Finite dimensional approximations and deformations of group C*-algebras

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By Andrew James Schneider

Entitled
Finite Dimensional Approximations and Deformations of Group C*-Algebras

For the degree of Doctor of Philosophy

Is approved by the final examining committee:

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Approved by Major Professor(s): Marius Dadarlat

Approved by: David Goldberg 4/12/2016

Head of the Departmental Graduate Program Date
FINITE DIMENSIONAL APPROXIMATIONS AND DEFORMATIONS
OF GROUP C*-ALGEBRAS

A Dissertation
Submitted to the Faculty
of
Purdue University
by
Andrew J. Schneider

In Partial Fulfillment of the
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of
Doctor of Philosophy

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ABSTRACT

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Quasidiagonality is a finite-dimensional approximation property of a C*-algebra which indicates that it has matricial approximations that capture the structure of the C*-algebra. We investigate when C*-algebras associated to discrete groups have such a property with particular emphasis on finding obstructions. In particular, we point out that groups with Kazhdan’s Property (T) and only finitely many unitary equivalence classes of finite dimensional representations do not produce quasidiagonal C*-algebras. We then observe and note interactions with Property (T) and other approximation properties.

Property (QH) is a related but stronger approximation property with deep connections to E-Theory and KK-Theory. In the case of groups, Property (QH) represents the property that the group not only has structure capturing matricial models but also that these models may be deformed to the trivial representation. In this sense, Property (QH) may then be considered a type of finite-dimensional deformation property. In joint work with Marius Dadarlat and Ulrich Pennig, we show the class of groups with Property (QH) is closed under wreath products, thus producing a new class of highly non-trivial groups with Property (QH) far from those currently known.
1. Introduction

Representation theory is a fundamental field of research; in particular, the study of finite-dimensional representations of groups has yielded applications throughout mathematics and other disciplines. Although a group may lack non-trivial finite-dimensional representations, one can relax the meaning of representation to maps which may only be approximately multiplicative and partially defined, yet such finite-dimensional approximations may still capture surprising amounts of both algebraic and topological information.

For example, Voiculescu [1] produced a sequence of asymptotically commuting unitary matrices that cannot be perturbed to commuting unitary matrices. This construction produces a family of partially defined and asymptotically homomorphic representations of $\mathbb{Z} \times \mathbb{Z}$ into finite-dimensional unitary matrices, which, by Voiculescu’s result, is far from any genuine representation. Most striking is that the obstruction to perturbation is topological in nature. Indeed, the root of the obstruction is that the family induces a homomorphism on the K-Theory of the torus capable of distinguishing the Bott element, an impossible feat for ordinary finite-dimensional representations. In fact, under mild assumptions, the existence of sufficiently many finite-dimensional approximations allows one to interpolate K-theoretic information completely [2].

Varying the precise meaning of “finite-dimensional approximation” yields characterizations of deep concepts throughout mathematics. Hyperlinear groups are precisely those satisfying Connes’ Embedding Problem for group von Neumann algebras [3] and sofic groups have roots in surjunctivity with applications to cellular automata [4]. Both hyperlinear and sofic groups have a characterization using finite-dimensional approximations as we recall in Def. 3.3.5.
In this work we seek to investigate when “finite-dimensional approximations” of groups exist with particular interest in determining whether group $C^*$-algebras are quasidiagonal or have Property (QH). As we will see, both of these concepts may be characterized as finite-dimensional approximations and deformations. Loosely speaking, a $C^*$-algebra is quasidiagonal if it has matricial approximations that capture the structure of the $C^*$-algebra. It is quasidiagonality which permits the previously mentioned K-theoretic interpolation. On the other hand, in the context of groups, Property (QH) represents not only structure capturing matricial approximations but imposes the condition that these approximations may be deformed to the trivial representation. Property (QH) also yields topological information with its relationship to E-Theory and KK-Theory.

Ultimately in this dissertation we point out that Rosenberg’s original argument (3.2.2) can be adapted to show that no infinite Property (T) group with only finitely many unitary equivalence classes of finite dimensional representations produces a quasidiagonal full group $C^*$-algebra. Notably, no other examples of groups which do not produce quasidiagonal full group $C^*$-algebras are currently known. For simple groups this was originally observed by Thom, Ozawa, and Yamashita as we will discuss in that chapter.

We will also provide a new class of groups satisfying Property (QH) as we will show Property (QH) passes to wreath products. It was recently shown by Dadarlat and Pennig that torsion-free nilpotent groups satisfy Property (QH), and, since wreath products are rarely nilpotent, the result in this dissertation provides contrasting examples.
2. Group $C^*$-Algebras

$C^*$-algebras associated to groups provide a convenient and concise framework for the study of representation theory. In this chapter we will review and establish basic properties and notation required for the concepts ahead. Unless stated otherwise, all groups shall be assumed to be countable and discrete so that the resulting $C^*$-algebras will be separable and unital.

2.1 Preliminaries

A natural starting point to construct $C^*$-algebras from groups is the group ring which is the $*$-algebra $\mathbb{C}[G]$ consisting of finite sums of the form $\sum_{g \in G} \alpha_g g$ with $\alpha_g \in \mathbb{C}$ endowed with multiplication given by

$$\left( \sum_{g \in G} \alpha_g g \right) \left( \sum_{h \in G} \beta_h h \right) = \sum_{g,h \in G} \alpha_g \beta_h gh$$

and involution given by

$$\left( \sum_{g \in G} \alpha_g g \right)^* = \sum_{g \in G} \overline{\alpha_g} g^{-1}.$$

Observe that unitary representations of $G$ extend uniquely to $*$-representations of $\mathbb{C}[G]$. Thus to produce a $C^*$-algebra one needs only produce an appropriate norm on $\mathbb{C}[G]$ and complete. There are two main faithful representations that are used to accomplish this: the canonical regular representation, and the universal representation.

Let $\lambda : G \rightarrow \mathcal{U}(\ell^2(G))$ be the left-regular representation given by $\lambda(g) = (\delta_h \mapsto \delta_{gh})$ where $\delta_g \in \ell^2(G)$ is the usual orthonormal basis vector corresponding to $g \in G$. Define the reduced group $C^*$-algebra of $G$, denoted $C^*_\lambda(G)$, to be the completion of $\mathbb{C}[G]$ with respect to the norm $\|x\| = \|\lambda(x)\|_{B(\ell^2(G))}$.Alternatively, $C^*_\lambda(G)$ is the $C^*$-algebra generated by the unitaries $\{\lambda(g)\}_{g \in G}$. 
The following is one of the most important facts regarding reduced group $C^*$-algebras. Recall a unital $C^*$-algebra $A$ is said to be stably finite if, for every $n \in \mathbb{N}$ and for every $a \in M_n(A)$, $a^*a = 1$ implies $aa^* = 1$. Also, a state $\tau$ on $A$ is said to be tracial if $\tau(ab) = \tau(ba)$ for every $a, b \in A$. For brevity, a tracial state will often simply be called a trace.

**Proposition 2.1.1** ([5, 2.5.3]) The vector state $x \mapsto \langle x\delta_e, \delta_e \rangle$ defines a faithful tracial state on $C^*_\lambda(G)$, denoted by $\tau_\lambda$, where $\delta_e \in \ell^2(G)$ denotes the usual orthonormal basis element corresponding to the identity $e \in G$. In particular, $C^*_\lambda(G)$ is stably finite.

On the other hand, to obtain the universal or full group $C^*$-algebra, denoted $C^*(G)$, one completes $\mathbb{C}[G]$ with respect to the norm $\|x\| = \sup_\pi \|\pi(x)\|$ where the supremum is taken over all cyclic (or other appropriate family) of $*$-representations $\pi : \mathbb{C}[G] \to B(\mathcal{H})$. The full group $C^*$-algebra is therefore characterized by the following universal property.

**Proposition 2.1.2** Let $\pi : G \to \mathcal{U}(\mathcal{H})$ a unitary representation of a group $G$, then $\pi$ extends uniquely to a $*$-representation $C^*(\pi) : C^*(G) \to B(\mathcal{H})$ such that $C^*(\pi)(g) = \pi(g)$ for $g \in G$.

For brevity, the extension of a representation $\pi : G \to \mathcal{U}(\mathcal{H})$ to $C^*(G)$ will also be denoted by $\pi$.

It is then not hard to see that the full group $C^*$-algebra construction actually defines a functor from the category of countable, discrete groups to unital, separable $C^*$-algebras. The reduced group $C^*$-algebra construction is not functorial in general. Both constructions, however, work favorably with inclusions. That is, if $H \subset G$ is a subgroup then there are canonical inclusions of $C^*$-algebras $C^*(H) \subset C^*(G)$ and $C^*_\lambda(H) \subset C^*_\lambda(G)$ extending the inclusion $H \subset G$ [5, 2.5.8, 2.5.9].

**Example 2.1.3** ([6, F.4.7]) Let $G$ be an abelian group. Then, by way of the Fourier transform, $C^*(G) \simeq C^*_\lambda(G) \simeq C(\hat{G})$ where $\hat{G}$ denotes the space of continuous homomorphisms from $G$ to $\mathbb{T}$, where $\mathbb{T}$ denotes the complex units with modulus one,
given the topology of pointwise convergence. \( \hat{G} \) is called the Pontryagin dual of \( G \). In particular, \( C^*(\mathbb{Z}) \cong C(\mathbb{T}) \).

In general, it is extremely difficult to determine how group properties correspond to operator algebraic properties on its associated \( C^* \)-algebras. For example, the following conjecture, although it has been proven for a wide class of groups, remains open in general.

**Conjecture 2.1.4** (Kadison-Kaplansky) *Let \( G \) be a torsion-free group. Then 1 and 0 are the only non-trivial idempotents in \( C^*_\lambda(G) \).*

One of the main ideas behind this work is to examine how finite-dimensional approximation and deformation properties of groups correspond to similar properties of the associated group \( C^* \)-algebras and vice versa. In particular, we seek to examine when such properties exist.

### 2.2 Some Representation Theory

The language of weak containment and tensor products of representations will be crucial to the material ahead.

**Definition 2.2.1** Let \( \pi \) and \( \rho \) be \( * \)-representations of \( C^*(G) \). \( \pi \) is said to be *weakly contained* in \( \rho \), written \( \pi \preceq \rho \), if \( \text{ker}(\rho) \subset \text{ker}(\pi) \). \( \pi \) is said to be contained in \( \rho \), written \( \pi \subset \rho \), if \( \pi \) is unitarily equivalent to a sub-representation of \( \rho \).

It is immediate that containment implies weak containment. It should be emphasized that kernels of representations will always be computed as kernels of \( * \)-representations on \( C^*(G) \) unless specified otherwise. Weak containment will be of particular importance when one representation is the trivial representation.

**Proposition 2.2.2** ([6, F.1.5]) *Let \( \pi : G \to \mathcal{U}(\mathcal{H}) \) be a unitary representation and let \( i_G \) denote the trivial representation of \( G \). \( i_G \preceq \pi \) if and only if for every \( F \subset G \) finite and \( \epsilon > 0 \) there exists a unit vector \( \xi \in \mathcal{H} \) such that \( \|\pi(g)\xi - \xi\| < \epsilon \) for every \( g \in F \).*
In this sense it is often said that $i_G \leq \pi$ if and only if $\pi$ has almost invariant vectors. Observe that $i_G \subset \pi$ if and only if $\pi$ has a nonzero invariant vector.

Let $\pi : G \to \mathcal{U}(\mathcal{H})$ and $\rho : G \to \mathcal{U}(\mathcal{L})$ be unitary representations. Define the tensor product of $\pi$ and $\rho$, denoted $\pi \otimes \rho$, to be the unitary representation on $G$ defined on $\mathcal{H} \otimes \mathcal{L}$ given by $(\pi \otimes \rho)(g)(\xi \otimes \eta) = \pi(g)\xi \otimes \rho(g)\eta$.

Alternatively, the Hilbert space $\mathcal{H} \otimes \mathcal{L}$ can be identified with the Hilbert space of Hilbert-Schmidt operators from $\bar{\mathcal{L}}$ to $\mathcal{H}$, denoted $HS(\bar{\mathcal{L}}, \mathcal{H})$ with inner product given by $\langle T, S \rangle = \text{Tr}(S^*T)$, where $\bar{\mathcal{L}}$ denotes the conjugate Hilbert space of $\mathcal{L}$. Then the representation $\pi \otimes \rho$ is given by $(\pi \otimes \rho)(g) = (T \mapsto \pi(g)T\bar{\rho}(g^{-1}))$ acting on the Hilbert space $HS(\bar{\mathcal{L}}, \mathcal{H})$, where $\bar{\rho}$ denotes on the conjugate representation on the conjugate Hilbert space $\bar{\mathcal{L}}$.

The particular case of a representation tensored by itself will be especially important. If $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation, then the unitary representation $\pi \otimes \bar{\pi}$ acts on the Hilbert space $HS(\mathcal{H})$, the Hilbert-Schmidt operators on $\mathcal{H}$, and is given by $(\pi \otimes \bar{\pi})(g) = (T \mapsto \pi(g)T\bar{\pi}(g^{-1}))$. The Hilbert-Schmidt norm of a Hilbert-Schmidt operator $T$ will be denoted $\|T\|_{HS}$ in order to distinguish it from the standard operator norm on $B(\mathcal{H})$.

A key property of the tensor product construction is that it converts finite-dimensional subrepresentations to invariant vectors and conversely.

**Proposition 2.2.3** ([6, A.1.12]) Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of a group $G$. Then $i_G \subset \pi \otimes \bar{\pi}$ if and only if $\pi$ contains a nonzero finite-dimensional subrepresentation of $G$.

**Proof.** Suppose $i_G \subset \pi \otimes \bar{\pi}$. Then $\pi \otimes \bar{\pi}$ has an invariant vector $0 \neq T \in HS(\mathcal{H})$. In other words, $\pi(g)T = T\pi(g)$ for all $g \in G$. By taking adjoints, note $\pi(g)$ also commutes with $T^*T$. Since $T^*T$ is Hilbert-Schmidt operator, it is also compact and so its spectrum is a discrete set except possibly at the 0. Let $P$ be any non-zero spectral projection corresponding to an isolated point in the spectrum of $T^*T$. Then $\pi(g)P = P\pi(G)$ and $g \mapsto P\pi(g)P$ is the desired finite-dimensional subrepresentation.
Conversely, suppose \( \pi \) contains a nonzero finite-dimensional subrepresentation. Then \( \pi \) has a finite-dimensional invariant subspace. If \( 0 \neq P \in B(\mathcal{H}) \) is the finite rank projection onto this subspace, then \( P\pi = \pi P \). \( P \) is Hilbert-Schmidt because it is finite rank, and so we then have \( P \in HS(\mathcal{H}) \) is an invariant vector for \( \pi \otimes \bar{\pi} \). \( \square \)

Prop. 2.2.3 is more general than stated here. A more refined argument using the same ideas allows one to reach an analogous conclusion for a representation of the form \( \pi \otimes \rho \). We will not require this added generality, however.

Finally, it will be useful to note that the regular representation has a striking tensorial absorption property: it absorbs any representation. First, if \( \pi \) is a representation and \( n \) is some cardinal then \( n \cdot \pi \) will denote direct sum of \( \pi \) with itself \( n \)-times.

**Theorem 2.2.4** (Fell’s Absorption Principle, [5, 2.5.5]) Let \( \pi \) be a representation of \( G \) on a Hilbert space \( \mathcal{H} \). Then \( \lambda \otimes \pi \) is unitarily equivalent to \( \dim(\mathcal{H}) \cdot \lambda \).

