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S. Khlebnikov

# Massive Schwinger model with a finite inductance: $\theta$ (in)dependence, the $U(1)$ problem, and low-energy theorems

S. Khlebnikov

Department of Physics, Purdue University, West Lafayette, Indiana 47907, USA  
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Gauge theories embedded into higher-dimensional spaces with certain topologies acquire inductance terms, which reflect the energy cost of topological charges accumulated in the extra dimensions. We compute topological susceptibility in the strongly-coupled two-flavor massive Schwinger model with such an inductance term and find that it vanishes, due to the contribution of a global low-energy mode (a “global axion”). This is in accord with the general argument on the absence of  $\theta$  dependence in such topologies. Because the mode is a single oscillator, there is no corresponding particle, and the solution to the  $U(1)$  problem is unaffected.

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## I. INTRODUCTION

An important aspect of the nontrivial vacuum structure of QCD [1–5] is the formation of coherent superpositions known as  $\theta$ -vacua. To each value  $-\pi < \theta \leq \pi$ , there corresponds a separate sector of the Hilbert space, and transitions between different sectors are prohibited by a selection rule. For all values of  $\theta$  except 0 and  $\pi$ , the theory breaks  $CP$  symmetry; the absence (or unnatural smallness) of this  $CP$  breaking in practice presents a naturalness problem, known as the strong  $CP$  problem.

The presence of a  $\theta$ -angle reflects the perfect degeneracy between states connected by topologically nontrivial (“large”) gauge transformations [4,5]. The axion solution [6,7] does not destroy this degeneracy but screens the value of  $\theta$  in a way somewhat similar to how the usual Higgs mechanism screens an electric field [8]. This solution gives rise to a new light particle—the axion [9,10] (for a recent review, see Ref. [11]).

On the other hand, if there are extra dimensions with suitable topology, the degeneracy is no longer protected by gauge invariance, and that gives a reason to think that in this way an axion-free solution to the strong  $CP$  problem can be achieved [12,13]. (Other extra-dimensional solutions have been proposed in Refs. [14,15].)

One may worry, however, that the solution proposed in Refs. [12,13] is in conflict with the low-energy theorems of QCD, such as those derived in Refs. [16–18]. These theorems connect the existence of observable  $\theta$ -dependencies to a successful solution to the  $U(1)$  problem—the absence of a light  $\eta'$  meson. In the present paper, we argue that this concern is not justified, using the massive two-flavor Schwinger model (two-dimensional electrodynamics), about which much is known [19], as an example.

The strongly-coupled Schwinger model provides a rather close analogy to QCD, including a solution to the  $U(1)$  problem. Of course, not all observable quantities of QCD have direct analogs in this model. An important one that does is the vacuum topological susceptibility, a low-energy relation for which was derived in Refs. [16,17].

Here we consider the essentially identical relation applicable in the Schwinger model.

The single (but crucial) new ingredient that we add to the model is a *finite inductance*, which reflects the energy of magnetic flux in the extra dimensions. Such a flux is the simplest example of a topological charge (instanton number), and a finite energy associated with accumulation of the topological charge in the extra dimensions appears naturally in the scenario of Ref. [13].

Thus, we consider the theory on a ring (Fig. 1) embedded in a higher-dimensional space. The length  $L$  of the ring is taken to be much larger than the inverse of either the fermion mass  $m$  or the gauge coupling  $g$ . The fermions are confined to move along the ring, but the gauge field can penetrate some distance into the bulk. Nevertheless, the gauge field is assumed localized, in the sense that the total number of modes available to it at low energies is the same

