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G^1 Interpolation using Piecewise Quadric and Cubic Surfaces

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Abstract

Algorithms are presented for constructing G^1 continuous meshes of degree two (quadric) and degree three (cubic) implicitly defined, piecewise algebraic surfaces, which exactly fit any given collection of points and algebraic space curves, of arbitrary degree. A combination of techniques are used from computational algebraic geometry and numerical approximation theory which reduces the problem to solving coupled systems of linear equations and low degree, polynomial equations.

1 Introduction

Interpolation provides an efficient way of generating G^1 continuous meshes of surface patches, necessary for the construction of accurate computer models of solid physical objects [2]. In this paper, we focus on the use of low degree, implicitly defined, algebraic surfaces in three dimensional real space $\mathbb{R}^3$. Modeled physical objects with algebraic surface patches of the lowest degree, lend themselves to faster computations in geometric design operations as well as in tasks such as computer graphics display, animation, and physical simulations, see for e.g. [1].

A real algebraic surface S in $\mathbb{R}^3$ is implicitly defined by a single polynomial equation $f(x, y, z) = 0$, where coefficients of f are over the real numbers $\mathbb{R}$. A real algebraic space curve can be defined by the intersection of two real algebraic surfaces and implicitly represented as a pair of polynomial equations ($f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$) with coefficients again over the real numbers $\mathbb{R}$. In modeling the boundary of physical objects it suffices to consider only space curves defined by the intersection of two algebraic surfaces. Space curves, in general, may be defined by the intersection of several surfaces [15].

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Why algebraic surfaces ? Manipulating polynomials, as opposed to arbitrary analytic functions, is computationally more efficient. Furthermore algebraic surfaces provide enough generality to accurately model almost all complicated rigid objects. Also as we show here, algebraic curves and surfaces lend themselves very naturally to the difficult problem of surface fitting.

Why implicit representations ? Most prior approaches to interpolation and surface fitting, have focused on the parametric representation of surfaces [9, 18, 24]. Contrary to major opinion and as we exhibit here, implicitly defined surfaces are also very appropriate. Additionally, while all algebraic surfaces can be represented implicitly, only a subset of them have the alternate parametric representation, with x, y and z given explicitly as rational functions of two parameters. Working with implicit algebraic surfaces of a fixed degree, thus provides a larger number of surfaces to design with. Furthermore, implicit algebraic curves and surfaces have compact storage representations and form a class which is closed under most common operations (boolean set operations, offsets etc.) required by a geometric design system.

The main problem we consider in this paper, is the following: Construct a G^1 continuous mesh of quadric and cubic surface patches which 'smoothly' interpolates a collection of points $\mathbf{p}$ in $\mathbb{R}^3$ and given space curves C in $\mathbb{R}^3$, with associated "normal" unit vectors varying along the entire span of the curves. Both points and space curves have an infinity of potential "normal" vector directions. While for points the normals may be chosen arbitrarily, for space curves the varying unit "normal" vectors are chosen to be always orthogonal to the tangent vector, along the entire curve. Our emphasis being algebraic space curves, the variance of the curves "normals" are restricted to polynomials of some degree. By smooth or G^1 interpolation we shall mean that the surface mesh contains the input points and curves and furthermore has its gradients in the same direction as the specified "normal" vectors. This is a natural generalization of Hermite interpolation, applied to fitting curves or surfaces through point data, and equating derivatives at those points.

There has been extensive prior work in surface fitting. Exact and approximate fitting of curves (primarily conics) has been considered by several authors, see for eg [7, 8, 13, 16, 19]. Paper [18] presents techniques for constructing a G^1 continuous surface of rectangular Bézier (parametric) surface patches, interpolating a net of cubic Bézier curves. Other approaches to parametric surface fitting and transfinite interpolation are also mentioned in that paper, as well as in [24]. An excellent exposition of exact and least squares fitting of algebraic surfaces through given data points, is presented in [17]. This paper generalizes the results of [17] in two ways. One, it considers exact fits of algebraic surfaces through given space curves as well as data points. Second, it also considers similar surface fits when derivative information ("normals") are also provided at the given data points and along the given data curves. Meshing of given algebraic surface patches using control techniques of joining Bézier polyhedrons is shown in [20]. Some of the results in [20] are extended in [4]. Paper [5] considers higher order surface fitting as well as least-squares approximations. Surface blending consisting of "rounding" and "filleting" surfaces (smoothing the intersection of two primary surfaces), a special case of surface fitting, has been considered for polyhedral models in [10] and for algebraic surface models in [3, 4, 5, 14, 23, 24]. The generalized techniques for G^1 continous surface meshes, presented in this paper, also provide algorithms to generate such blending and joining surfaces.

