1990

Conversion Methods Between Parametric and Implicit Curves and Surfaces

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Report Number:
90-975

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CSD-TR-975
April 1990

Conversion Methods Between Parametric and Implicit Curves and Surfaces*

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Abstract

We present methods for parameterizing implicit curves and surfaces and for implicitizing parametric curves and surfaces, based on computational techniques from algebraic geometry. After reviewing the basic mathematical facts of relevance, we describe and illustrate state-of-the-art algorithms and insights for the conversion problem.

Keywords: Parametric and implicit curves and surfaces, parameterization, implicitization, elimination, birational maps, projection. Algebraic geometry, symbolic computation, Gröbner bases, monoids, resultants.

*Notes for the course *Unifying Parametric and Implicit Surface Representations*, at SIGGRAPH '90.

†Supported in part by NSF Grant CCR 86-19817 and ONR Contract N00014-90-J-1599.

1 Introduction

These notes contain the slides of the course *Unifying Parametric and Implicit Surface Representations for Computer Graphics*, given at SIGGRAPH 90. They begin with commentaries and references that introduce the material to individuals who have not attended this course.

Of necessity, the subject is mathematical, using many concepts from elementary algebraic geometry. The concepts do have an intuitive interpretation that can guide the reader using those concepts even though he or she may not be aware of some of the finer points. Those intuitions are only sketched here, but can be found in much detail, for instance, in [13], in Chapters 5 through 7.

The material is organized as follows: After reviewing some basic facts from mathematics, methods for parameterizing implicit curves and surfaces are presented, concentrating on how to deal with monoids. Monoids allow an especially easy conceptual approach, but one can appreciate some of the technical complexities when studying the method in the case of cubic curves.

Thereafter, we discuss how to convert from parametric to implicit form. Many authors have discussed these techniques, and only a very limited perspective is developed here. A broader description of the subject is found in [13], in Chapters 5 and 7, and also in [15], where the problem of *faithfulness* is conceptualized and discussed. In particular, resultants are covered in detail in the theses by Sederberg [17] and by Chionh [9]. We do not discuss *multivariate* resultant formulations, but refer to [9] for further reading on the subject.

Conversion between implicit and parametric form is, in general, an expensive computation. It is therefore worth considering alternatives. One such approach is to view parametric curves and surfaces as manifolds in higher-dimensional spaces. Such a view no longer has to distinguish between implicit and parametric representations, and the methods it develops apply to both equally well. We will not discuss this approach here, and the reader is referred for details to [14, 15].

Basic Mathematical Facts

Recall that a plane *parametric curve* is defined by two functions

$$\begin{aligned} x &= h_1(s) \\ y &= h_2(s) \end{aligned}$$

and that a *parametric surface* is defined by three functions

$$\begin{aligned} x &= h_1(s,t) \\ y &= h_2(s,t) \\ z &= h_3(s,t) \end{aligned}$$

We can think of a parametric curve as a map from a straight line with points s to a curve in the (x,y)-plane, and of a parametric surface as a map from a plane with points (s,t) to a surface in (x,y,z)-space.

The functions h_k will be polynomials or ratios of polynomials in s and t. Accordingly, we speak of *integral* or *rational* parametric curves and surfaces whenever the distinction is critical. Ordinarily, the curves or surfaces are restricted in the literature to a *domain*; e.g., to the interval $[0,1]$ or to the square $[0,1] \times [0,1]$. Here, we do not so restrict parametric curves and surfaces.

Typically, the functions h_k are presented in a particular basis: for instance, in the *Bernstein-Bezier* basis, and this allows relating the coefficients of the functions h_k with an intuitive understanding of the shape of the curve they define. A suitable basis also affords a wealth of techniques for combining patches of parametric curves or surfaces into larger surfaces, and to modify the shape of the larger surface locally or globally, in an intuitive manner. See, e.g., [5, 11].

An *implicit curve* is defined by a single equation

$$f(x,y) = 0$$

and an *implicit surface* is defined by a single equation

$$f(x,y,z) = 0$$

Thus, the curve or surface points are those that satisfy the implicit equation, so that we no longer think of curves and surfaces as the result of a mapping. We will restrict the function f to polynomials.

Since we restrict the h_k to polynomials, or ratios of polynomials, and restrict the f to polynomials, we are dealing with *algebraic* curves and surfaces. Algebraic curves and surfaces include virtually all surfaces studied and used in geometric and solid modeling, and in computer-aided geometric design. Algebraic geometry provides us with the following key facts about algebraic curves; e.g., [20]:

> *Every plane parametric curve can be expressed as an implicit curve. Some, but not all, implicit curves can be expressed as parametric curves.*

Similarly, we can state of algebraic surfaces

> *Every plane parametric surface can be expressed as an implicit surface. Some, but not all, implicit surfaces can be expressed as parametric surfaces.*

This means, that the class of parametric algebraic curves and surfaces is smaller than the class of implicit algebraic curves and surfaces. There is even a rigorous characterization of what distinguishes a parameterizable algebraic curve or surface from one that is not parameterizable. Roughly speaking, a curve is parameterizable if it has many singular points; that is, many points at which the curve intersects itself or has cusps. We will not go

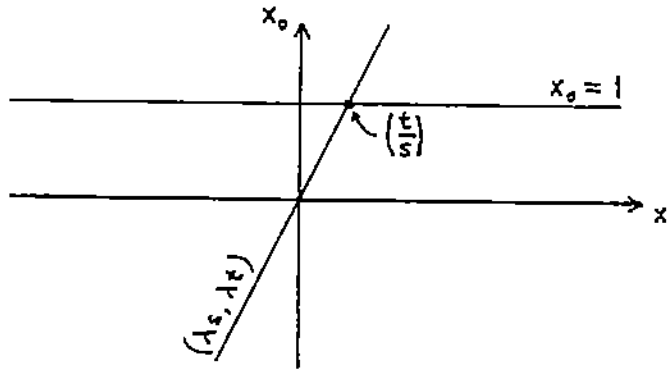

Figure 1: The Projective Line

into those details, because the characterization is very technical, and the computations that would be needed to test whether a curve or surface is parameterizable are quite complex and time-consuming; [3].

We will discuss special curves and surfaces that can be parameterized fairly easily. They include the following cases.

- All conic sections and all quadratic surfaces are parameterizable.
- Cubic curves that have a singular point are parameterizable.
- Monoids are parameterizable.[1]

All parametric curves and surfaces have an implicit form, and we will discuss several approaches for finding the implicit forms. However, it is possible that a parametric surface does not contain certain points found on the corresponding implicit surface. Some of the missing points can be recovered by considering the surface parameterization projectively, but not all missing points can be so recovered. Except for certain special cases, the conversion between implicit and parametric form is expensive, and one does not invoke the conversion algorithms lightly.

Ordinarily, we deal with *affine* spaces in which points may be fixed using Cartesian coordinates. On the affine line, a point has the coordinate (x_1); on the affine plane, a point has the coordinates (x_1, x_2); and in affine space, a point has the coordinates (x_1, x_2, x_3). In contrast, projective spaces add another coordinate x_0, and consider a point defined by the *ratio* of its coordinates. For example, the projective line has points (x_0, x_1), and for all $\lambda \neq 0$ both (x_0, x_1) and $(\lambda x_0, \lambda x_1)$ are the same point. The coordinate tuple $(0,0)$ is not allowed.

The projective line can be visualized as the pencil of lines through the origin, embedded in the affine plane, as shown in Figure 1. Here, the projective point (s, t) corresponds to the

[1] Monoids are defined later.

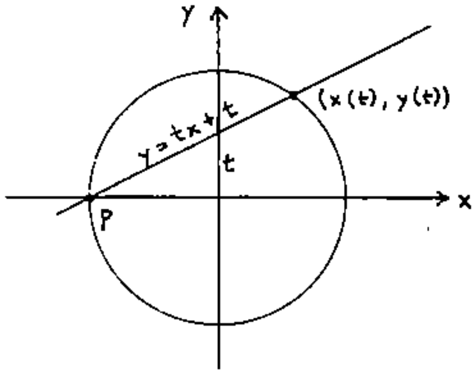

Figure 2: Parameterizing the Unit Circle

line

$$
\begin{aligned}
x_0 &= s\lambda \\
x_1 &= t\lambda
\end{aligned}
$$

To each affine line point (x_1), there corresponds the projective line point $(1, x_1)$. Visually, the affine line is viewed as the points obtained by intersecting the pencil with the line $x_0 = 1$. The projective line has *one* additional point, with coordinates $(0,1)$, corresponding to the line $x_0 = 0$. This point is said to be *at infinity*.

The projective plane has the points (x_0, x_1, x_2), $(0,0,0)$ is not allowed. For all $\lambda \neq 0$, (x_0, x_1, x_2) and $(\lambda x_0, \lambda x_1, \lambda x_2)$ are the same point. Again, one may embed the projective plane into affine 3-space by considering the projective plane as the bundle of all lines, in 3-space, through the origin. The affine plane is a subset, obtained by intersecting the bundle with the plane $x_0 = 0$. The additional points correspond to the pencil of lines through the origin that lie in the plane $x_0 = 0$, and form the *line at infinity*. The coordinates of these points are of the form $(0, x_1, x_2)$.