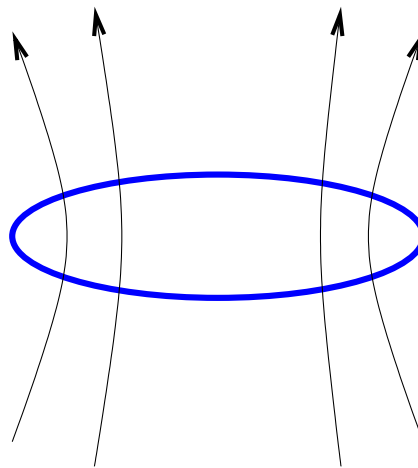


FIG. 1 (color online). A ring threaded by a magnetic flux. Instanton transitions in the gauge theory on the ring correspond to changing the value of the flux, and  $\theta$ -vacua—to a steady current of flux into the ring. One expects that the energy cost associated with increasing the flux (a finite inductance) will prevent such steady currents.

as in the two-dimensional (one space and one time) theory. (In particular, the Coulomb law is the same as in two dimensions.) The precise mechanism of localization will not be important here. (For a variant of such a mechanism, see Ref. [13].)

A finite inductance lifts the degeneracy between vacua corresponding to different values of magnetic flux through the ring, and it stands to reason that this will prevent formation of  $\theta$ -vacua. We stress that the system is not confined to a vicinity of a specific value of the flux: transitions between different values are fully allowed, and the true vacuum is a superposition of different flux states. It is just that this superposition is now unique and is no longer characterized by a value of “quasimomentum” (a  $\theta$ -angle).

In the present paper we show that this  $\theta$  independence is in complete accord with the low-energy theorem and does not in any way upset the solution to the  $U(1)$  problem. What happens is that the global topological mode, which was originally frozen by the superselection rule, is now liberated and contributes precisely the right amount to satisfy the theorem.

These results are derived in Secs. II and III. Considerations of Sec. II are general, while in Sec. III we restrict our attention to the strong-coupling limit, where the analogy with QCD applies. In that limit, we explicitly compute the topological susceptibility and find that it is zero.

The topological mode is a single oscillator: there is no particle associated with it (it is a “global axion” in the terminology of Ref. [13]). Thus, the price one pays for such a solution to the strong  $CP$  problem is a (weak) violation of Lorentz invariance.

## II. THE ROLE OF FINITE INDUCTANCE

The model contains two species of fermions of equal masses  $m$ , interacting with an Abelian gauge field  $A_\mu$ ,  $\mu = 0, 1$ . The action is

$$S = \int dx dt \left\{ \frac{1}{2} F_{01}^2 - \frac{g\theta}{2\pi} F_{01} + \sum_{f=1,2} \bar{\psi}_f [i\gamma^\mu (\partial_\mu - igA_\mu) - m] \psi_f \right\} - \frac{1}{2I} \int dt \Phi^2, \quad (1)$$

where  $F_{01} = \partial_0 A_1 - \partial_1 A_0$ ,  $\Phi$  is the magnetic flux through the ring:

$$\Phi = \int_0^L dx A_1,$$

and  $I$  is a finite inductance. The usual two-flavor massive Schwinger model is recovered in the limit  $I \rightarrow \infty$ .

Following Ref. [19], we will make use of the bosonization technique. However, we will need to bosonize on a circle, as opposed to the infinite line. Bosonization on a

circle has a sizable literature on the particle-theory side of the problem (see Ref. [20] and references therein) and an even larger one on the condensed-matter side (see the review [21]). For developments along different lines, see Ref. [22] (on the relation to conformal field theory) and Ref. [23] (on lattice results).

We summarize key bosonization formulas in Appendix A, where we also indicate simplifications obtained by restricting to the sector with zero total fermionic charges (the only sector relevant to the present problem). In particular, the crucial relation for the fermionic currents, in terms of two scalars  $\phi_{1,2}$ , in the zero-charge sector does not acquire any  $1/L$  corrections and reads exactly as on the infinite line:

$$\bar{\psi}_f \gamma^\mu \psi_f = \frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \phi_f. \quad (2)$$

(A gauge-invariant regularization of the left-hand side is discussed in the Appendix.)