### 2.3 Amenability

The class of amenable groups was introduced by von Neumann and later named as such by Day. This class admits many equivalent definitions which yield both desirable group theoretic and operator algebraic properties. As we will see, countable discrete amenable groups always produce separable, stably finite, and nuclear \( C^* \)-algebras thus providing important test cases throughout the subject.

Before introducing the definition of amenability we require one more piece of terminology.

**Definition 2.3.1** A finitely additive probability measure \( \mu \) on a discrete group \( G \) is called an *invariant mean* if \( \mu(gS) = \mu(S) \) for every \( S \subseteq G \) and \( g \in G \).

As a different interpretation, an invariant mean on \( G \) defines a translation invariant state on \( \ell^\infty(G) \) and conversely.

**Theorem 2.3.2** ([5, 2.6.8]) Let \( G \) be a discrete group. The following are equivalent:
1. $G$ admits an invariant mean;

2. (Følner’s Condition) For every $E \subset G$ finite and $\epsilon > 0$ there exists a finite 
subset $F \subset G$ such that $\frac{|sF \triangle F|}{|F|} < \epsilon$ for every $s \in E$;

3. $i_G \leq \lambda$;

4. $\lambda : C^*(G) \rightarrow C^*_\lambda(G)$ is injective and thus $C^*(G) \simeq C^*_\lambda(G)$;

5. $C^*_\lambda(G)$ or $C^*(G)$ is nuclear.

**Definition 2.3.3** A countable discrete group $G$ is said to be *amenable* if it satisfies 
any of the equivalent conditions of Theorem 2.3.2.

If $G$ is not necessarily discrete, then $i_G \leq \lambda$ will be used as the definition of 
amenability. Due to Example 2.1.3, it is then not hard to see that finite groups and 
abelian groups are amenable and that amenability is closed under extensions and 
passes to subgroups, quotients, direct limits, and direct sums.

**Example 2.3.4** ([7, VII.2.4]) The free group $\mathbb{F}_2$ is not amenable as it admits what 
is known as a *paradoxical decomposition*. Suppose $\mathbb{F}_2$ is generated by free generators 
a and b. Set $A_0$ and $A_1$ to be the set of reduced words starting with even and odd 
powers of $a$, respectively, and set $B_0$, $B_1$, and $B_2$ to be the set reduced words starting 
with $b$ with exponent congruent to 0, 1, and 2 modulo 3, respectively. Assume $\mu$ is 
an invariant mean on $\mathbb{F}_2$. Then by additivity

$$1 = 2\mu(A_0) = 3\mu(B_0),$$

but

$$\frac{1}{2} = \mu(A_1) \leq \mu(B_0) = \frac{1}{3}$$

since $A_1 \subset B_0$.

Since $\mathbb{F}_n$ contains $\mathbb{F}_2$ for every countable $\infty \geq n \geq 2$, no non-abelian free group 
is amenable. It was an open question of Day and von Neumann whether or not 
non-amenable groups were characterized by having free subgroups. A non-amenable
group with no free subgroups was exhibited by Ol’shankii in 1980 [8]. This question of Day and von Neumann is answered affirmatively for finitely generated linear groups, however, as such groups are either virtually solvable or contain a non-abelian free group by the Tits alternative.

As we will see later in Cor. 3.2.7, the notion of amenability will play a critical role in determining when reduced group $C^*$-algebras are quasidiagonal.
3. Quasidiagonality

In this chapter we will review different characterizations of quasidiagonal C*-algebras and how these characterizations relate specifically to group C*-algebras. In particular, the relationship between amenability, quasidiagonality, and finite-dimensional approximations of the group will be explored.

3.1 Definition and Examples

The concept of quasidiagonality was originally introduced by Halmos [9] in 1970 in the context of a single operator. To motivate the terminology, an operator $T \in B(\mathcal{H})$ is said to be quasidiagonal if it can be written as $T = D + K$ where $D$ is a block diagonal operator and $K$ is a compact operator. In other words, $T$ is a compact perturbation of a block diagonal operator $D$. Using another characterization provided by Halmos, the notion of quasidiagonality was extended to an arbitrary collection of operators as in the next definition.

**Definition 3.1.1** Let $\mathcal{H}$ be a Hilbert space and $S \subset B(\mathcal{H})$. $S$ is called a quasidiagonal set if for each $F \subset S$ finite, $X \subset \mathcal{H}$ finite, and $\epsilon > 0$ there exists a finite-rank projection $P \in B(\mathcal{H})$ such that $\|PT - TP\| < \epsilon$ for all $T \in F$ and $\|Pv - v\| < \epsilon$ for all $v \in X$.

It is easy to see that if $S$ is a quasidiagonal set, then the smallest $C^*$-algebra containing $S$ is also a quasidiagonal set. The notion of quasidiagonality may then be naturally extended to $C^*$-algebras as a $C^*$-algebra can always be concretely represented on some Hilbert space.
**Definition 3.1.2** Let $A$ be a $C^*$-algebra and $\pi: A \to B(\mathcal{H})$ a $*$-homomorphism. $\pi$ is said to be a *quasidiagonal representation* if $\pi(A)$ is a quasidiagonal set. $A$ is said to be *quasidiagonal* if it has a faithful, quasidiagonal representation.

If $A$ is quasidiagonal then it may not be the case that $\pi(A)$ is a quasidiagonal set for an arbitrary faithful representation $\pi: A \to B(\mathcal{H})$. However, due to the work of Voiculescu, if we assume that the representation is *essential*, that is $\pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, where $\mathcal{K}(\mathcal{H})$ denotes the compact operators on $\mathcal{H}$, then this is no longer the case. Note that any representation may be associated to an essential representation by taking its infinite inflation, that is, taking a direct sum with itself infinitely many times.

**Theorem 3.1.3** ([5, 7.2.5]) Let $A$ be a $C^*$-algebra. If $A$ is quasidiagonal, then every faithful, essential representation is quasidiagonal.

Since we will be primarily considering separably represented $C^*$-algebras the following reformulation will also be useful.

**Proposition 3.1.4** ([5, 7.2.3]) Let $\mathcal{H}$ be a separable Hilbert space and $S \subset B(\mathcal{H})$ a separable quasidiagonal set. Then there exists an increasing sequence of finite-rank projections $P_1 \leq P_2 \leq \cdots$ converging strongly to the identity with $\|P_n T - T P_n\| \to 0$ for all $T \in S$.

By cutting down a quasidiagonal representation by the projections obtained from quasidiagonality, one obtains contractive completely positive (ccp) maps which are approximately multiplicative and isometric. In fact, the existence of these finite dimensional approximations provides a representation free characterization of quasidiagonality.

**Theorem 3.1.5** ([5, 7.1.3]) $A$ is quasidiagonal if and only if for every $F \subset A$ finite and $\epsilon > 0$, there exists a ccp map $\varphi: A \to \mathbb{M}_n(\mathbb{C})$ such that

$$\|\varphi(ab) - \varphi(a)\varphi(b)\| < \epsilon$$
and
\[ \|a\| - \epsilon < \|\varphi(a)\| \]
for all \( a, b \in F \).

With this new characterization it easily follows that quasidiagonality passes to subalgebras and inductive limits with injective connecting maps. Since finite dimensional algebras are quasidiagonal, it then follows that AF-algebras are quasidiagonal.

If \( A \) is unital, then the ccp maps occurring in Theorem 3.1.5 may be chosen to be unital and completely positive (ucp) [5, 7.1.4]. Such approximately multiplicative and isometric maps will be called \((F, \epsilon)\)-multiplicative and \((F, \epsilon)\)-isometric, respectively.

If \( A \) is separable and quasidiagonal, then we may obtain a sequence of ccp maps \( \{\varphi_n : A \to M_{k(n)}(C)\}_n \) such that
\[
\lim_n \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0
\]
and
\[
\lim_n \|\varphi_n(a)\| = \|a\|
\]
for every \( a, b \in A \). Such a sequence of ccp maps satisfying these conditions will be called \textit{asymptotically multiplicative} and \textit{asymptotically isometric}, respectively.

If we are given such an asymptotically multiplicative and asymptotically isometric sequence of ccp maps there is then an embedding
\[
\Phi : A \hookrightarrow \prod_n M_{k(n)}(C)/\sum_n M_{k(n)}(C)
\]
given by
\[
\Phi(a) = \pi((\varphi_n(a))_n)
\]
where \( \sum_n M_{k(n)}(C) \) denotes the completion of the \(*\)-algebra \( \bigoplus_n M_{k(n)}(C) \) in the \( C^*\)-algebra \( \prod_n M_{k(n)}(C) \) and \( \pi \) is the canonical quotient map. Note that the ideal \( \sum_n M_{k(n)}(C) \) is the collection of sequences converging to zero. \( \Phi \) is indeed an embedding because
\[
\|\Phi(a)\| = \limsup_n \|\varphi_n(a)\|
\]
and the sequence \( \{ \varphi_n \}_n \) is asymptotically isometric.

If such an embedding

\[
A \hookrightarrow \prod_n \mathbb{M}_{k(n)}(\mathbb{C}) / \sum_n \mathbb{M}_{k(n)}(\mathbb{C})
\]

exists then \( A \) is called a MF-algebra. The MF condition is strictly weaker than quasidiagonality in general. However, the two notions are equivalent if one assumes the existence of a completely positive lift of the embedding. That occurs, for example, if \( A \) is nuclear.

**Proposition 3.1.6** ([5, Ex.7.1.3]) Let \( A \) be a separable \( C^* \)-algebra. \( A \) is quasidiagonal if and only if there exists an embedding

\[
A \hookrightarrow \prod_n \mathbb{M}_{k(n)}(\mathbb{C}) / \sum_n \mathbb{M}_{k(n)}(\mathbb{C})
\]

which has a ccp lift \( A \to \prod_n \mathbb{M}_{k(n)}(\mathbb{C}) \).

**Proof.** We have already shown that quasidiagonality guarantees the existence of such an embedding. To prove the converse, let \( \varphi_n : A \to \mathbb{M}_{k(n)}(\mathbb{C}) \) denote the ccp map obtained from the \( n \)th component of the ccp lift \( f : A \to \prod_n \mathbb{M}_{k(n)}(\mathbb{C}) \).

Since \( \Phi(x) = \pi(f(x)) = \pi((\varphi_n(x))_n) \) where \( \pi \) is the quotient map, we then have \( (\varphi_n(ab) - \varphi_n(a)\varphi_n(b))_n \in \sum_n \mathbb{M}_{k(n)}(\mathbb{C}) \) for every \( a, b \in A \). Equivalently,

\[
\lim_n \| \varphi_n(ab) - \varphi_n(a)\varphi_n(b) \| = 0
\]

and so we have asymptotic multiplicativity. Now we need to show \( \lim_n \| \varphi_n(a) \| = \| a \| \), but we only know \( \| a \| = \limsup_n \| \varphi_n(a) \| \). We will need to produce a new sequence.

By separability we can write \( A = \bigcup_{i=1}^{\infty} F_i \) with \( F_i \subset A \) finite and \( F_i \subset F_{i+1} \) for every \( i \). For each \( a \in F_i \) there exists \( n_a \in \mathbb{N} \) such that \( \varphi_{n_a} \) is \((F_i, 1/i)\)-multiplicative and \( \| a \| - 1/i \leq \| \varphi_{n_a}(a) \| \). Define

\[
\psi_i = \bigoplus_{a \in F_i} \varphi_{n_a} : A \to \bigoplus_{a \in F_i} \mathbb{M}_{k(n_a)}(\mathbb{C})
\]

and then \( \psi_i \) is also \((F_i, 1/i)\)-isometric. Therefore the sequence \( \{ \psi_i \}_i \) is asymptotically multiplicative and asymptotically isometric as desired. \( \square \)
**Definition 3.1.7** Let $A$ be a $C^*$-algebra. If there exists an embedding

$$A \hookrightarrow \prod_{i \in I} M_{k(i)}(\mathbb{C})$$

then $A$ is said to be *residually finite dimensional* or RFD.

If $A$ is RFD, then $A$ is quasidiagonal. The converse need not hold even for UHF algebras. Since there exists amenable groups which do not have a separating family of finite dimensional representations, counterexamples can be realized from the class of group $C^*$-algebras as well.

Indeed, let $S$ be the finitary symmetric group defined as the directed union $S = \cup_{n=1}^{\infty} S_n$, where $S_n$ denotes the symmetric group on $n$ letters. Then, being defined by a direct limit of finite groups with injective connecting maps, $S$ is amenable and $C^*(S)$ is quasidiagonal. Again by the direct limit structure, the only nontrivial normal subgroup of $S$ is the index two subgroup $A$, the finitary alternating group. Let $\pi : S \to U(n)$ be an irreducible representation. Then, computing kernels as a group homomorphism, either $\text{ker}(\pi) = A$ or $\text{ker}(\pi) = \{e\}$. If $\text{ker}(\pi) = A$, then $\pi$ is actually the sign representation. On the other hand, if $\text{ker}(\pi) = \{e\}$, then $\pi$ restricts to a faithful representation of $S_k$ for every $k \in \mathbb{N}$. However, this contradicts the fact that the smallest faithful representation of $S_k$ is on $U(k-1)$ for $k \geq 5$ [10]. Hence, $C^*(S)$ is not residually finite dimensional as its only nontrivial finite dimensional representation is the sign representation.

From a representation perspective, $A$ is RFD if and only if it has a faithful, quasidiagonal representation for which the projections commute exactly. Thus every abelian $C^*$-algebra is RFD and therefore quasidiagonal via point evaluations.

The following is an important obstruction to quasidiagonality and will provide many examples of separable and nuclear $C^*$-algebras which are not quasidiagonal such as the Cuntz algebras $\mathcal{O}_n$ or the Toeplitz algebra $\mathcal{T}$.  

**Proposition 3.1.8** ([5, 7.1.15]) *Every quasidiagonal $C^*$-algebra is stably finite.*

The converse of Prop. 3.1.8 has motivated many questions surrounding quasidiagonality. Most notable is the following question of Blackadar and Kirchberg:
**Question** ([11]) Is every stably finite nuclear $C^*$-algebra quasidiagonal?

It is said that one reason quasidiagonality continues to be not well understood is that it has deep topological content. To state this, we require the language of homotopy for $C^*$-algebras.

**Definition 3.1.9** Let $\varphi : A \to B$ and $\psi : A \to B$ be $*$-homomorphisms of $C^*$-algebras. $\varphi$ and $\psi$ are said to be *homotopic* if there exists a $*$-homomorphism $\Phi_t : A \to B$ for $t \in [0,1]$ with $t \mapsto \Phi_t(a)$ continuous for each $a \in A$ and $\Phi_0 = \varphi, \Phi_1 = \psi$. Equivalently, if there exists a $*$-homomorphism $\Phi : A \to C[0,1] \otimes B$ with $\text{ev}_0 \circ \Phi = \varphi$ and $\text{ev}_1 \circ \Phi = \psi$.

**Definition 3.1.10** $C^*$-algebras $A$ and $B$ are said to be *homotopy equivalent* if there exists $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\varphi \circ \psi$ is homotopic to $\text{id}_B$ and $\psi \circ \varphi$ is homotopic to $\text{id}_A$

The topological nature is then most profoundly captured by the following theorem due to Voiculescu.

**Theorem 3.1.11** ([12]) Quasidiagonality is a homotopy invariant. That is, if $A$ and $B$ are homotopy equivalent $C^*$-algebras and $A$ is quasidiagonal, then $B$ is quasidiagonal.