Projective geometry leads to simplifying many theorems by eliminating special cases. For our purposes, projective curve parameterization has the advantage that all points on a parametric curve can be reached with finite parameter values without exception. This will not necessarily be the case for projectively parameterized surfaces.

2 Parameterization

2.1 Plane Algebraic Curves

A Geometric View of Parameterization

The geometric idea underlying parameterizing a plane algebraic curve is illustrated by the unit circle, $x^2 + y^2 - 1 = 0$. We pick one point on the circle, say $p_1 = (-1,0)$, and consider all lines through this point. A line through p_1 has the equation $y = tx + t$, where t is the intercept with the y-axis. The lines are indexed by t: that is, to each value of t corresponds a specific line through p_1. Each line intersects the circle in p_1 and in one additional point $p(t)$.[2] The coordinates of $p(t)$ are obtained in three steps:

1. Substitute $tx + t$ for y in the circle's equation, obtaining the equation

$$x^2(1 + t^2) + 2t^2x + t^2 - 1 = 0$$

2. Solve this equation for x, obtaining

$$x_1 = -1 \qquad\qquad x_2 = \frac{1 - t^2}{1 + t^2}$$

3. Observe that x_1 is the abscissa of the point p_1, and that x_2 the abscissa of the point p_2. Clearly, x_2 is a function of t. The corresponding ordinate y_2 is obtained from the line equation as $y_2 = tx_2 + t$ and is

$$y_2 = \frac{2t}{1 + t^2}$$

Note that (x_2, y_2) is the familiar parameterization of the unit circle.

The method just illustrated can be thought of as the following procedure. Let $f(x, y) = 0$ be the curve to be parameterized.

Simple Parameterization Algorithm 1

1. Pick a point p_1 on the curve f, and consider all lines through p_1, indexed by the parameter t. These lines form a *pencil*.
2. For each line of the pencil, compute the "other" intersection point with f, expressing its coordinates as functions of t. These functions are the parameterization.

This simple method works unchanged for every conic section, but not necessarily for curves of higher algebraic degree. The reason it works for conics is a consequence of Bezout's Theorem that states that a line intersects a conic in just two points. With one of these two points fixed as p_1, there is just one additional point, and that point is uniquely associated with t. If f were a cubic curve, a line would intersect f in general in three points. Fixing

[2] As a fine point, observe that the line $x + 1 = 0$ also contains the point $p_1 = (-1, 0)$ and intersects the circle only in p_1. Here, p_1 counts as a *double* point, since the line is tangent to the circle. The point p_1 is therefore the additional point in which this line intersects the circle, and corresponds to $t = \pm\infty$. Only a projective parameterization reaches this point from a finite parameter value.

one of them on the curve, would leave two additional intersection points, p_2 and p_3, both associated with the same value of t, and so the curve would not be properly parameterized by our algorithm.

To make the method work for cubics requires choosing a special point: that is, a point such that lines through it intersect the cubic in just one additional point. Such a special point exists only on singular cubics, and is actually the singular point itself. Nonsingular cubics do not have such a point, and cannot be so parameterized. Indeed, it can be proved that a nonsingular cubic cannot be parameterized at all.

Modified in this way, choosing a special curve point p_1, will our simple method of parameterizing curves work for all those curves that possess a parametric form? Not unless other modifications are made that we do not discuss. But there is an interesting class of curves that can be so parameterized, namely the class of *monoids*, to be discussed later.

Parameterization of Conics

Conic sections are parameterized by the method described before, using any curve point. However, it may be inconvenient to pick an arbitrary point on the conic. The computations simplify if the point is picked such that it is the origin, or else at infinity, in a principal direction.[3] This may require a change of coordinates.

Let the conic equation be

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0$$

We first look for a point "at infinity" by computing the roots of the quadratic form

$$a_{11}x^2 + a_{22}y^2 + 2a_{12}xy = 0$$

The roots are given by

$$\overline{x} = -a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}} \qquad \overline{y} = a_{11}$$

They are either two real roots, possibly equal, or they are conjugate complex. In the complex case, the conic does not have points at infinity, and is an ellipse or a circle. In that case we must find a point at finite distance, and use thereafter the method for monoids described later. Otherwise we have a point at infinity, and proceed as follows.

Let (u, v) be one of the two real roots. We substitute

$$x = x_1 + uy_1 \qquad y = vy_1$$

in the conic equation. It is easy to see that the resulting equation is of the form

$$y_1(cx_1 + d) + q(x_1) = 0$$

[3] See [1], or [13], p. 177.

where $q(x_1)$ is a quadratic polynomial in x_1 alone. This conic is parameterized by

$$x_1(t) = t \qquad y_1(t) = \frac{-q(t)}{ct + d}$$

Because of the substitution, therefore, the original conic is parameterized by

$$x(t) = t + u y_1(t) \qquad y(t) = v y_1(t)$$

There is another method for parameterizing conics, based on linear algebra, due to Jacobi in the previous century. We sketch it briefly; see [13] p. 170 for details. The advantage of Jacobi's algorithm is that it generalizes directly to quadratic surfaces, and that it is well behaved numerically.

The conic equation,

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0$$

can be written as the bilinear form

$$(x \;\; y \;\; 1) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

An iterative algorithm can be devised that applies rotations to the coefficient matrix and diagonalizes it. After diagonalization, the resulting conic has a standard parameterization, of the form[4]

$$x(t) = \mu_1 \frac{1 - t^2}{1 + t^2} \qquad y(t) = \mu_2 \frac{2t}{1 + t^2}$$

The numbers μ_1 and μ_2 depend on the entries b_{kk} of the diagonalized matrix; that is, $\mu_1 = \sqrt{|b_{11}/b_{33}|}$ and $\mu_2 = \sqrt{|b_{22}/b_{33}|}$. By applying the inverse rotations, this standard parameterization is mapped to a parameterization of the original conic.

Parameterization of Cubic Curves

The difficulty of parameterizing a cubic curve is to find out whether the curve has a singular point, and if so, where it is. The singularity could be apparent and at the origin, as in the cuspidal cubic

$$y^2 - x^3 = 0$$

[4] More precisely, the parameterization depends on the signs of the diagonal entries. We assume here that the first two diagonal entries have equal sign opposite to the sign of the third entry.

that is parameterized by

$$x(t) = t^2 \qquad y(t) = t^3$$

The singularity could be at infinity, as in the case of

$$y - x^3 + x^2 - x = 0$$

parameterized by

$$x = t \qquad y = t^3 - t^2 + t$$

But the cubic could also be nonsingular and have no parameterization, as in the case of

$$y^2 - x^3 + x = 0$$

The following algorithm from [2] solves the parameterization problem of cubic curves. Despite the relative complexity, the algorithm is related to the simple parameterization procedure discussed in geometric terms before. We illustrate the method with an example from [13] p. 181.

We are given the cubic

$$f = 28y^3 + 26xy^2 + 7x^2y + x^3/2 + 28y^2 + 16xy + 7y + 3x/2$$

Its *degree form* consists of all cubic terms. The first step is to find a real root of the degree form, so as to apply a coordinate transformation to f that eliminates the y^3-term. Since the degree form is cubic, it has at least one real root. Here, $(-2, 1)$ is a root of $28y^3 + 26xy^2 + 7x^2y + x^3/2$.

In the second step, we use the root (u, v) to perform the substitution

$$x = x_1 + uy_1 \qquad y = vy_1$$

Note that this substitution is exactly as in the conic parameterization. In our example, we obtain

$$4(x_1 - 1)y_1^2 + 4(x_1^2 + 4x_1 + 1)y_1 + (x_1^3 + 3x_1)/2 = 0$$

The transformed curve equation has the form

$$h_1(x_1)y_1^2 + h_2(x_1)y_1 + h_3(x_1) = 0$$

where h_1 is a linear polynomial, h_2 a quadratic polynomial, and h_3 a cubic polynomial. In step 3, we multiply with h_1 to bring the equation into the form

$$[h_1(x_1)y_1 + h_2(x_1)]^2 + h_4(x_1) = 0$$

where $h_4 = h_1 h_3 - h_2^2/4$. Note that h_4 has degree 4. So, we have

$$4(x_1-1)^2 y_1^2 + 4(x_1-1)(x_1^2+4x_1+1)y_1 + (x_1^4 - x_1^3 + 3x_1^2 - 3x_1)/2 = 0$$

which can be rewritten as

$$[2(x_1-1)y_1 + (x_1^2+4x_1+1)]^2 - (x_1^2+4x_1+1)^2 + (x_1^4 - x_1^3 + 3x_1^2 - 3x_1)/2 = 0$$