We now fix the gauge  $\partial_1 A_1 = 0$ , substitute Eq. (2) into Eq. (1), and separate the action into two parts: the part containing the dependence on the spatially constant (“zero”) mode of  $A_1$  and the remainder. The first of these equals

$$S_A = L \int dt \left\{ \frac{1}{2} (\partial_0 A_1)^2 - \frac{g\theta}{2\pi} \partial_0 A_1 - \mu \partial_0 \varphi A_1 - \frac{L}{2I} A_1^2 \right\}, \quad (3)$$

where  $\mu = g\sqrt{2/\pi}$ , and  $\varphi$  denotes the zero mode of  $\phi_+ = (\phi_1 + \phi_2)/\sqrt{2}$  ( $L$  is the length of the ring).

Upon integrating out  $A_1$ , Eq. (3) gives rise to the following term in the effective action for  $\varphi$ :

$$S_1 = \frac{1}{2} L \mu^2 \int dt dt' \varphi'(t) G(t-t') \varphi'(t') \quad (4)$$

where  $\varphi' = \varphi - \theta/2\sqrt{2\pi}$ , and

$$G(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{\omega^2}{\omega_{LC}^2 - \omega^2};$$

$\omega_{LC} = \sqrt{L/I}$  is the  $LC$  frequency. (The terminology comes about because the first term in Eq. (3) can be thought of as capacitive energy.)

We can now appreciate the role of finite inductance. If we formally set  $I$  to infinity, Eq. (4) becomes a mass term for  $\varphi'$ . This mass term pulls  $\varphi$  towards the value  $\theta/2\sqrt{2\pi}$ , and that value then shows up in the remainder of the action. This is the source of the  $\theta$  dependence described in Ref. [19]. On the other hand, if keep  $\omega_{LC}^2$  is finite and consider the modes with  $\omega^2 \ll \omega_{LC}^2$ , Eq. (4) is not a mass but a kinetic term. As a result,  $\theta$  disappears from the theory. Its role is taken over by the initial value of  $\varphi$ .

As we will see in the next section, interaction with  $\phi_- = (\phi_1 - \phi_2)/\sqrt{2}$  produces an effective potential for  $\varphi$  with a minimum at  $\varphi = 0$ . If  $\varphi$  has a chance to relax to

this minimum (either through a decay into light particles or through friction effects at a finite temperature), the memory of its initial value will be lost, and no counterpart of  $\theta$  will be left.

This conversion of the  $\theta$ -angle into a global time-dependent degree of freedom (a ‘‘global axion’’) is the essence of the solution to the strong  $CP$  problem proposed in Ref. [13]. The solution does not seem to do much damage to the known QCD phenomenology, and so it is natural to think that the low-energy theorems will somehow work out correctly. To help alleviate possible doubts, we explicitly address here the fate of one of those theorems—the relation for the topological susceptibility [16,17].

### III. TOPOLOGICAL SUSCEPTIBILITY

We now restrict ourselves to the strong-coupling limit  $g \gg m$ , the one that provides an adequate analogy to QCD. In this case, the nonzero modes of  $\phi_+$  are heavy, with a mass of order  $g$ , and decouple at low energy. The low-energy theory contains the light field  $\phi_-$ , described by the Hamiltonian [19]

$$H = N_m \int dx \left\{ \frac{1}{2} p^2 + \frac{1}{2} (\partial_1 \phi_-)^2 - 2cm^{3/2} \mu^{1/2} \cos(\sqrt{2\pi}\varphi) \cos(\sqrt{2\pi}\phi_-) \right\}, \quad (5)$$

and the zero mode  $\varphi$  of  $\phi_+$ , described by the effective action [cf. Equation (4)]

$$S_0 + S_1 = \frac{1}{2} L \int \frac{d\omega}{2\pi} \left\{ 1 + \frac{\mu^2}{\omega_{LC}^2 - \omega^2} \right\} \omega^2 |\tilde{\varphi}(\omega)|^2. \quad (6)$$

In this section, we compute topological susceptibility of the vacuum with  $\theta = 0$ ; this will determine the response of that vacuum to a small  $\theta$ -term. Accordingly,  $\tilde{\varphi}$  in Eq. (6) is simply the Fourier transform of  $\varphi$ . In Eq. (5),  $N_m$  denotes normal ordering with respect to the mass  $m$ , and  $c$  is a numerical constant inherited from the bosonization procedure.