The rest of the paper is as follows. In section 2, we first show that the problem of G^1 interpolation of points and curves with a single algebraic surface, reduces to solving a linear system. These results are extended in section 3 to the construction of G^1 continuous meshes of quadric and cubic surface patches, which together smoothly interpolate the input points and curves. Here one needs to solve a system of low degree polynomial equations. As applications of these characterizations of interpolation and G^1 continuous fits with algebraic surfaces, we exhibit, in sections 2 and 3, interesting examples of geometric design.

2 G^1 Interpolation Matrices

For any multivariate polynomial f, partial derivatives are written by subscripting, for example, $f_x = \partial f/\partial x$, $f_{xy} = \partial^2 f/(\partial x \partial y)$, and so on. Since we consider algebraic curves and surfaces, we have $f_{xy} = f_{yx}$ etc. The *gradient* of $f(x,y,z)$ is the vector $\nabla f = (f_x, f_y, f_z)$.

Bajaj and Ihm in [3, 4], present a simple constructive characterization of the real algebraic surface $f(x,y,z) = 0$ of degree n which *smoothly contains* any given number of points and algebraic space curves, with associated "normal" directions. This characterization, called *Hermite interpolation* or G^1 interpolation, deals with the containment and matching normals at the input points or varying along the entire span of the input space curves. To summarize:

1. Linear equations are generated from the smooth containment of a point $\mathbf{p} = (a,b,c)$ with an associated "normal" $\mathbf{m} = (m_x, m_y, m_z)$, viz., $f(\mathbf{p}) = f(a,b,c) = 0$ and $\nabla f(\mathbf{p}) = \alpha \mathbf{m}$, for some

nonzero α.

2. Linear equations are also generated from the smooth containment of a space curve $C : (f_1(x, y, z) = 0, f_2(x, y, z) = 0)$ of degree d, together with associated "normals" defined implicitly by the triple $\mathbf{n}(x, y, z) = (n_x(x, y, z), n_y(x, y, z), n_z(x, y, z))$ where n_x, n_y and n_z are polynomials of maximum degree m, defined for all points $\mathbf{p} = (x, y, z)$ along the curve C. In fact, it suffices to satisfy the point containment condition of 1. at $nd + 1$ points of C and the point gradient matching condition of 1. at $(n + m - 1)d + 1$ points of C. These follow directly from a form of Bezout's theorem, stated below (See also, for example, [21].)

   Theorem 2.1 *An algebraic curve C of degree d intersects an algebraic surface S of degree n in at most nd points, or C must intersect S infinitely often, that is, a component of C must lie entirely on S.*

For an algebraic surface S of degree n, G^1 interpolation above, generates a homogeneous linear system $\mathbf{M_I x} = \mathbf{0}$ where $\mathbf{x}$ is a $\binom{n+3}{3}$-vector[1] of the coefficients of the algebraic surface S. All nontrivial vectors, if any, in the nullspace of $\mathbf{M_I}$ forms a family of all algebraic surfaces, satisfying the input constraints and whose coefficients are expressible in terms of p-parameters, where p is the rank of the nullspace.

Example 2.1 *G^1 interpolation with quadric and quartic surface patches*

Consider the following wireframe of a solid model consisting of two circles (the intersections of planes with a sphere), $C_1 : ((x^2 + y^2 + z^2 - 25 = 0, x = 0)$, and $C_2 : ((x^2 + y^2 + z^2 - 25 = 0, y = 0)$. Each circle has an associated "normal" direction which is chosen in the same direction as the gradients of the sphere, viz., $\mathbf{n}_1(x, y, z) = (0, 2y, 2z)$, and $\mathbf{n}_2(x, y, z) = (2x, 0, 2z)$. The wireframe has 4 faces: $face_1 = (x \geq 0, y \geq 0)$, $face_2 = (x \geq 0, y \leq 0)$, $face_3 = (x \leq 0, y \leq 0)$, and $face_4 = (x \leq 0, y \geq 0)$. In Figure 1, $face_1$ and $face_3$ are filled with the patches taken from the sphere $x^2 + y^2 + z^2 - 25 = 0$ (yellow patches). A designer, who wishes to *smoothly* flesh the remaining faces with quartic (degree 4) surface patches, applies the above Hermite interpolation method to C_1 and C_2.This results in an 11-parameter (10 independent) family of quartic $\mathbf{G}^1$ interpolating surfaces, which is given by $f(x, y, z) =$

