERIALS

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Yu-Chi Su

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

May 2016

Purdue University

West Lafayette, Indiana
To my teachers, friends, and family.
ACKNOWLEDGMENTS

First, I wish to express my gratitude to my advisor, Dr. Chin-Teh Sun, for his support and guidance through these years. Studying abroad has been a precious experience in my life, and I am very appreciative of Professor Sun for giving me this opportunity to learn. I would also like to thank Dr. J. Stuart Bolton, Dr. Weinong Chen, and Dr. Vikas Tomar for serving on my thesis committee and providing valuable advice.

Next, I would like to acknowledge CML members for their suggestions in weekly group meetings. I would also like to thank my friends, Andy, Bang-Shiuh, Benjamin, Chiao-Ling, Chien-Hao, Chih-Hua, Cindy, Enyi, Eric, Fang-Yu, Gowtham, Harshad, Hsun, John, Jui-Mien, Kurt, Nitin, Sabrina, Sai, Tejas, Tiffany, Veronica, Waterloo, William, Yi, Yi-Chuang, Yizhou, Yuan, and Zenki for their friendship and support, which have carried me over mountains and through valleys during my PhD program.

Special thanks to my teachers both in NTU and NCKU, Dr. Chien-Ching Ma (M.S. advisor), Dr. Jing-Tang Yang, Dr. Wen-Fang Wu, Dr. Jiann-Quo Tarn, and Dr. Hsuan-Teh Hu, for their encouragement and support, which have helped me get through many difficulties. I am also thankful my aunts, Pao-Hui Wang and Pao-Li Wang, for their selfless help and care. Finally, I would like to thank my family and many people who have helped me in Taiwan.

This work is supported by Air Force Office of Scientific Research #FA9550-10-1-0061. Dr. Byung-Lip (Les) Lee is the program manager.
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SYMBOLS

\( E \)  Young’s modulus
\( \rho \)  mass density
\( \nu \)  poisson’s ratio
\( \lambda \)  lamé constant
\( \mu \)  shear modulus
\( k \)  spring constant
\( m \)  mass
\( L \)  lattice constant
\( q \)  wave number
\( A_0 \)  amplitude for incident wave
\( A_1 \)  amplitude for reflected longitudinal wave
\( A_2 \)  amplitude for reflected transverse wave
\( A_3 \)  amplitude for transmitted longitudinal wave
\( A_4 \)  amplitude for transmitted transverse wave
\( \theta_0 \)  angle of incidence
\( \theta_1 \)  reflected angle of longitudinal wave
\( \theta_2 \)  reflected angle of transverse wave
\( \theta_3 \)  transmitted angle of longitudinal wave
\( \theta_4 \)  transmitted angle of transverse wave
\( u_1 \)  in-plane displacement in \( x_1 \) direction
\( u_2 \)  in-plane displacement in \( x_2 \) direction
\( u_3 \)  out-of-plane displacement in \( x_3 \) direction
\( \tau_{ij} \)  stress tensor
\( \epsilon_{ij} \)  strain tensor
$Q_{ij}$ elastic moduli
$F$ force
$A$ area of unit cell
$A_c$ metamaterial cross-sectional area
$\omega$ circular frequency
$\omega^*$ circular frequency where effective Young’s modulus equals zero
$p_j$ direction of wave propagation
$\vec{d}^{(n)}$ direction of particle movement
$C_L$ longitudinal wave speed in medium 1
$C_L^B$ longitudinal wave speed in medium 2
$C_T$ transverse wave speed in medium 1
$C_T^B$ transverse wave speed in medium 2
$C_p$ phase velocity
$C_g$ group velocity
ABBREVIATIONS

DP double positivity
DN double negativity
FEA finite element analysis
NMD negative mass density
NYM negative Young’s modulus
SN single negativity
Su, Yu-Chi PhD, Purdue University, May 2016. Design of elastic metamaterials. Major Professor: C.T. Sun.

This study focused on the design and fabrication of a double negativity and three broadband single negativity elastic metamaterials using a 3D printer. We investigated dispersion curves and dynamic material properties of the metamaterials. Negative phase velocity in the double negativity metamaterial was also demonstrated.

For metamaterials with single negativity, three types of broadband metamaterials were designed from parametric studies. A comparison showed that using frame bending/stretching mode is more effective than applying beam bending mode to broaden bandgap. Furthermore, it is found that adding internal resonant components could enlarge the bandgap. The single negativity metamaterials were validated by numerical simulations of dispersion curves and attenuation factors.

Moreover, an effective continuum model was derived and utilized to investigate the wave propagation in the elastic metamaterials. The effective continuum model works for long wavelength, and is able to accurately determine the band-gap region. In addition, the applications of double negativity metamaterials such as negative refraction and interface mode conversion were demonstrated using the effective continuum model.

Finally, tunable characteristics of metamaterials were employed to select the propagation speed of a pulse. Potential applications such as signal delay were identified and simulated.
1. INTRODUCTION

Metamaterials are materials with man-made microstructures that can exhibit unusual material properties that cannot be found in nature. Although we review nonlinear metamaterials [1–8], and active metamaterials [9–16], this study focuses on the design of passive linear elastic metamaterials, and the following review is limited to this area.

1.1 Single Negativity Acoustic/Elastic Metamaterials

Materials with periodic structures are well known for their use in filtering waves. However, since its underlying principle for blocking waves is to apply impedance variation, the wavelength of the gap must be of the same order as the lattice constant of the periodic structure. In the field of mechanics, blocking low-frequency waves is generally more desirable than blocking those with high frequencies, thus rendering the periodic structure impractical due to its large lattice constant. To overcome the limitation of low-frequency wave mitigation in periodic structures [17], Liu et al. first applied local resonance to fabricate an acoustic metamaterial with negative effective mass [18]. Using lead balls coated with silicone rubber as resonators, they found that the metamaterial could filter waves with wavelengths two orders larger than its lattice constant. Since then, acoustic metamaterials have become a hot topic among researchers. Yang et al. [19] theoretically and experimentally investigated a membrane-type acoustic metamaterial that possess negative effective mass. Fang et al. [20] proposed a metamaterial with negative effective modulus achieved with Helmholtz resonators. Lee et al. [21] used a tube with side holes to produce an acoustic metamaterial with negative effective modulus. Huang and Sun [22] proposed a negative effective Young’s modulus metamaterial using rigid and massless truss members to connect two resonators. Oudich et al. [23,24] used pillars as resonators to
achieve an elastic metamaterial with negative effective mass density. Although many metamaterials have been proposed, the narrow working frequencies of metamaterials have limited their applications [25–27]. As such, several studies have proposed to solve the narrow band-gap problem. However, most were focused on optical metamaterials [28–33]; there is little in the literature regarding the development of acoustic/elastic broadband metamaterials. Zigoneanu et al. [34] used strong anisotropic acoustic metamaterials to broaden the bandgap up to 500 - 3000 Hz. Meng et al. [35] applied a genetic algorithm to achieve an optimized broadband sound absorption at 800 - 2500 Hz. Zhu et al. [36,37] designed two distributed elastic metamaterial sections as one beam for broadband vibration suppression at 400 - 900 Hz.

In this study, we focus on a metamaterial design to improve the bandwidth up to 3370 Hz starting with a bandgap lower bound at 292 Hz. Three metamaterial models with single negativity (SN) are proposed. Parametric studies and comparisons are made to obtain a metamaterial with the broadest bandgap. It is noted that for a phononic crystal with a similar lattice constant (the lattice constant for our metamaterial designs are 40 mm), the lowest band-gap region begins at a scale of ten thousand hertz [38–41]; while the broadband single negativity metamaterials in this study begin at a scale of hundred hertz.

1.2 Double Negativity Acoustic/Elastic Metamaterials

By combining negative effective mass and negative effective modulus, one can obtain "double negativity metamaterials". Although a number of double negativity (DN) acoustic metamaterials have been investigated [42–53], there are few reports in the literature on DN elastic metamaterials. Among the elastic metamaterials that have been published, Wang [54] investigated the mechanism of negative effective mass and negative effective modulus of lumped mass models. Wu et al. [55] proposed a metamaterial with negative effective shear modulus and mass density. Lai et al. [56] designed a metamaterial with negative effective bulk modulus, mass density, and
shear modulus. Liu et al. [57] proposed the idea of elastic chiral metamaterials, and Zhu et al. [58] created a single-phase solid based on the lumped mass model. The metamaterial design in this study is based on the lumped mass model proposed by Huang and Sun [59], and fabricated it as a single-phase solid using a 3D printer.

1.3 Applications of Acoustic/Elastic Metamaterials

Applications of acoustic/elastic metamaterials can be classified into six different areas. The first is to use metamaterials to overcome diffraction limit [25,60–66] or to produce flat lens [67–70]. The second is to trap waves utilizing negative refraction or defect localization [67,69]. The third is applying metamaterials to satisfy the material properties for cloaking based on transformation method [71–74]. The forth is to create an acoustic black hole by arranging metamaterials to develop a gradient index shell, and guide the waves into the energy dissipating metamaterial core [75,76]. The fifth is to use the tunable characteristics of metamaterials for collimation [77]. The sixth is to suppress vibrations [37,78,79]. These potential applications make metamaterials a novel research direction and generate considerable attention.

1.4 Design Concepts for Elastic Metamaterials

This work was based on Huang and Sun [22,59,80]. Table 1.1 lists the resonators designed in this thesis. We applied structural vibration modes such as beam bending mode and frame bending/stretching mode for resonator design. The design concepts are to use resonators unidirectional motion to generate negative effective mass, while to apply resonators inward-outward motion to produce negative effective modulus (details of calculation for effective continuum model are discussed in Chapter 4). Figs. 1.1(a)-(c) show the ideas to create negative effective Young’s modulus. Red arrows indicate the motion of resonators, while red-dash arrows denote the corresponding movement on the side edges. When a pair of symmetrical force applied, shown in purple arrows, with frequencies close to local resonance of these resonators,
the models have negative effective modulus. Figs. 1.1(d) and (e) show the ideas to produce negative effective mass density. The movements of resonators, indicated by red arrows, induce side edges motion shown by red-dash arrows. When the forces applied along the direction of purple arrows, the models have negative effective mass density. The other types of design concepts can be found in [56,81].

Fig. 1.1.: Design concepts: (a) negative Young’s modulus resonator; (b) negative Young’s modulus resonator; (c) negative Young’s modulus resonator; (d) negative mass density resonator; (e) negative mass density resonator.
Table 1.1.: Resonators designed in this study.

<table>
<thead>
<tr>
<th>type of resonator</th>
<th>mode shape</th>
<th>mode description</th>
</tr>
</thead>
<tbody>
<tr>
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<td><img src="image" alt="mode shape" /></td>
<td>horizontal beam bending mode</td>
</tr>
<tr>
<td>negative Young’s modulus</td>
<td><img src="image" alt="mode shape" /></td>
<td>frame bending/stretching mode</td>
</tr>
<tr>
<td>negative Young’s modulus</td>
<td><img src="image" alt="mode shape" /></td>
<td>internal masses symmetrical mode</td>
</tr>
<tr>
<td>negative mass density</td>
<td><img src="image" alt="mode shape" /></td>
<td>internal masses anti-symmetrical mode</td>
</tr>
<tr>
<td>negative mass density</td>
<td><img src="image" alt="mode shape" /></td>
<td>internal masses anti-symmetrical mode</td>
</tr>
</tbody>
</table>
2. DESIGN OF DOUBLE NEGATIVITY ELASTIC METAMATERIALS

The DN metamaterial design in this chapter is based on the lumped mass model [22]. Dispersion curves and dynamic responses of the model are investigated numerically. In the double negative frequency region, it is demonstrated that the phase velocity is negative. In addition, by taking parts of single-unit cell to analyze the steady-state response, band-gap region of the metamaterial can be accurately determined. The design is fabricated by an Eden 350 3D printer.