We will see that fluctuations of  $\varphi$  in the ground and low excited states are small, so in these states  $\cos(\sqrt{2\pi}\varphi)$  deviates little from unity. For our purposes, it will be sufficient to consider only the terms of the zeroth and second order in  $\varphi$ . The first of these is the low-energy limit of the usual two-flavor massive Schwinger model and determines the properties of the  $\phi_-$  subsystem, in particular, the ‘‘quark condensate’’

$$C = - \left\langle \sum_f \bar{\psi}_f \psi_f \right\rangle = \langle 2cm^{1/2} \mu^{1/2} N_m \cos(\sqrt{2\pi}\phi_-) \rangle. \quad (7)$$

Dependence of  $C$  on  $m$  can be understood by normal reordering with respect to the mass  $M = [2cm\mu^{1/2}]^{2/3}$ , which then becomes the only mass parameter in the Hamiltonian of  $\phi_-$  [19]. As a result,

$$mC = \langle M^2 N_M \cos(\sqrt{2\pi}\phi_-) \rangle \sim M^2,$$

which scales as  $m^{4/3}$ .

The second-order term

$$H_2 = 2\pi c m^{3/2} \mu^{1/2} \varphi^2 \int dx N_m \cos(\sqrt{2\pi}\phi_-)$$

is an interaction between  $\varphi$  and  $\phi_-$ . It affects little  $\phi_-$  but significantly  $\varphi$ : averaging it over the vacuum of  $\phi_-$  gives rise to an effective potential  $\mathcal{E}(\varphi) = \pi m C \varphi^2$ . The full effective action of  $\varphi$  in this approximation is quadratic:

$$S_{\text{eff}} = S_0 + S_1 - L \int dt \mathcal{E}(\varphi), \quad (8)$$

where  $S_0 + S_1$  is given by Eq. (6).

We can now read off the frequency of small oscillations of  $\varphi$ :

$$\omega_0^2 \approx \mathcal{E}''(0) \left( 1 + \frac{\mu^2}{\omega_{LC}^2} \right)^{-1}.$$

The approximation sign corresponds to the limit  $\omega_0 \ll \omega_{LC}$ , which as we see is always justified. Depending on the relation between  $\omega_{LC}$  and  $\mu$ ,  $\omega_0$  can be either much smaller than  $M$  or of the same order. It is important to note that it always belongs to the low-energy sector.

Equation (8) also shows that the strength of zero-point fluctuations of  $\varphi$  is controlled by the parameter

$$\alpha = \frac{\omega_0}{L \mathcal{E}''(0)}. \quad (9)$$

Given that  $L$  is macroscopic, this parameter is always small,  $\alpha \ll 1$ . This justifies the expansion in powers of  $\varphi$ . We will find, however, that, although the fluctuations of  $\varphi$  are suppressed by a power of  $L$ , their contribution to the topological susceptibility is not.

Topological susceptibility is defined as

$$K = i \int \left\langle T \frac{2g}{\pi} F_{01}(x) \frac{2g}{\pi} F_{01}(0) \right\rangle d^2x,$$

where  $T$  denotes the  $T$ -product, and the averaging is over the true vacuum of the entire system.  $K$  determines the average electric field in the vacuum induced by a small  $\theta$ -term in the Lagrangian. It satisfies a relation virtually identical to the relation derived in Refs. [16,17] for the topological susceptibility in QCD. In our case, the relation is

$$K = 4m \left\langle \sum_f \bar{\psi}_f \psi_f \right\rangle + 4im^2 \int \langle T O(x) O(0) \rangle d^2x, \quad (10)$$

where  $O = i \sum_f \bar{\psi}_f \gamma_5 \psi_f$  is the pseudoscalar density,  $\gamma_5 = \gamma^0 \gamma^1$ . Derivation of Eq. (10) is completely parallel to that in the case of QCD and will not be repeated here.