[1] There are $\binom{n+3}{3}$ coefficients in $f(x, y, z)$ of degree n

$r_1z^4+(r_2y+r_6x+5r_4)z^3+(r_3y^2+(r_7x+5r_8)y+r_{10}x^2+5r_{11}x-25r_9-25r_1)z^2+(r_2y^3+(r_6x+5r_4)y^2+(r_2x^2-25r_2)y+r_6x^3+5r_4x^2-25r_6x-125r_4)z+(r_3-r_1)y^4+(r_7x+5r_8)y^3+(r_5x^2+5r_{11}x-25r_9-25r_3+25r_1)y^2+(r_7x^3+5r_8x^2-25r_7x-125r_8)y+(r_{10}-r_1)x^4+5r_{11}x^3+(-25r_9-25r_{10}+25r_1)x^2-125r_{11}x+625r_9$. An instance of this family is $f(x,y,z) = -1250 - x^4 - y^4 - x^2z^2 - y^2z^2 + 50z^2 + 75y^2 + 75x^2$ which is used to flesh $face_2$ and $face_4$ in Figure 1 (red patches). □

3 Quadric and Cubic Surface Patches

Solving a linear system of equations plays a key role in G^1 interpolation of the previous section. In what follows, we give another approach of algebraic surface design where a *nonlinear* system of polynomial equations needs to be solved. In interpolation, the linear equations generated, represent the constraints to be met by a single interpolating surface. The larger the number of independent containment and tangency constraints, the higher the degree of the resulting interpolating surface. The total number of constraints depends largely on the degrees of the given curves and their "normals". Since the number of terms in an algebraic surface increases as the cube of its degree, computation with high degree algebraic surfaces gets expensive and error prone. Hence, for good reasons we are advised to keep the degrees of our designed surfaces as low as possible.

The problem considered in this section is to G^1-interpolate, curves in space with (not necessarily one), but a combination of quadric and cubic surface patches which themselves meet smoothly along their intersection curves. Such "smooth" meshing has been largely addressed by [18, 20, 22] amongst others, using the Bézier representations of surfaces.

The technique we now explain is primarily based on Bezout's surface intersection theorem see [25]

Theorem 3.1 *If an algebraic surface S of degree n intersects an algebraic surface T of degree m in a curve of degree d with intersection multiplicity i, then $i * d \leq nm$.*

and a theorem from [22]

Theorem 3.2 *If surfaces f(x,y,z) = 0 and g(x,y,z) = 0 intersect transversally in a single irreducible curve[2] C, then any algebraic surface h(x,y,z) = 0 contains C with G^k continuity must be of the form $h(x,y,z) = \alpha(x,y,z)f(x,y,z) + \beta(x,y,z)g^{k+1}(x,y,z)$. Furthermore, the degree of $\alpha(x,y,z)f(x,y,z) \leq$ degree of h(x,y,z) and the degree of $\beta(x,y,z)g^{k+1}(x,y,z) \leq$ degree of h(x,y,z).*

[2]More precisely surfaces f(x,y,z) =0 and g(x,y,z) =0 intersect properly and share no common components at infinity

Another theorem that we need, relates continuity with the intersection multiplicity of smooth algebraic surfaces, see [11, 12].

Theorem 3.3 *Two smooth algebraic surfaces $S_1 : f(x,y,z) = 0$ and $S_2 : g(x,y,z) = 0$ meet with G^k continuity along a curve C if and only if S_1 and S_2 intersect with multiplicity $k+1$ along C.*

From theorem 3.2 we obtain the following special case lemma

Lemma 3.1 *Let $S : f(x,y,z) = 0$ be an irreducible quadric surface, and $Q : q(x,y,z) = 0$ be a plane which intersects S in a conic C. Then, another quadric surface $S_1 : f_1(x,y,z)$ is tangent to S along C if and only if there exists nonzero constants α, β (possibly complex) such that $f_1 = \alpha f + \beta q^2$.*

Since we are interested in surface fitting with real surfaces, we may restrict α and β to be real numbers. A related theorem can be derived for the quadric surface interpolation of two conics in space.