In particular, the cone and suspension of any $C^*$-algebra, $CA = C_0[0,1) \otimes A$ and $SA = C_0(0,1) \otimes A$ respectively, are both quasidiagonal. Indeed, $SA \subseteq CA$ and $\Phi : CA \to C[0,1] \otimes CA$ given by $\Phi_t(f) = (s \mapsto f((1-t)s))$ defines a homotopy between $\text{id}_{CA}$ and the zero map. Therefore, $CA$ is homotopy equivalent to the zero $C^*$-algebra for any $C^*$-algebra $A$. $C^*$-algebras which are homotopy equivalent to 0 are said to be contractable. We also note that every $C^*$-algebra is a quotient of a quasidiagonal $C^*$-algebra.
3.2 Reduced Group $C^*$-Algebras

With the basics of amenability and quasidiagonality we are now prepared to examine when group $C^*$-algebras are quasidiagonal. Recall that $C^*_\lambda(G)$ is stably finite and nuclear if $G$ is amenable, and so these algebras are test cases for the question of Blackadar and Kirchberg.

**Lemma 3.2.1** Let $G$ be an infinite, discrete group. Then the left-regular representation $\lambda : C^*_\lambda(G) \to B(\ell^2(G))$ is essential. In particular, if $C^*_\lambda(G)$ is quasidiagonal, then $\lambda(G)$ is a quasidiagonal set.

**Proof.** If $\lambda$ is essential and $C^*_\lambda(G)$ is quasidiagonal, then $\lambda(G)$ is a quasidiagonal set by Theorem 3.1.3.

If $\lambda$ is not essential, then there exists $0 \neq T \in C^*_\lambda(G) \cap K(\ell^2(G))$. Since $T$ is compact, the spectrum of $T^*T$ is discrete except possibly at zero. Let $P \in C^*_\lambda(G)$ be any non-zero spectral projection of $T^*T$ corresponding to an isolated point. Since $P$ is compact, $P$ is finite rank. Let $g \in G$ and let $\delta_g \in \ell^2(G)$ denote the canonical orthonormal basis element corresponding to $g$, then

$$\langle P\delta_g, \delta_g \rangle = \langle P\lambda(g)\delta_e, \lambda(g)\delta_e \rangle$$

$$= \langle \lambda(g)^*P\lambda(g)\delta_e, \delta_e \rangle$$

$$= \tau_\lambda(\lambda(g)^*P\lambda(g))$$

$$= \tau_\lambda(P) > 0$$

since $\tau_\lambda$ is a faithful trace. Note that $P$ is trace-class [13, 2.4.13] as it’s a finite rank projection, but, by the previous computation, the trace-class norm is given by

$$\|P\|_1 = \sum_{g \in G} \langle P\delta_g, \delta_g \rangle = \sum_{g \in G} \tau_\lambda(P)$$

which is nonsense if $G$ is infinite since $\tau_\lambda(P) > 0$. Therefore, if $G$ is infinite, then $\lambda$ is essential. \qed

The following foundational observation was made by Rosenberg in 1987. The proof provided here uses the same main idea as Rosenberg’s original argument.
Theorem 3.2.2 ([14]) If $C^*_\lambda(G)$ is quasidiagonal, then $G$ is amenable.

Proof. If $G$ is finite, then it is automatically amenable. We then assume $G$ is infinite. By the previous lemma, $\{\lambda(g)\}_{g \in G}$ is a quasidiagonal set. Let $F \subset G$ be finite and $\epsilon > 0$. By assumption, there exists a non-zero finite-rank projection $P$ such that $\|\lambda(g)P\lambda(g)^* - P\| < \frac{\epsilon}{2}$ for every $g \in F$. Since $P$ is a projection,

$$\|\lambda(g)P\lambda(g)^* - P\|_{HS} \leq \|P - P\lambda(g)P\lambda(g)^*\|_{HS} + \|(P\lambda(g)P - \lambda(g)P)\lambda(g)^*\|_{HS}$$

$$\leq \|P(P - \lambda(g)P\lambda(g)^*)\|_{HS} + \|P\lambda(g)P - \lambda(g)P\|_{HS}$$

$$\leq \|P\|_{HS}\|P - \lambda(g)P\lambda(g)^*\| + \|P\lambda(g) - \lambda(g)P\|\|P\|_{HS}$$

$$\leq 2\|P\|_{HS}\|\lambda(g)P\lambda(g)^* - P\| \leq \epsilon\|P\|_{HS}$$

where we have used properties of the Hilbert-Schmidt norm: $\|KT\|_{HS} \leq \|K\|_{HS}\|T\|$, $\|TK\|_{HS} \leq \|K\|_{HS}\|T\|$, and $\|uK\|_{HS} = \|Ku\|_{HS} = \|K\|_{HS}$ where $T$ is an arbitrary operator, $K$ is Hilbert-Schmidt, and $u$ is a unitary [13, 2.4.10]. Therefore, for any $F \subset G$ finite and $\epsilon > 0$ we can find a finite-rank projection $P$ such that

$$\left\|\left(\lambda \otimes \bar{\lambda}\right)(g)\left(\frac{P}{\|P\|_{HS}}\right) - \frac{P}{\|P\|_{HS}}\right\|_{HS} < \epsilon$$

for every $g \in F$ and so we conclude using Prop. 2.2.2 that $i_G \leq \lambda \otimes \bar{\lambda}$.

By Fell’s Absorption Principle (2.2.4), $\lambda \otimes \bar{\lambda}$ is unitarily equivalent to an infinite multiple of $\lambda$, which we will denote by $\lambda_\infty$. Since $i_G \leq \lambda \otimes \bar{\lambda}$, we then have that $i_G \leq \lambda_\infty$ by unitary equivalence. We now endeavor to show that this implies $i_G \leq \lambda$ which is equivalent to amenability.

Let $F \subset G$ and $\epsilon > 0$. By Prop. 2.2.2, there exists a unit vector $\xi$ such that

$$\|\lambda_\infty(g)\xi - \xi\| < \frac{\epsilon}{2}$$

for every $g \in F$. By Voiculescu’s theorem [5, 1.7.5], the representations $\lambda$ and $\lambda_\infty$ are approximately unitarily equivalent as they are both faithful and essential. That is, there exists a unitary $u : \ell^2(G)_\infty \to \ell^2(G)$, where $\ell^2(G)_\infty$ denotes the representation space for $\lambda_\infty$, such that

$$\|\lambda(g) - u\lambda_\infty(g)u^*\| < \frac{\epsilon}{2}$$
for \( g \in F \). Combining these estimates,

\[
\|\lambda(g)u\xi - u\xi\| \leq \|\lambda(g)u\xi - u\lambda_\infty(g)\xi\| + \|u\lambda_\infty(g)\xi - u\xi\|
\]

\[
\leq \|\lambda(g)u - u\lambda_\infty(g)\| + \|\lambda_\infty(g)\xi - \xi\| \leq \epsilon
\]

for \( g \in F \). Once again using Prop. 2.2.2, we conclude that \( i_G \preceq \lambda \), which completes the proof.

**Remark 3.2.3** Theorem 3.2.2 shows that if \( G \) is not amenable, say \( G = \mathbb{F}_2 \), then \( C^*_\lambda(G) \) is not quasidiagonal.

Rosenberg also asks whether the converse of Theorem 3.2.2 is true. This problem became known as Rosenberg’s Conjecture and remained unsolved until 2015 due to the work of Tikuisis, White and Winter [15]. Some terminology is required to introduce their main result. Recall that given a sequence of states \( \{\phi_n\} \) on \( A \) we say \( \phi_n \) converges to \( \phi \) in the weak-* topology, written \( \phi_n \to \phi \), if for each \( a \in A \) we have

\[
\lim_{n} \phi_n(a) = \phi(a).
\]

**Definition 3.2.4** ([16, 3.3.1]) Let \( A \) be a unital, separable \( C^* \)-algebra. A trace \( \tau : A \to \mathbb{C} \) is said to be a quasidiagonal if there exists a sequence of ucp maps \( \{\phi_n : A \to M_{k(n)}(\mathbb{C})\}_n \) such that \( \text{tr}_{k(n)}\circ\phi_n \to \tau \) in the weak-* topology and \( \|\phi_n(ab) - \phi_n(a)\phi_n(b)\| \to 0 \) for all \( a, b \in A \).

Note that every quasidiagonal \( C^* \)-algebra has a quasidiagonal trace. The converse is not true, however, as characters always define a quasidiagonal trace. If a \( C^* \)-algebra has a faithful, quasidiagonal trace then quasidiagonality does follow.

**Proposition 3.2.5** ([16, 4.1.3]) Let \( A \) be a unital, separable \( C^* \)-algebra and \( \tau \) a faithful, quasidiagonal trace on \( A \). Then \( A \) is quasidiagonal.

**Proof.** Since \( \tau \) is quasidiagonal, there exists a sequence of ucp maps \( \{\phi_n : A \to M_{k(n)}(\mathbb{C})\} \) with

\[
\lim_{n} \|\phi_n(ab) - \phi_n(a)\phi_n(b)\| = 0
\]
and $\tau(a) = \lim_n \text{tr}(\varphi_n(a))$ for every $a, b \in A$. \{\varphi_n\} then defines a $\ast$-homomorphism

$$\Phi : A \to \prod_n \frac{M_k(n)(\mathbb{C})}{\sum_n M_k(n)(\mathbb{C})}$$

with $\|\Phi(a)\| = \lim \sup_n \|\varphi_n(a)\|$ for every $a \in A$.

Using Prop. 3.1.6 we need only show $\Phi$ is injective. Indeed, suppose $0 \neq a \in A$ and $\Phi(a) = 0$. Then

$$0 = \|\Phi(a)\| = \|\Phi(a)\|^2 = \|\Phi(a^* a)\| = \lim \sup_n \|\varphi_n(a^* a)\|$$

but, since $\tau$ is faithful,

$$0 < \tau(a^* a) = \lim_n \text{tr}(\varphi_n(a^* a))$$

Hence, for $n$ sufficiently large $\|\varphi_n(a^* a)\| > 0$, but this implies

$$0 = \|\Phi(a)\| = \lim \sup_n \|\varphi_n(a^* a)\| > 0$$

which is a contradiction. Therefore, $\Phi$ is injective. \hfill \Box

Recall that a separable $C^*$-algebra is said to satisfy the universal coefficient theorem (UCT) if it is KK-equivalent to a commutative $C^*$-algebra [17]. It was shown by Tu [18] that $C^*_\lambda(G)$ satisfies the UCT if $G$ is amenable.

**Theorem 3.2.6** ([15, Thm. A]) Let $A$ be a separable, nuclear $C^*$-algebra which satisfies the UCT. Then every faithful trace on $A$ is quasidiagonal.

Since the canonical trace $\tau_\lambda$ on $C^*_\lambda(G)$ is faithful by Prop. 2.1.1, the following corollary is established and Tikuisis, White and Winter resolve Rosenberg’s conjecture.

**Corollary 3.2.7** ([15, Cor. C]) Let $G$ be a countable, discrete group. $C^*_\lambda(G)$ is quasidiagonal if and only if $G$ is amenable.

It is even shown that $C^*_\lambda(G)$ is AF-embeddable in a trace preserving way. Theorem 3.2.6 also establishes that the question of Blackadar and Kirchberg is answered
affirmatively for simple, nuclear $C^*$-algebras satisfying the UCT as nuclearity implies the existence of a trace [19] which is faithful by simplicity.

In light of these results, the question of quasidiagonality of reduced group $C^*$-algebras is entirely resolved. The case of the full group $C^*$-algebra is less clear. Indeed, $C_\chi^*(\mathbb{F}_2)$ is not quasidiagonal, but $C^*(\mathbb{F}_2)$ is residually finite dimensional. It is non-trivial to exhibit an example of a countable, discrete group $G$ for which $C^*(G)$ is not quasidiagonal. Exploring this will be the focus and motivation of the next chapter.

3.3 Other Finite-Dimensional Approximations

The following finite-dimensional matricial approximation property for groups was introduced in [20] as a tool to determine when a group produced a quasidiagonal $C^*$-algebra. We introduce a similar but logically stronger property and highlight intriguing symmetries with other well-known finite-dimensional approximations of groups.

**Definition 3.3.1** A countable, discrete group $G$ is MF if for any finite subset $F \subset G$ and $\epsilon > 0$ there exists $n \in \mathbb{N}$ and a map $\theta : G \to U(n)$ such that

1. $\|\theta(gh) - \theta(g)\theta(h)\| < \epsilon$ whenever $g, h \in F$
2. $\theta(e) = 1_n$ if $e \in F$
3. $\|\theta(g) - 1_n\| \geq \frac{1}{4}$ whenever $e \neq g \in F$

where $U(n)$ denotes $n \times n$ unitary matrices and $\|\cdot\|$ denotes the operator norm.

These maps in the Def. 3.3.1 are often referred to as $(F, \epsilon)$-almost homomorphisms. The precise value of the constant $\frac{1}{4}$ is not important and has only minor restrictions, but the fact that it is uniform over the entire group $G$ distinguishes this definition from the definition given in [20]. However, $\frac{1}{4}$ has been strategically chosen to make concise definitions for the other finite-dimensional approximation properties that we
shall soon discuss. First a perturbation lemma which allows us to perturb almost unitaries to unitaries.

**Lemma 3.3.2** Let $A$ be a unital $C^*$-algebra and $\epsilon < \frac{1}{2}$. If $x \in A$ and $\|xx^*-1\| < \epsilon$ and $\|x^*-1\| < \epsilon$, then there exists a unitary $u \in \mathcal{U}(A)$ such that $\|x-u\| < 2\epsilon$.

**Proof.** By assumption, both $x^*$ and $xx^*$ are invertible and so $x$ is invertible as well. Then, by polar decomposition, $u = x|x|^{-1}$ is a unitary and so $\|x-u\| \leq \|x\|\|1-(x^*)^{-1/2}\|$. By assumption, $\text{sp}(x^*) \subset (1-\epsilon, 1+\epsilon)$ and so $\|x\|^2 = \|x^*x\| \leq 1 + \epsilon$. By functional calculus,

$$\|1-(x^*)^{-1/2}\| \leq \frac{\sqrt{1+\epsilon} - 1}{\sqrt{1-\epsilon}}$$

and combining these estimates yields

$$\|x-u\| \leq \sqrt{1+\epsilon}\frac{\sqrt{1+\epsilon} - 1}{\sqrt{1-\epsilon}} \leq \frac{\epsilon}{\sqrt{1-\epsilon}} \leq 2\epsilon$$

so long as $\epsilon < \frac{1}{2}$. \qed

**Proposition 3.3.3** If $C^*(G)$ is quasidiagonal, then $G$ is MF.

**Proof.** Let $F \subset G \subset C^*(G)$ be finite and $\frac{1}{2} > \epsilon > 0$. Assume without loss of generality that $e \in F$. Define $F^2 = \{ab : a, b \in F\}$, $F^{-1} = \{a^{-1} : a \in F\}$ and $E = \{ab : a, b \in F^2 \cup (F^2)^{-1}\}$. Then quasidiagonality and Theorem 3.1.5 provide a $(E, \epsilon)$-multiplicative map $\varphi : G \to M_n(\mathbb{C})$ with $\varphi(e) = 1$ and $\|g-e\| - \epsilon \leq \|\varphi(g)-1\|$ whenever $e \neq g \in F$. Since $g-e \in \mathbb{C}[G]$ and the norm on $C^*(G)$ is the largest $C^*$-norm completing $\mathbb{C}[G]$ we have

$$\sqrt{2} = \|\lambda(g)-\lambda(e)\| \leq \|g-e\|$$

Thus, $\sqrt{2} - \epsilon \leq \|\varphi(g)-1\|$ whenever $e \neq g \in F$.