The correlator appearing in Eq. (10) can be expressed as

$$i \int \langle TO(x)O(0) \rangle d^2x = 2L \sum_s \frac{1}{E_s} |\langle s|O(0)|0 \rangle|^2, \quad (11)$$

where  $|0\rangle$  is the vacuum of the system, and  $|s\rangle$  are excited states, with energies  $E_s$  relative to the vacuum. The operator  $O$  has the quantum numbers of an  $\eta'$  meson. One relevant quantum number is  $G$ -parity [19], under which  $\phi_-$  is even, while  $\phi_+$  and  $O$  are odd. If there were no low-energy excitations with odd  $G$ -parity, the second term in Eq. (10) would scale as  $m^2$  at small  $m$ . Meanwhile, the first term scales as  $m^{4/3}$  (the corresponding term in QCD scales linearly with the quark masses). Hence, the first term would win and make  $K$  nonzero.

We have seen, however, that in the presence of a finite inductance there is a low-energy oscillator mode,  $\varphi$ . It has precisely the right quantum numbers and therefore needs to be included in (11). Because this mode is a single oscillator, it does not correspond to a new particle, and so the solution to the  $U(1)$  problem (the absence of a light  $\eta'$ ) is unaffected.

The pseudoscalar density can be bosonized in the usual way:

$$i \sum_f \bar{\psi}_f \gamma_5 \psi_f = 2cmN_m \cos(\sqrt{2\pi}\phi_-)N_m \sin(\sqrt{2\pi}\phi_+),$$

but the sine needs to be reordered, to get rid of the tadpoles of the heavy field  $\phi_+$  (cf. Ref. [19]). Upon doing that,

$$\begin{aligned} i \sum_f \bar{\psi}_f \gamma_5 \psi_f &\approx 2cm^{3/2}\mu^{1/2} \sin(\sqrt{2\pi}\varphi)N_m \cos(\sqrt{2\pi}\phi_-) \\ &\approx -\sqrt{2\pi}\varphi \sum_f \bar{\psi}_f \psi_f, \end{aligned}$$

where we have used the fact that fluctuations of  $\varphi$  are small. Thus, the sum in (11) is saturated by the first excited state of  $\varphi$ :

$$i \int \langle TO(x)O(0) \rangle d^2x = \frac{2\pi L\alpha}{\omega_0} C^2,$$

and using Eq. (9) for  $\alpha$  we see that the second term in Eq. (10) precisely cancels the first. We obtain  $K = 0$ , our main result.

#### IV. CONCLUSION

The overall picture that emerges from the present study is of the world as a giant resonator, whose global oscillating mode is coupled to a continuum branch (the familiar elementary particles). In the massive two-flavor Schwinger model, analytical methods take us quite far, even in the strongly-coupled limit, which has many analogies to QCD. For that case, we have explicitly computed the topological susceptibility and seen that the global oscillator contributes just so as to banish the vacuum  $\theta$  dependence.

We find it remarkable that the cancellation of the topological susceptibility occurs even though the coupling of the global mode to the continuum is suppressed by the ‘‘size of the universe’’ (the length of the ring in our example). This mode has the quantum numbers of  $\eta'$  but, being a single oscillator, does not correspond to a new particle. Therefore, the solution to the  $U(1)$  problem is not affected.

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#### APPENDIX: BOSONIZATION ON A CIRCLE

We follow the approach popular in the condensed-matter literature (for a review, see Ref. [21], whose notation we mostly follow). In this approach, Bose operators are constructed in terms of the operators of a massless free Fermi field. Since these are equal-time operator relations, they are true regardless of what the actual fermion dynamics is (i.e., even if the fermion is massive and interacting, as in our case). In the particle-theory language, this approach corresponds to normal ordering (a gauge-invariant version thereof) with respect to a mass of order  $1/L$ . Once all the requisite operators are constructed in this way, they can be reordered with respect to any mass by using the reordering relations of Ref. [24]. To simplify notation, we set  $L = 2\pi$ .