2.1 Metamaterial Models and Design Process

Fig. 2.1 shows the design process for determining the dimensions of the negative effective Young’s modulus (NYM) resonator. To change the lumped mass model into a single-phase solid for 3D printing, we used beams to replace vertical springs. After performing parametric studies on the length and thickness of the horizontal beam, we found that a beam with a length of 30 mm and width of 1 mm can provide resonance at frequencies of interest. It is noted that due to geometric complexity, it is impractical to use theoretical analysis to calculate local resonance. Instead, we numerically estimate the local resonances by the finite element analysis (FEA). Since the inclined frame is only used for transmitting force, the width of the frame is designed as 1.5 mm to make the mode of frame bending/stretching occur at higher frequencies. For the mass of the NYM resonators, we found that the arrow shape gives the best spatial usage. After determining the geometry, we proceeded to adjust the material properties in each part of the metamaterial model. Furthermore, since the

thickness is converted into effective material properties in the simulations, the range of effective material properties is set to be $E = 5 - 20 \, GPa$, $\rho = 1300 - 5200 \, kg/m^3$ (we used high-temperature photopolymer ink for 3D printing, which has material properties: $E = 5 \, GPa$, $\rho = 1300 \, kg/m^3$, and $\nu = 0.49$). Fig. 2.1(b) and Table 2.1 show the finalized parts of the unit cell of the metamaterial that produces NYM. The characteristic length of the unit cell, $L$, is 40 mm.

**Fig. 2.1.:** Design of NYM resonator based on the lumped mass model: (a) NYM lumped mass model; (b) NYM submodel.

**Table 2.1.:** Material parameters for the NYM submodel.

<table>
<thead>
<tr>
<th>marked area</th>
<th>mass density ($kg/m^2$)</th>
<th>Young’s modulus ($GPa$)</th>
<th>poisson’s ratio</th>
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<tr>
<td>green</td>
<td>1742</td>
<td>6.7000</td>
<td>0.49</td>
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</tbody>
</table>

In the design of the negative effective mass density (NMD) resonator, as shown in Fig. 2.2, we used the same procedure to investigate the length of the vertical
beam, which serves as the horizontal spring of the lumped mass model. The shape of the NMD resonator is also determined according to the efficiency of spatial usage. Fig. 2.2(b) and Table 2.2 show the finalized parts of the unit that produces NMD. The vertical beams supporting two triangular-shaped masses serve as two horizontal springs. The top and bottom horizontal beams are restrained from vertical motion. The characteristic length of the unit cell, $L$, is also 40 mm.

Fig. 2.2.: Design of NMD resonator based on the lumped mass model: (a) NMD lumped mass model; (b) NMD submodel.

Table 2.2.: Material parameters for the DN metamaterial and NMD submodel.

<table>
<thead>
<tr>
<th>marked area</th>
<th>mass density ($kg/m^2$)</th>
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<tr>
<td>green</td>
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<td>6.7000</td>
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<tr>
<td>red</td>
<td>3354</td>
<td>12.9000</td>
<td>0.49</td>
</tr>
</tbody>
</table>

The DN model is the same as the NMD model except that the horizontal beams are free to move vertically. The material parameters for the DN model are also listed
in Table 2.2. In order to have a DN frequency region, the dimensions for the NYM and NMD resonators are designed to have overlapping bandgaps.

### 2.2 Dispersion Curves for the DN Model

Dispersion curves show the frequencies at which harmonic waves can propagate without amplitude attenuation. Frequencies between two dispersion curves locate the stopping band.

Considering 1D longitudinal harmonic wave propagation, the displacement can be expressed as

\[ u(x,t) = Ae^{iq(x-C_p t)} = Ae^{i qx} e^{-i \omega t} \]  \hspace{1cm} (2.1)

where \( C_p \) is phase velocity, and \( q \) is wave number. By selecting two locations as nodes (zero displacement) in simulation, Eqn. 2.1 can be treated as a free vibration problem of a finite length body with fixed end conditions. As demonstrated in Fig. 2.3, if the harmonic wave denoted by the blue line is the longest wavelength propagating within the infinite number of metamaterial unit cells, it can be modeled by setting the harmonic waves with fixed end condition at the left and right edges of either 400 unit cell, 200 unit cell, or 100 unit cell metamaterial strip. In this study, we found that the 100 unit cell strip is long enough to model the dispersion curves of the infinite metamaterial chain.

To model the infinite number of unit cells metamaterial by the 100 unit cells, we constrained the horizontal movements of the left and right boundaries of the 100 unit cell metamaterial strip, so that these two boundaries became nodal points. The result of the free vibration analysis yields the natural frequencies and mode shapes, from which the corresponding wave numbers were obtained.

We used ABAQUS commercial software for finite element analysis. Since beam bending is used as the resonator’s stiffness, accuracy for the beam bending mode estimation is crucial. For element-type selection, the incompatible mode element (CPS4I)
Fig. 2.3.: (a) Infinite long metamaterial chain with a harmonic wave (blue line); (b) 400 unit cell metamaterial strip modeling; (c) 200 unit cell metamaterial strip modeling; (d) 100 unit cell metamaterial strip modeling.
was used because of its efficiency for bending estimation. Four CPS4I elements are placed along the thickness of the beam to achieve element number convergence.

Fig. 2.4 shows the dispersion curves of the DN metamaterial. Each point on the dispersion curve represents an eigenmode (calculated by ABAQUS) of the 100 unit cell long metamaterial strip. The DN region is recognized by negative phase velocity [59]. It is noted that the narrow passing band, $579.28 - 579.37$ Hz, is due to a rotational mode (horizontal beams rotate around the joint).

![Dispersion curves for the DN metamaterial.](image)

We’ve also compared this multi-cell method with Floquet-Bloch analysis in COMSOL Multiphysics (using one unit cell), as shown in Fig. 2.5. The two methods yield the same dispersion curves, which implies the accuracy of simulations.
Fig. 2.5.: Comparison of dispersion curves for the DN metamaterial.
2.3 Wave Propagation in the DN Model

Negative phase velocity is one of the characteristics of DN metamaterials and can be observed from wave propagation. Hence, we investigated the transient responses of the DN metamaterial to demonstrate the negative phase velocity.

Since a large number of finite elements are used for beam bending calculation in each unit cell, accumulated numerical error is an issue for ABAQUS Dynamic Explicit packet. To deal with this numerical problem, Dynamic Implicit version is used instead. A long metamaterial strip (200 unit cells) was built to delay wave reflection, and a sinusoidal displacement with a frequency of 685 Hz was applied at the first unit cell. As before, the vertical displacements of the upper and lower boundaries of the unit cells are constrained because of plane wave assumption. Negative phase velocity is observed by tracking the responses of three proximal unit cells.

Since the initial condition is quiescent before the application of the sinusoidal motion, the steady-state response is reached after a period of time. As shown in Fig. 2.6, negative phase velocity is observed after 0.186 s. It is noted that the wave amplitude has no decay since 685 Hz is within the DN frequency range.
Fig. 2.6.: Demonstration of negative phase velocity.
2.4 Determination of Band-Gap Region by Single-Unit Cell

The band-gap region in metamaterials is not purely caused by local resonance. Liu et al. [62] used a mass-in-mass metamaterial to demonstrate that the extremely large mass, induced by local resonance, would result in a strong spatial oscillation of wave fields within the periodic structures, giving rise to a Bragg gap $qL - \pi$ just below resonance frequency. This low-frequency Bragg gap might be different from a common one because it occurs at sub-wavelength scale. Therefore, in order to accurately determine the total band-gap region in metamaterials and further to ensure that the negative phase velocity of the DN model is caused by overlapping NYM and NMD band gaps, we present a procedure for band-gap prediction using single-unit cells (NYM submodel and NMD submodel). Since lattice models are more flexible in adjusting parameters, in the following sections, we first use lattice models for parametric studies, and then move to practical models for estimation of bandgaps by a single-unit cell.

2.4.1 NYM submodel and its lattice model

Fig. 2.1(a) shows the NYM lumped mass model; the two masses $m_1$ and $m_2$ in the unit cell are influenced by each other. Therefore, if $m_1$ and $m_2$ do not differ too much, the Bragg gap below resonance frequency would be more dominant. As shown in Fig. 2.7, the band-gap region below the red dash line is the Bragg gap ($\omega^*$ is the circular frequency where the effective Young's modulus equals zero [22]).
In view of this, we used the following method to evaluate the total band-gap region. The NYM submodel is shown in Fig. 2.1(b) and Table 2.1. The frequency-dependent effective modulus of a metamaterial can be obtained by applying symmetrical loadings $F_0 \cos \omega t$ on the unit cell, as shown in Fig. 2.8. The side edge of the unit cell are maintained straight and their displacements are $+u_0 \cos \omega t$ and $-u_0 \cos \omega t$, respectively. The change of the unit cell dimension is $2u_0$, and the dynamic stiffness of the unit cell is given by

$$k_{eff} = \frac{F_0}{2u_0} \quad (2.2)$$
It is noted that the stiffness of the unit cell defined by Eqn. 2.2 is proportional to the effective Young's modulus of the metamaterial. The NYM band-gap region is defined as the frequency range in which $k_{eff}$ is negative. By using this procedure, the band-gap region is 588.6 – 704.0 Hz (see Fig. 2.9).

![NYM unit cell subjected to symmetrical loadings for testing dynamic stiffness (effective modulus).](image)

For comparison, we also used 100 unit cells to calculate dispersion curves for the NYM submodel, as shown in Fig. 2.10. The NYM band-gap region is 588 – 704 Hz, which agrees with the bandgap obtained from the unit cell. The narrow passing band, 576.01 – 576.61 Hz, is the horizontal beam rotational mode. In Fig. 2.9, the lower bound of the NYM bandgap shows an out-of-phase motion on the left and right edges of unit cell, which indicates a short wavelength corresponding to a dispersion curve (the shortest wavelength in dispersion curve is $2L$). In contrast, the upper bound of the NYM band-gap shows an in-phase motion (zero displacement) on side edges of the unit cell, which corresponds to the longest wavelength of the optical branch in the dispersion curves.
Fig. 2.9.: Displacement frequency spectrum.

Fig. 2.10.: Dispersion curves for the NYM submodel.
2.4.2 NMD submodel and its lattice model

The NMD lumped mass model is shown in 2.2(a). The two springs $k_1$ and $k_3$ within the unit cell will interact with each other. As shown in Fig. 2.11, when $k_1$ and $k_3$ are at the same scale, the low-frequency Bragg gap is dominant.