Since $\varphi$ is ucp we have $\varphi(g)^* = \varphi(g^{-1})$ for every $g \in G$. Then for $g \in F^2$, $\|\varphi(g)\varphi(g^{-1}) - 1\| < \epsilon$ and $\|\varphi(g^{-1})\varphi(g)-1\| < \epsilon$ by multiplicativity on $E$. Now using Lemma 3.3.2 for each $e \neq g \in F^2$ we may find $u_g \in \mathcal{U}(n)$ with $\|\varphi(g) - u_g\| < 2\epsilon$ and put $u_e = 1$. Since $\|\varphi(g)\| \leq 1$ for every $g \in F$ we then have for $g, h \in F$

$$\|u_{gh} - u_gu_h\| \leq \|u_{gh} - \varphi(gh)\| + \|\varphi(gh) - \varphi(g)\varphi(h)\| + \|\varphi(g)\varphi(h) - u_gu_h\|$$
\[
\leq 3\epsilon + \|\varphi(g)\varphi(h) - \varphi(g)u_h\| + \|\varphi(g)u_h - u_gu_h\|
\]
\[
\leq 3\epsilon + \|\varphi(g) - u_h\| + \|\varphi(g) - u_g\| \leq 5\epsilon
\]

and
\[
\|u_g - 1\| = \|u_g - \varphi(g) + \varphi(g) - 1\|
\]
\[
\geq \|\varphi(g) - 1\| - \|u_g - \varphi(g)\| \geq \sqrt{2} - 3\epsilon
\]

for \(e \neq g \in F\). We then define \(\theta : G \to \mathcal{U}(n)\) given by \(\theta(g) = u_g\) for \(g \in F^2\) and define \(\theta\) arbitrarily otherwise. Then \(\theta\) satisfies the conditions in the MF definition. Since \(F\) and \(\epsilon\) are arbitrary we conclude that \(G\) is MF.

Recall if \(G\) is amenable, then \(\mathcal{C}_\lambda^*(G) = \mathcal{C}^*(G)\) is quasidiagonal by Cor. 3.2.7 and so the next corollary is immediate.

**Corollary 3.3.4** If \(G\) is a countable amenable discrete group, then \(G\) is MF.

If it is not known if there exists a MF group \(G\) for which \(\mathcal{C}^*(G)\) is not quasidiagonal. It is also not known if there exists a group \(G\) which is not MF.

Insisting on a global constant \(\frac{1}{4}\) in the MF definition as we have allows us to give the definition of sofic and hyperlinear groups in a concise and symmetrical way.

**Definition 3.3.5** Let \(G\) be a countable discrete group.

1. ([21]) \(G\) is said to be **hyperlinear** if it satisfies the same conditions as Def. 3.3.1 with the operator norm replaced by the normalized Hilbert-Schmidt norm given by \(\|T\|_2 = \sqrt{\frac{1}{n} \text{Tr}(T^*T)}\).

2. ([22], [23]) \(G\) is said to be **sofic** if it satisfies the same conditions as Def. 3.3.1 with \(\mathcal{U}(n)\) replaced \(S_n\), the symmetric group on \(n\) letters, and the operator norm replaced by the Hamming distance given by \(d_{\text{Hamm}}(\sigma, \tau) = \frac{1}{n}|\{i : \sigma(i) \neq \tau(i)\}|\).

Sofic groups were originally introduced by Gromov who showed that they satisfy Gottschalk’s surjunctivity conjecture. Elek and Szabo later provided the above characterization in terms of almost representations. Due to the work of Kirchberg, Radulescu, Ozawa and others, a countable group \(G\) is hyperlinear if and only if the group...
von Neumann algebra $L(G)$ satisfies Connes’ Embedding Problem for groups. Further motivation and many alternative characterizations may be found in [24] and [25].

**Proposition 3.3.6** Let $G$ be a group.

1. ([24]) If $G$ is amenable, then $G$ is sofic.

2. ([23]) If $G$ is sofic, then $G$ is hyperlinear.

**Proof.** Suppose $G$ is amenable. Using Følner’s Condition, for every $E \subset G$ finite and $\epsilon > 0$ there exists a finite set $F \subset G$ such that for each $s \in E$, $|sF \triangle F| < \epsilon |F|$. The key observation is that for $s \in E$ the Følner’s Condition implies the map $x \mapsto sx$ is well-defined bijection on a subset of $F$ of size at least $(1 - \epsilon)|F|$. This map may then be arbitrarily extended to a bijection on $F$ and produces $\theta(s) = (x \mapsto sx)$ a $(F, 2\epsilon)$-almost homomorphism on $S_{|F|}$.

Now suppose $G$ is sofic. Observe that $S_n$ embeds into $U(n)$ as permutation matrices and this inclusion satisfies $d_{\text{Hamm}}(\sigma, \tau) = \frac{1}{2} \|U_\sigma - U_\tau\|_2^2$. Thus $G$ is also hyperlinear.

Residually finite groups are sofic hence sofic groups need not be amenable. It is unknown if every hyperlinear group is sofic. It is also unknown if there exists a group which is not hyperlinear. Similarly, it is unknown what relationship, if any, there is between the class of MF and the class of hyperlinear or sofic groups. Thus far, the only unifying theme appears to be amenability.
4. Kazhdan’s Property (T)

It is known that infinite, simple groups with Kazhdan’s Property (T) do not produce quasidiagonal full group $C^*$-algebras as explained by A. Thom, N. Ozawa, and M. Yamashita via two separate arguments [26]. In this chapter we point out that Rosenberg’s original argument (Theorem 3.2.2) can be adapted to show that no infinite Property (T) group with only finitely many unitary equivalence classes of finite dimensional representations produces a quasidiagonal full group $C^*$-algebra. We then begin by reviewing background information on Kazhdan’s Property (T).

4.1 Definition and Examples

Property (T) was introduced by D. Kazhdan in 1967 [27] in order to show many lattices are finitely generated. Since this original three page paper, use of Property (T) has expanded with applications throughout mathematics. From the perspective of this thesis, Property (T) represents a finite-dimensional rigidity condition antithetical to finite-dimensional approximation.

For the purposes of this section, it will be advantageous to relax our standing assumption that all groups are countable and discrete to allow for general locally compact groups. [6] will be the primary reference for this section.

**Definition 4.1.1** Let $\pi : G \to \mathcal{U}(H_{\pi})$ be a unitary representation of a locally compact group $G$, $Q \subset G$, and $\kappa > 0$.

1. A vector $\xi \in H_{\pi}$ is said to be $(Q, \kappa)$-invariant if $\|\pi(s)\xi - \xi\| < \kappa$ for every $s \in Q$. 
2. The pair \((Q, \kappa)\) is called a \textit{Kazhdan Pair} if for every unitary representation \(\pi\) of \(G\) with a \((Q, \kappa)\)-invariant unit vector there exists \(0 \neq x \in \mathcal{H}_\pi\) with \(\pi(g)x = x\) for all \(g \in G\).

3. \(G\) is said to have \textit{Kazhdan’s Property \((T)\)} if there exists a Kazhdan pair \((Q, \kappa)\) with \(Q \subset G\) compact.

Producing examples of infinite groups with property \((T)\) is non-trivial. Even for compact groups, some work is required.

\textbf{Proposition 4.1.2} ([6, 1.1.5]) \textit{Let \(G\) be a topological group. Then \((G, \sqrt{2})\) is a Kazhdan Pair. In particular, compact groups have property \((T)\).}

Before introducing non-compact examples, we will record some other basic facts that will be required.

\textbf{Proposition 4.1.3} \textit{Let \(G\) be a locally compact group with Property \((T)\).}

1. \(G\) is compactly generated.

2. Property \((T)\) passes to quotients.

In particular, a discrete group \(G\) with property \((T)\) is finitely generated and has finite abelianization. However the most important hereditary property is that property \((T)\) is inherited by lattices, a fact originally discovered by Kazhdan.

\textbf{Theorem 4.1.4} ([6, 1.7.1]) \textit{Let \(G\) be a locally compact group, and let \(H\) be a closed subgroup of \(G\) such that \(G/\Lambda\) has a finite invariant regular Borel measure. The following are equivalent:}

1. \(G\) has Property \((T)\);

2. \(H\) has property \((T)\),

In particular, if \(G\) is discrete and \(H \subset G\) is a finite index subgroup, then \(G\) has Property \((T)\) if and only if \(H\) has Property \((T)\).
Theorem 4.1.5 ([6]) Simple real Lie groups of real rank at least two have Property (T). In particular, $SL_n(\mathbb{R})$ and $SL_n(\mathbb{Z})$ have Property (T) for $n \geq 3$. Furthermore, the rank one groups $Sp(n, 1)$ have Property (T) for $n \geq 2$.

We then have a wealth of examples of both discrete and general locally compact groups with Property (T), but examples other than those listed here are also numerous. Note $SL_2(\mathbb{Z})$ does not have Property (T). Suppose $SL_2(\mathbb{Z})$ did have Property (T), then Theorem 4.1.4 implies that $F_2$ has Property (T) as $F_2$ is a finite index subgroup of $SL_2(\mathbb{Z})$. However, $F_2$ has infinite abelianization which contradicts Prop. 4.1.3.

We now observe that Property (T) may be rephrased in terms of weak containment of the trivial representation.

Proposition 4.1.6 ([6, 1.2.1]) Let $G$ be a locally compact group. $G$ has Property (T) if and only if for every unitary representation $\pi$ of $G$ whenever $i_G \preceq \pi$, then $i_G \subset \pi$.

Since amenability is characterized by $\lambda$ weakly containing $i_G$ and $\lambda$ has an invariant vector if and only if $G$ is compact [6, A.5.1], the following corollary is immediate.

Corollary 4.1.7 Let $G$ be an amenable group with Property (T). Then $G$ is compact.

It is not accurate to consider Property (T) as a negation to amenability for non-compact groups. Rather, the negation of Property (T) is better considered to be the Haagerup property which is strictly weaker than amenability. For example, $F_2$ has the Haagerup property. For completeness, we provide a definition.

Definition 4.1.8 ([5, 12.2.1]) A discrete group $G$ has the Haagerup Property if there is a net of states $\{\varphi_n\}$ on $C^*(G)$ such that the restrictions $\varphi_n|_G$ converge to 1 pointwise on $G$ and each $\varphi_n|_G$ vanishes at infinity.

4.2 Quasidiagonality

We begin by using a theorem of Dauns and Hoffman to explain and review Kazhdan projections which will provide a crucial tool for factoring out finite dimensional
representations from group $C^*$-algebras associated to groups with Property (T). From there we will begin addressing quasidiagonality. From now on, all groups considered will again be countable and discrete.

Kazhdan projections will ultimately be obtained from the interaction of Property (T) with the primitive spectrum of $C^*(G)$. Let $A$ be a unital $C^*$-algebra. Define the set $\text{Prim}(A)$ to be the collection of kernels of irreducible $*$-representations of $A$. This set is topologized via the closure operation:

$$\bar{S} = \{\ker(\rho) : \rho \preceq \bigoplus_{\ker(\pi) \in S} \pi\}$$

$$= \{\ker(\rho) : \bigcap_{\ker(\pi) \in S} \ker(\pi) \subset \ker(\rho)\}$$

This topology is often called the Jacobson or Hull-Kernel topology which then provides $\text{Prim}(A)$ with the structure of a quasi-compact, $T_0$-space [13]. Since Property (T) can be characterized in terms of weak containment, it then has a topological characterization on $\text{Prim}(A)$.

**Theorem 4.2.1** ([6, 1.2.5]) Let $G$ be a countable, discrete group. The following are equivalent:

1. $G$ has Property (T);
2. $\{\ker(i_G)\}$ is isolated in $\text{Prim}(C^*(G))$;
3. $\{\ker(\pi)\}$ is isolated in $\text{Prim}(C^*(G))$ for every finite-dimensional, irreducible representation $\pi$ of $G$.

It should be noted the content of Theorem 4.2.1 is that the sets $\{\ker(i_G)\}$ and $\{\ker(\pi)\}$ are open. These sets are always closed as they correspond to finite dimensional irreducible representations.

**Theorem 4.2.2** (Dauns-Hoffman [28, A.34]) Let $A$ be a $C^*$-algebra. For each $P \in \text{Prim}(A)$ let $\pi_P : A \to A/P$ be the quotient map. Then there is an isomorphism
\[ \varphi : C_b(\text{Prim}(A)) \to Z(M(A)), \text{ the center of the multiplier algebra of } A, \text{ such that} \]
for all \( f \in C_b(\text{Prim}(A)) \) and \( a \in A \),
\[ \pi_P(\varphi(f)a) = f(P)\pi_P(a) \]
for every \( P \in \text{Prim}(A) \).

Suppose has \( G \) has Property (T). Let \( S \subset \text{Prim}(C^*(G)) \) and let \( \chi_S \) denote the indicator function on \( S \) defined by
\[ \chi_S(x) = \begin{cases} 
1 & x \in S \\
0 & x \notin S 
\end{cases} \]
then Theorem 4.2.1 implies \( \chi_{\{\ker(\pi)\}} \in C_b(\text{Prim}(C^*(G))) \) and defines a projection for every irreducible, finite-dimensional representation \( \pi \) of \( G \). For each such \( \pi \) we then define the central projection \( P_\pi \in Z(C^*(G)) \) by \( \varphi(\chi_{\{\ker(\pi)\}}) = P_\pi \), where \( \varphi \) is the isomorphism provided by Theorem 4.2.2. Identifying the quotient map \( \pi_{\{\ker(\pi)\}} \) with \( \pi \) we have \( \pi(P_\pi a) = \pi(a) \) for all \( a \in C^*(G) \). The central projection \( P_\pi \) defines the Kazhdan projection associated to \( \pi \). Note in general, if \( A \) is a \( C^* \)-algebra and \( P \in A \) is a central projection, then \( A \) decomposes as a direct sum \( A = PAP \oplus (1 - P)A(1 - P) \).

Finally, recall if \( \pi \) and \( \sigma \) are finite-dimensional, irreducible representations, then \( \pi \) and \( \sigma \) have the same kernel if and only if they are unitarily equivalent. We now summarize these observations in the following theorem.

**Theorem 4.2.3** Let \( G \) be a group, \( \pi : C^*(G) \to \mathbb{M}_n(\mathbb{C}) \) be a finite-dimensional, irreducible representation and \( P_\pi \) be its associated central Kazhdan projection. Then,

1. \( \pi(P_\pi) = 1_n \);
2. \( P_\pi C^*(G)P_\pi \simeq \pi(C^*(G)) \) and \( (1 - P_\pi)C^*(G)(1 - P_\pi) = \ker(\pi) \);
3. \( C^*(G) \simeq \mathbb{M}_n(\mathbb{C}) \oplus \ker(\pi) \).
4. \( P_\pi P_\sigma = 0 \) if \( \sigma \) is any other irreducible, finite-dimensional representation not equivalent to \( \pi \).
Proof. The first statement follows immediately from the previous discussion. The third follows from the previous discussion and the second statement.

We now prove the second statement. Define the surjection \( \beta : C^\ast(G) \to P C^\ast(G) P \) by \( \beta(a) = p_\pi a \). It then suffices to show that \( \ker(\beta) = \ker(\pi) \) since \( \pi \) is also surjective.

First suppose \( a \in \ker(\beta) \) so that \( P_\pi a = 0 \). However, \( \pi(a) = \pi(P_\pi a) = 0 \). Thus, \( a \in \ker(\pi) \) and \( \ker(\beta) \subset \ker(\pi) \). Conversely, suppose \( \pi(a) = 0 \) for some \( 0 \neq a \in C^\ast(G) \). Assume \( 0 \neq \beta(a) = P_\pi a \). Choose an irreducible representation \( \sigma \) of \( C^\ast(G) \) for which \( \sigma(P_\pi) \neq 0 \). In particular, \( \sigma(P_\pi) \neq 0 \). Since \( 0 = \pi(a) = \pi(P_\pi a) \), \( \ker(\sigma) \neq \ker(\pi) \).