We use the chiral representation of the  $\gamma$ -matrices, in which they are  $\gamma^0 = \sigma_1$  and  $\gamma^1 = -i\sigma_2$ , so that  $\gamma_5 = \gamma^0\gamma^1 = \sigma_3$ . The upper and lower components of a Dirac fermion are expanded in Fourier series as follows

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \sum_k \alpha_k e^{ikx} \\ \sum_k \beta_k^\dagger e^{ikx} \end{pmatrix}.$$

The operators  $\alpha$  and  $\beta$  obey the usual equal-time anticommutation relations. If  $\psi$  really were a free massless fermion, these operators would correspond to right- and left-movers. The momentum  $k$  takes the values

$$k = n_k - \delta,$$

where  $n_k$  are integers, and  $0 \leq \delta < 1$  is a fraction depending on the boundary conditions. We will consider a general  $\delta$ , even though compactifications that lead to theories with finite inductances typically produce a special case—anti-periodic fermions ( $\delta = \frac{1}{2}$ ) [13].

In a gauge theory, we identify  $\psi$  with the fermion in the Coulomb gauge  $\partial_1 \mathcal{A}_1 = 0$ . So, in what follows, only the large gauge transformations, which change  $\mathcal{A}_1$  by a constant, will be of relevance. (In this Appendix, we use the calligraphic  $\mathcal{A}$  to denote the gauge field.)



Define Bose operators ( $q$  is a *positive* integer)

$$A_q = -\frac{i}{\sqrt{q}} \sum_k \alpha_{k-q}^\dagger \alpha_k, \quad B_q = -\frac{i}{\sqrt{q}} \sum_k \beta_{k-q}^\dagger \beta_k.$$

$A_q$  and  $A_q^\dagger$  (and  $B_q$  and  $B_q^\dagger$ ) obey the usual Bose commutation relation. Then, the density operators are

$$\begin{aligned} \sum_{q,k} e^{iqx}: \alpha_{k-q}^\dagger \alpha_k &:= i \sum_{q>0} \sqrt{q} \{A_q e^{iqx} - A_q^\dagger e^{-iqx}\} + N_\alpha, \\ \sum_{q,k} e^{iqx}: \beta_{k-q} \beta_k^\dagger &:= i \sum_{q>0} \sqrt{q} \{B_q^\dagger e^{iqx} - B_q e^{-iqx}\} - N_\beta. \end{aligned}$$

Note that the sums on the left are over all integer  $q$ , including  $q = 0$ , but each has been reorganized into a sum over  $q > 0$  plus a  $q = 0$  term, proportional to the particle number.

The normal ordering in these expressions is defined so as to include the effect of the spectral flow, i.e., it is with respect to the vacuum  $|0\rangle$  such that

$$\begin{aligned} \alpha_k |0\rangle &= \beta_k |0\rangle = 0, & k - g\mathcal{A}_1 > 0, \\ \alpha_k^\dagger |0\rangle &= \beta_k^\dagger |0\rangle = 0, & k - g\mathcal{A}_1 \leq 0, \end{aligned}$$

where  $\mathcal{A}_1$  is the constant (zero-momentum) component of the gauge field, and  $g$  is the gauge coupling. The resulting expressions for the densities are invariant under the large gauge transformations. Once again, we stress that the vacuum thus defined does not have to be the true vacuum of the system.