Similarly, we performed a single-unit cell test to predict the band-gap region. The NMD submodel was made by suppressing the bending motions of the two green horizontal beams, as shown in Fig. 2.2(b). The effective mass density of the NMD submodel can be obtained from the dynamic response of the unit cell loaded as shown in Fig. 2.12. The side edges are maintained straight and their displacements are given by $u_0 \cos \omega t$. The effective mass of the unit cell is given by

$$m_{eff} = \frac{-F_0}{u_0 \omega^2}$$ (2.3)

From the result of the finite element simulation, Fig. 2.13 shows the displacement spectrum. The band-gap region is defined as the frequency range where $u_0$ becomes positive, 625.3 – 703.7 Hz, in which $m_{eff}$ has a negative value.

For validation, we also used 100 unit cells of the NMD submodel to evaluate the band-gap region from dispersion curves, as shown in Fig. 2.14. The NMD band-gap region is 625 – 705 Hz. Again, this agrees with that obtained from the unit cell of the NMD submodel. In Fig. 2.13, the lower bound of NMD bandgap shows an out-of-phase motion on the side edges of the NMD unit cell, which indicates the shortest wavelength in the acoustic branch of dispersion curves. In contrast, the upper bound of the NMD bandgap shows an in-phase motion on the side edge of the unit cell, which corresponds to the longest wavelength in the optical branch of dispersion curves.

2.4.3 Comparison of band-gap region estimation

We compared the band-gap regions obtained from the effective material properties for one unit cells and dispersion curves for 100 unit cells (Sec. 2.4.1 and Sec. 2.4.2).
Fig. 2.11.: Parametric study for the NMD lattice model.
Constrain the vertical motion along the red dashed line

\[ \frac{F_0}{2} \cos \omega t \]

\[ \frac{F_0}{2} \cos \omega t \]

Fig. 2.12.: NMD unit cell subjected to anti-symmetrical loadings for testing effective mass.

**Negative Effective Mass Density Model**

- Short wavelength 625.3 Hz
- Long wavelength 703.7 Hz

Fig. 2.13.: Displacement frequency spectrum.
Fig. 2.14.: Dispersion curves for the NMD submodel.
As shown in Table 2.3, the analysis for a single-unit cell can accurately predict the band-gap region.

Table 2.3.: Comparison of band-gap region estimation.

<table>
<thead>
<tr>
<th>model type</th>
<th>band-gap region prediction by single-unit cell</th>
<th>band-gap region prediction by dispersion curves</th>
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<tr>
<td>NYM submodel</td>
<td>569 – 678 (Hz)</td>
<td>568 – 679 (Hz)</td>
</tr>
<tr>
<td>NMD submodel</td>
<td>611 – 648 (Hz)</td>
<td>611 – 647 (Hz)</td>
</tr>
</tbody>
</table>

It is noted that in the effective material test of the NYM unit cell, the rotational mode is absent due to its anti-symmetrical characteristic.

2.5 3D Printing Fabrication

The metamaterial proposed in this study was fabricated by a 3D printer. Since 3D printing is three-dimensional with varying thicknesses, we used the following principles to convert the 2D metamaterial model with a uniform thickness to a 3D specimen.

1. The effective Young's modulus in a 2D model is proportional to the thickness of the specimen.

2. Mass density is proportional to the thickness of the specimen.

Fig. 2.15 shows the DN metamaterial specimen, which was fabricated using an Eden 350 3D printer, based on the calculations in this study. The fabrication process can be divided into three steps: preprocessing, production, and support removal.

In the preprocessing step, we built the specimen into a 3D form of various thicknesses using the CAD software, CATIA, and input the CAD file into the Eden 350. The 3D printing jet then dripped liquid photopolymer layer-by-layer to build the 3D model. We used high-temperature material RGD 525 as the photopolymer ink to
Fig. 2.15.: A DN metamaterial made with a 3D printer.
produce the specimen. The liquid photopolymer was then cured by UV light during processing. It is noted that the 3D printer also dripped removable gel-like supporting materials on the parts with weaker structures to stabilize them. Finally, we used a water jet to remove the supporting materials and obtain a metamaterial specimen with a 16 $\mu m$ layer resolution and 0.1 $mm$ accuracy.
Three types of broadband single negativity (SN) designs are introduced in this chapter. We optimized these models in parametric studies and made comparisons to determine the elastic metamaterial with the broadest bandgap. It is evident that frame bending/stretching is more appropriate than beam bending serving as a metamaterial resonator. In addition, by designing several resonators within the metamaterial unit cell, bandgaps can be made even broader. We validated the SN metamaterial models by numerical simulations and found that by taking the unit cell to analyze the steady-state response, the band-gap region can be accurately determined. We fabricated the metamaterial designs using an Eden 350 3D printer. A design guide for three SN metamaterials is also proposed in this study.

3.1 Three Types of Single Negativity Metamaterial Models

As shown in Figs. 3.1 and 3.2, the SN metamaterials were fabricated by the Eden 350 3D printer with photopolymer ink: high-temperature material RGD 525. The specimens are three-dimensional with varying thickness. However, for the convenience of numerical simulations, we vary the material properties to account for the thickness variation in each part of the metamaterial. By doing so, the 3D metamaterial can be simulated as a two-dimensional material with varying material properties. Tables 3.1 – 3.3 list the effective material constants for the 2D model, and Figs. 3.3 - 3.4 show the geometries and effective material properties of the three metamaterial designs.

1This chapter is published and can be found in Y.C. Su and C.T. Sun, "Design of Broadband Elastic Metamaterials." Theories and Designs of Acoustic Metamaterials, SPIE, 2015.
Fig. 3.1.: Photograph, 3D printed specimen of the type 2 SN metamaterial.

Fig. 3.2.: Photograph, 3D printed specimen of the type 3 SN metamaterial.

Fig. 3.3.: Type 1 SN metamaterial model.
Fig. 3.4.: Type 2 SN metamaterial model.
Fig. 3.5.: Type 3 SN metamaterial model.
Table 3.1.: Effective material properties for the type 1 SN metamaterial.

<table>
<thead>
<tr>
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Table 3.2.: Effective material properties for the type 2 SN metamaterial.

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Table 3.3.: Effective material properties for the type 3 SN metamaterial.

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</table>

Fig. 3.3 shows the unit cell of the type 1 SN metamaterial model. The two horizontal beams with a length of 30 mm and a width of 1 mm serve as two vertical springs with the desired frequency to form NYM resonators. In addition, a red vertical beam serving as two horizontal springs to support two triangular-shaped masses is
used as a NMD resonator. The characteristic length $L$ of the metamaterial design is 40 mm. The thickness variation in the specimen translates to different material properties; these effective properties are listed in Table 3.1. The material properties are chosen to broaden the band-gap region by combining the bandgaps of the NYM and NMD.

Instead of using beam bending, a type 2 SN metamaterial applies frame bending/stretching as stiffness of the resonator. As shown in Fig. 3.4, the inclined frame serves as two vertical springs supporting purple masses to achieve NYM local resonance. Since the type 2 SN model only has NYM resonator, it is a NYM metamaterial design. The characteristic length $L$ of the type 2 SN metamaterial is also 40 mm, and its material parameters are listed in Table 3.2.

The type 3 SN metamaterial is obtained by adding two resonators to the type 2 SN model. As shown in Fig. 3.5, the two red vertical beams serve as two sets of horizontal springs and support orange masses to achieve NMD local resonance when they move in-phase. In addition, the frame bending/stretching resonator and the symmetrical mode of the internal masses (when the two red vertical beams move 180° out-of-phase) form two sets of NYM resonators. The band-gap region is broadened by combining multiple NMD and NYM bangaps. Table 3.3 lists the material properties for the type 3 metamaterial chosen from a parametric study. The characteristic length $L$ for the type 3 SN model is also 40 mm.

### 3.2 Mode Shapes and Dispersion Curves

Using the methodology stated in Sec. 2.2, we obtain the dispersion curves shown in Figs. 3.6 – 3.8 for the type 1 to type 3 SN metamaterials. Each passing band corresponds to a distinct unit cell vibration mode, as shown in Tables 3.4 and 3.5. Because of symmetry, only the upper half of the unit cell is taken into account for simulation.
The dispersion curves for the type 1 SN metamaterial are shown in Fig. 3.6, for which the NMD and NYM resonators are optimized by minimizing the passing band between two adjacent bandgaps (the narrow passing band is in the frequency range of 509.96 – 510.17 Hz). The other narrow passing band in the frequency range of 463.35 – 463.47 Hz is caused by the rotational mode of the horizontal beams. The overall band-gap region is 403 to 554 Hz, first caused by NMD and then by NYM. The two local resonances in the type 1 SN metamaterial both originate from beam bending.

![Dispersion curves for the type 1 SN metamaterial.](image)

Fig. 3.6.: Dispersion curves for the type 1 SN metamaterial.

Fig. 3.7 shows the dispersion curves for the type 2 SN metamaterial. If the narrow passing band from the mass rotational mode (505.41 to 506.90 Hz) is neglected, the overall band gap-region is 345 to 3097 Hz. This bandgap is caused by frame bending/stretching only. It is evident that the frame bending/stretching mode can make a broader bandgap than the beam bending mode.

Dispersion curves for the type 3 SN model are shown in Fig. 3.8. There are two narrow passing bands in the overall band-gap region: mass rotational mode, i.e., 752.40 to 755.24 Hz, and a double negativity region, i.e., 1064.8 to 1097.3 Hz. The
overall band-gap region, 292 to 3694 Hz, is caused by a frame bending/stretching mode (NYM), an internal masses symmetrical mode (NYM), and an internal masses antisymmetrical mode (NMD). The type 3 SN model broadens the bandgap of the type 2 SN model by adding two extra resonators.

Fig. 3.7.: Dispersion curves for the type 2 SN metamaterial.

Table 3.4.: Free vibration of the type 2 SN metamaterial.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frame bending/stretching</td>
<td>340.85 Hz</td>
</tr>
<tr>
<td>Mass rotational mode</td>
<td>501.93 Hz</td>
</tr>
</tbody>
</table>
Fig. 3.8.: Dispersion curves for the type 3 SN metamaterial.
Table 3.5.: Free vibration of the type 3 SN metamaterial.

<table>
<thead>
<tr>
<th>Frame bending/stretching mode</th>
<th>Mass rotational mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>287.05 Hz</td>
<td>746.23 Hz</td>
</tr>
<tr>
<td>Internal masses symmetrical mode</td>
<td>Internal masses anti-symmetrical mode</td>
</tr>
<tr>
<td>1062.0 Hz</td>
<td>1089.8 Hz</td>
</tr>
</tbody>
</table>

3.3 Steady-State Analysis of Unit Cells

Other than utilizing dispersion curves to obtain the band-gap region, we also investigate the possibility of using steady-state analysis of the unit cell. Taking type 3 SN model as an example, for long wave motions, the frequency dependent effective modulus of the metamaterial is obtained by applying symmetrical loadings $F_0 \cos \omega t$ on the unit cell, shown in Fig. 3.9(a). The side edges of the unit cell are maintained straight and their displacements are given by $+u_0 \cos \omega t$ and $-u_0 \cos \omega t$, respectively. The change of the unit cell dimension is $2u_0$, and the dynamic stiffness of the unit cell is expressed as

$$ k_{eff} = \frac{F_0}{2u_0} $$ (3.1)

The stiffness of the unit cell defined by Eqn. 3.1 is proportional to the effective Youngs modulus of the metamaterial.
The band-gap region is defined as the frequency range where $k_{eff}$ becomes negative, i.e., 294 to 866 and 1064 to 3694 Hz. From Eqn. 3.1, we observe that the dynamic stiffness $k_{eff}$ has the same sign as $u_0$. Therefore, we present the displacement frequency spectrum in Fig. 3.9(b). The lower bound of each NYM bandgap is at 294 and 1064 Hz, respectively.