Dauns-Hoffman then implies that \( \sigma((1 - P_\pi)a) = \sigma(a) \) for every \( a \in C^\ast(G) \). But this means \( 0 \neq \sigma(P_\pi) = \sigma((1 - P_\pi)P_\pi) = \sigma(0) = 0 \), a contradiction.

For the fourth statement, if \( \pi \) and \( \sigma \) are unitarily inequivalent representations, then they have distinct kernels. Therefore, \( \chi_{\{\ker(\pi)\}} \chi_{\{\ker(\sigma)\}} = 0 \) and the corresponding statement for their Kazhdan projections follows from the isomorphism provided by Dauns-Hoffman.

Kazhdan projections allow finite-dimensional representations to be detected and decomposed inside the ambient algebra. They also provide a useful tool for detecting when representations have finite-dimensional sub-representations and for factoring finite-dimensional representations out of existing representations.

Noticing that the proof of Theorem 3.2.2 is much more general than stated brings us to the following theorem.

**Theorem 4.2.4** Let \( \pi \) be a quasidiagonal representation of \( C^\ast(G) \). Then \( i_G \) is weakly contained in \( \pi \otimes \bar{\pi} \).

Proof. Let \( F \subset G \) be finite and \( \epsilon > 0 \). Since \( \pi(G) \) is a quasidiagonal set there exists a non-zero finite-rank projection \( P \) such that \( \|\pi(g)P\pi(g)^* - P\| < \frac{\epsilon}{2} \) for every \( g \in F \).

Since \( P \) is a projection,

\[
\|\pi(g)P\pi(g)^* - P\|_{HS} \leq \|P - P\pi(g)P\pi(g)^*\|_{HS} + \|(P\pi(g)P - \pi(g)P)\pi(g)^*\|_{HS} \\
\leq \|P(P - \pi(g)P\pi(g)^*)\|_{HS} + \|P\pi(g)P - \pi(g)P\|_{HS}
\]

Therefore, for every \( g \in F \),

\[
\|\pi(g)P\pi(g)^* - P\|_{HS} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Thus, \( \pi \otimes \bar{\pi} \) is weakly contained in \( i_G \).
\[ \leq \|P\|_{HS}\|P - \pi(g)P\pi(g)^*\| + \|P\pi(g) - \pi(g)P\|\|P\|_{HS} \]
\[ \leq 2\|P\|_{HS}\|\pi(g)P\pi(g)^* - P\| \leq \epsilon \|P\|_{HS} \]

where we have used properties of the Hilbert-Schmidt norm: \(\|KT\|_{HS} \leq \|K\|_{HS}\|T\|\), \(\|TK\|_{HS} \leq \|K\|_{HS}\|T\|\), and \(\|uK\|_{HS} = \|K\|_{HS}\|u\|_{HS} = \|K\|_{HS}\) where \(T \in B(H)\) is an arbitrary operator, \(K\) is Hilbert-Schmidt, and \(u\) is a unitary [13, 2.4.10]. Therefore, for any \(F \subset G\) finite and \(\epsilon > 0\) we can find a finite-rank projection \(P\) such that

\[ \left\| (\pi \otimes \bar{\pi})(g)\left(\frac{P}{\|P\|_{HS}}\right) - \frac{P}{\|P\|_{HS}} \right\|_{HS} < \epsilon \]

for every \(g \in F\) and so we conclude using Prop. 2.2.2 that \(i_G \leq \pi \otimes \bar{\pi} \).

Corollary 4.2.5 If \(G\) has Property \((T)\) and \(\pi\) is a quasidiagonal representation of \(C^*(G)\), then \(\pi\) contains a finite dimensional sub-representation.

Proof. By Theorem 4.2.4, \(i_G\) is weakly contained in \(\pi \otimes \bar{\pi}\). By Property \((T)\) and Prop. 4.1.6, \(i_G \subset \pi \otimes \bar{\pi}\). By Prop. 2.2.3, \(\pi\) contains a finite dimensional sub-representation. \(\square\)

With these facts in hand, we are now able to state the main theorem of this chapter.

Theorem 4.2.6 Let \(G\) be a countably infinite, discrete group with Property \((T)\). If \(G\) has only finitely many unitary equivalence classes of finite dimensional unitary representations, then \(C^*(G)\) is not quasidiagonal.

Proof. Assume \(C^*(G)\) is quasidiagonal. Let \(F\) be the finite collection of Kazhdan projections in bijective correspondence with the unitary equivalence classes of irreducible, finite dimensional unitary representations of \(G\). Set \(Q = \sum_{P \in F} P\), which is also a central projection since the projections in \(F\) are mutually orthogonal. We then decompose \(C^*(G)\) using \(Q\) to obtain \(C^*(G) = QC^*(G)Q \oplus (1 - Q)C^*(G)(1 - Q)\). We claim \((1 - Q)C^*(G)(1 - Q)\) has no finite dimensional representations. If this were the case, then Cor. 4.2.5 produces a contradiction. Indeed, since \(C^*(G)\) is quasidiagonal,
so is \((1 - Q)C^*(G)(1 - Q)\) as it’s a subalgebra. We then find a faithful, quasidiagonal representation \(\pi\) of \((1 - Q)C^*(G)(1 - Q)\). Since \((1 - Q)C^*(G)(1 - Q)\) is also a direct summand, we may view \(\pi\) as a quasidiagonal representation on \(C^*(G)\). By Cor. 4.2.5, \(\pi\) would then have a finite dimensional subrepresentation which contradicts \((1 - Q)C^*(G)(1 - Q)\) having no finite dimensional representations.

We now show \((1 - Q)C^*(G)(1 - Q)\) has no finite dimensional representations. Suppose it did have a finite dimensional representation \(\pi\). By decomposing \(\pi\) into irreducible components we may assume \(\pi\) is irreducible and let \(P_\pi\) be its associated Kazhdan projection. Once again we view \(\pi\) as a representation of \(C^*(G)\) and so \(\pi(Q) = 0\) using the direct sum decomposition. But, by construction, \(P_\pi\) occurs in the sum \(Q\) since \(P_\pi \in F\). By positivity, \(\pi(P_\pi) = 0\) which contradicts the first part of Theorem 4.2.3.

**Example 4.2.7** ([29], [30]) Any lattice in \(Sp(n, 1)\) for \(n \geq 2\) has uncountably many infinite quotients which are simple and torsion. Any of these quotients have no non-trivial finite-dimensional representations. Indeed, let \(G\) be such a quotient group. Then by simplicity any non-trivial finite-dimensional representation of \(G\) would be necessarily faithful and, hence, \(G\) is a linear group. By the Tits alternative and Property (T) it would contain a free group contradicting torsion. Thus, none of these quotients produce group \(C^*\)-algebras which are quasidiagonal.

Unfortunately, many other groups with Property (T), such as \(SL_n(\mathbb{Z})\) for \(n \geq 3\), are residually finite and Theorem 4.2.6 cannot determine whether or not those groups produce quasidiagonal \(C^*\)-algebras.

While quasidiagonality provides a lower bound for the number of finite dimensional representations, it is interesting to note that Property (T) also provides an upper bound for finite-dimensional representations.

**Theorem 4.2.8** ([31, 3.1.3]) Let \(N \in \mathbb{N}\). There are at most finitely many unitary equivalence classes of \(N\)-dimensional irreducible representations of a group \(G\) with
Property (T). In particular, $G$ has at most countably many unitary equivalence classes of irreducible finite-dimensional representations.

4.3 Stronger Approximation Properties

While the issue of quasidiagonality for $C^*(SL_n(\mathbb{Z}))$ for $n \geq 3$ remains open, some progress has been made regarding conditions stronger than quasidiagonality.

**Theorem 4.3.1** ([32]) $C^*(SL_n(\mathbb{Z}))$ for $n \geq 3$ and $C^*(Sp_n(\mathbb{Z}))$ for $n \geq 2$ are not residually finite dimensional.

The statement of Theorem 4.3.1 in [32] is more general than what is stated here, but these particular examples will suffice for our use. It should be noted that the case for $SL_2(\mathbb{Z})$ is entirely different. Since free groups produce residually finite-dimensional full group $C^*$-algebras and $SL_2(\mathbb{Z})$ contains a finite-index free subgroup, it follows that $C^*(SL_2(\mathbb{Z}))$ is also residually finite-dimensional via induced representations.

We now record that Theorem 4.3.1 is enough to show such groups are not inner quasidiagonal, a condition stronger than quasidiagonality.

**Theorem 4.3.2** ([33]) Let $A$ be a separable $C^*$-algebra. The following are equivalent:

1. For every $F \subset A$ finite and $\epsilon > 0$, there is a representation $\pi : A \rightarrow B(\mathcal{H})$ and a finite-rank projection $P \in \pi(A)^\prime\prime$ such that $\|P\pi(x) - \pi(x)P\| < \epsilon$ and $\|P\pi(x)P\| > \|x\| - \epsilon$ for every $x \in F$.

2. $A$ has a separating family of irreducible quasidiagonal representations.

**Definition 4.3.3** ([33]) Let $A$ be a separable $C^*$-algebra. $A$ is said to be **inner quasidiagonal** if it satisfies either equivalent condition of Theorem 4.3.2.

Clearly every inner quasidiagonal $C^*$-algebra is also quasidiagonal. Also, every residually finite dimensional $C^*$-algebra is inner quasidiagonal.

**Corollary 4.3.4** Let $G$ be a countable, discrete group with Property (T). $C^*(G)$ is inner quasidiagonal if and only if $C^*(G)$ is residually finite dimensional. In particular, $C^*(SL_n(\mathbb{Z}))$ for $n \geq 3$ is not inner quasidiagonal.
**Proof.** Assume $C^*(G)$ is inner quasidiagonal. By Cor. 4.2.5, an irreducible quasidiagonal representation of $C^*(G)$ is necessarily finite-dimensional. Since this family is separating by assumption, $C^*(G)$ is therefore residually finite dimensional.

Bekka later improves Theorem 4.3.1 by showing that $C^*(SL_n(\mathbb{Z}))$ for $n \geq 3$ does not have a faithful trace [34]. The main ingredient for this improvement and the main result of [34] is a complete characterization of the traces on $C^*(SL_n(\mathbb{Z}))$ for $n \geq 3$. Recall a tracial state $\tau$ is said to be indecomposable if it is an extreme point of the tracial state space. Equivalently, $\tau$ is indecomposable if and only the corresponding GNS representation generates a finite factor.

A group $G$ is said to have the infinite conjugacy class property (ICC) if for every $e \neq g \in G$ the set $\{hgh^{-1} : h \in G\}$ is infinite. If $G$ is ICC, then $\lambda(G)$ generates a $II_1$-factor and so $\tau_\lambda$ is indecomposable.

**Theorem 4.3.5** ([34, Thm. 3]) Suppose $\tau$ is an indecomposable tracial state of $C^*(SL_n(\mathbb{Z}))$ for $n \geq 3$. Then $\tau = tr \circ \pi$ for some finite-dimensional representation $\pi$ or $\tau$ is induced from the trivial extension of a character on the center of $SL_n(\mathbb{Z})$.

This result has been generalized to other similar groups in [35]. With an argument distinct from Bekka’s, we observe that this basic set up is enough to show such groups do not have a faithful trace, and so this approach also applies to [35]. The core observation will be that if $C^*(G)$ has a faithful, amenable trace, then $C^*(G)$ is residually finite dimensional.

**Definition 4.3.6** ([5, 6.2.7]) Let $A$ be a unital, separable $C^*$-algebra and $\tau$ a tracial state on $A$. $\tau$ is said to be amenable if there exists a sequence of ucp maps $\{\varphi_n : A \to \mathcal{M}_{k(n)}(\mathbb{C})\}_n$ such that

$$\tau(a) = \lim_n tr \circ \varphi_n(a)$$

and

$$\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\|_2 \to 0$$

for all $a, b \in A$ where $\|a\|_2 = \sqrt{tr(a^*a)}$ and $tr$ is the usual normalized trace on matrices.
Observe every quasidiagonal trace and every finite-dimensional trace are amenable by Def. 3.2.4, and so any unital, quasidiagonal $C^*$-algebra has an amenable trace as well. Also, it is known that the set of amenable traces form a weak-$*$ closed and convex face of the tracial state space [5, 6.3.7].

It is also important to note that amenability of a trace is highly dependent on its domain. For example, $\tau_\lambda$ is amenable as a trace on $C^*_\lambda(G)$ if and only if $G$ is amenable [5, 6.3.3]. Amenability of $\tau_\lambda$ on $C^*(G)$ is much more complicated.

**Proposition 4.3.7** ([5, 3.7.10, 6.4.3]) *If $G$ is residually finite, then $\tau_\lambda$ is an amenable trace on $C^*(G)$.***

Kirchberg shows that if $G$ has Property (T), then the converse holds as well [36]. The following theorem is the main technical tool that is used for this.

**Theorem 4.3.8** ([5, 6.4.10]) *Let $G$ be a group with Property (T) and $\tau$ a tracial state on $C^*(G)$. $\tau$ is amenable if and only if there exists a sequence of finite-dimensional representations $\pi_n : C^*(G) \to M_{k(n)}(\mathbb{C})$ such that $\lim_n \text{tr} \circ \pi_n(a) = \tau(a)$ for every $a \in C^*(G)$.***

**Corollary 4.3.9** Let $G$ be a group with Property (T). $C^*(G)$ is residually finite dimensional if and only if $C^*(G)$ has a faithful, amenable trace $\tau$.

**Proof.** In general, if $A$ is a residually finite dimensional $C^*$-algebra, then $A$ has a faithful amenable trace by taking a limit of convex combinations of finite-dimensional traces.

Conversely, Theorem 4.3.8 implies there exists a sequence of finite-dimensional representations $\pi_n : C^*(G) \to M_{k(n)}(\mathbb{C})$ such that $0 < \tau(a^*a) = \lim_n \text{tr} \circ \pi_n(a^*a)$ for $0 \neq a \in C^*(G)$. Therefore, $\pi(a^*a) \neq 0$ for $n$ large enough and therefore $C^*(G)$ is residually finite dimensional.

For simplicity we will use $C^*(\text{SL}_n(\mathbb{Z}))$ for $n$ odd as a model for our next theorem. In this case, the center of $\text{SL}_n(\mathbb{Z})$ is trivial and so Theorem 4.3.5 implies the only trace induced from a central character is the trace coming from the trivial representation
on the center. That is, an indecomposable trace on $C^*(SL_n(Z))$ for $n \geq 3$ and $n$ odd is either from a finite-dimensional representation or is $\tau_\lambda$, the canonical trace from the regular representation.

**Theorem 4.3.10** Let $G$ be a residually finite group with Property (T) and suppose $C^*(G)$ has a faithful trace. Suppose further that every indecomposable trace on $C^*(G)$ is either of the form $\tau_\lambda$ or finite-dimensional. Then $C^*(G)$ is residually finite dimensional. In particular, $C^*(SL_3(Z))$ does not have a faithful trace.

*Proof.* Since $G$ is residually finite, $\tau_\lambda$ is amenable and so every indecomposable trace on $C^*(G)$ is amenable. Since the collection of amenable traces is weak-$*$ closed and convex, it follows every trace is amenable. Cor. 4.3.9 now applies. \qed

**Remark 4.3.11** M. Yamashita observed in [26] that Theorem 4.3.8 proves Theorem 4.2.6 when $G$ is simple. The same idea can be used to provide an alternative proof of Theorem 4.2.6 in the general case. The key point is that Yamashita uses Theorem 4.3.8 rather than Theorem 4.2.4 to produce a non-zero finite dimensional representation of the unital direct summand $(1 - Q)C^*(G)(1 - Q)$.