In Step 4, we substitute y_2^2 for the quadratic form $(h_1(x_1)y_1 + h_2(x_1))^2$. In the example, $y_2 = 2(x_1-1)y_1 + (x_1^2+4x_1+1)$, whence

$$y_2^2 = (x_1^4 + 17x_1^3 + 33x_1^2 + 19x_1 + 2)/2$$

The righthand side of this equation is a polynomial of degree up to 4. It can be proved that the original cubic is nonsingular if and only if the righthand side is of degree 3 or 4 and has no multiple roots. Note that a nonsingular cubic cannot be parameterized. Otherwise, the multiple root λ can be used to transform the equation into a quadratic equation, by the substitution

$$y_3 = \frac{y_2}{x_1 - \lambda}$$

In Step 5, therefore, we investigate the roots of the righthand side. If a double root is present, we make this substitution. In the example, the righthand side has the double root $x_1 = -1$, so we set $y_3 = y_2/(x_1+1)$ and obtain

$$2y_3^2 = x_1^2 + 15x_1 + 2$$

If Step 5 succeeds, the substitution transforms the equation into a quadratic one, and this quadratic equation is parameterized in Step 6 using the methods discussed before. Thereafter, the various substitutions are inverted, transforming the conic parameterization to a parameterization of the original cubic. In our example, the conic is parameterized by

$$x_1(t) = \frac{t^2-2}{2t+15} \qquad y_3(t) = -\frac{t^2+15t+2}{\sqrt{2}(2t+15)}$$

Since $y_3 = y_2/(x_1+1)$, we thus obtain

$$y_2 = -\frac{(t^2+15t+2)(t^2+2t+13)}{\sqrt{2}(2t+15)^2}$$

Now $y_2 = 2(x_1-1)y_1 + x_1^2 + 4x_1 + 1$, from which we obtain

$$y_1 = -\frac{(\sqrt{2}+1)t^4 + (8\sqrt{2}+17)t^3 + (60\sqrt{2}+45)t^2 + (44\sqrt{2}+199)t + (109\sqrt{2}+26)}{\sqrt{2}(4t^3 + 22t^2 - 128t - 510)}$$

Note that we can cancel $(t - 1 + 3\sqrt{2})$, whence

$$y_1 = \frac{(\sqrt{2}+1)t^3 + (6\sqrt{2}+12)t^2 + (30\sqrt{2}+21)t + (11\sqrt{2}+40)}{\sqrt{2}(4t^2 - (12\sqrt{2}-26)t - (90\sqrt{2}+30))}$$

From this parameterization, we finally obtain the parameterization of the original cubic f.

Parameterizing Monoids

A curve f of degree n with a point of multiplicity $n-1$ is a *monoid*. Every conic section is a monoid, and every singular cubic curve is a monoid. An analogous definition for monoidal surfaces is discussed later.

Monoids are especially easy to parameterize, provided we know where the $(n-1)$-fold point is and have brought it to the origin, [17]; or else if the point is at infinity, in a principal direction. We explain the parameterization method assuming the singularity is at the origin. Examples of monoids in this form include the circle

$$x^2 + y^2 - 2x = 0$$

through the origin, the cuspidal cubic,

$$x^3 - y^2 = 0$$

and the alpha curve,

$$x^3 + x^2 - y^2 = 0$$

both with the singularity at the origin.

When the $(n-1)$-fold point is at the origin, the implicit monoid equation is

$$h_n(x,y) - h_{n-1}(x,y) = 0$$

where h_n has only terms of degree n, and h_{n-1} has only terms of degree $n-1$. This is readily verified in the three examples.

It is easy to see that the parameterization of the monoid is given by

$$x(s,t) \quad = \quad s\,\frac{h_{n-1}(s,t)}{h_n(s,t)} \qquad\qquad y(s,t) \quad = \quad t\,\frac{h_{n-1}(s,t)}{h_n(s,t)}$$

This is a projective parameterization that is changed to the normal parametric form by either setting $s = 1$ or setting $t = 1$.

The monoid parameterization is derived by considering a pencil of lines through the origin. Every line can be expressed parametrically as

$$x(\lambda) \quad = \quad s\lambda \qquad\qquad y(\lambda) \quad = \quad t\lambda$$

and is determined by a unique ratio $s:t$. With $s = 1$, we obtain the usual form

$$y = tx$$

Now each line intersects the monoid in one additional point, and this point is therefore uniquely associated with the ratio $s:t$ of the line.

In the example of the alpha curve, $x^3 + x^2 - y^2 = 0$, we have

$$\begin{aligned} h_3(x,y) &= x^3 \\ h_2(x,y) &= y^2 - x^2 \end{aligned}$$

Its (projective) parameterization is therefore

$$x(s,t) = s\frac{t^2 - s^2}{s^3} \qquad y(s,t) = t\frac{t^2 - s^2}{s^3}$$

With $s = 1$ we obtain the usual parameterization

$$x(t) = t^2 - 1 \qquad y(t) = t^3 - t^2$$

Because the parameterization is so easy to find, monoids have also been called *dual forms* in the CAGD literature; e.g., [17].

2.2 Algebraic Surfaces

The parameterization of implicit algebraic surfaces is much more complicated than curve parameterization. For one, a characterization of when an implicit surface has a parametric form is technically quite complicated, and is not readily explained in geometric terms. There is no known general algorithm for determining whether a given implicit surface can be parameterized, and if so, how. Fortunately, monoidal surfaces can be parameterized in a very simple manner with a clear geometric intuition, and these surfaces include all quadrics.

Monoidal Surfaces

An algebraic surface $f(x,y,z) = 0$ of degree n that has an $(n-1)$-fold point is a monoidal surface. Monoidal surfaces include all quadrics, cubic surfaces with a double point, and Steiner surfaces, [17].

Bezout's theorem states that a line intersects a surface of degree n in exactly n points, assuming we admit complex intersections and intersections at infinity, and account for intersection multiplicity. It follows that a monoid could be parameterized by an extension of the simple parameterization algorithm described before:

Simple Parameterization Algorithm 2

1. Let p_1 be a point on the monoidal surface of multiplicity $n-1$, and consider all lines through p_1. Each line is determined by a pair (s,t) of slopes in two principal directions, or, alternatively, by a unique ratio $r:s:t$ of direction cosines.

2. Determine $(x(s,t), y(s,t), z(s,t))$, the additional intersection point of each line with the monoidal surface, as function of s and t, thereby deriving a parameterization.

Algebraically, things are extremely simple when the singular point is at the origin, for then the implicit equation has the form

$$h_n(x,y,z) - h_{n-1}(x,y,z) = 0$$

where h_n has only terms of degree n, and h_{n-1} has only terms of degree $n-1$. The parameterization is then

$$\begin{aligned}
x(r,s,t) &= r\,\frac{h_{n-1}(r,s,t)}{h_n(r,s,t)} \\
y(r,s,t) &= s\,\frac{h_{n-1}(r,s,t)}{h_n(r,s,t)} \\
z(r,s,t) &= t\,\frac{h_{n-1}(r,s,t)}{h_n(r,s,t)}
\end{aligned}$$

This is a projective parameterization that is changed to the normal parametric form by setting one of the parameters r, s, or t to 1.

As an example, consider the sphere with radius 1 and center $(1,0,0)$

$$x^2 + y^2 + z^2 - 2x = 0$$

It is parameterized by

$$\begin{aligned}
x(r,s,t) &= \frac{2r^2}{r^2+s^2+t^2} \\
y(r,s,t) &= \frac{2rs}{r^2+s^2+t^2} \\
z(r,s,t) &= \frac{2rt}{r^2+s^2+t^2}
\end{aligned}$$

Setting $r = 1$, we obtain

$$\begin{aligned}
x(s,t) &= \frac{2}{1+s^2+t^2} \\
y(s,t) &= \frac{2s}{1+s^2+t^2} \\
z(s,t) &= \frac{2t}{1+s^2+t^2}
\end{aligned}$$

Quadric Surfaces

Quadrics can be parameterized as monoidal surfaces. Any point on the quadric suffices. The algebraic method discussed for conic sections can be generalized [1]: First, we find a point on the quadric at infinity, next, we change coordinates to move this point into a special position, and then we pick up a standard parameterization that is structurally like the one given before, for monoidal surfaces whose singularity is at the origin. The inverse coordinate transformation, finally, maps the standard parameterization to a parameterization of the original quadric surface.

Jacobi's algorithm can also be used for quadric surfaces. Here, the coefficients of the quadric surface

$$a_{11}x^2 + 2a_{12}xy + 2a_{13}xz + 2a_{14}x + a_{22}y^2 + 2a_{23}yz + 2a_{24}y + a_{33}z^2 + 2a_{34}z + a_{44} \quad = \quad 0$$

are written as the symmetric matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}$$

and this matrix is diagonalized using the usual rotations. From the diagonal form, a standard parameterization is determined that is mapped back to a parameterization of the original quadric by the inverse rotations. The standard parameterization depends on the signs of the diagonal entries and on the rank of the matrix. For details see [13], p. 180.