The effective mass density of the metamaterials can be obtained from the dynamic response of the unit cell loaded in Fig. 3.9(a). The side edges of the unit cell are maintained straight and their displacements are given by $u_0 \cos \omega t$. The effective mass of the unit cell is

$$m_{eff} = -\frac{F_0}{u_0 \omega^2} \quad (3.2)$$

The effective mass $m_{eff}$ frequency spectrum is shown in Fig. 3.10(b). The band-gap region is defined as the frequency range where $m_{eff}$ becomes negative, i.e., 867 to 1099 Hz. The small peak in Fig. 3.10(b) at the frequency 755 Hz is caused by the mass rotational mode, which can also be observed in the dispersion curves shown in Fig. 3.8.

The bandgaps predicted using unit cells are listed in Tables 3.6 and 3.7 together with the bandgaps obtained from the dispersion curves using 100 unit cells. It is evident that the steady-state analysis of the unit cell is accurate in terms of bandgap estimation. Note that the frequency range 1065 to 1097 in Table 3.7 of dispersion curves is the double negativity region (bandgaps overlap of NMD and NYM).

Table 3.6.: Comparison of band-gap region predictions for the type 2 SN metamaterial.

<table>
<thead>
<tr>
<th>method</th>
<th>band-gap region estimation (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>dispersion curves</td>
<td>345 to 3097</td>
</tr>
<tr>
<td>unit cell steady-state analysis</td>
<td>347 to 3097</td>
</tr>
</tbody>
</table>
Fig. 3.9.: (a) Testing dynamic stiffness (effective Youngs modulus). (b) Displacement frequency spectrum.
Fig. 3.10.: (a) Testing effective mass density. (b) Effective mass frequency spectrum.
Table 3.7.: Comparison of band-gap region predictions for the type 3 SN metamaterial.

<table>
<thead>
<tr>
<th>method</th>
<th>band-gap region estimation (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>dispersion curves</td>
<td>292 to 1065</td>
</tr>
<tr>
<td></td>
<td>1097 to 3694</td>
</tr>
<tr>
<td>unit cell steady-state analysis</td>
<td>294 to 866 (from unit cell NYM test)</td>
</tr>
<tr>
<td></td>
<td>867 to 1099 (from unit cell NMD test)</td>
</tr>
<tr>
<td></td>
<td>1064 to 3694 (from unit cell NYM test)</td>
</tr>
</tbody>
</table>

3.4 Optimization Process

In order to make the 3D printed specimen sufficiently strong and less of an out-of-plane effect, the thickness of the specimen is restricted to the range 4 to 16 mm. Furthermore, since the thickness is converted into effective material properties in the simulations, the range of effective material properties is set to be $E = 5$ to 20 GPa and $\rho = 1300$ to 5200 kg/m$^2$. We note that $E = 5$ GPa and $\rho = 1300$ kg/m$^2$ are the material properties of the high-temperature photopolymer ink that was used for 3D printing.

For the optimization process in simulations, we first perform parametric studies on the geometry and then on different material properties in each section of the metamaterial model. The optimization process is to find a metamaterial with the largest band-gap region at low frequencies.

For the type 1 SN metamaterial, since the design is to combine the NMD and NYM bandgaps caused by beam bending, the geometry of the model is chosen to have the largest double negativity frequency region to assure the capacity for overlapping bandgaps. After determination of geometry, we optimize the material properties in each section to broaden bandgaps with minimum overlapping.
Fig. 3.11 shows an example of in the shape of the NYM resonator. We found that an arrow-shaped design can provide the best spatial usage for the resonator. The size of the arrow-shaped resonator was then adopted in a parametric study.

(a)

(b)

(c)

Fig. 3.11.: Geometric determination of the NYM resonator for the type 1 SN model (a) solid line: rectangular-shaped of undeformed NYM resonator; dash line: arrow-shaped of undeformed NYM resonator; (b) mode shape of the arrow-shaped resonator; (c) mode shape of the rectangular-shaped resonator.

For the type 2 SN metamaterial, as shown in Fig. 3.12, we investigate the effect of $t$ and $l$ in order to find the broadest band gap in the low-frequency range. A parametric study in material properties (corresponding to different thicknesses in the printed specimen) in each section is also carried out after the geometric investigation. Since the principle of broadening the band gap in type 3 SN metamaterial is to combine multiple bandgaps, we adjusted the size of the resonators and then the
material properties in each section to enlarge the overall band gap region in low frequencies for the type 3 SN metamaterial.

![Diagram](image)

Fig. 3.12.: Geometry determination of the NYM resonator for the type 2 SN model.

3.5 Design Guide for Three SN Metamaterials

For the type 1 SN metamaterial, a smaller mass and larger stiffness will broaden the bandgap of the NYM resonator, whereas a large mass and large stiffness generally results in a wider bandgap. Furthermore, a large bandgap in high-frequency range is more easily produced by the NYM than NMD. Therefore, the appropriate design is to use a large mass and large stiffness for the NMD resonator to forbid low-frequency waves, while applying a small mass and large stiffness of the NYM resonator to block high-frequency waves. By combining the NMD and NYM bandgaps, a metamaterial for filtering low-frequency waves in a wider range is made.

For the type 2 SN metamaterial, a large mass and small stiffness of the NYM resonator are both effective for shifting the band-gap region to the low-frequency range. However, the bandwidth shrinks drastically with the increment of the mass of the resonator. As a result, for the type 2 SN metamaterial, a medium-size mass of the NYM resonator is recommended.

For the type 3 SN metamaterial, although a large mass of the two NYM resonators (the frame bending/stretching mode and internal mass symmetrical motion are the two NYM resonators) can bring down the lower bound of bandgap, it also narrows the bandwidth. Therefore, medium-size masses of the NYM resonators are more
appropriate. Since the internal masses serve as NYM and NMD resonators, once the material properties of the NYM resonator are set, the NMD resonator is also set.

### 3.6 Wave Attenuation in SN Metamaterials

We also investigated the wave attenuation within the band-gap region by FEA. Since a large number of finite elements is used for beam bending calculation in each unit cell, accumulated numerical error is an issue for the ABAQUS Dynamic Explicit packet. To deal with this numerical problem, Dynamic Implicit version is used instead. A long metamaterial strip (500 unit cells) model similar to Fig. 3.1 is employed for dynamic wave propagation simulations without wave reflections. A harmonic displacement excitation with a frequency of interest is applied at the first unit cell. Fig. 3.13 shows the profiles of the wave amplitudes generated with different frequencies. The wave attenuates drastically within the metamaterial.

![Fig. 3.13. Attenuation of the type 2 SN metamaterial.](image)
3.7 Comparison of the Three SN Metamaterials

Table 3.8 compares the three types of SN metamaterial designs. The type 1 SN model applies the beam bending mode as the stiffness of the resonators and combines bandgaps from the NMD and NYM to form a relatively broad bandgap in the low-frequency range. The type 2 SN model simply uses frame bending/stretching as the stiffness of the NYM resonator. Frame bending/stretching mode is more appropriate than the beam bending mode in the design of a broadband metamaterial.

The advantages of using a frame bending/stretching can be attributed to the following two factors. First, beam bending requires an inclined frame or another structural component to transmit force to the side edges of each unit cell. Since these transmitted structural components are not entirely rigid, they store strain energy. However, frame bending/stretching plays dual roles for directly connecting to side edges and serves as a resonator, therefore making it more efficient. In addition, a frame bending/stretching design leaves more spaces for placing the large resonator’s mass, which also enhances the effectiveness of spatial usage to broaden the bandgap in the low-frequency range.

The type 3 SN model adds internal masses inside the inclined frame to enlarge the bandgap. Depending on the motion, the internal masses serve as either an NYM or NMD resonator. When the internal masses move in-phase, they are NMD resonators, whereas when the internal masses move out-of-phase, they become NYM resonators. The internal masses broaden the band-gap region by combining multiple resonators of different resonances, although extra narrow passing bands are also generated.

Comparing the weight of the three SN metamaterials, it is observed that the type 3 metamaterial is the heaviest, whereas the 2 metamaterial has the least weight. Therefore, for weight-sensitive vibration suppression such as in aircraft structures, the type 2 metamaterial is considered to be the best choice. However, in the mechanical or civil engineering field, the type 3 model may be the most efficient.
Table 3.8.: Comparison of three SN metamaterial designs.

<table>
<thead>
<tr>
<th></th>
<th>type 1</th>
<th>type 2</th>
<th>type 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>total mass of unit cell</td>
<td>$6.36 \times 10^{-4} kg$</td>
<td>$5 \times 10^{-4} kg$</td>
<td>$7.74 \times 10^{-4} kg$</td>
</tr>
<tr>
<td>band-gap region</td>
<td>403 to 554 Hz</td>
<td>345 to 3097 Hz</td>
<td>292 to 1065 Hz, 1097 to 3694 Hz</td>
</tr>
<tr>
<td>bandwidth</td>
<td>151 Hz</td>
<td>2752 Hz</td>
<td>3370 Hz</td>
</tr>
</tbody>
</table>
4. EFFECTIVE MATERIAL PROPERTIES FOR ELASTIC METAMATERIALS

Determination of dynamic effective material properties for acoustic/elastic metamaterials is a hot topic in the literature [56, 57, 82–93]. In this chapter, we present a procedure to determine the effective material properties of the metamaterial designs and use them to compute the dispersion curves. Equation of motion for 1D orthotropic material is expressed by

$$\frac{\partial \tau_{11}}{\partial x_1} = \rho_{\text{eff}} \frac{\partial^2 u_1}{\partial t^2}$$  \hspace{1cm} (4.1)

In addition, stress-strain relation for orthotropic material gives

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \end{bmatrix} = \begin{bmatrix} Q_{\text{eff}11} & Q_{\text{eff}12} & 0 \\ Q_{\text{eff}21} & Q_{\text{eff}22} & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \end{bmatrix}$$  \hspace{1cm} (4.2)

We take a unit cell to determine effective elastic material properties. In Fig. 4.1, harmonic concentrated forces $\pm F_1 \cos \omega t$ are symmetrically applied on the side edges of unit cell. We restrain the top and bottom edges from vertical movement, and keep the side edges to be straight under loading. The corresponding displacements of the left and right edges are $-u_1 \cos \omega t$ and $+u_1 \cos \omega t$, respectively. The positive or negative sign indicates direction. We also calculate normal stress $\tau_{22}$ on the top and bottom edges. Substituting the force amplitude $F_1$, its corresponding displacement $u_1$, and normal stress $\tau_{22}$ into stress-strain relation, we have

$$\tau_{11} = \frac{F_1}{A_c} = Q_{\text{eff}11} \frac{\partial u_1}{\partial x_1}$$

$$\tau_{22} = Q_{\text{eff}21} \frac{\partial u_1}{\partial x_1}$$  \hspace{1cm} (4.3)

where $A_c$ is the cross-sectional area for metamaterial unit cell with the in-plane dimensions $L$ and $L_2$; for metamaterial with unit thickness, $A_c = L_2 \times 1$. Effective elastic moduli $Q_{\text{eff}11}$ and $Q_{\text{eff}21}$ are then obtained from Eqn. 4.3.
Fig. 4.2 shows the method utilized to determine effective mass \( m_{\text{eff}} \). The harmonic forces are now applied in the same direction. We calculate the corresponding acceleration, and the effective mass is obtained through Newton’s second law. Effective mass density \( \rho_{\text{eff}} \) is then defined by

\[
\rho_{\text{eff}} = \frac{m_{\text{eff}}}{A}
\]  

(4.4)

where \( A = L \times L_2 \) is the area of metamaterial with the in-plane dimensions \( L \) and \( L_2 \).