Unfortunately little is known in the general case when, if ever, an infinite group with Property (T) produces a residually finite dimensional $C^*$-algebra. In fact, Lubotzky and Shalom raise this question in slightly different terminology.

**Question** ([37, 6.5]) Does there exist an infinite discrete group $G$ with Property (T) such that $C^*(G)$ is residually finite dimensional?

Note $C^*(G)$ is residually finite dimensional if and only if the set of kernels of finite-dimensional representations in $Prim(C^*(G))$ is dense. Indeed, $\bigcap_\pi \ker(\pi) = \{0\}$, where this intersection is taken over all irreducible, finite dimensional representations, if and only if finite dimensional representations separate points. On the other hand, Property (T) means that each kernel of an irreducible finite-dimensional representation is isolated in $Prim(C^*(G))$. While these two viewpoints seem mutually exclusive, it remains unclear how to proceed.
However, it is interesting to note that, due to Kazhdan projections, \( C^*(G) \) being residually finite dimensional has a purely algebraic characterization. Recall an ideal \( J \triangleleft A \) is said to be essential if \( aJ = 0 \) implies \( a = 0 \) for \( a \in A \).

**Proposition 4.3.12** Let \( G \) be a group with Property (T) and let \( J_f \) denote the ideal of \( C^*(G) \) generated by all Kazhdan projections. The following are equivalent:

1. \( C^*(G) \) is residually finite dimensional;
2. \( J_f \triangleleft C^*(G) \) is an essential ideal.

**Proof.** Assume \( C^*(G) \) is residually finite dimensional and suppose \( aJ_f = 0 \) for some \( 0 \neq a \in C^*(G) \). Then, in particular, \( aP = 0 \) for every Kazhdan projection \( P \). Since \( C^*(G) \) is residually finite dimensional, there exists a finite dimensional representation \( \pi \) such that \( \pi(a) \neq 0 \) and so the Kazhdan projection \( P_\pi a \neq 0 \), a contradiction. Therefore, \( a = 0 \) and \( J_f \) is essential.

Conversely, if \( J_f \) is essential then for every \( 0 \neq a \in C^*(G) \) there exists a Kazhdan projection \( P_\pi \) such that \( P_\pi a \neq 0 \) and thus \( \pi(a) \neq 0 \). Therefore, \( C^*(G) \) is residually finite dimensional. \( \square \)

**Proposition 4.3.13** Let \( G \) be a group with Property (T) and let \( J_f \) denote the ideal of \( C^*(G) \) generated by all Kazhdan projections. If \( C^*(G) \) is residually finite dimensional, then \( C^*(G)/J_f \) is MF.

**Proof.** If \( G \) is finite, then \( J_f = C^*(G) \) and there is nothing to prove, so we may assume \( G \) is infinite. By assumption and Prop. 4.3.12, \( J_f \subset C^*(G) \) is an essential ideal and so \( C^*(G) \hookrightarrow \mathcal{M}(J_f) \), where \( \mathcal{M}(J_f) \) denotes the multiplier algebra of \( J_f \). This then induces an inclusion into the corona algebra \( C^*(G)/J_f \hookrightarrow \mathcal{M}(J_f)/J_f \).

Let \( \{ P_n \} \) be the sequence of all Kazhdan projections in \( C^*(G) \). By Theorem 4.2.3 we may identify \( P_nC^*(G) \) with \( \mathbb{M}_{k(n)}(\mathbb{C}) \). By definition of \( J_f \) and using the fact that Kazhdan projections are mutually orthogonal, we may then identify \( J_f \) with the orthogonal direct sum \( \sum_n P_nC^*(G) \simeq \sum_n \mathbb{M}_{k(n)}(\mathbb{C}) \). Therefore,

\[
\mathcal{M}(J_f) \simeq \prod_n P_nC^*(G) \simeq \prod_n \mathbb{M}_{k(n)}(\mathbb{C})
\]
and passing to the corona algebra

\[ C^*(G)/J_f \hookrightarrow \mathcal{M}(J_f)/J_f \cong \frac{\prod_n P_n C^*(G)}{\sum_n P_n C^*(G)} \cong \frac{\prod_n \mathcal{M}_{k(n)}(\mathbb{C})}{\sum_n \mathcal{M}_{k(n)}(\mathbb{C})} \]

which completes the proof.

As seen by investigating the proof of the above proposition, something stronger than the MF property is being shown as the matricial approximations are entirely internal to the algebra. It’s not clear how to use this extra strength to our advantage.

While it is often not clear what properties $C^*(G)$ may or may not have for Property (T) groups, it is possible to produce pathological $C^*$-algebras from these group $C^*$-algebras. For example, one can start with a residually finite group with Property (T) and produce a MF algebra which is not quasidiagonal as shown in [38], see also [5, 17.3.3].
5. Property (QH)

We begin this chapter by reviewing the definition and fundamental properties of Property (QH). We will then provide a new class of groups with Property (QH) by showing that wreath products of groups with Property (QH) also has Property (QH).

5.1 Definition and Examples

Property (QH) is introduced by Dadarlat and Pennig in [39] after continued study of the possibility of unsuspending E-Theory, which we will now briefly review. Let $A$ and $B$ be separable $C^*$-algebras. An asymptotic morphism is a family of linear maps $\{\varphi_t : A \rightarrow B\}_t$ parametrized by $t \in [0, \infty)$ such that $t \mapsto \varphi_t(a)$ is continuous for each $a \in A$ and the family $\{\varphi_t\}_{t \in [0,\infty)}$ is asymptotically multiplicative and asymptotically $*$-preserving. Equivalently, a family $\{\varphi_t\}_{t \in [0,\infty)}$ is an asymptotic morphism if it defines a $*$-homomorphism $\Phi : A \rightarrow C_b([0,\infty),B)/C_0([0,\infty),B)$. A homotopy of asymptotic homomorphisms $(\varphi_t)$ and $(\psi_t)$ is an asymptotic morphism $\Psi_t : A \rightarrow C[0,1] \otimes B$ with $ev_0 \circ \Psi_t = \varphi_t$ and $ev_1 \circ \Psi_t = \psi_t$. Let $[[A,B]]$ denote the homotopy classes of asymptotic morphisms from $A$ to $B$. E-Theory as introduced by Connes and Higson [40] is then defined as the group $E(A,B) = [[[SA,SB \otimes K]]]$, where $K$ denotes the compact operators on a separable, infinite dimensional Hilbert space and $SA$ denotes the suspension of $A$ given by $SA = C_0(0,1) \otimes A$. If $A$ is nuclear, then $E(A,B)$ is isomorphic to Kasparov’s $KK(A,B)$.

The stabilization via the compact operators in the second variable of $E(A,B) = [[[SA,SB \otimes K]]]$ introduces a monoid structure via direct sum and the suspensions provide the remaining group structure. The suspensions do more, however, since the quasidiagonality of $SA$ guarantees the existence of asymptotic morphisms. The
question is raised in [41] when $[[A, B \otimes K]] \simeq E(A, B)$, that is, they ask when E-theory can be ‘unsuspended.’

**Definition 5.1.1** ( [41]) A separable $C^*$-algebra $A$ is said to be homotopy symmetric if $[id_A] \in [[A, A \otimes K]]$ is invertible.

**Theorem 5.1.2** ( [41]) Let $A$ be a separable $C^*$-algebra. If $A$ is homotopy symmetric, then whenever $B$ is a separable $C^*$-algebra, the natural map $[[A, B \otimes K]] \to E(A, B)$ is an isomorphism.

In this sense, the homotopy symmetric condition allows ‘unsuspension’ of E-theory. Unfortunately it is difficult to check when a $C^*$-algebra is homotopy symmetric. Much later, Dadarlat and Pennig introduce the notion of Property (QH) to assist with this difficulty.

Before progressing we require some terminology. We say an asymptotically multiplicative sequence of ccp maps between $C^*$-algebras $\{\varphi_n : A \to B_n\}_n$ is injective if the induced $*$-homomorphism

$$\Phi : A \to \prod_n B_n / \sum_n B_n$$

is injective. Equivalently, if $\limsup_n \|\varphi_n(a)\| = \|a\|$ or, since a $*$-homomorphisms is isometric if and only if it is injective, $\limsup_n \|\varphi_n(a)\| \neq 0$ for every $a \in A$.

Two asymptotically multiplicative sequences of ccp maps $\{\varphi_n : A \to B_n\}_n$ and $\{\psi_n : A \to B_n\}_n$ are said to be homotopic if there exists an asymptotically multiplicative sequence of ccp maps $\{\Psi : A \to C[0,1] \otimes B_n\}_n$ with $ev_0 \circ \Psi_n = \varphi_n$ and $ev_1 \circ \Psi_n = \psi_n$ for every $n$.

An asymptotically multiplicative sequence of ccp maps $\{\varphi_n\}_n$ is said to be null-homotopic if it is homotopic to the zero map. That is, there is an asymptotically multiplicative sequence of ccp maps $\{\Psi_n : A \to C_0[0,1] \otimes B_n\}_n$ with $ev_0 \circ \Psi_n = \varphi_n$ for every $n$. Recall $CA = C_0[0,1] \otimes A$, called the cone over $A$, is a contractible, quasidiagonal $C^*$-algebra.

**Theorem 5.1.3** ( [39]) Let $A$ be a separable $C^*$-algebra. The following are equivalent:
1. There exists a null-homotopic, asymptotically multiplicative, and injective sequence of ccp maps \( \{ \eta_n : A \to \mathcal{K} \} \);

2. There exists a null-homotopic, asymptotically multiplicative, and injective sequence of ccp maps \( \{ \gamma_n : A \to B(\mathcal{H}) \} \);

3. There exists an injective \( \ast \)-homomorphism
\[
\eta : A \to \frac{\prod_n CB(\mathcal{H})}{\sum_n CB(\mathcal{H})}
\]
which is liftable to a contractive, completely positive map \( A \to \prod_n CB(\mathcal{H}) \);

4. There exists an injective \( \ast \)-homomorphism
\[
\eta : A \to \frac{\prod_n CK}{\sum_n CK}
\]
which is liftable to a contractive, completely positive map \( A \to \prod_n CK \),

where \( \mathcal{K} \) denotes the compact operators on a separable, infinite dimensional Hilbert space.

**Definition 5.1.4** ([39, 2.5]) A separable \( C^\ast \)-algebra \( A \) is said to have Property (QH) if it satisfies either equivalent condition of Theorem 5.1.3.

Notably, this new definition has striking similarity to previous approximation properties such as quasidiagonality or MF. Indeed, Property (QH) implies quasidiagonality. Moreover, Property (QH) provides an alternative access point to study homotopy symmetry.

**Theorem 5.1.5** ([39, 3.1]) Let \( A \) be a separable, nuclear \( C^\ast \)-algebra. \( A \) has Property (QH) if and only if \( A \) is homotopy symmetric.

Theorem 5.1.5 then has many immediate consequences not obvious from the definition of homotopy symmetry such as homotopy symmetry passing to subalgebras, minimal tensor products, and direct limits [39, 3.3]. This also provides many examples of homotopy symmetric, equivalently Property (QH), \( C^\ast \)-algebras such as \( C_0(X \setminus \{x_0\}) \) for a compact, connected, metrizable space \( X \).
Another immediate consequence from Theorem 5.1.3 is that Property (QH) implies there are no non-zero projections and so connectedness in the previous example is necessary.

Theorem 5.1.3 also indicates why Property (QH) is thought of as a finite dimensional deformation property. Indeed, given \( \{\eta_n : A \to K\}_n \) as in the first part of Theorem 5.1.3 we use quasidiagonality of \( K \) to produce a sequence \( \{P_n\}_n \) of finite rank projections converging strongly to 1 which produces a new asymptotically multiplicative, null-homotopic and injective sequence of ccp maps \( \{\varphi_n : A \to M_{k(n)}(\mathbb{C})\}_n \) via \( \varphi_n = P_n \eta_n P_N \). We then record this as the next proposition.

**Proposition 5.1.6** ([39, 2.2]) Let \( A \) be a separable \( C^\ast \)-algebra. \( A \) has Property (QH) if and only if there exists an asymptotically multiplicative, null-homotopic and injective sequence of ccp maps \( \{\varphi_n : A \to M_{k(n)}(\mathbb{C})\}_n \).

Although it does represent a finite dimensional deformation property, one of the unique strengths of Property (QH) is that finite dimensional deformations are equivalent to infinite dimensional ones. Of course, in infinite dimensions one has more freedom.

Before progressing it will be advantageous to clear up a technical issue with the relationship between an injective sequence and an asymptotically isometric sequence. Note being asymptotically isometric implies injectivity, but the converse does not hold. This defect can largely be corrected.

**Proposition 5.1.7** Let \( A \) be a separable \( C^\ast \)-algebra. \( A \) has Property (QH) if and only if there exists an asymptotically multiplicative sequence of ccp maps \( \{\psi_n : A \to B(\mathcal{H})\}_n \) that is null-homotopic and asymptotically isometric.

**Proof.** Since being asymptotically isometric implies injective, we only need to prove the converse.

Assume \( A \) has Property (QH). For brevity we will denote the composition \( \text{ev}_t \circ \varphi \) as \( \varphi^t \). By definition there exists an asymptotically multiplicative sequence of ccp maps \( \{\varphi_n : A \to C_0[0,1] \otimes B(\mathcal{H})\}_n \) with \( \{\varphi^0_n\}_n \) injective. The same argument used
to prove Prop. 3.1.6 allows us to define for a ccp map $\psi^0_k(a) = \bigoplus_{i \in F_k} \varphi^0_i(a)$ for $k \in \mathbb{N}$ where $F_k \subset \mathbb{N}$ is an appropriate finite set of indices as provided by the proof of 3.1.6. We then have that the sequence $\{\psi^0_k : A \to \bigoplus_{i \in F_k} B(\mathcal{H})\}_k$ is asymptotically multiplicative and asymptotically isometric.

We now check that the null-homotopic property is preserved. Define $\psi^t_k(a) = \bigoplus_{i \in F_k} \varphi^t_i(a)$ for $a \in A$. Since $\psi^1_k = 0$, we only need to show for fixed $a \in A$ that the map $t \mapsto \psi^t_k(a)$ is continuous. This follows from the observation that

$$\|\psi^t_k(a) - \psi^s_k(a)\| = \max_{i \in F_k} \|\varphi^t_i(a) - \varphi^s_i(a)\|$$

combined with continuity of $t \mapsto \varphi^t_i(a)$ for each $i \in F_k$. Summarizing, we have constructed an asymptotically multiplicative sequence of ccp maps

$$\{\psi_k : A \to C_0[0,1] \otimes \bigoplus_{i \in F_k} B(\mathcal{H})\}_k$$

with the sequence $\{\psi^0_k\}_k$ injective. Choosing a sufficiently large Hilbert space $\mathcal{H}'$ such that $\bigoplus_{i \in F_k} B(\mathcal{H}) \subset B(\mathcal{H}')$ for every $k$ then completes the proof. \qed

5.2 Group $C^*$-Algebras

To produce non-commutative examples of $C^*$-algebras with Property (QH) we turn to $C^*$-algebras associated to discrete, countable, torsion-free, amenable groups. Specifically, we seek to study when $I(G) = \ker(i_G)$ has Property (QH). Observe that if $G$ is not torsion-free and $g^n = e$, then $1 - \frac{1}{n} \sum_{i=0}^{n-1} g^i \in I(G)$ defines a non-zero projection and so the torsion-free assumption is necessary.