Using the above expressions for the component densities, we can express the total charge density as

$$\rho =: \psi_R^\dagger \psi_R: +: \psi_L^\dagger \psi_L: = \frac{1}{\sqrt{\pi}} \partial_x \phi + \frac{1}{2\pi} (N_\alpha - N_\beta), \quad (\text{A1})$$

where

$$\phi = \frac{1}{\sqrt{2\pi}} \sum_{q>0} \frac{1}{\sqrt{2q}} \{(A_q + B_q^\dagger) e^{iqx} + (A_q^\dagger + B_q) e^{-iqx}\} \quad (\text{A2})$$

is a scalar field. It has not yet acquired a zero ( $q = 0$ ) mode but for now, as only the derivative of  $\phi$  occurs in Eq. (A1), that does not matter. The zero-charge sector is defined by the condition

$$N_\alpha - N_\beta \approx 0. \quad (\text{A3})$$

We use Dirac's  $\approx$  notation to indicate that this is a condition imposed on states, rather than operators. In this sector, Eq. (2) becomes the  $\mu = 0$  component of Eq. (2) (for each  $f = 1, 2$  separately).

Similarly, the electric current density can be expressed as

$$j =: \psi_R^\dagger \psi_R: -: \psi_L^\dagger \psi_L: = -\frac{1}{\sqrt{\pi}} p_\phi + \frac{1}{2\pi} (N_\alpha + N_\beta), \quad (\text{A4})$$

where

$$p_\phi = -\frac{i}{2\sqrt{\pi}} \sum_{q>0} \sqrt{q} \{(A_q - B_q^\dagger) e^{iqx} + (B_q - A_q^\dagger) e^{-iqx}\}.$$

Comparing this to Eq. (A2), we see that the Fourier modes of  $p_\phi$  are canonical momenta conjugate to the (nonzero) modes of  $\phi$ . For the zero mode, however, we need to be more careful: our normal ordering of fermions depends on the gauge field, and so  $N_\alpha + N_\beta$  is not the zero-mode canonical momentum ( $p_0$ ) of the scalar. The correct expression is

$$N_\alpha + N_\beta = -\frac{1}{\sqrt{\pi}} \left( p_0 + \frac{g}{\sqrt{\pi}} \Phi \right),$$

where  $\Phi = \oint \mathcal{A}_1 dx$  is the flux through the circle. Equation (A4) then becomes the  $\mu = 1$  component of Eq. (2).

In this notation, the bosonization identities [21] read

$$\psi_R(x) = \frac{1}{\sqrt{2\pi}} F_\alpha e^{i(N_\alpha - \delta)x}: e^{i \sum_{q>0} (1/\sqrt{q}) [A_q e^{iqx} + A_q^\dagger e^{-iqx}]}:, \quad (\text{A5})$$

$$\psi_L(x) = \frac{1}{\sqrt{2\pi}} e^{i(N_\beta - \delta)x} F_\beta^\dagger: e^{-i \sum_{q>0} (1/\sqrt{q}) [B_q e^{-iqx} + B_q^\dagger e^{iqx}]}:, \quad (\text{A6})$$

where  $F_\alpha$  and  $F_\beta$  are the so-called Klein factors (or ‘‘ladder operators’’), which are unitary, anticommuting ( $\{F_\alpha, F_\beta\} = \{F_\alpha, F_\beta^\dagger\} = 0$ ) and satisfy these relations:

$$F_\alpha e^{iN_\alpha \xi} = e^{i(N_\alpha + 1)\xi} F_\alpha$$

and the same with  $\alpha \rightarrow \beta$ ;  $\xi$  is an arbitrary  $c$ -number. Again, a simplification occurs in the zero-charge sector: from Eqs. (A5) and (A6) we obtain, for example,

$$\psi_L^\dagger \psi_R \approx \frac{1}{2\pi} F_\beta F_\alpha: e^{2i\sqrt{\pi}\phi}:,$$

where  $\phi$  is the field (A2). Moreover, since  $F_\beta F_\alpha$  is a unitary Bose operator, we can express it as  $F_\beta F_\alpha = -e^{2i\sqrt{\pi}\phi_0}$ , where  $\phi_0$  is a single degree of freedom which now becomes the zero-momentum mode of the field  $\phi$ .

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