3 Implicitizing Rational Curves and Surfaces

Existence of the Implicit Equation

Given a parametric curve with rational coordinate functions $(h_1(s), h_2(s))$, its implicit equation should be a polynomial $f(x, y) = 0$ such that $f(h_1(s), h_2(s)) \equiv 0$. Moreover, the degree of f should be as small as possible. To understand the existence of the implicit equation, it is useful to learn about transcendental field extensions.

Let K be a subfield of a larger field E.[5] Consider a set S of elements in E that are not in K. The *extension field* $K' = K(S)$ generated by S is the smallest subfield of E containing S and K. If every element of S is the root of a polynomial $f(x)$ with coefficients in K, then K' is an *algebraic extension*. Otherwise, K' is a *transcendental extension*. For example, let K be the field of rational numbers, E the field of real numbers, and let S contain only the

[5] A field is a set with addition, subtraction, multiplication, and division. For the properties that must be satisfied see [19].

number π. Then the extension field $K(\pi)$ is a transcendental extension. On the other hand, with $S = \{\sqrt{2}\}$ we obtain the algebraic extension field $K(\sqrt{2})$, since $\sqrt{2}$ is a root of $x^2 - 2$. See also [13], Chapter 7.

Consider a transcendental extension of K by a finite set of elements of E, say $S = \{x_1, x_2, \ldots, x_r\}$. It is then known that every element in the extension field can be expressed as a rational polynomial expression in the x_i. This permits us to consider purely symbolic extensions of K in which the elements of S are symbols. The resulting extension field is the *rational function field* of K, and is denoted $K(x_1, \ldots, x_n)$.

The cardinality of S is called the *transcendency degree* of the extension, and a theorem from algebra asserts that the transcendency degree is unique. This implies that if $u_1, \ldots, u_{r+1}$ are arbitrary elements in the extension field, then they must satisfy a polynomial equation $f(u_1, \ldots, u_{r+1}) = 0$ where the coefficients of f are in K.

We explain why the implicit equation of a rational curve exists, following [9]. Consider the rational function field $\mathbf{C}(s)$ of the complex numbers $\mathbf{C}$ of the transcendental element s. Because the curve is given by

$$\begin{aligned} x &= h_1(s) \\ y &= h_2(s) \end{aligned}$$

we should think of x and y as elements of $\mathbf{C}(s)$. But the transcendency degree of $\mathbf{C}(s)$ is 1, so x and y must satisfy a polynomial equation $f(x, y) = 0$. The simplest such equation is the implicit equation of the rational curve.

The same argument shows the existence of an implicit equation for a rational surface: Let

$$\begin{aligned} x &= h_1(s, t) \\ y &= h_2(s, t) \\ z &= h_3(s, t) \end{aligned}$$

be the parametric surface. Then x, y, and z are elements of the field $\mathbf{C}(s, t)$, of transcendency degree 2. So, they must satisfy a polynomial equation $f(x, y, z) = 0$.

Implicit Forms Vs. Base Points

We usually think of a parametric curve as a mapping from the line to a plane curve, and of a parametric surface as a mapping from the plane to a surface in 3-space. Suppose we have started with a parametric surface, found its implicit equation, and compare the point sets defined by each. Then it turns out that every point on the parametric surface also lies on the implicit surface, but not necessarily vice-versa. Consider as example the parametric surface

$$(st, st^2, s^2)$$

The implicit equation of the surface is

$$x^4 - y^2 z = 0$$

Since

$$(st)^4 - (st^2)^2(s^2) \equiv 0$$

every point of the parametric surface is also on the implicit surface. Now the points

$$(0, u, 0)$$

are on the implicit surface. But when $u \neq 0$, those points are not on the parametric surface, because $z = 0$ forces $s = 0$, and therefore also $y = 0$. It follows that the parametric surface could be a proper subset of the implicit surface.

In general, it is known that the implicit surface may contain finitely many isolated points and curves that are "missing" in the parametric form of the surface. Some of these points can be recovered by changing the parameterization to a projective one. We did this for the unit circle, in order to reach the point $(-1, 0)$, and for the unit sphere, to reach $(-1, 0, 0)$. But not all points can be so recovered, and, in particular, not the points missing in the example just shown.

Consider the rational surface

$$\left(\frac{h_1(s,t)}{h_0(s,t)}, \frac{h_2(s,t)}{h_0(s,t)}, \frac{h_3(s,t)}{h_0(s,t)}\right)$$

The polynomials h_k define four curves on the (s,t)-plane. A *base point* is a common intersection (s_0, t_0) of the four curves. Such parameter values (s_0, t_0) do not define a surface point. Chionh [9] discusses the relationship between base points and "missing points" on the parametric surface.

Sylvester's Resultant

Sylvester's resultant is a simple method for eliminating a variable from two algebraic equations, and it can be used to find the implicit equation of a parametric curve or surface. Given two polynomials

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \\ g(x) &= b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0 \end{aligned}$$

it can be shown that f and g have a common root iff the $(m+n)\times(m+n)$ determinant

$$R = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & a_n & \cdots & a_1 & a_0 & \cdots & 0 \\ \vdots & & \ddots & & & \ddots & \\ 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_0 \\ b_m & b_{m-1} & \cdots & b_0 & 0 & \cdots & 0 \\ 0 & b_m & \cdots & b_1 & b_0 & \cdots & 0 \\ \vdots & & \ddots & & & \ddots & \\ 0 & \cdots & 0 & b_m & b_{m-1} & \cdots & b_0 \end{vmatrix}$$

is zero: see [19], section 130. The determinant is the Sylvester resultant. Here, the a_k and b_k are assumed to be numbers.

The Sylvester resultant can be used for two multivariate equations: that is, under the assumption that the a_k and b_k are polynomials in the remaining variables. In this case, the Sylvester resultant R is a polynomial in these remaining variables. The solutions of the equation $R = 0$ either extend to a common solution of the original equations $f = 0$, $g = 0$, or are a solution of the system $a_n = 0$, $b_m = 0$, or else are a common solution of the coefficient polynomials of f or of g.

The Sylvester resultant can be used for implicitizing parametric curves and surfaces, subject to certain limitations. Briefly, the rational curve

$$\left(\frac{h_1(s)}{h_0(s)}, \frac{h_2(s)}{h_0(s)}\right)$$

is considered as the intersection of two surfaces in 3-space given by

$$\begin{aligned} xh_0(s) - h_1(s) &= 0 \\ yh_0(s) - h_2(s) &= 0 \end{aligned}$$

Elimination of s using the Sylvester resultant delivers the implicit equation, or a multiple of the implicit equation due to the possibility that the lead coefficients vanish.

Parametric surfaces could be implicitized similarly, forming the equations

$$\begin{aligned} xh_0(s,t) - h_1(s,t) &= 0 \\ yh_0(s,t) - h_2(s,t) &= 0 \\ zh_0(s,t) - h_3(s,t) &= 0 \end{aligned}$$

and eliminating first s, say, and then t. Alternatively, a number of other resultant formulations have been proposed for eliminating both variables at once. See [9] for details.

Gröbner Bases

For integral rational curves and surfaces, Gröbner bases algorithms provide an alternative for implicitizing parametrics without the introduction of extraneous factors. These algorithms are very sophisticated and are very general: see [6, 7, 13, 15, 16]. Certain specializations exist that improve their performance significantly.

We sketch some of the ideas that go into Gröbner bases algorithms. For a more detailed explanation and many examples and applications see [13], Chapter 7. Consider the set $K[x_1, \ldots, x_r]$ of all multivariate polynomials in $x_1, \ldots, x_r$ and with coefficients in the field K. For our purposes, we consider the field of complex numbers. An *ideal* I is a subset of polynomials in $K[x_1, \ldots, x_r]$ that is closed under addition and subtraction, and also under multiplication with other polynomials that are not necessarily in I. A basic theorem asserts that we can find a finite set $f_1, \ldots, f_n$ of polynomials in I such that every other polynomial g of I can be written as an algebraic combination of the f_k; that is,

$$g = u_1 f_1 + u_2 f_2 + \cdots + u_n f_n, \qquad u_k \in K[x_1, \ldots, x_r]$$

There are many ideal bases, not necessarily of the same cardinality. A Gröbner basis is an ideal basis with special properties that permit answering basic questions about the ideal using simple algorithms. Every ideal has a Gröbner basis, and this basis depends on certain orderings of terms. For example, in the *elimination ordering*[6] we first arrange the variables in a fixed sequence, say

$$x_1 \prec x_2 \prec \cdots \prec x_r$$

and declare that a term u comes earlier in the ordering than another term v provided that v contains a variable that is later in the variable sequence than every variable occurring in u. So, with $x \prec y \prec z$, the term $u = x^4 y^2$ would precede the term $v = xyz$. If the highest occurring variables in the two terms are the same, then the degree of that variable determines the order, and ties are broken by recursively considering subterms derived by deleting the highest variable from both terms. Thus $x^{10} y^2 \prec xy^3$ and $x^2 yz^2 \prec xy^2 z^2$.