Combining Eqn. 4.1-4.2 and using \( \epsilon_{ij} = \partial u_i / \partial x_j \), we have wave equation as follows.

\[
\rho_{\text{eff}} \frac{\partial^2 u_1}{\partial t^2} = Q_{\text{eff}11} \frac{\partial^2 u_1}{\partial x_1^2}
\]

(4.5)

Assuming displacement is a harmonic plane wave \( u_1 = A m \cos(qx_1 - \omega t) \) and substituting it into Eqn. 4.5 gives dispersion relation as follows

\[
\rho_{\text{eff}} \omega^2 - Q_{\text{eff}11} q^2 = 0
\]

(4.6)

We use 1D dispersion relation in Eqn. 4.6 and obtain the relation between wave number and frequency: \( q = \omega \sqrt{\rho_{\text{eff}} / Q_{\text{eff}11}} \). Substituting \( Q_{\text{eff}11} \) and \( \rho_{\text{eff}} \) into the dispersion relation, dispersion curves are obtained.

Fig. 4.1.: Test for effective elastic moduli \( Q_{\text{eff}11} \) and \( Q_{\text{eff}21} \).
Fig. 4.2.: Test for effective mass $m_{eff}$. 
4.1 DN Model and Submodels

The double negativity metamaterial in Chapter 2 is composed of NYM submodel (Fig. 2.1(b) and Table 2.1) and NMD submodel (Fig. 2.2(b) and Table 2.2). The $m_{eff}$ frequency spectrum of the NMD submodel is shown in Fig. 4.3, while the $Q_{eff11}$ and $Q_{eff21}$ frequency spectrum of the NYM submodel are shown in Figs. 4.4 and 4.5, respectively.

The estimation of dispersion curves for the NMD submodel can be obtained by substituting $m_{eff}$ and $Q_{eff11}$ into wave equation. Fig. 4.6 shows the estimation of the effective continuum model and actual dispersion curves. Good agreement indicates the accuracy of the two approaches.

![Graph showing effective mass $m_{eff}$ for the NMD submodel.](image)

Fig. 4.3.: Effective mass $m_{eff}$ for the NMD submodel.

Fig. 4.7 shows dispersion curves for the NYM submodel. It is noted that the effective continuum model only works for very long wavelength. In addition, since the test can only excite symmetrical modes, the narrow passing band caused by rotational mode (anti-symmetrical mode), 579.28 – 579.37 Hz, cannot be estimated.
Fig. 4.4.: Effective modulus $Q_{\text{eff}11}$ for the NYM submodel.

Fig. 4.5.: Effective modulus $Q_{\text{eff}21}$ for the NYM submodel.
Fig. 4.6.: Comparison of dispersion curves for the NMD submodel.

Fig. 4.7.: Comparison of dispersion curves for the NYM submodel.
From $m_{\text{eff}}$ and $Q_{\text{eff11}}$ of the NMD and NYM submodels, we obtained the dispersion curves for the DN metamaterial (introduced in Chapter 2). As shown in Fig. 4.8, the estimation of dispersion curves from effective continuum model works well for long wavelengths.

Fig. 4.8.: Comparison of dispersion curves for the DN metamaterial.
4.2 Type 2 SN Model

Figs. 4.9-4.10 show $Q_{eff11}$ and $Q_{eff21}$ frequency spectrums. Since the type 2 SN metamaterial is a NYM model, $m_{eff}$ is frequency independent. We compared the dispersion curves in Fig. 4.11. It is evident that the effective continuum model works in long wavelengths.

Fig. 4.9.: Effective modulus $Q_{eff11}$ for the type 2 SN model.
Fig. 4.10.: Effective modulus $Q_{eff21}$ for the type 2 SN model.

Fig. 4.11.: Comparison of dispersion curves for the type 2 SN model.
4.3 Type 3 SN Model

Figs. 4.12-4.14 show the effective mass and elastic modulus frequency spectrums. Fig. 4.15 shows the comparison of dispersion curves. The estimations from effective continuum model match the exact dispersion curves in long wavelengths.

Fig. 4.12.: Effective mass $m_{\text{eff}}$ for the type 3 SN model.
Fig. 4.13.: Effective modulus $Q_{eff11}$ for the type 3 SN model.

Fig. 4.14.: Effective modulus $Q_{eff21}$ for the type 3 SN model.
Fig. 4.15.: Comparison of dispersion curves for the type 3 SN model.
5. APPLICATIONS OF DOUBLE NEGATIVITY METAMATERIALS

It is difficult to design an isotropic metamaterial, especially for high frequencies. The best approach for isotropic metamaterial designs are based on hexagonal lattice [94,95]. Although the most common acoustic/elastic metamaterial designs in the literature belong to orthotropic materials, for the convenience of theoretical derivation, we adopted equivalent isotropic material properties of metamaterials to investigate their potential applications in this chapter.

5.1 Refraction and Reflection in Two Elastic Media

For any three-dimensional waves, we can always find a plane to decouple them into SH wave, P wave, and SV wave. Therefore, in this section, we decouple them into three cases: incident SH wave, incident P wave, and incident SV wave to analyze wave propagation in elastic metamaterials.

5.1.1 Incident SH wave

Consider the case of incident SH wave, as shown in Fig. 5.1. An incident wave generates one reflected SH wave and one transmitted SH wave. The angles $\theta_0, \theta_2, \theta_4$ are in the range of $[-\pi/2, \pi/2]$ to include left-handed phenomena. Using harmonic wave assumption, the out-of-plane displacement $u_3$ and shear stress $\tau_{23}$ in each medium can be written as follows [96]:

\[
\begin{align*}
    u_{3}^{(n)} &= A_n e^{i q_n (x_j p_j^{(n)} - C_T t)} \\
    \tau_{23}^{(n)} &= i \mu q_n p_j^{(n)} A_n e^{i q_n (x_j p_j^{(n)} - C_T t)}
\end{align*}
\]  

(5.1)
Fig. 5.1.: Incident SH wave.
\[
\begin{align*}
\left\{ \begin{array}{l}
  u_3^{(n)} = A_n e^{iq_n(x_j p_j - C_T^{(n)} t)} \\
  \tau_{23}^{(n)} = i \mu^B q_n p_j^{(n)} A_n e^{iq_n(x_j p_j - C_T^{(n)} t)}
\end{array} \right. 
\tag{5.2}
\end{align*}
\]

where \( n = 0, 2, 4 \). \( A_n \) denotes the magnitude of each wave, \( p_j \) indicates the direction of wave propagation, and \( q_n \) represents the wave number. \( C_T \) denotes the phase velocity of SH wave in medium 1, while \( C_T^B \) denotes the phase velocity of SH wave in medium 2. Also, \( \mu \) is the shear modulus in medium 1, while \( \mu^B \) is the shear modulus in medium 2.

The displacement \( u_3 \) and the stress \( \tau_{23} \) satisfy the continuity condition at the interface \( x_2 = 0 \), which results in the following relations.

\[
\begin{align*}
  q_0 \sin \theta_0 &= q_2 \sin \theta_2 = q_4 \sin \theta_4 \\
  q_0 C_T &= q_2 C_T = q_4 C_T^B \\
  A_0 + A_2 &= A_4 \\
  \mu q_0 \cos \theta_0 A_0 - \mu q_2 \cos \theta_2 A_2 &= \mu^B q_4 \cos \theta_4 A_4
\end{align*}
\tag{5.3–5.6}
\]

Substituting Eqns. 5.1–5.2 into Eqns. 5.3–5.6, we obtain the out-of-plane displacement and shear stress for every position and time of each medium. The complete solutions for transmitted waves are expressed by

\[
\begin{align*}
  u_3^{(4)} &= A_4 e^{iq_4(x_1 \sin \theta_4 + x_2 \cos \theta_4 - C_T^{(4)} t)} \\
  \tau_{23}^{(4)} &= i \mu^B q_4 \cos \theta_4 A_4 e^{iq_4(x_1 \sin \theta_4 + x_2 \cos \theta_4 - C_T^{(4)} t)}
\end{align*}
\tag{5.7–5.8}
\]

### 5.1.2 Incident P or SV wave

For the case of incident P wave or SV wave, as shown in Fig. 5.2, two reflected waves (P and SV waves) and two transmitted waves (P and SV waves) are generated. The angles \( \theta_0, \theta_1, \theta_2, \theta_3, \theta_4 \) are in the range of \([-\pi/2, \pi/2]\). Assume the incident wave is a harmonic wave, the transmitted and reflected waves are then to be harmonic waves as follows [96].

\[
\tilde{u}^{(n)} = A_n \tilde{d}^{(n)} e^{i\eta_n}
\tag{5.9}
\]
Fig. 5.2.: Incident P wave or SV wave.
waves satisfy the displacement and stress continuity conditions in Eqn. 5.10. For the case of incident P wave, by substituting Eqn. 5.9 into Eqn. 5.10, Snell’s law (Eqn. 5.11) and wave magnitude relations (Eqn. 5.12) are expressed by

\[
\begin{align*}
q_0 \sin \theta_0 &= q_1 \sin \theta_1 = q_2 \sin \theta_2 = q_3 \sin \theta_3 = q_4 \sin \theta_4 \\
q_0 C_L &= q_1 C_L = q_2 C_T = q_3 C_L^B = q_4 C_T^B
\end{align*}
\]  

(5.11)

\[
A_0 \begin{bmatrix}
\sin \theta_0 \\
\cos \theta_0 \\
\sin 2\theta_0 \\
\frac{C_L^2}{C_T^2} \cos 2\theta_0
\end{bmatrix} = \begin{bmatrix}
- \sin \theta_1 & - \cos \theta_2 & \sin \theta_3 & - \cos \theta_4 \\
\cos \theta_1 & - \sin \theta_2 & \cos \theta_3 & \sin \theta_4 \\
\sin 2\theta_1 & C_L \cos 2\theta_2 & \frac{\mu B C_L}{\mu C_L^2} & \cos 2\theta_3 & - \frac{\mu B C_L}{\mu C_L^2} \cos 2\theta_4 \\
- \frac{C_L^2}{C_T^2} \cos 2\theta_2 & C_L \sin 2\theta_2 & \frac{\mu B C_L C_T^2}{\mu C_L^2 C_T^2} & \cos 2\theta_4 & \frac{\mu B C_L}{\mu C_L^2} \sin 2\theta_4
\end{bmatrix}
\]  