Lemma 3.2 *Consider quadrics $S_1 : f_1 = 0$, $S_2 : f_2 = 0$ and planes $Q_1 : q_1 = 0$, $Q_2 : q_2 = 0$. Let $C_1 : (f_1 = 0, q_1 = 0)$ and $C_2 : (f_2 = 0, q_2 = 0)$ be two conics in space. Then C_1 and C_2 can be Hermite interpolated by a quadric surface S if and only if there exist nonzero constants α_1, α_2, β_1, and β_2 (possibly complex) such that $\alpha_1 f_1 + \beta_1 q_1^2 - \alpha_2 f_2 - \beta_2 q_2^2 = 0$.*

Proof: Trivial. (Just apply Lemma 3.1 twice.) ♠

This lemma is constructive, in that, it again yields a system of linear equations and a direct way of computing a G^1-interpolating quadric surface. Furthermore a solution to the above equations, linear in the α's and β's, exists if and only if such an interpolating quadric surface exists. Again, when real surfaces are favorable, we require α_1, α_2, β_1, and β_2 to be real numbers.

Example 3.1 *Suppose $C_1 : (x^2 + z^2 - 1 = 0, 3x + y = 0)$, and $C_2 : (y^2 + z^2 - 1 = 0, x + 3y = 0)$. We get the following equation from Lemma 3.2: $(\alpha_1 + 9\beta_1 - \beta_2)x^2 + (\beta_1 - \alpha_2 - 9\beta_2)y^2 + (\alpha_1 - \alpha_2)z^2 + (6\beta_1 - 6\beta_2)xy + (\alpha_1 - \alpha_2) = 0$. This implies $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\alpha_1 = -8\beta_1$. When $\alpha_1 = -8$ and $\beta_1 = 1$, the interpolating surface is $x^2 + y^2 - 8z^2 + 6xy + 8 = 0$.*

In the Lemma 3.2 and the example, the two conics on the given quadric surfaces, S_1 and S_2, were fixed. If we have freedom to choose different intersecting planes Q_1 and Q_2 then we may be able to find a family of quadric interpolating surfaces. In this case, the equations of planes Q_1 and Q_2 would have unknown coefficients and the use of Lemma 3.2 would result in a nonlinear system of equations, linear in terms of α_1, α_2, β_1 and β_2, and quadratic in terms of the unknowns of the plane's equations.

Now, rather than trying to find a single quadric surface, we can also extend the above Lemma 3.2, to construct two or more quadrics which smoothly contain two given conics in space, and furthermore themselves intersect in a smooth fashion. The following Lemma 3.2, which is constructive tells us how to go about this.

Lemma 3.3 *Let $C_1 : (f_1 = 0, q_1 = 0)$ and $C_2 : (f_2 = 0, q_2 = 0)$ be two conics in space. These two curves can be smoothly contained by two "smoothly intersecting" quadrics $S_1 : g_1 = a_1 f_1 + b_1 q_1^2 = 0$ and $S_2 : g_2 = a_2 f_2 + b_2 q_2^2$ if and only if there exist nonzero constants a_1, a_2, b_1, b_2, α, β, and a plane $Q : q(x,y,z) = 0$ such that $a_1 f_1 + b_1 q_1^2 - \alpha(a_2 f_2 + b_2 q_2^2) - \beta q^2 = 0$.*

Proof: From theorem 3.3 we note that two quadrics that intersect smoothly (at least G^1), must intersect with multiplicity at least two. It follows then from Bezout's theorem 3.1 for surface intersection, that the two quadrics S_1 and S_2 must meet in a plane curve (either an irreducible conic or straight lines). Let the intersection curve lie on the unknown plane Q, then just apply Lemma 3.1 three times. ♠

The final equation of the above Lemma results in a nonlinear (cubic) system of equations which is linear in terms of the unknowns a_1, a_2, b_1, b_2, α, and β, and quadratic in terms of the unknown coefficients of the plane $Q : q = 0$. *Note, that in Lemma 3.3, the quadric surfaces S_1 and S_2 need not be in the form given (as constructed via Lemma 3.1), but may instead be an m-parameter family of solutions, obtained by G^1 interpolation of input curves with possibly "normal" data, as explained in the previous section 2.*