Given a set of algebraic equations, the Gröbner basis of the ideal generated by the occurring polynomials, with respect to the elimination ordering, defines an equivalent system that is in triangular form and can be solved much more easily. The basis will contain the implicit form of a parametric curve or surface, provided the surface is integrally parameterized.

The term ordering influences the time required to construct a Gröbner basis. The elimination ordering just discussed produces a basis best-suited to many CAGD applications, but requires more time than basis construction with respect to certain other orderings. Basis conversion algorithms exist that allow first computing a Gröbner basis F with respect to any ordering, and then post-processing F to reveal some of the information explicit in the basis F' with respect to the elimination ordering. Combined, the two steps often are much more efficient than the outright construction of the basis F'. The approach is especially

[6] also called lexicographic ordering

appropriate for implicitizing parametric curves and surfaces. For details see [13], Section 7.8.

Experience with Elimination Algorithms

Many resultant formulations have not been implemented so that no experimental data can be cited in support of their possible practicality. We have experimented with implicitization of curves and surfaces using

1. Sylvester's resultant,
2. Gröbner bases with the elimination order, and
3. Basis conversion.

Three surface implicitization problems were solved, using integral parametric surfaces of degree two, three, and bicubic. The parametric quadric is

$$\begin{aligned} x &= 3t^2 + 4s^2 + st - 2s - 5t + 4 \\ y &= 6s^2 - st + 8t + 7 \\ z &= 9st + 12s - 15t + 34 \end{aligned}$$

The parametric cubic is

$$\begin{aligned} x &= -t^3 + 3st + s^3 + s \\ y &= ts^2 - 3t + 1 \\ z &= 2t^3 - 5st + t - s^3 \end{aligned}$$

The bicubic surface is

$$\begin{aligned} x &= 3t(t-1)^2 + (s-1)^3 + 3s \\ y &= 3s(s-1)^2 + t^3 + 3t \\ z &= -3s(s^2 - 5s + 5)t^3 - 3(s^3 + 6s^2 - 9s + 1)t^2 \\ &\quad + t(6s^3 + 9s^2 - 18s + 3) - 3s(s-1) \end{aligned}$$

The running times are shown in the table. All computations were done on a Symbolics 3650 Lisp machine with 16MB main memory and 120 MB virtual memory. Note that the hardware speed of the machine is less than one MIP. Methods 1 and 2 are the standard implementations of resultants and Gröbner bases offered by Macsyma 414.62. Method 3 was written at Purdue. An entry ∞ indicates that the computation could not be completed due to insufficient virtual memory. The table shows clearly that Method 3 improves efficiency significantly, but overall the times are much slower than one would require for routine applications. Future work is required to improve the situation.

Problem	Method 1	Method 2	Method 3
quadratic	21	22	6
cubic	10^5	∞	315
bicubic	∞	∞	10^5

Table 1: Implicitization Times in Seconds

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5 The Slides

I. Basic Facts

I.A. Definitions Recalled

I.B. Theorems on Conversion

I.C. Projective Parameterizations

Plane Parametric Curve

$$x = h_1(s) \qquad y = h_2(s)$$

Parametric Surface

$$x = h_1(s,t) \qquad y = h_2(s,t) \qquad z = h_3(s,t)$$

The h_k are polynomials or ratios of polynomials.

Implicit Curve

$$f(x,y) = 0$$

Implicit Surface

$$f(x,y,z) = 0$$

Typically f is a polynomial in the power base.

Nota Bene

- Parametrics are often restricted to a *domain*, but not here.
- Many properties of parametrics depend on the *basis* in which the h_k have been expressed, and are valid within a given domain only. We assume a basis in which the h_k are uniquely written.

Basic Theorems

- Every plane parametric curve has an implicit form.
- Every parametric surface has an implicit form.
- Not every implicit plane curve has a parametric form.
- Not every implicit surface has a parametric form.

so...

The class of parametric curves and surfaces is smaller than the class of implicit curves and surfaces.

but...

An exact characterization of parameterizability is not simple.

Some Parameterizable Implicits

- All lines and all conic sections
- All planes and all quadratic surfaces
- Singular cubic curves
- All monoids

Methods for Implicitizing Parametrics

- Variable elimination via resultants.
- Elimination ideals via Grōbner bases.

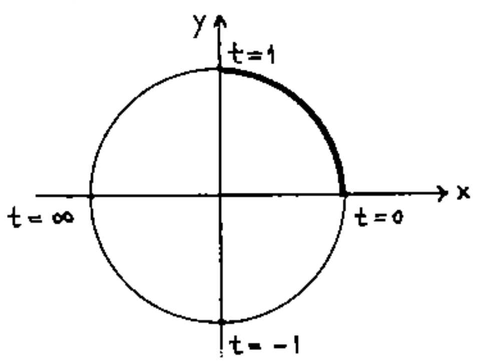
All methods are expensive except in certain special cases.

Example

The circle $x^2 + y^2 - 1 = 0$ is a conic section and can be parameterized. The parametric form is

$$x(t) = \frac{1 - t^2}{1 + t^2} \qquad y(t) = \frac{2t}{1 + t^2}$$

Parameter/Curve Point Correspondence



Other parameterizations of the circle can be obtained with a fractional linear transformation of t; e.g.,

$$t = \frac{3s - 1}{s + 1}$$

yields

$$x(s) = \frac{-4s^2 + 4s}{5s^2 - 2s + 1} \qquad y(s) = \frac{3s^2 + 2s - 1}{5s^2 - 2s + 1}$$

Parameter/Curve Point Correspondence

The parameterization does not "reach" the point $(-1,0)$ unless it is changed to a *projective* parameterization, by homogenizing the functions h_k.

$$x = \frac{s^2 - t^2}{s^2 + t^2} \qquad y = \frac{2st}{s^2 + t^2}$$

On surfaces, not all "missing points" may be so recovered.

Parameter/Curve Point Correspondence

Missing Points Example

The parametric surface

$$x = st \qquad y = st^2 \qquad z = s^2$$

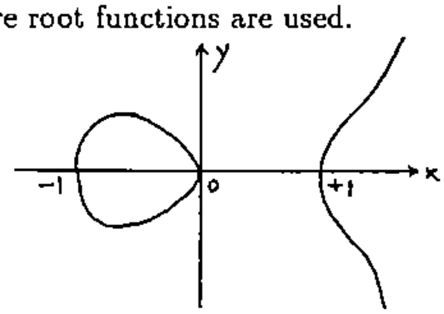
has the implicit form

$$x^4 - y^2 z = 0$$

This surface contains the line $x = z = 0$ that is not reached by the parametric form.

Nonparameterizability Example

The cubic $y^2 - x^3 + x = 0$ cannot be parameterized unless square root functions are used.



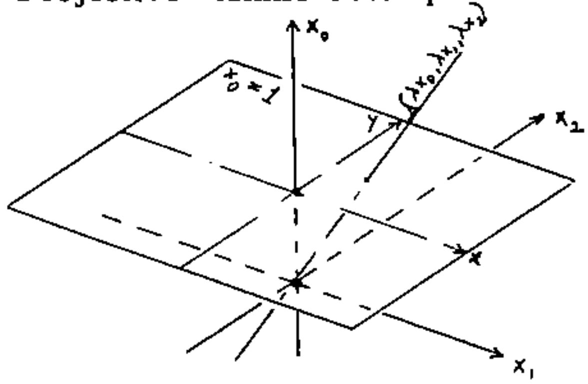
Projective Coordinates

The affine point (x_1, x_2) corresponds to the projective point $(\lambda x_0, \lambda x_1, \lambda x_2)$, where $x_0 = 1$ and λ is not zero.

This sets up a correspondence between lines in 3-space and points in the Cartesian plane.

Points with $x_0 = 0$ are permitted, but not the point $(0, 0, 0)$. Such points are called *points at infinity.*

Projective - Affine Correspondence



Having projective coordinates simplifies many theorems by eliminating special cases.

The affine implicit curve $f(x, y) = 0$ corresponds to the projective implicit curve $F(w, x, y) = 0$, where

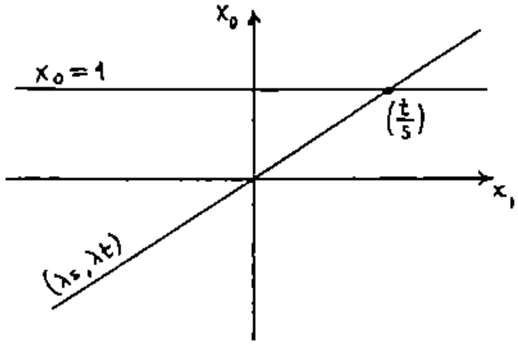
$$F = w^n f(\frac{x}{w}, \frac{y}{w})$$

and n is the degree of f.