(5.12)

The final transmitted displacement solution is obtained by substituting Eqns. 5.11 – 5.12 into Eqn. 5.13.

\[
\begin{align*}
u_1^{(3)} &= A_3 \sin \theta_3 e^{i q_3 (x_1 \sin \theta_3 + x_2 \cos \theta_3 - C_L^B t)} \\
u_1^{(4)} &= -A_4 \cos \theta_4 e^{i q_4 (x_1 \sin \theta_4 + x_2 \cos \theta_4 - C_T^B t)} \\
u_2^{(3)} &= A_3 \cos \theta_3 e^{i q_3 (x_1 \sin \theta_3 + x_2 \cos \theta_3 - C_L^B t)} \\
u_2^{(4)} &= A_4 \sin \theta_4 e^{i q_4 (x_1 \sin \theta_4 + x_2 \cos \theta_4 - C_T^B t)}
\end{align*}
\]  

(5.13)
For the case of incident SV wave, Snell’s law and wave magnitude relations are presented as

\[
q_0 \sin \theta_0 = q_1 \sin \theta_1 = q_2 \sin \theta_2 = q_3 \sin \theta_3 = q_4 \sin \theta_4 \tag{5.14}
\]

\[
q_0 C_T = q_1 C_L = q_2 C_T = q_3 C_L^B = q_4 C_T^B
\]

\[
A_0 \begin{bmatrix}
\cos \theta_0 \\
\sin \theta_0 \\
\cos 2\theta_0 \\
\sin 2\theta_0
\end{bmatrix}
= \begin{bmatrix}
\sin \theta_1 & \cos \theta_2 & -\sin \theta_3 & \cos \theta_4 \\
\cos \theta_1 & -\sin \theta_2 & \cos \theta_3 & \sin \theta_4 \\
-\frac{C_L}{C_T} \sin 2\theta_1 & -\cos 2\theta_2 & -\frac{\mu B C_T}{\mu C_L} \sin 2\theta_3 & \frac{\mu B C_T}{\mu C_L} \cos 2\theta_4 \\
-\frac{C_T(\lambda + 2\mu \cos \theta_1^2)}{C_L \mu} & \sin 2\theta_2 & \frac{\mu B C_T}{\mu C_L} \sin 2\theta_3 & \frac{C_T \mu B}{C_L \mu} \cos 2\theta_4
\end{bmatrix} \begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
A_4
\end{bmatrix}
\tag{5.15}
\]

The final transmitted displacement can be obtained by solving Eqns. 5.14 – 5.15 and substituting into the following equations.

\[
\begin{align*}
u_1^{(3)} &= A_3 \sin \theta_3 e^{iq_3(x_1 \sin \theta_3 + x_2 \cos \theta_3 - C_L^B t)} \\
u_1^{(4)} &= -A_4 \cos \theta_4 e^{iq_4(x_1 \sin \theta_4 + x_2 \cos \theta_4 - C_T^B t)} \\
u_2^{(3)} &= A_3 \cos \theta_3 e^{iq_3(x_1 \sin \theta_3 + x_2 \cos \theta_3 - C_L^B t)} \\
u_2^{(4)} &= A_4 \sin \theta_4 e^{iq_4(x_1 \sin \theta_4 + x_2 \cos \theta_4 - C_T^B t)}
\end{align*}
\tag{5.16}
\]

### 5.2 Anomalous Transmitted Waves

To thoroughly investigate anomalous refraction in metamaterials, four sets of material combinations are discussed in this section. The material properties listed in Table 5.1 – Table 5.4 are analyzed with incident anti-plane and in-plane waves.

Table 5.1.: Material properties for the DP medium to DP medium.

<table>
<thead>
<tr>
<th>Medium</th>
<th>(E_{\text{eff}})</th>
<th>(\nu_{\text{eff}})</th>
<th>(\rho_{\text{eff}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Medium 1</td>
<td>3.5 GPa</td>
<td>0.3</td>
<td>1 kg/m(^3)</td>
</tr>
<tr>
<td>Medium 2</td>
<td>6.9 GPa</td>
<td>0.3</td>
<td>1 kg/m(^3)</td>
</tr>
</tbody>
</table>
Table 5.2.: Material properties for the DN medium to DN medium.

<table>
<thead>
<tr>
<th></th>
<th>$E_{\text{eff}}$</th>
<th>$\nu_{\text{eff}}$</th>
<th>$\rho_{\text{eff}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Medium 1</td>
<td>$-3.5 \text{ GPa}$</td>
<td>0.3</td>
<td>$-1 \text{ kg/m}^3$</td>
</tr>
<tr>
<td>Medium 2</td>
<td>$-6.9 \text{ GPa}$</td>
<td>0.3</td>
<td>$-1 \text{ kg/m}^3$</td>
</tr>
</tbody>
</table>

Table 5.3.: Material properties for the DP medium to DN medium.

<table>
<thead>
<tr>
<th></th>
<th>$E_{\text{eff}}$</th>
<th>$\nu_{\text{eff}}$</th>
<th>$\rho_{\text{eff}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Medium 1</td>
<td>$3.5 \text{ GPa}$</td>
<td>0.3</td>
<td>$1 \text{ kg/m}^3$</td>
</tr>
<tr>
<td>Medium 2</td>
<td>$-6.9 \text{ GPa}$</td>
<td>0.3</td>
<td>$-1 \text{ kg/m}^3$</td>
</tr>
</tbody>
</table>

Table 5.4.: Material properties for the DN medium to DP medium.

<table>
<thead>
<tr>
<th></th>
<th>$E_{\text{eff}}$</th>
<th>$\nu_{\text{eff}}$</th>
<th>$\rho_{\text{eff}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Medium 1</td>
<td>$-3.5 \text{ GPa}$</td>
<td>0.3</td>
<td>$-1 \text{ kg/m}^3$</td>
</tr>
<tr>
<td>Medium 2</td>
<td>$6.9 \text{ GPa}$</td>
<td>0.3</td>
<td>$1 \text{ kg/m}^3$</td>
</tr>
</tbody>
</table>
5.2.1 Results for incident SH wave

We adopt the same settings shown in Fig. 5.1 to perform numerical simulation. Therefore, \( x_2 = 0 \) m in Fig. 5.3 stands for the interface between two elastic media. An incident SH wave applied as a plane wave in medium 1 generates a reflected SH wave. A transmitted SH wave can be observed in medium 2. Fig. 5.3 shows the snapshot at \( t = 2 \times 10^{-5} \) sec for an incident SH wave with frequency 10000 Hz in four sets of elastic media. The angle of incidence \( \theta_0 = 28^\circ \) is smaller than the critical angle \( \theta_{cSH} = 45^\circ \). The result of DP medium to DP medium has the same reflection and transmission coefficients (\( A_2 = -0.0892m, A_4 = 0.9108m \)) and the same reflected and transmitted angle (\( \theta_2 = 28^\circ, \theta_4 = 41^\circ \)) because two flipped signs make no difference in Snell’s law. The only discrepancy of the these two cases is the wave phase. Since double negativity metamaterial has negative phase velocity (as demonstrated in Sec. 2.3), wave propagation in DN medium is in the opposite direction of that in DP medium. Therefore, phase difference is observed in the comparison of Fig. 5.3(a) and Fig. 5.3(b). For the cases of DP medium to DN medium and DN medium to DP medium, the reflected angle, reflection and transmission coefficients (\( \theta_2 = 28^\circ, A_2 = -0.0892m, A_4 = 0.9108m \)) remain the same as previous cases. However, the transmitted angle switches its sign to negative (\( \theta_4 = -41^\circ \)). This is so-called ”negative refraction”. Different from positive refraction, the transmitted wave appears on the same side as the incident wave of the interface normal in negative refraction, as shown in Fig. 5.3(c) and Fig. 5.3(d). Note that the phase difference between Fig. 5.3(c) and Fig. 5.3(d) is caused by negative phase velocity of the DN media. The application of negative refraction includes wave focusing [97] and open cavity [98].
Fig. 5.3.: Out-of-plane displacement $u_3$ at $t = 2 \times 10^{-5}$ sec for incident SH wave (a) from the DP medium to DP medium; (b) from the DN medium to DN medium; (c) from the DP medium to DN medium; (d) from the DN medium to DP medium.
In order to validate the theoretical analysis, we performed finite element simulations using COMSOL Multiphysics. The modeling of negative refractive index in electromagnetic waves can be found in the COMSOL Multiphysics Model Library Manual. However, we found an issue that comes along with simulations in the Solid Mechanics Module.

For modeling SH wave, a three-dimensional rectangular solid with large height is built. A SH plane wave with displacement in $x_3$ direction is applied at the vertical plane $x_2 = -10 \, m$. In order to model plane wave propagation in half space, two sets of Floquet periodicity are applied on the vertical planes $x_1 = -10 \, m$ and $x_2 = 10 \, m$. An absorbing material such as a PML layer or a low-reflection boundary is used to model wave propagation in half space beyond $x_2 = 10 \, m$. It is noted that since there is no corresponding transition boundary condition (in the Wave Optics Module) in the Solid Mechanics Module, the flux direction at the interface of DN material cannot be modeled well. At this stage, we switched the sign of incident angle to correct its direction ($\theta_0 = -28^\circ$ in Fig. 5.4(b); $\theta_0 = 28^\circ$ in Fig. 5.5(b); $\theta_0 = 28^\circ$ in Fig. 5.6(b); $\theta_0 = -28^\circ$ in Fig. 5.7(b)). The simulation issue at the interface may also be responsible for the numerical error in FEA.

Figs. 5.4 – 5.7 show the comparisons of analytical solutions and FEA simulation. Although the simulations are not very accurate, the trends are the same. The causes of numerical error in COMSOL Multiphysics can be attributed to two factors. First, we used three-dimensional solids to simulate SH waves. The finite height of the solid may disturb the result of out-of-plane displacement $u_3$. In addition, no function to adjust the flux of elastic waves at the interface in COMSOL Multiphysics may also influence the accuracy of the simulation.
(a) Theoretical result at $t = 2 \times 10^{-5}$ sec

(b) COMSOL simulation result

Fig. 5.4.: Out-of-plane displacement $u_3$ for incident SH wave from the DP medium to DP medium.

(a) Theoretical result at $t = 1.05 \times 10^{-4}$ sec

(b) COMSOL simulation result

Fig. 5.5.: Out-of-plane displacement $u_3$ for incident SH wave from the DN medium to DN medium.
(a) Theoretical result at $t = 1 \times 10^{-4}$ sec 
(b) COMSOL simulation result

Fig. 5.6.: Out-of-plane displacement $u_3$ for incident SH wave from the DP medium to DN medium.