In specializing Property (QH) to groups it will be useful for technical reasons to characterize it on $C^*(G)$ rather than $I(G)$. Notice the unitization of $I(G)$ is $C^*(G)$. Since passing to unitizations produces ucp maps from ccp maps [5, 2.2.1]. Hence, combined with Prop. 5.1.7, the next proposition amounts to the easy verification that being multiplicative and isometric is preserved.
Proposition 5.2.1 Let $G$ be a group. $I(G)$ has Property (QH) if and only if there is an asymptotically multiplicative and asymptotically isometric sequence of ucp maps \( \{ \pi_n : C^*(G) \to M_{k(n)}(\mathbb{C}) \}_n \) which are homotopic to multiples of the trivial representation. Equivalently, $M_{k(n)}(\mathbb{C})$ may be replaced by $B(H)$ for some separable, infinite dimensional Hilbert space $H$.

Recall in the abelian case, $C^*(G) \simeq C(\hat{G})$ where $\hat{G}$ denotes the Pontryagin dual of $G$, and $\hat{G}$ is connected if and only if $G$ is torsion-free. Therefore, if $G$ is torsion-free and abelian, then $I(G)$ has Property (QH). Using this fact and considerations of continuous fields, Dadarlat and Pennig then prove the next theorem.

Theorem 5.2.2 ([39, 4.3]) Let $G$ be a countable, torsion-free, nilpotent group. Then $I(G)$ has Property (QH).

With many parallels between quasidiagonality and Property (QH), then following question then seems natural and was conjectured to have a positive answer by Dadarlat in [2].

Question ([2]) Does $I(G)$ have Property (QH) for every torsion-free, amenable group $G$?

To this end we will show in the next section that wreath products of groups with Property (QH) also have Property (QH). These groups are essentially never nilpotent and serve as a convenient contrast with Theorem 5.2.2.

5.3 Wreath Products

The main result of this section and of this chapter is that Property (QH) passes to wreath products. Throughout this section, all groups are assumed be amenable, discrete, and torsion-free. The content of this section is joint work with Marius Dadarlat and Ulrich Pennig.
Definition 5.3.1 Let $G$ and $H$ be groups. The reduced wreath product or simply wreath product of $G$ and $H$, denoted $G \wr H$, is defined to be the semi-direct product
$$(\bigoplus_H G) \rtimes_{\beta} H$$
where $\beta(h_0)(gh)_h = (g_{h_0^{-1}}h)_h$.

Theorem 5.3.2 Let $G$ and $H$ be discrete, amenable groups. If $I(G)$ and $I(H)$ have Property (QH), then $I(G \wr H)$ has Property (QH).

To prove and explain this theorem we will first require the language of crossed products to perform a necessary reduction.

Since $G \wr H$ is given by a semi-direct product, $C^*(G \wr H)$ is then given by a crossed product, a construction which generalizes the group $C^*$-algebra construction and may be thought as a group $C^*$-algebra with coefficients. We now recall this construction and some key properties.

Let $A$ be a $C^*$-algebra and $G$ a group acting on $A$ by automorphisms, that is, we have a homomorphism $\alpha : G \to \text{Aut}(A)$. We then seek to complete the group ring with coefficients in $A$, the $*$-algebra $A[G]$, consisting of finite sums of the form $\sum_{g \in G} a_g g$ with $a_g \in A$ endowed with multiplication given by

$$
\left( \sum_{g \in G} a_g g \right) \left( \sum_{h \in G} b_h h \right) = \sum_{g,h \in G} a_g \alpha_g(b_h) g h
$$

and with involution given by

$$
\left( \sum_{g \in G} a_g g \right)^* = \sum_{g \in G} \alpha_g(a_g^*) g^{-1}.
$$

A covariant $*$-representation is a pair $\pi : A \to B(\mathcal{H})$ and a unitary representation of $G, g \mapsto u_g \in U(\mathcal{H})$ such that $u_g \pi(a) u_g^* = \pi(\alpha_g(a))$. Just as in the group $C^*$-algebra construction, there is a reduced and universal completion using covariant representations. Due to amenability, we need not distinguish the two and the crossed product will be denoted by $A \rtimes_\alpha G$ [5, 4.2.6]. Rather than provide an exact construction, it will suffice for our purposes to observe the crossed product $A \rtimes_\alpha G$ is the $C^*$-algebra with a dense subset comprised of finite sums of the form $\sum_{g \in G} a_g u_g$, $a_g \in A$, with relations $u_g u_h = u_{gh}$ and $u_g a u_g^* = \alpha_g(a)$. Then $A \rtimes_\alpha G$ contains an isomorphic copy.
of $A$ and contains $C^*(G)$ if $A$ is unital. Also note $\mathbb{C} \rtimes G \simeq C^*(G)$, where the action is trivial.

Finally, given an action $\alpha : H \to \text{Aut}(N)$ on groups, it then lifts to an action on $C^*(N)$ and so we have that $C^*(N \rtimes H) \simeq C^*(N) \rtimes_{\alpha} H$ as they satisfy the same universal property.

**Lemma 5.3.3** Let $G = N \rtimes H$ be an amenable group. If $I(H)$ and $I(N) \rtimes H$ both have Property (QH), then $I(G)$ has Property (QH).

**Proof.** First observe the action of $H$ on $N$ induces an action on $I(N)$ and so the crossed product $I(N) \rtimes H$ is well-defined. Now we have a split exact sequence:

$$0 \to I(N) \to C^*(N) \xrightarrow{i_N} \mathbb{C} \to 0$$

Since $H$ is amenable and, in particular, exact we obtain a new split exact sequence

$$0 \to I(N) \rtimes H \to C^*(G) \xrightarrow{\pi} C^*(H) \to 0$$

where $\pi : G \to H$ denotes the quotient map using [5, 5.1.10]. Let $\pi' : I(G) \to I(H)$ denote its well-defined restriction. A diagram chase then yields that there is a well-defined inclusion $\ker(\pi') \hookrightarrow I(N) \rtimes H$ and so $\ker(\pi')$ has Property (QH) by assumption. Since the sequence

$$0 \to \ker(\pi') \to I(G) \xrightarrow{\pi'} I(H) \to 0$$

is also split exact, then [39, 3.3.(d)] implies $I(G)$ has Property (QH) since we have assumed $I(H)$ has Property (QH). \qed

By Lemma 5.3.3, to prove Theorem 5.3.2 it suffices to show $I(\bigoplus_H G) \rtimes_{\beta} H$ has Property (QH). Since they share the same universal property, we note that $C^*(\bigoplus_H G) \simeq \bigotimes_H C^*(G)$ where the infinite tensor product $\bigotimes_H C^*(G)$ is defined as the direct limit of finite tensor products with inclusions given by extending $a = \bigotimes_{h \in F} a_h$ to $a = \bigotimes_{h \in H} a_h$ via $a_h = 1$ for $h \notin F$. Elements of $\bigotimes_H C^*(G)$ of the form $a = \bigotimes_{h \in H} a_h$ with $a_h = 1$ for all but finitely many $h \in H$ will be called elementary
tensors. Due to the direct limit structure, the linear span of all elementary tensors is dense in \( \bigotimes_H C^*(G) \).

We now collect a few useful and straightforward estimates. The first says that if a map is almost multiplicative on \( F \), then it’s almost multiplicative on linear combinations of elements in \( F \). The second and third will be useful for performing estimates on elementary tensors.

**Proposition 5.3.4**

1. Let \( \varphi : A \rightarrow B \) be a \((F, \epsilon)\)-multiplicative ucp map. Let \( F_{N,R} \subset A \) denote a collection of elements \( x \in A \) of the form \( x = \sum_{n=1}^{N} \lambda_i x_i \) with \( x_i \in F, \lambda_i \in \mathbb{C} \) and \( |\lambda_i| \leq R \). Then \( \varphi \) is \((F_{N,R}, N^2R^2\epsilon)\)-multiplicative.

2. Let \( A_1, \ldots, A_N \) be \( C^* \)-algebras, \( a_n, b_n \in A_n \) for each \( n \), and \( M \) a constant greater than \( \|a_n\| \) and \( \|b_n\| \) for every \( n \). Then
   \[
   \| \bigotimes_{n=1}^{N} a_n - \bigotimes_{n=1}^{N} b_n \| \leq M^{N-1} \sum_{n=1}^{N} \|a_n - b_n\|
   \]

3. With the same assumptions as (2), let \( \{\varphi_n : A_n \rightarrow B_n\}_{n=1}^{N} \) be a finite collection of ucp maps between \( C^* \)-algebras. Then
   \[
   \| \bigotimes_{n=1}^{N} \varphi_n(a_n b_n) - \bigotimes_{n=1}^{N} \varphi_n(a_n) \varphi_n(b_n) \| \leq M^{2N-2} \sum_{n=1}^{N} \| \varphi_n(a_n b_n) - \varphi_n(a_n) \varphi_n(b_n) \|
   \]

**Proof.**

1. Let \( x, y \in F_{N,R} \) and write \( x = \sum_{n=1}^{N} \lambda_i x_i, y = \sum_{n=1}^{N} \mu_i y_i \) with \( x_i, y_i \in F, \lambda_i, \mu_i \in \mathbb{C} \) and \( |\lambda_i|, |\mu_i| \leq R \). Then by linearity,
   \[
   \| \varphi(xy) - \varphi(x) \varphi(y) \| = \| \sum_{i,j=1}^{N} \lambda_i \mu_j \varphi(x_i y_j) - \sum_{i,j=1}^{N} \lambda_i \mu_j \varphi(x_i) \varphi(y_j) \|
   \]
   \[
   = \| \sum_{i,j=1}^{N} \lambda_i \mu_j (\varphi(x_i y_j) - \varphi(x_i) \varphi(y_j)) \|
   \]
   \[
   \leq \sum_{i,j=1}^{N} |\lambda_i| |\mu_j| \| \varphi(x_i y_j) - \varphi(x_i) \varphi(y_j) \| \leq N^2 R^2 \epsilon
   \]

2. Let \( c = \bigotimes_{n=2}^{N} a_n \) and \( d = \bigotimes_{n=2}^{N} b_n \). Then,
   \[
   \|a_1 \otimes c - b_1 \otimes d\| \leq \|a_1 \otimes c - b_1 \otimes c\| + \|b_1 \otimes c - b_1 \otimes d\|
   \]
\[ \leq \|a_1 - b_1\|\|c\| + \|b_1\|\|c - d\| \]

since the tensor product norm is cross \([5, 3.4.10]\). Since \(\|c\| \leq M^{N-1}\) the claim then follows by induction.

3. This follows from (2).

\[\square\]

**Lemma 5.3.5** ([42, 4.1.9]) Let \(A\) and \(B\) be \(C^*\)-algebras and \(0 \neq D \subset A \otimes_{\min} B\) be a hereditary subalgebra. Then there is \(0 \neq x \in A \otimes_{\min} B\) such that \(xx^* \in D\) and \(x^*x = a \otimes b\) for some \(a \in A\) and \(b \in B\).

**Proposition 5.3.6** Let \(\{\varphi_n : A \to C\}_n\) and \(\{\psi_n : B \to D\}_n\) be sequences of asymptotically multiplicative \(\text{ucp}\) maps. Then the sequence of \(\text{ucp}\) maps \(\{\varphi_n \otimes \psi_n : A \otimes_{\min} B \to C \otimes_{\min} D\}_n\) is asymptotically multiplicative. If \(\{\varphi_n\}_n\) is also asymptotically isometric and \(\{\psi_n\}_n\) is also injective, then \(\{\varphi_n \otimes \psi_n\}_n\) is also injective.

**Proof.** The map \(\varphi_n \otimes \psi_n\) is \(\text{ucp}\) for every \(n\) by \([5, 3.5.3]\). The sequence \(\{\varphi_n \otimes \psi_n\}\) is asymptotically multiplicative due to Prop. 5.3.4 and linearity.

We now further assume that \(\{\varphi_n\}_n\) is asymptotically isometric and \(\{\psi_n\}_n\) is injective. Since \(\{\varphi_n \otimes \psi_n\}\) is asymptotically multiplicative, this defines a \(*\)-homomorphism

\[ \Phi : A \otimes_{\min} B \to \frac{\prod_n C \otimes_{\min} D}{\sum_n C \otimes_{\min} D} \]

with

\[ \|\Phi(x)\| = \limsup_n \|(\varphi_n \otimes \psi_n)(x)\| \]

By definition, injectivity of \(\Phi\) is equivalent to injectivity of the sequence \(\{\varphi_n \otimes \psi_n\}_n\).

Let \(J\) denote the kernel of \(\Phi\) and assume \(J \neq 0\). Since \(J\) is an ideal of \(A \otimes_{\min} B\), it is also a hereditary subalgebra by \([13, 3.2.3]\). Let \(x\) be the element produced by Lemma 5.3.5 applied to \(J\). That is, \(0 \neq x \in A \otimes_{\min} B\), \(xx^* \in J\), and \(x^*x = a \otimes b\). We then have

\[ 0 = \|\Phi(xx^*)\| = \|\Phi(x)\|^2 = \|\Phi(x^*x)\| \]
\[
= \| \Phi(a \otimes b) \| = \limsup_n \| \varphi_n(a) \otimes \psi_n(b) \|
\]
\[
= \limsup_n \| \varphi_n(a) \| \| \psi_n(b) \|
\]
\[
= \| a \| \limsup_n \| \psi_n(b) \|
\]
\[
= \| a \| \| b \|
\]
since \( \lim_n \| \varphi_n(a) \| = \| a \| \). But if \( a \) or \( b \) were zero, then \( x \) would be zero, which is a contradiction. Therefore, \( J = 0 \) and the sequence \( \{ \varphi_n \otimes \psi_n \}_n \) is injective. \( \square \)

Applying Prop. 5.3.6 inductively yields the following corollary with minimal tensor products.

**Corollary 5.3.7** Let \( A \) and \( D \) be \( C^* \)-algebras and \( \{ \varphi_n : A \rightarrow D \} \) be an asymptotically multiplicative sequence of ucp maps. Then the sequence of ucp maps \( \{ \bigotimes_F \varphi_n : \bigotimes_F A \rightarrow \bigotimes_F D \}_n \) is also asymptotically multiplicative for any finite index set \( F \). If \( \{ \varphi_n \}_n \) is also asymptotically isometric, then \( \{ \bigotimes_F \varphi_n \}_n \) is injective.

We now have all the tools to prove the main technical lemma.

**Lemma 5.3.8** Let \( G \) and \( H \) be amenable groups. If \( I(G) \) has Property \( (QH) \), then there exists an asymptotically multiplicative sequence of ucp maps \( \{ \sigma_n : \bigotimes_H C^*(G) \rightarrow C[0,1] \otimes \bigotimes_H B(H) \}_n \) with the following properties:

1. The asymptotically multiplicative sequence of ucp maps \( \{ ev_0 \circ \sigma_n \}_n \) is injective;
2. \( ev_1 \circ \sigma_n = \bigotimes_H i_G \) for every \( n \);
3. \( \sigma_n(\beta_h(a)) = \alpha_h(\sigma_n(a)) \) for every \( n, h \in H \) and \( a \in \bigotimes_H C^*(G) \),

where \( \alpha \) is the shift action on \( C[0,1] \otimes \bigotimes_H B(H) \) defined by \( \alpha_k(f \otimes (\otimes_h b_h)) = f \otimes (\otimes_h b_{k-1}h) \).

**Proof.** Using Prop. 5.2.1 we have an asymptotically multiplicative sequence of ucp maps \( \{ \varphi_n : C^*(G) \rightarrow C[0,1] \otimes B(H) \}_n \) with the properties \( \{ ev_0 \circ \varphi_n \}_n \) is asympt-
totically isometric and \( ev_1 \circ \varphi_n = i_G \) for every \( n \). For brevity we will denote the composition \( ev_t \circ \varphi_n \) by \( \varphi_n^t \). For every \( t \in [0, 1] \) and \( n \) define

\[
\rho_n^t = \bigotimes_H \varphi_n^t
\]

so we have

\[
\rho_n^t : \bigotimes_H C^*(G) \to \bigotimes_H B(H)
\]

is a well-defined, ucp map. Cor. 5.3.7 then gives that for each \( t \in [0, 1] \) the sequence \( \{\rho_n^t\}_n \) is asymptotically multiplicative and \( \{\rho_n^0\}_n \) is also injective.