The projective line has points $(\lambda s, \lambda t)$, where $\lambda \neq 0$, and s and t not both zero.

A *projective* curve parameterization is one in which the coordinate functions are homogeneous in s and in t.

Projective - Affine Correspondence



Example

The unit circle is parameterized by

$$\begin{aligned} w(s,t) &= s^2 + t^2 \\ x(s,t) &= s^2 - t^2 \\ y(s,t) &= 2st \end{aligned}$$

When s and t are integer, we obtain the rational points of the unit circle.

A projective surface parameterization is a map from the projective plane to the surface.

There may be some points that are not reached by a projective surface parameterization; they are called *base points.*

Valuations are a method for reaching all surface points.

End of Part I

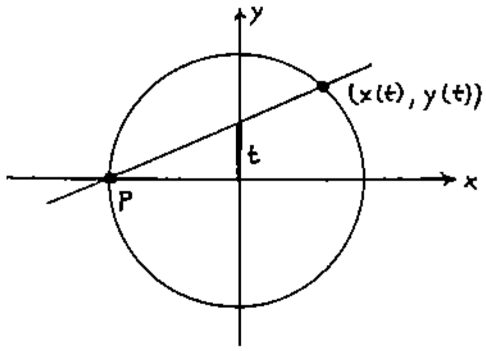
II. Parameterizing Implicit Algebraics

II.A. Curves:
Geometric Idea
Conics, Cubics
Monoids

II.B. Surfaces:
Monoids, Quadrics

II.A Curve Parameterization

How to Parameterize A Circle



Line equations are $y = t(x+1)$
Substitution into circle yields

$$x^2(1+t^2) + 2t^2x + t^2 - 1 = 0$$

Solutions are -1 and

$$x(t) = \frac{1-t^2}{1+t^2}$$

Resulting Parameterization is

$$x(t) = \frac{1-t^2}{1+t^2}$$
$$y(t) = \frac{2t}{1+t^2}$$

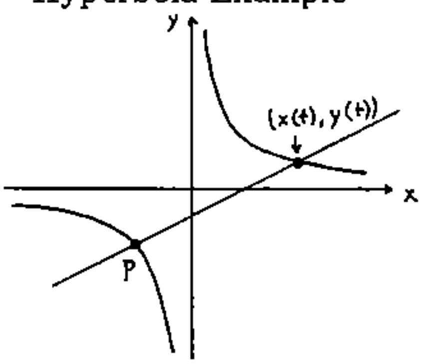
Algorithm

1. Fix a point p on the conic. Consider the pencil of lines through p. Formulate the line equations.

2. Substitute for y in the conic equation, solve for $x(t)$.

3. Use the line equations to determine $y(t)$.

Hyperbola Example



Implicit equation:	$xy - 1 = 0$
Lines are:	$y = (x+1)t - 1$
Substitution yields:	$x(x+1)t - x - 1 = 0$
Interesting root:	$x(t) = 1/t$
Parameterization:	$x(t) = 1/t$
	$y(t) = t$

How to Find a Point on a Conic

Intersect with a line – requires solving a quadratic equation.

Easiest with line at infinity, but that may yield complex points.

Otherwise, find extrema by intersecting with a partial.

Algebraic Method for Conics

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0$$

1. Find the roots of degree form

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2$$

They are

$$\begin{aligned} x &= -a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}} \\ y &= a_{11} \end{aligned}$$

2. If (u, v) is a real root, then substitute

$$\begin{aligned} x &= x_1 + uy_1 \\ y &= vy_1 \end{aligned}$$

The effect is to cancel the y^2 term.

3. Set $x(t) = t$. Compute $y(t)$ for transformed conic.

4. Backtransform.

Example

Given: $x^2 - y^2 + 2x + y + 4$

Degree form: $x^2 - y^2$

Root: $(-1, 1)$

Substitution: $x = x_1 - y_1$

$y = y_1$

Result: $x_1^2 + 2x_1 + 4 - y_1(2x_1 + 1) = 0$

Parameterization: $x_1 = t$

$$y_1 = (t^2 + 2t + 4)/(2t + 1)$$

Backtransformation: $x(t) = \dfrac{t^2 - t - 4}{2t + 1}$

$$y(t) = \frac{t^2 + 2t + 4}{2t + 1}$$

Jacobi's Method

1. Write the conic as the bilinear form

$$(x \;\; y \;\; 1) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

2. Diagonalize by applying rotations $B = RAR^{\mathrm{T}}$
3. Backtransform a standard parameterization of the diagonal form.

Cancel p in the submatrix

$$\begin{pmatrix} m & p \\ p & n \end{pmatrix}$$

with

$$R = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

where

$$\tan(2\alpha) = \frac{2p}{m-n}$$

But observe...

Suppose R has canceled a_{12} with matrix R. Then a subsequent rotation R', canceling a_{13} say, may reintroduce a nonzero a_{12}.

It can be proved that $a_{12}^2 + a_{13}^2 + a_{23}^2$ is reduced with every rotation.

Let

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

be the final diagonal form.
If some of the λ_k are zero, then the original conic is degenerate.
If all λ_k have the same sign, the original conic is imaginary.

Otherwise, assume without loss of generality that $\lambda_1 > 0$. If λ_2 is also positive, then the parameterization is

$$\begin{aligned} x(t) &= \mu_1 \frac{1-t^2}{1+t^2} \\ y(t) &= \mu_2 \frac{2t}{1+t^2} \end{aligned}$$

where

$$\mu_1 = \sqrt{|\lambda_1/\lambda_3|} \qquad \mu_2 = \sqrt{|\lambda_2/\lambda_3|}$$

If λ_2 is negative, then the parameterization is

$$\begin{aligned} x(t) &= \mu_1 \frac{1+t^2}{1-t^2} \\ y(t) &= \mu_2 \frac{2t}{1-t^2} \end{aligned}$$

Again

$$\mu_1 = \sqrt{|\lambda_1/\lambda_3|} \qquad \mu_2 = \sqrt{|\lambda_2/\lambda_3|}$$

Example of Jacobi's Method

Parabola $y^2 - 2x = 0$

$$P = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Rotation angle $\pi/4$ gives matrix

$$P = \begin{pmatrix} a & 0 & -a \\ 0 & 1 & 0 \\ a & 0 & a \end{pmatrix}$$

where $a = \sqrt{2}/2$.

Then

$$RPR^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with parameterization

$$\begin{aligned} x_1 &= s^2 - t^2 \\ y_1 &= 2st \\ z_1 &= s^2 + t^2 \end{aligned}$$

but

$$(x_1\, y_1\, z_1)R = (x\, y\, z)$$

so the parabola is parameterized by

$$\begin{aligned} x &= \sqrt{2}s^2 \\ y &= 2st \\ z &= \sqrt{2}t^2 \end{aligned}$$

In affine coordinates, and with $t = 1$, therefore,

$$\begin{aligned} x &= s^2 \\ y &= \sqrt{2}s \end{aligned}$$

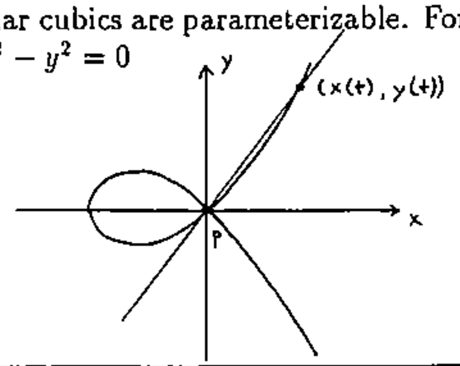
The more familiar form

$$\begin{aligned} x &= s^2/2 \\ y &= s \end{aligned}$$

is obtained with $t = \sqrt{2}$

Cubic Curves

Only singular cubics are parameterizable. For example, $x^3 + x^2 - y^2 = 0$



Line equations: $y = tx$

Substitution: $x^3 + x^2(1 - t^2) = 0$

Roots: $x = 0$

$x(t) = t^2 - 1$

Therefore: $y(t) = t^3 - t$

Problems with Cubic Curves

An algebraic algorithm for parameterizing cubics has to

1. determine if there is a singularity,
2. and if so, find it.

Algebraic Method for Cubic Curves

1. Eliminate y^3 term.
2. Transform cubic to the form

   $$y^2 = h_4(x)$$

   where h_4 is degree 4.
3. If h_4 has a double root, then cubic can be parameterized.

1. How to Eliminate the y^3-term

a) Find a real root of the degree form. say (u, v).

b) Substitute

$$
\begin{aligned}
x_1 &= x_1 - vy_1 \\
y_1 &= uy_1
\end{aligned}
$$

2. Transformation to $y^2 = h_4(x)$

Cubic has form $h_1(x)y^2 + h_2(x)y + h_3(x) = 0$

a) Multiply with $h_1(x)$:

$$h_1^2 y^2 + h_1 h_2 y + h_1 h_3 = 0$$

$$(h_1 y + \tfrac{1}{2} h_2)^2 - \tfrac{1}{4} h_2^2 + h_1 h_3 = 0$$

b) Substitute

$$y_2 = h_1 y + \frac{1}{2} h_2$$

3. Parameterizing $y^2 = h_4(x) = (x - \lambda)^2 g(x)$

a) Substitute $y_3 = y/(x - \lambda)$.

b) Parameterize the conic $y_3^2 = g(x)$.

c) Backtransform the parameterization.