(a) Theoretical result at $t = 8 \times 10^{-5}$ sec 
(b) COMSOL simulation result

Fig. 5.7.: Out-of-plane displacement $u_3$ for incident SH wave from the DN medium to DP medium.
5.2.2 Results for incident P or SV wave

We followed the settings shown in Fig. 5.2 for numerical simulations. An incident in-plane wave is more complicated than an anti-plane wave since it generates two reflected and two transmitted waves. However, the physical principle remains the same. Figs. 5.8 – 5.9 show a snapshot at $t = 2 \times 10^{-5} \text{sec}$ in-plane displacements for incident P wave with frequency $10000 \text{Hz}$ in different combinations of elastic media. The angle of incidence $\theta_0 = 28^\circ$ is smaller than the critical angles $\theta_{cP} = 45^\circ$ and $\theta_{cSV} = 90^\circ$. Four sets of elastic media share the same reflected angles ($\theta_1 = 28^\circ, \theta_2 = 15^\circ$). Negative transmitted angles in the case of DP medium to DN medium and DN medium to DP medium ($\theta_3 = -41^\circ, \theta_4 = -21^\circ$) indicate negative refraction. The phase difference between the case of DP medium to DP medium and the case of DN medium to DN medium (also the phase difference between the case of DP medium to DN medium and DN medium to DP medium) results from negative phase velocity.

Figs. 5.10–5.11 show a snapshot at $t = 2 \times 10^{-5} \text{sec}$ in-plane displacements for incident SV wave with frequency $10000 \text{Hz}$ in four sets of elastic media. The angle of incidence $\theta_0 = 20^\circ$ is smaller than the critical angles $\theta_{cP} = 22^\circ$ and $\theta_{cSV} = 45^\circ$. Similar to the case of incident P wave, the reflected angles from an incident SV wave are the same in four material combinations ($\theta_1 = 40^\circ, \theta_2 = 20^\circ$). Negative refraction is observed from the transmitted angles in DP to DN and DN to DP ($\theta_3 = -64^\circ, \theta_4 = -29^\circ$). In addition, negative phase velocity causes phase difference between the case of DP medium to DP medium and the case of DN medium to DN medium (also the phase difference between DP to DN and DN to DP).
Fig. 5.8.: In-plane displacement $u_1$ at $t = 2 \times 10^{-5}$ sec for incident P wave (a) from the DP medium to DP medium; (b) from the DN medium to DN medium; (c) from the DP medium to DN medium; (d) from the DN medium to DP medium.
Fig. 5.9.: In-plane displacement $u_2$ at $t = 2 \times 10^{-5} \text{ sec}$ for incident P wave (a) from the DP medium to DP medium; (b) from the DN medium to DN medium; (c) from the DP medium to DN medium; (d) from the DN medium to DP medium.
Fig. 5.10.: In-plane displacement $u_1$ at $t = 2 \times 10^{-5} \text{ sec}$ for incident SV wave (a) from the DP medium to DP medium; (b) from the DN medium to DN medium; (c) from the DP medium to DN medium; (d) from the DN medium to DP medium.
Fig. 5.11.: In-plane displacement $u_2$ at $t = 2 \times 10^{-5}$ sec for incident SV wave (a) from the DP medium to DP medium; (b) from the DN medium to DN medium; (c) from the DP medium to DN medium; (d) from the DN medium to DP medium.
5.2.3 Interface mode conversion

Mode conversion in one elastic medium is allowed in double positivity material [96]. However, for two elastic media, interface mode conversion can only be achieved by negative refraction [55]. The interface mode conversions can be classified into the following cases.

Case 1: Incident P wave from DP medium to DN medium.
Case 2: Incident P wave from DN medium to DP medium.
Case 3: Incident SV wave from DP medium to DN medium.
Case 4: Incident SV wave from DN medium to DP medium.

The proof of Case 3 and Case 4 can be found in [55]. Therefore, we focus on the investigation of incident P wave in Case 1 here. Note that Case 2 can be regarded as a time reversal case of the Case 1.

As shown in Fig. 5.12, displacement continuity in $u_1$ and $u_2$ requires

\[
A_0 \sin \theta_0 = -A_4 \cos \theta_4 \\
A_0 \cos \theta_0 = A_4 \sin \theta_4
\] (5.17)

From Eqn. 5.17, we obtain $A_0 = -A_4$, $\theta_4 = -90^\circ + \theta_0$ for two of the conditions for interface mode conversion. $\theta_4$ is negative indicating that the mode conversion can only occur in double negativity metamaterials.

At the interface, waves also satisfy the stress continuity, which are expressed by

\[
A_0 \sin 2\theta_0 = -\frac{\mu^B C_L}{\mu^C_T} A_4 \cos 2\theta_4 \\
\frac{\lambda + 2\mu \cos^2 \theta_0}{\mu} A_0 = \frac{\mu^B C_L}{\mu^C_T} \sin 2\theta_4 A_4
\] (5.18)

Hence, the requirements for interface mode conversion are presented as

\[
\frac{\mu^B C_L}{\mu^C_T} = -\tan 2\theta_0 \\
\lambda + 2\mu \cos^2 \theta_0 = \frac{\mu^B C_L}{C_T^B} \sin 2\theta_0
\] (5.19)
Fig. 5.12.: Investigation of interface mode conversion with an incident P wave.
Note that in interface mode conversion, the amplitudes of incident and transmitted waves are of the same value. Moreover, the difference between the incident and transmitted angle is 90°.

Zhu et al. [99] numerically demonstrated a case of interface mode conversion. Similar to their findings, we used the material settings listed in Table 5.5 to investigate interface mode conversion.

Table 5.5: Material properties for interface mode conversion.

<table>
<thead>
<tr>
<th></th>
<th>$E_{\text{eff}}$</th>
<th>$\nu_{\text{eff}}$</th>
<th>$\rho_{\text{eff}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Medium 1</td>
<td>4.6 $GPa$</td>
<td>0.3</td>
<td>1 $kg/m^3$</td>
</tr>
<tr>
<td>Medium 2</td>
<td>−6.9 $GPa$</td>
<td>0.3</td>
<td>−1 $kg/m^3$</td>
</tr>
</tbody>
</table>

As shown in Fig. 5.13, the amplitude of reflected waves and transmitted P waves drop to zero at the angle of incidence $\theta_0 = 56.789°$, which is exactly the angle of interface mode conversion. Fig. 5.14 shows the reflected and transmitted angles versus angle of incidence. At the angle of incidence $\theta_0 = 56.789°$, the difference between the angle of incidence and transmitted angle is 90°.
Fig. 5.13.: Amplitude of reflected and transmitted waves.

Fig. 5.14.: Reflected and transmitted angles.
6. SLOW GROUP VELOCITY

Metamaterials have a tunable feature that can be used to produce bandgaps and double negativity. In addition, wave speed can also be manipulated. In this chapter, we investigate a Gaussian pulse with slow group velocity using several metamaterial lattice models. Numerical simulations are demonstrated and compared with theoretical analyses in 1D cases. The theoretical formulation can be used to predict wave distortion due to differences in phase velocities.

6.1 Theoretical Analysis for Wave Distortion

Group velocity represents the speed of modulation and indicates the direction of energy flow. It can be obtained by calculating tangent slope of dispersion curves. To estimate 1D wave distortion, we adopted the theoretical formulation given by Brillouin [100]. Assume the group velocity $C_g$ is a constant $a$, as shown in the dispersion curve, see Fig. 6.1. The phase velocity is expressed by

$$C_p = a + \frac{2\pi b}{q}$$

(6.1)

where $b$ is also a constant.

The wave number $q$ is positive if waves propagate in double positivity materials, while $q$ is negative if waves propagate in double negativity metamaterials. Now we restricted our case in double positivity materials for derivation. $\omega_0 - \omega \geq 2\pi b$ since the dispersion curve in Fig. 6.1 does not transmit frequencies below $2\pi b$.

If a frequency range of the passing band in Fig. 6.1 is chosen for a pulse and input to the edge of material I, then we have the wave packet with a carrier frequency $\omega_0$ that corresponds to $q_0$, and frequencies $\omega_0 + \omega$ correspond to $q_0 + \omega/C_g$ traveling in material I.
Fig. 6.1.: Dispersion curves for the material I.
Now we assume a wave packet carried by a carrier-frequency $\omega_0$ and characterized by a modulation curve $C(t)$. The wave form at $x = 0$ is expressed by

$$C_1(t, 0) = C(t) \cos \omega t$$  \hspace{1cm} (6.2)

We analyze modulation $C(t)$ by Fourier integral, assuming that this modulation has a finite spectrum extending from 0 to $\omega_m$

$$C(t) = \int_{\omega=0}^{\omega=\omega_m} B_\omega \cos(\omega t + \phi_\omega) d\omega$$  \hspace{1cm} (6.3)

where $B_\omega$ is the amplitude, and $\phi_\omega$ is the phase of the $\omega$ component. Using Eqn. 6.3, the wave packet is represented by the Fourier integral

$$C_1(t, 0) = \left[ \int_{\omega=0}^{\omega=\omega_m} B_\omega \cos(\omega t + \phi_\omega) d\omega \right] \cos \omega_0 t$$

$$= \frac{1}{2} \int_{\omega=0}^{\omega=\omega_m} B_\omega \{ \cos[ (\omega_0 + \omega) t + \phi_\omega ] + \cos[ (\omega_0 - \omega) t - \phi_\omega ] \} d\omega$$  \hspace{1cm} (6.4)

The resulting spectrum now extends from $\omega_0 - \omega_m$ to $\omega_0 + \omega_m$ and thus covers a band $2\omega_m$. When the wave packet arrives at point $x$, simply by replacing in Eqn. 6.4, $\omega_0 t$ by $\omega_0 t - q_0 x$ and $(\omega_0 + \omega) t$ by $(\omega_0 + \omega) t - [q_0 + \omega/C_g].x$. Then

$$C_1(t, x) = \frac{1}{2} \int_{\omega=0}^{\omega=\omega_m} B_\omega [ \cos(\theta_0 + \theta) + \cos(\theta_0 - \theta) ] d\omega$$

$$= \cos \theta_0 \int_{\omega=0}^{\omega=\omega_m} B_\omega \cos \theta d\omega$$

$$= \cos \theta_0 C(t - \frac{x}{C_g})$$  \hspace{1cm} (6.5)

where $\theta_0 = \omega_0 t - q_0 x$; $\theta = \omega(t - x/C_g) + \phi_\omega$. It is noted that the last transformation in Eqn. 6.5 results from Eqn. 6.3. Finally, the wave packet reaching distance $x$ is given by

$$C_1(t, x) = C(t - \frac{x}{C_g}) \cos \omega_0 (t - \frac{x}{C_p})$$  \hspace{1cm} (6.6)

From the derivation in this section, we have proven the following two points:

1. Modulation $C(t - \frac{x}{C_g})$ propagates without distortion and yields the group velocity $C_g$. 

2. Carrier $\omega_0$ exhibits its own phase velocity $C_p$.

Therefore, using these two points, we have the theoretical solution for evaluation of wave distortion. For example, if we input a wave packet modulated by a Gaussian pulse

$$C_1(t, 0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} \cos \omega t$$  \hspace{1cm} (6.7)

Then, applying the above principles, the exact transient full-field solution of this wave packet gives

$$C_1(t, x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-x/C_p)^2}{2\sigma^2}} \cos \omega (t - \frac{x}{C_p})$$  \hspace{1cm} (6.8)

The modulation $e^{-(t^2/2\sigma^2)}/\sqrt{2\pi\sigma^2}$ only contains frequencies in a specific range. The carrier wave $\cos \omega t$ is composed of one single frequency $\omega$. We also note that although the analytical solution can be used to predict a wave packet that contains various phase velocities, the solution is no longer valid if the material is dispersive with group velocity at desired frequencies. It is because the derivation is with an assumption that $C_g$ is a constant.