We now endeavor to show that for each \( a \in \bigotimes_H C^*(G) \) and \( n \) the map \( t \mapsto \rho_n^t(a) \) is continuous. Since \( \bigotimes_H C^*(G) \) has the structure of a direct limit we may, because of linearity, assume \( a = \bigotimes_h a_h \) with \( a_h = 1 \) whenever \( h \not\in F \) where \( F \subset H \) is some finite subset. Choose a constant \( M > \max_{h \in H} \|a_h\| \), then Prop. 5.3.4 implies

\[
\|\rho_n^s(a) - \rho_n^t(a)\| = \|\bigotimes_{h \in F} \varphi_n^s(a_h) - \bigotimes_{h \in F} \varphi_n^t(a_h)\| \\
\leq M|F|^{-1} \sum_{h \in F} \|\varphi_n^s(a_h) - \varphi_n^t(a_h)\|
\]

and hence the continuity of \( t \mapsto \rho_n^t(a) \) follows from the continuity of \( t \mapsto \varphi_n^t(a_h) \) for each \( h \in F \).

Thanks to continuity we may now define

\[
\sigma_n : \bigotimes_H C^*(G) \to C[0, 1] \otimes \bigotimes_H B(H)
\]

by

\[
\sigma_n(a) = (t \mapsto \rho_n^t(a))
\]

and this produces a well-defined, ucp map due to Stinespring’s Dilation Theorem [5, 1.5.3]. We now check that the sequence \( \{\sigma_n\}_n \) satisfies all the desired properties.

First we check multiplicativity. Let \( a, b \in \bigotimes_H C^*(G) \). We begin by assuming we can write \( a = \bigotimes_h a_h \) and \( b = \bigotimes_h b_h \) with \( a_h, b_h = 1 \) whenever \( h \not\in F \) where \( F \subset H \)
is some finite subset. Choose a constant $M > \max_{h \in H} \|a_h\|$ and $M > \max_{h \in H} \|b_h\|$.

Then by compactness and Prop. 5.3.4,

$$\|\sigma_n(ab) - \sigma_n(a)\sigma_n(b)\| = \sup_{t \in [0,1]} \|\rho^t_n(ab) - \rho^t_n(a)\rho^t_n(b)\|$$

$$= \|\rho^0_n(ab) - \rho^0_n(a)\rho^0_n(b)\|$$

$$= \|\bigotimes_{h \in F} \varphi^0_n(a_h b_h) - \bigotimes_{h \in F} \varphi^0_n(a_h)\varphi^0_n(b_h)\|$$

$$\leq M^{2|F|-2} \sum_{h \in F} \|\varphi^0_n(a_h b_h) - \varphi^0_n(a_h)\varphi^0_n(b_h)\|$$

$$\leq M^{2|F|-2} \sum_{h \in F} \|\varphi_n(a_h b_h) - \varphi_n(a_h)\varphi_n(b_h)\|$$

and so we conclude that the sequence $\{\sigma_n\}_n$ is asymptotically multiplicative on elementary tensors since the sequence $\{\varphi_n\}_n$ is asymptotically multiplicative. This is sufficient to conclude $\{\sigma_n\}_n$ is asymptotically multiplicative without restrictions. Indeed, the first part of Prop. 5.3.4 says asymptotic multiplicativity on elementary tensors implies asymptotic multiplicativity on linear combinations on such tensors, but the collection of all such linear combinations is dense in $\bigotimes H C^*(G)$.

Next $\sigma^0_n = \rho^0_n$ and so the sequence $\{\sigma_n\}_n$ is injective. Also, $\sigma^1_n = \rho^1_n = \bigotimes H i_G$ for every $n$. Only equivariance remains, but this is also easy. Let $a'$ denote the shift action on $\bigotimes H B(H)$. Then for each $t \in [0,1]$ and $n$ we have

$$\rho^t_n(\beta_k(\otimes h a_h)) = \rho^t_n(\otimes h a_{k-1} h) = \otimes h \varphi^t_n(a_{k-1} h) = \alpha'_k(\otimes h \varphi^t_n(a_h)) = \alpha'_k(\rho^t_n(\otimes h a_h))$$

and, by linearity, this is enough to conclude

$$\rho^t_n(\beta(a)) = \alpha'_k(\rho^t_n(a))$$

for every $a \in \bigotimes H C^*(G)$. The action $\alpha'$ extends to $C[0,1] \otimes \bigotimes H B(H)$ via $\alpha_k(f \otimes (\otimes h b_h)) = f \otimes (\otimes h b_{k-1} h)$ and so we have $\sigma_n(\beta_h(a)) = \alpha_h(\sigma_n(a))$ for every $a \in \bigotimes H C^*(G)$, which completes the proof.

Proof of Theorem 5.3.2. Let $\{\sigma_n\}_n$ be as in the conclusion of Lemma 5.3.8. For each $n$, $\sigma_n$ is equivariant and so $\sigma_n$ extends to a ucp map

$$\sigma_n : (\bigotimes H C^*(G)) \rtimes_\beta H \to (C[0,1] \otimes \bigotimes H B(H)) \rtimes_\alpha H$$
defined by
\[ \tilde{\sigma}_n(\sum_{h \in H} a_h h) = \sum_{h \in H} \sigma(a_h) h \]
by [5, Ex. 4.1.4]. By definition of the action \( \alpha \) on \( C[0, 1] \otimes \bigotimes_H B(\mathcal{H}) \), [5, Ex. 4.1.3] gives the identification
\[ (C[0, 1] \otimes \bigotimes_H B(\mathcal{H})) \rtimes_{\alpha} H \simeq C[0, 1] \otimes (\bigotimes_H B(\mathcal{H}) \rtimes_{\alpha} H) \]
and so \( \tilde{\sigma} \) may then be realized as
\[ \tilde{\sigma}_n : (\bigotimes_H C^*(G)) \rtimes_{\beta} H \to C[0, 1] \otimes (\bigotimes_H B(\mathcal{H}) \rtimes_{\alpha} H). \]
\( \tilde{\sigma} \) then restricts to
\[ \tilde{\sigma}_n : I(\bigoplus_H G) \rtimes_{\beta} H \to C_0[0, 1] \otimes (\bigotimes_H B(\mathcal{H}) \rtimes_{\alpha} H) \]
and so we are done if we show that the sequence of ucp maps \( \{\tilde{\sigma}_n\}_n \) is asymptotically multiplicative and that the sequence \( \{\text{ev}_0 \circ \tilde{\sigma}_n\}_n \) is injective by Lemma 5.3.3.

First we show multiplicativity. By density it will suffice to show that the sequence \( \{\tilde{\sigma}_n\} \) is asymptotically multiplicative on elements \( a, b \in (\bigotimes_H C^*(G)) \rtimes_{\beta} H \) of the form \( a = \sum_h a_h h \) and \( b = \sum_h b_h h \) with \( a_h = b_h = 0 \) for \( h \not\in F \) for some finite subset \( F \subset H \). Note
\[ \tilde{\sigma}_n(\sum_{h \in F} a_h h) \cdot \tilde{\sigma}_n(\sum_k b_k k) = \sum_{h,k} \sigma_n(a_h) \sigma_n(\beta_h(b_k)) h k \]
and
\[ \tilde{\sigma}_n((\sum_h a_h h)(\sum_k b_k k)) = \sum_{h,k} \sigma(a_h \beta_k(b_k)) h k \]
Hence,
\[ \|\tilde{\sigma}_n(ab) - \tilde{\sigma}_n(a)\tilde{\sigma}_n(b)\| = \| \sum_{h,k \in F} \sigma_n(a_h) \sigma_n(\beta_h(b_k)) h k - \sum_{h,k \in F} \sigma(a_h \beta_k(b_k)) h k \|
\leq \sum_{h,k \in F} \|\sigma_n(a_h) \sigma_n(\beta_h(b_k)) - \sigma(a_h \beta_k(b_k))\| \]
Therefore the sequence \( \{ \tilde{\sigma}_n \} \) is asymptotically multiplicative because the sequence \( \{ \sigma_n \} \) is also asymptotically multiplicative.

Next we must show \( \{ \tilde{\sigma}_n \} \) is injective. Let

\[
\Phi : \bigotimes_H C^*(G) \rtimes_{\beta} H \to \prod_H \bigotimes_H B(\mathcal{H}) \rtimes_{\alpha} H
\]

be the \( * \)-homomorphism associated to the sequence \( \{ \sigma_n \} \), let \( 0 \neq a \in \bigotimes_H C^*(G) \rtimes_{\beta} H \), and suppose \( \Phi(a) = 0 \). Since

\[
0 = \| \Phi(a) \| = \| \Phi(a^*a) \|
\]

we may assume \( a \) is positive.

Since the crossed product is defined as the closure of finite sums, there exists a sequence \( \{ a^{(n)} \} \subset (\bigotimes_H C^*(G)) \rtimes_{\beta} H \) with \( \lim_{n \to \infty} a^{(n)} = a \) where for each \( n \) we have that \( a^{(n)} = \sum_{h \in H} a_h^{(n)} h \) and \( a_h^{(n)} = 0 \) for all but finitely many \( h \in H \). By [5, 4.1.9], there exists a faithful conditional expectation

\[
E : \bigotimes_H C^*(G) \rtimes_{\beta} H \to \bigotimes_H C^*(G)
\]

given by

\[
E\left( \sum_{h \in H} a_h h \right) = a_e
\]

on finite sums and

\[
\| a_e \| \leq \| \sum_{h \in H} a_h h \|.
\]

Since \( E \) is faithful and \( a \) is positive we have \( E(a) > 0 \). By continuity,

\[
0 < \| E(a) \| = \lim_{n \to \infty} \| E(a^{(n)}) \| = \lim_{n \to \infty} \| a^{(n)} \|
\]

and now since \( \{ \sigma_n^0 \} \) is injective we have

\[
0 = \| \Phi(a) \| = \lim_{n \to \infty} \| \Phi(a^{(n)}) \|
\]

\[
= \lim_{n \to \infty} \limsup_{k \to \infty} \| \tilde{\sigma}_k^0 (a^{(n)}) \|
\]

\[
= \lim_{n \to \infty} \limsup_{k \to \infty} \left\| \sum_{h \in H} \sigma_k^0 (a_h^{(n)}) h \right\|
\]
\[
\geq \lim_{n \to \infty} \limsup_{k \to \infty} \|\sigma_k^0(a_e^{(n)})\| \\
= \lim_{n \to \infty} \|a_e^{(n)}\| \\
= \|E(a)\| > 0
\]

which is a contradiction. Therefore \(\Phi\) is injective and the sequence \(\{\tilde{\sigma}_n^0\}\) is injective as well, by definition.

\[\square\]

**Remark 5.3.9** Theorem 5.3.2 is more general than stated here as the action need not be the shift action. Indeed, the same proof of the theorem and all supporting statements will hold for generalized wreath products. If \(S\) is a countable set with an action of \(H\) given by \(\beta\) then the associated action on \(\bigoplus_S G\), also denoted by \(\beta\), is defined by \(\beta_k(\bigoplus_s g_s) = \bigoplus_s g_{\beta_k(s)}\). The associated semi-direct product \((\bigoplus_S G) \rtimes_{\beta} H\) is called a generalized wreath product. In summary, if \(G\) and \(H\) are amenable groups, \(I(G)\) and \(I(H)\) have Property (QH) then \(I((\bigoplus_S G) \rtimes_{\beta} H)\) also has Property (QH) by Theorem 5.3.2 and its proof.
6. Open Questions

In this brief chapter we will recap and collect the questions which arose throughout this text or within the context of this work.

In the context of quasidiagonality, the following question of Blackadar and Kirchberg is foundational:

**Question 1** ([11]) Is every stably finite nuclear $C^*$-algebra quasidiagonal?

Reduced group $C^*$-algebras are a natural test case for this question as they are automatically stably finite. Indeed, as Cor. 3.2.7 indicates, this case has been recently resolved with quasidiagonality being equivalent to amenability of the group. For full group $C^*$-algebras the question of quasidiagonality appears to not have such a succinct answer. On one hand, there exists non-amenable groups which have residually finite dimensional full group $C^*$-algebras such as $\mathbb{F}_2$. On the other hand, as indicated in Example 4.2.7 certain groups with Property (T) are the only known examples to not produce quasidiagonal algebras. Since free groups have the Haagerup property, Def. 4.1.8, the following might be an appropriate middle ground.

**Question 2** If $G$ has the Haagerup property, is $C^*(G)$ quasidiagonal?

Unfortunately, the relationship between Property (T) and quasidiagonality appears to be poorly understood at this time. In light of available evidence, it might be reasonable to suspect the answer to the following question is ‘no.’

**Question 3** If $G$ is an infinite group with Property (T), is $C^*(G)$ quasidiagonal? In particular, is $C^*(SL_3(\mathbb{Z}))$ quasidiagonal?

This question appears difficult in general as even the residually finite dimensional case is not known as indicated by Lubotzky and Shalom.
**Question 4** ([37, 6.5]) Does there exist an infinite discrete group $G$ with Property (T) such that $C^*(G)$ is residually finite dimensional?

Once again, available evidence would seem to suggest the answer is ‘no.’ It should also be noted that it is not clear if $C^*(G)$ is stably finite if $G$ is not amenable, an important obstruction for quasidiagonality.

Questions regarding Property (QH) naturally follow a similar line to those regarding quasidiagonality. For example, the case of torsion-free amenable groups seems to be a promising place to start.

**Question 5** ([2]) If $G$ is amenable and torsion-free, does $I(G)$ have Property (QH)?

A natural place to begin investigating this question would be consideration of quotients or extensions. Examples of groups which are torsion-free but $I(G)$ does not have Property (QH) are abundant, however. For example, if $G$ torsion-free and has Property (T), then $I(G)$ has many non-trivial projections due to Kazhdan projections.

Outside of groups, determining appropriate conditions when crossed products have Property (QH) would also be beneficial.

**Question 6** If $A$ and $I(G)$ have Property (QH), when does $A times G$ have Property (QH)?

This question is likely too general as stated. A better starting point would be to consider the following:

**Question 7** What conditions must be placed on the action $\alpha$ so that $C_0(X \setminus \{x_0\}) \rtimes_{\alpha} \mathbb{Z}$ has Property (QH)?

A careful consideration of Theorem 5.3.2 and its proof may provide some guidance. One should also observe that this situation is analogous to determining if $I(G \rtimes \mathbb{Z})$ has Property (QH) for an abelian group $G$.

Finally, we come to questions regarding other types of finite-dimensional approximations of groups. The classes of MF, sofic and hyperlinear groups are all characterized by very similar approximation properties, however the relationship between these classes is largely unknown.
**Question 8** What is the relationship between the class of MF groups, the class of sofic groups, and the class hyperlinear groups?

Recall every amenable group is sofic, and every sofic group is hyperlinear. Therefore, if one shows every hyperlinear group is MF then this will produce a new proof of Cor. 3.2.7. This may indicate the depth and difficulty in determining if any relationships exist. In general, all of these classes are somewhat mysterious and represent highly non-trivial properties not indicated by their simple characterizations. For example, hyperlinear groups are precisely those groups satisfying Connes’ embedding problem and every group for which $C^*(G)$ is quasidiagonal is also MF. It is not even clear if these classes of groups are even restrictive in some way.

**Question 9** Does there exist a group which is not MF, sofic, or hyperlinear?
REFERENCES


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