A Worked Example

$$28y^3+26xy^2+7x^2y+x^3/2+28y^2+16xy+7y+3x/2 = 0$$

Degree form is

$$28y^3 + 26xy^2 + 7x^2y + x^3/2$$

and has the real root $(-2, 1)$.

Substitute

$$\begin{aligned} x &= x_1 - 2y_1 \\ y &= y_1 \end{aligned}$$

Yields

$$4(x_1 - 1)y_1^2 + 4(x_1^2 + 4x_1 + 1)y_1 + (x_1^3 + 3x_1)/2 = 0$$

Multiply with $(x_1 - 1)$ and regroup, obtaining

$$\begin{aligned} &(2(x_1 - 1)y_1 + (x_1^2 + 4x_1 + 1))^2 \\ - &\ (x_1^2 + 4x_1 + 1)^2 + (x_1^4 - x_1^3 + 3x_1^2 - 3x_1)/2 \end{aligned}$$

Substitute

$$y_2 = 2(x_1 - 1)y_1 + (x_1^2 + 4x_1 + 1)$$

Result is

$$y_2^2 = (x_1^4 + 17x_1^3 + 33x_1^2 + 19x_1 + 2)/2$$

Here, -1 is double root of righthand side. Substitute

$$y_3 = \frac{y_2}{x_1 + 1}$$

Result is

$$2y_3^2 = x_1^2 + 15x_1 + 2$$

which is parameterized by

$$x_1 = \tfrac{t^2-2}{2t+15} \qquad y_3 = -\tfrac{t^2+15t+2}{\sqrt{2}(2t+15)}$$

Backtransformation yields

$$y_2 = -\frac{(t^2 + 15t + 2)(t^2 + 2t + 13)}{\sqrt{2}(2t + 15)^2}$$

and $y_1 =$

$$-\frac{(\sqrt{2}+1)t^4+(8\sqrt{2}+17)t^3+(60\sqrt{2}+45)t^2+(44\sqrt{2}+199)t+(109\sqrt{2}+26)}{\sqrt{2}(4t^3+22t^2-128t-510)}$$

y_1 simplifies to

$$\frac{(\sqrt{2}+1)t^3+(6\sqrt{2}+12)t^2+(30\sqrt{2}+21)t+(11\sqrt{2}+40)}{\sqrt{2}(4t^2-(12\sqrt{2}-26)t-(90\sqrt{2}+30))}$$

because of the common factor $(t - 1 + 3\sqrt{2})$.

Generalizations

Conceptually, we parameterize using a pencil of lines,

$$(y - a) = t(x - b)$$

through the curve point $p = (a, b)$.

Cubics mandate that p is a special curve point. Will the method work in general, when p is suitably chosen?

No, but there is a class of algebraic curves that may be treated in this way. These are *monoids*.

Monoid Definition

A *monoid* is an algebraic curve of degree n that has a point of multiplicity $n - 1$.

- All conics are monoids.
- All singular cubics are monoids.

If the special point is known, monoids are easy to parameterize.

Implicit Equation of Monoids

If the $(n - 1)$-fold point is at the origin, the implicit equation of the monoid is of the form

$$h_n(x, y) - h_{n-1}(x, y) = 0$$

where h_n has terms only of degree n, and h_{n-1} has terms only of degree $n - 1$

Examples

Circle:	$x^2 + y^2 - (2x) = 0$
Hyperbola:	$xy - (x + y) = 0$
Parabola:	$y^2 - (x) = 0$
Alpha curve:	$x^3 - (y^2 - x^2) = 0$
Cusp:	$x^3 - (y^2) = 0$

Parameterizing a Monoid

Monoid equation $\quad h_n(x, y) - h_{n-1}(x, y) = 0$

Parameterization $\quad x(t) = \dfrac{h_{n-1}(1, t)}{h_n(1, t)}$

$$y(t) = t\,\frac{h_{n-1}(1, t)}{h_n(1, t)}$$

So easy, that monoids are also called *dual forms.*

Example

The circle through origin

$$(x^2 + y^2) - 2x = 0$$

has the parameterization

$$\begin{aligned} x(t) &= \frac{2}{1 + t^2} \\ y(t) &= \frac{2t}{1 + t^2} \end{aligned}$$

II.B. Surface Parameterization

The general problem is algorithmically unsolved.

The pencil-of-lines approach generalizes to a bundle-of-lines approach, where the bundle centered at $p = (a, b, c)$ consists of all lines, through p, indexed by s and t:

$$\begin{aligned} y - b &= s(x - a) \\ z - c &= t(x - a) \end{aligned}$$

Jacobi's algorithm also generalizes to quadrics.

Bundle-of-Lines Idea

Pick a (special) point, on the surface, as center of line bundle.

Determine the additional intersection as function of s and t.

Example

The unit sphere has the equation

$$x^2 + y^2 + z^2 - 1 = 0$$

Choosing $p = (-1, 0, 0)$, we substitute $y = s(x+1)$ and $z = t(x+1)$, obtaining

$$x^2(1 + s^2 + t^2) + 2x(s^2 + t^2) - (1 - s^2 - t^2) = 0$$

Resulting Parameterization

$$x(s,t) = \frac{1 - s^2 - t^2}{1 + s^2 + t^2}$$
$$y(s,t) = \frac{2s}{1 + s^2 + t^2}$$
$$z(s,t) = \frac{2t}{1 + s^2 + t^2}$$

Monoids

The bundle-of-lines method works for any surface on which there is a point such that (almost) every line intersects the surface in one additional point. Such surfaces are *monoids*.

A monoid is an algebraic surface of degree n with an $(n-1)$-fold point on it.
Monoids include all quadrics. cubics with a double point. and Steiner surfaces.

Implicit Equation of Monoids

If the $(n-1)$-fold point is at the origin, the implicit equation of the monoid is of the form

$$h_n(x,y,z) - h_{n-1}(x,y,z) = 0$$

where h_n has terms only of degree n, and h_{n-1} has terms only of degree $n-1$

Parameterizing a Monoid

Monoid $\quad h_n(x,y,z) - h_{n-1}(x,y,z) = 0$

Parameterization
$$x(s,t) = \frac{h_{n-1}(1,s,t)}{h_n(1,s,t)}$$
$$y(s,t) = s\,\frac{h_{n-1}(1,s,t)}{h_n(1,s,t)}$$
$$z(s,t) = t\,\frac{h_{n-1}(1,s,t)}{h_n(1,s,t)}$$

Example

$$(x^2+y^2+z^2) - 2x = 0$$

has the parameterization

$$\begin{aligned} x(s,t) &= \frac{2}{1+s^2+t^2} \\ y(s,t) &= \frac{2s}{1+s^2+t^2} \\ z(s,t) &= \frac{2t}{1+s^2+t^2} \end{aligned}$$

End of Part II

Part III

III.A. Existence of an Implicit Form

III.B. Sylvester's Resultant

III.C. Gröbner Bases Methods

III.D. Some Experiments

Why should there be an implicit form?

Indeed, given rational functions $h_1(s)$ and $h_2(s)$, why is there a polynomial $q(x,y)$ such that

$$q(h_1, h_2) \equiv 0\,?$$

Fields

A *field* is a set of "numbers" which we can add, subtract, multiply and divide. Examples: Complex numbers, real numbers, rational numbers.

If a field K is a subset of another field E, then E is an *extension field* of K.

Extension by Adjoining an Element

Let $K \subset E$, and $s \in E - K$. We construct the extension field $K(s)$ by adding to K all elements *required* to make $K \cup \{s\}$ a field.

The extension is either *algebraic* or *transcendental*.

Transcendental Field Extensions

$K(s)$ is transcendental if there is no polynomial $p(x)$ with coefficients in K such that s is a root.

For instance, $\mathbf{R}(\pi)$ is a transcendental extension.

Of course, we can extend K with several transcendentals, e.g., $K(s_1, s_2, ..., s_m)$.

The general element in the transcendental extension $K(s_1, ..., s_m)$ has the form

$$u = \frac{p(s_1, ..., s_m)}{q(s_1, ..., s_m)}$$

where p and q are polynomials with coefficients in K.

Algebraic Dependence

Let K' be a transcendental extension of K, obtained by adjoining finitely many s_k.

The elements $u_1, ..., u_r$ in $K' - K$ are *algebraically dependent* if there is a polynomial $q(x_1, ..., x_r)$ with coefficients in K such that the u_k are a root of q.

Otherwise the u_k are *algebraically independent.*

Transcendency Degree

The *transcendency degree* of K' is a number d such that any $d + 1$ elements in $K' - K$ are algebraically dependent.