6.2 Comparison of Theoretical Analysis and Numerical Simulation in 1D Metamaterial Lattices

Because of the tunable feature of metamaterials, we can easily manipulate the group velocity of a wave packet by adjusting the mass and stiffness of resonators. In this section, we use three lattice models to demonstrate slow group velocities. Analytical predictions of wave distortion are also provided and compared with numerical simulations.

We first investigate wave propagation in lattice model I, as shown in Fig. 6.2 [22]. $k_1 = 1000 \text{ N/m, } m_1 = 0.001 \text{ kg, } k_2 = 1 \text{ N/m, } m_2 = 1 \text{ kg, } L = 40 \text{ mm, } D = 40 \text{ mm}$ were used as parameter settings. The dispersion curves calculated by the theoretical analysis and finite element simulation are shown in Fig. 6.3.

A frequency range of 3.5 – 6.5 Hz was chosen as the Gaussian modulated wave packet, as shown in Fig. 6.4, and input at the first unit cell of lattice model I. From
the dispersion curves, calculation was done for group velocity $C_g = 37.911 \text{ m/s}$ at frequencies $3.5 \text{–} 6.5 \text{ Hz}$, and phase velocity $C_p = 37.92 \text{ m/s}$ at $5 \text{ Hz}$. Finite element analysis was performed using long chain metamaterials (2000 unit cells) to prevent reflected waves during the time of interest. Figs. 6.5 and 6.6 show a comparison of the theoretical prediction and the finite element simulation at the first and 400th unit cell, respectively. The good agreement in results indicates the accuracy of the theoretical analysis.

Fig. 6.2.: Lattice model I.
Fig. 6.3.: Dispersion curves for the lattice model I.

Fig. 6.4.: Gaussian modulated pulse for the lattice model I.
Fig. 6.5.: Comparison of theoretical analysis and numerical simulation at the first unit cell of lattice model I.

Fig. 6.6.: Comparison of theoretical analysis and numerical simulation at the 400th unit cell of lattice model I.
Secondly, lattice model II [80] was chosen, as shown in Fig. 6.7, to investigate slow group velocity and wave distortion. We used $k_1 = 600 \, N/m$, $m_1 = 1 \, kg$, $k_2 = 3 \, N/m$, $m_2 = 0.001 \, kg$ as parameter settings. The corresponding dispersion curves are shown in Fig. 6.8. A frequency range of $0 – 4$ Hz was chosen for Gaussian pulse modulation, as shown in Fig. 6.9. The Gaussian modulated wave packet was the input at the first unit cell of lattice model II, and the transient responses at the first and 50th unit cell were calculated, as shown in Figs. 6.10 and 6.11. Since the group velocity is very slow, it takes two seconds for the wave packet to travel $1.96 \, m$. The theoretical formulation is able to accurately predict wave distortion.

Now, we turn to a case where the theoretical analysis may fail. Because group velocity is assumed to be a constant in derivation (no restriction on phase velocity), if the components of the wave packet have different group velocities, then the theoretical formulation is no longer accurate. Therefore, we chose a wave packet with a frequency range of $3.5 – 7.5$ Hz, as shown in Fig. 6.12 and input at the first unit cell of lattice model II. As expected, the theoretical analysis matches the numerical simulation at the first unit cell, as shown in Fig. 6.13. However, after the wave packet travels to the 50th unit cell, as shown in Fig. 6.14, the theoretical analysis no longer accurately describe the distortion. This arises from the wave packet containing a frequency range with varying group velocities.

Finally, we used lattice model III [59], as shown in Fig. 6.15, to demonstrate special group velocities of Gaussian modulated wave packets. Because lattice model III has two sets of resonators, by adjusting their parameters, the model can exhibit
Fig. 6.8.: Dispersion curves for the lattice model II.

Fig. 6.9.: Gaussian modulated wave packet I for the lattice model II.
Fig. 6.10.: Comparison of theoretical analysis and numerical simulation at the first unit cell of lattice model II.

Fig. 6.11.: Comparison of theoretical analysis and numerical simulation at the 50th unit cell of lattice model II.
Fig. 6.12.: Gaussian modulated wave packet II for lattice model II.

Fig. 6.13.: Comparison of theoretical analysis and numerical simulation at the first unit cell.
Fig. 6.14.: Comparison of theoretical analysis and numerical simulation at the 50th unit cell.
double positivity, single negativity, and double negativity material properties. In the following, two sets of parameter settings were utilized to investigate wave propagation in double positivity and double negativity metamaterials.

For the first set of parameter settings, we used \( k_1 = 1 \text{ N/m}, \ k_2 = 1 \text{ N/m}, \ k_3 = 0.1 \text{ N/m}, \ m_1 = 10^{-4} \text{ kg}, \ m_2 = 10^{-4} \text{ kg}, \ m_3 = 1.5 \times 10^{-4} \text{ kg}, \ L = 40 \text{ mm}, \ D = 24 \text{ mm}, \) and the corresponding dispersion curves are shown in Fig. 6.16. A frequency range of 20 – 23 Hz was chosen for the Gaussian modulated wave packet, as shown in Fig. 6.17. In this frequency range, group velocity \( C_g = 3.8 \text{ m/s} \) is almost a constant, although phase velocities do not remain at the same value. A theoretical solution was applied to predict the wave packet after traveling a distance. Figs. 6.18 and 6.19 show good agreement between theoretical analysis and numerical simulation, which implied the accuracy of the theoretical estimation. In addition, since it takes 1.04 sec for the wave packet to travel 3.96 m, the group velocity is relatively small.

Fig. 6.15.: Lattice model III.

Group velocity is no longer a constant in a frequency range of 16 – 19 Hz for the Gaussian modulated wave packet; therefore, analytical solution can not be applied. Fig. 6.20 shows the wave packet at the first and 100th unit cells. We observe that the wave distorts more drastically. In addition, it takes 1.7 sec for waves to propagate 3.96 m. Thus, the group velocity is very small.
Fig. 6.16.: Dispersion curves for the lattice model III under the first set of parameter settings.
Gaussian Pulse \( \sigma = 0.3 \text{ s} \)

Fig. 6.17.: Gaussian modulated pulse for the lattice model III under the first set of parameter settings.
Fig. 6.18.: Comparison of theoretical analysis and numerical simulation at the first unit cell of the lattice model III under the first set of parameter settings.

Fig. 6.19.: Comparison of theoretical analysis and numerical simulation at the 100th unit cell of the lattice model III under the first set of parameter settings.
Fig. 6.20.: Transient response of the 1st and 100th unit cell for the first set of parameter settings in lattice model III.
To investigate wave propagation in double negativity metamaterials, we used $k_1 = 1 \text{ N/m}$, $k_2 = 0.25 \text{ N/m}$, $k_3 = 0.3 \text{ N/m}$, $m_1 = 10^{-4} \text{ kg}$, $m_2 = 0.4 \times 10^{-4} \text{ kg}$, $m_3 = 2.25 \times 10^{-4} \text{ kg}$, $L = 40 \text{ mm}$, $D = 8 \text{ mm}$ for the second set of parameter settings. The corresponding dispersion curves are shown in Fig. 6.21. A frequency range of 8.46 – 9.54 Hz was chosen for the Gaussian modulated wave packet. Positive group velocity and negative phase velocity are observed from Figs. 6.22–6.23. Fig. 6.23 also demonstrates slow group velocity, which makes the wave packet take 3.3 sec to travel 3.96 m.

![Graph showing dispersion curves](image)

Fig. 6.21.: Dispersion curves for the lattice model III under the second set of parameter settings.
Fig. 6.22.: Transient response of the 1st and 2nd unit cell for the second set of parameter settings in lattice model III.

Fig. 6.23.: Transient response of the 1st and 100th unit cell for the second set of parameter settings in lattice model III.
7. SUMMARY AND RECOMMENDATIONS

A DN and three SN metamaterials were designed in this study. We demonstrated negative phase velocity of the DN model by dispersion curves and transient simulations. For the three types of SN metamaterials, parametric studies were made to obtain the metamaterial with the broadest bandgap in a low-frequency range. We found that the frame bending/stretching mode is more effective than beam bending mode when designing a broadband SN metamaterial. Moreover, by adding internal mass components, the band-gap region can also be enlarged. The broadband designs can solve the issue of narrow working frequencies for elastic metamaterials.

The elastic metamaterial designs were fabricated using a 3D printer by converting distinct material properties into different thicknesses in the individual parts of the metamaterial. The equivalent material properties were also investigated using a numerical effective medium theory. Combining these results with wave equation, dispersion curves were obtained. The numerical effective medium theory is appropriate to use for long wavelengths.

For applications of metamaterials, we analyzed wave propagation in elastic media by different material combinations. Negative refraction was demonstrated theoretically and numerically. Moreover, we found that the reflected and transmitted angles from the DP medium to DP medium are the same as that from the DN medium to DN medium because the interaction of two media at the interface is governed by Snell’s law. Interface mode conversion was also investigated in this study. A double negativity metamaterial is the only material that makes interface mode conversion possible.

The tunable feature of metamaterials provides flexibility in wave speed selection. We investigated slow group velocity using Gaussian modulated pulses and lattice models. The slow group velocity can be used to delay signals.
The research of metamaterials originates from electromagnetism, then acoustics and solid mechanics. Although the coupling of shear and longitudinal waves in solid mechanics generates more interesting and complicated physics than the other fields, wave propagations share similar features. Therefore, many research which have developed in electromagnetism and acoustics but have not yet been investigated in solid mechanics are identified as future work. For example, open cavity [69,98], which applies negative refraction in double negativity metamaterials, can be used to constrain the region of wave propagation. In general, for wave guide in periodic structures, defects are required to restrain waves propagate in the desired direction. However, employing negative refraction, which has investigated in this study, we can design the track of wave propagation and constrain it in certain area. Open cavity provides empty space for utilization since it does not need a closed region for waveguide. In addition, reverse Doppler effect is also a research topic which has not yet been explored in solid mechanics. When the wave source moves, wavelength differs depending on the moving speed and the location of receiver. Applying the negative phase velocity simulated in Chapter 2 and Chapter 6, reverse Doppler effect is expected since wave propagates in the opposite direction in DN metamaterials.
REFERENCES


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Yu-Chi Su was born in Taipei, Taiwan. In 2007, she obtained her B.S. degree in Civil Engineering, from National Cheng Kung University. Subsequently, she earned her M.S. degree in Mechanical Engineering from National Taiwan University (NTU) in 2009. After graduation, she continued as a full-time research assistant at NTU for two years, funded by the National Science Council of Taiwan. In 2011, she joined the PhD program in Aeronautics and Astronautics at Purdue University to work with Prof. Chin-Teh Sun in the design of elastic metamaterials and its applications, funded by the Air Force Office of Scientific Research.