Theorem

The transcendency degree of K' is unique.

Existence of Implicit Form

Given the rational functions

$$\begin{aligned} x &= h_1(s) \\ y &= h_2(s) \end{aligned}$$

then x and y are elements in $\mathbf{R}(s)$, the transcendental extension of $\mathbf{R}$ by s. But $\mathbf{R}(s)$ has transcendency degree 1, so x and y are algebraically dependent; i.e.,

$$q(x, y) = 0$$

The minimum degree q is the implicit form.

Similarly, given the parametric functions

$$x = h_1(s,t) \qquad y = h_2(s,t) \qquad z = h_3(s,t)$$

then x, y, and z are in $\mathbf{R}(s,t)$, of transcendency degree 2, so they are algebraically dependent; i.e.,

$$q(x, y, z) = 0$$

Nota bene...

The implicit form may contain points not found on the parametric form.

Example

The parametric surface

$$x = st \qquad y = st^2 \qquad z = s^2$$

has the implicit form

$$x^4 - y^2 z = 0$$

This surface contains the line $x = z = 0$ that is not on the parametric surface, except for the point $(0, 0, 0)$.

III.B. Sylvester's Resultant

Variable Elimination

Given two polynomials

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \\ g(x) &= b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0 \end{aligned}$$

we want a criterion for a common solution.

Form the determinant R

$$R = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & a_n & \cdots & a_1 & a_0 & \cdots & 0 \\ \vdots & & \ddots & & & \ddots & \\ 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_0 \\ b_m & b_{m-1} & \cdots & b_0 & 0 & \cdots & 0 \\ 0 & b_m & \cdots & b_1 & b_0 & \cdots & 0 \\ \vdots & & \ddots & & & \ddots & \\ 0 & \cdots & 0 & b_m & b_{m-1} & \cdots & b_0 \end{vmatrix}$$

The determinant R is the *Sylvester resultant.*

Theorem

$R = 0$ iff the two polynomials have a common root.

The Sylvester resultant "eliminates" the variable x.

The resultant can be used for multivariate polynomials, but then it introduces extraneous factors:

Theorem

$R = 0$ iff there is a common solution of the two equations, or of the two lead coefficients, or of all coefficients of one or the other polynomial.

Example

$$x = st \qquad y = st^2 \qquad z = s^2$$

Step 1: Eliminate s

$$\begin{vmatrix} -t & x & 0 \\ 0 & -t & x \\ -1 & 0 & z \end{vmatrix} = zt^2 - x^2$$

$$\begin{vmatrix} -t^2 & y & 0 \\ 0 & -t^2 & y \\ -1 & 0 & z \end{vmatrix} = zt^4 - y^2$$

So, we now have two polynomials

$$zt^2 - x^2$$
$$zt^4 - y^2$$

Step 2: Eliminate t

$$\begin{vmatrix} z & 0 & -x^2 & 0 & 0 & 0 \\ 0 & z & 0 & -x^2 & 0 & 0 \\ 0 & 0 & z & 0 & -x^2 & 0 \\ 0 & 0 & 0 & z & 0 & -x^2 \\ z & 0 & 0 & 0 & -y^2 & 0 \\ 0 & z & 0 & 0 & 0 & -y^2 \end{vmatrix} = z^2(y^2z - x^4)^2$$

So, we obtain

$$z^2(y^2z - x^4)^2 = 0$$

as implicit form.
Here, $z^2(y^2z - x^4)$ is an *extraneous factor*.

III.C. Gröbner Bases

Computations in Ideals

Intuition

Given a system of linear equations, manipulations such as LU-decomposition derive an equivalent linear system that is easier to solve.

Likewise, given a system of nonlinear equations, a Gröbner basis is an equivalent system of nonlinear equations that is easier to solve.

Technically

Gröbner bases deal with polynomial ideals. Ideals come up as follows.

What is a Unique Surface Representation?

$$f(x,y,z) = 0?$$

$$5f(x,y,z) = 0?$$

$$g(x,y,z)f(x,y,z) = 0?$$

The unique representation is an ideal...

Also true for curves, surface intersections, and so on.

Ideals

An *ideal* I is a set of polynomials such that

1. If p and q are in I, then so is $p - q$.
2. If $p \in I$, and q is any polynomial, then pq is also in I.

Ideal Bases

All ideals are finitely generated; that is, there are polynomials

$$f_1, f_2, \ldots, f_m$$

in I such that every other polynomial in I can be written

$$q = u_1 f_1 + u_2 f_2 + \cdots u_m f_m$$

where the u_i are polynomials.

A Gröbner basis is an ideal basis with special properties.

In particular, a GB with respect to the "elimination ordering" allows solving a system of nonlinear equations in an especially simple manner.

Example 1

We are given the nonlinear system

$$f_1: \; x^2 + y^2 - 1$$
$$f_2: \; y^2 + z^2 - 1$$
$$f_3: \; z^2 + x^2 - 1$$

These polynomials generate an ideal I with a GB

$$g_1: \; x^2 - 2$$
$$g_2: \; y^2 - 2$$
$$g_3: \; z^2 - 2$$

So, $x = \pm\sqrt{2}/2$, $y = \pm\sqrt{2}/2$, $z = \pm\sqrt{2}/2$.

Example 2

Given

$$\begin{aligned} z^2 + 2yz + 2xz + y^2 + 2xy + x^2 - 1 &= 0 \\ z^2 - 2yz - 2xz + y^2 + 2xy + x^2 - 1 &= 0 \\ z^2 - 2yz + 2xz + y^2 - 2xy + x^2 - 1 &= 0 \\ z^2 + 2yz - 2xz + y^2 - 2xy + x^2 - 1 &= 0 \\ z^2 + y^2 - x - 1 &= 0 \end{aligned}$$

GB is

$$\begin{aligned} x^2 + x &= 0 \\ xy &= 0 \\ y^3 - y &= 0 \\ xz &= 0 \\ yz &= 0 \\ z^2 + y^2 - x - 1 &= 0 \end{aligned}$$

$x = -1$:
Substitution gives $y = 0$, which in turn gives $z = 0$

$x = 0$:
$y^3 - y = 0$, so $y = 0, -1, +1$. Each (x, y) pair extends to one or more solutions in z.

Final set is

$$(-1,0,0),\ (0,1,0),\ (0,-1,0),\ (0,0,1),\ (0,0,-1)$$

Implicitization with Gröbner Bases

Gröbner bases can be used to construct the implicit form of integrally parameterized curves and surfaces.

The method does not introduce extraneous factors.

Example

Given

$$x - st \qquad y - st^2 \qquad z - s^2$$

the Gröbner basis wrt the elimination ordering is

$$\begin{aligned}&x^4 - y^2 z,\\ &tx - y, tyz - x^3, t^2 z - x^2,\\ &sy - x^2, sx - tz, st - x, s^2 - z\end{aligned}$$

The Gröbner basis discloses the implicit form, plus inversion formulae that show that the surface is faithfully parameterized.

Gröbner bases are always wrt a particular term ordering.

For some orderings, basis computations can be much faster, but the resulting basis does not reveal as much information explicitly.

Given some Gröbner basis, there are *conversion algorithms* that reconstruct the missing information.

Such conversion algorithms are extremely important for efficiency.

III.D. Some Experiments

How expensive is implicitization?

Comparison of three implicitization algorithms:

1. Sylvester's resultant,
2. Gröbner bases with the elimination order, and
3. Basis conversion.

Implicitization for a parametric quadric, a parametric cubic, and a bicubic.

All computations on a Symbolics 3650 under Genera 7.2, Macsyma 414.62, with 16MB main memory and 120MB swap space.

The parametric quadric is

$$
\begin{aligned}
x &= 3t^2 + 4s^2 + st - 2s - 5t + 4 \\
y &= 6s^2 - st + 8t + 7 \\
z &= 9st + 12s - 15t + 34
\end{aligned}
$$

The parametric cubic is

$$
\begin{aligned}
x &= -t^3 + 3st + s^3 + s \\
y &= ts^2 - 3t + 1 \\
z &= 2t^3 - 5st + t - s^3
\end{aligned}
$$

The bicubic surface is

$$
\begin{aligned}
x &= 3t(t-1)^2 + (s-1)^3 + 3s \\
y &= 3s(s-1)^2 + t^3 + 3t \\
z &= -3s(s^2 - 5s + 5)t^3 - 3(s^3 + 6s^2 - 9s + 1)t^2 \\
&\quad + t(6s^3 + 9s^2 - 18s + 3) - 3s(s-1)
\end{aligned}
$$

Implicitization Times in Sec.

Problem	Method 1	Method 2	Method 3
quadratic	21	22	6
cubic	10^5	∞	315
bicubic	∞	∞	10^5

Conclusions

We need both faster machines and faster algorithms.

Basis conversion improvement seems to suggest a route of specialization, and of paring down the information that is computed.

End of Part III