Example 3.2 *Let conic C_1 be given by $f_1 = x^2 + y^2 - z^2 + 4xy + 4x + 4y + 3 = 0$ (a hyperboloid of one sheet) and $q_1 = x + y + 1 = 0$. Similarly, let conic C_2 be given by $f_2 = 19x^2 + 10y^2 - 9z^2 + 38xy - 114x - 114y + 180 = 0$ (a hyperbooid of one sheet), $q_2 = x + y - 3 = 0$, and let the unknown plane be $P : ax + by + cz + d = 0$. Then the equation for the system of smooth interpolating quadrics $a_1 f_1 + b_1 q_1^2 - \alpha(a_2 f_2 + b_2 q_2^2) = \beta(ax + by + cz + d)^2$ results in a nonlinear system of 10 equations: $-\beta c^2 + 9a_2\alpha - a_1 = 0$, $-2b\beta c = 0$, $-2a\beta c = 0$, $-2\beta cd = 0$, $-b^2\beta - \alpha b_2 + b_1 - 10a_2\alpha + a_1 = 0$, $-2ab\beta - 2\alpha b_2 + 2b_1 - 38a_2\alpha + 4a_1 = 0$, $-2b\beta d + 6\alpha b_2 + 2b_1 + 114a_2\alpha + 4a_1 = 0$, $-a^2\beta - \alpha b_2 + b_1 - 19a_2\alpha + a_1 = 0$, $-2a\beta d + 6\alpha b_2 + 2b_1 + 114a_2\alpha + 4a_1 = 0$, and $-\beta d^2 - 9\alpha b_2 + b_1 - 180a_2\alpha + 3a_1 = 0$. This nonlinear* *system has a nontrivial solution (in the sense that a_1, a_2, and α are nonzero) :* $a_1 = -a^2\beta$, $b_1 = 2a^2\beta$, $a_2 = -\frac{a^2\beta}{9\alpha}$, $b_2 = \frac{19a^2\beta}{9\alpha}$, *and* $b = c = d = 0$.[3] *Hence, the two conics C_1 and C_2 are smoothly contained by quadrics $g_1 = 0$ and $g_2 = 0$, respectively, and which in turn, smoothly intersect in a conic in the plane Q. The real quadric $g_1 = x^2 + y^2 + z^2 - 1 = 0$ is a sphere, while the other real quadric $g_2 = y^2 + z^2 - 1$ is a cylinder. Note that the above solution implies that there is only one pair of real quadric surfaces which smoothly contain the given conics. Also, for this case, it can be shown that neither a single quadric nor a single cubic surface can Hermite interpolate the two given conics. Geometrically then, the two hyperboloids of one sheet are smoothly joined by a sphere and a cylinder. See figure 2 at the end of the paper.*

The above method of Lemma 3.3 can straightforwardly be extended to finding a G^1 continuous mesh of k quadric surfaces which smoothly contain k conics in space.

Theorem 3.4 *Let $C_1 : (f_1 = 0, q_1 = 0)$, $C_2 : (f_2 = 0, q_2 = 0)$ $\ldots$ $C_k : (f_k = 0, q_k = 0)$ be k conics in space. These curves can be smoothly contained by k quadrics $S_1 : g_1 = a_1 f_1 + b_1 q_1^2 = 0$, $S_2 : g_2 = a_2 f_2 + b_2 q_2^2$, $\ldots$, $S_k : g_k = a_k f_k + b_k q_2^k$ which themselves "smoothly intersect" if and only if there exist nonzero constants a_1, a_2, $\ldots$, a_k, b_1, b_2, $\ldots$, b_k, α_1, $\ldots$, α_{k-1}, β_1, $\ldots$, β_{k-1} and planes $R_1 : r_1(x, y, z) = 0$, $\ldots$, $R_{k-1} : r_{k-1}(x, y, z) = 0$ such that*

$$
\begin{aligned}
a_1 f_1 + b_1 q_1^2 - \alpha_1(a_2 f_2 + b_2 q_2^2) - \beta_1 r_1^2 &= 0 \\
a_2 f_2 + b_2 q_2^2 - \alpha_2(a_3 f_3 + b_3 q_3^2) - \beta_2 r_2^2 &= 0 \\
&\cdots \\
a_{k-1} f_{k-1} + b_{k-1} q_{k-1}^2 - \alpha_{k-1}(a_k f_k + b_k q_k^2) - \beta_{k-1} r_{k-1}^2 &= 0
\end{aligned} \tag{1}
$$

Proof: Direct applications of Lemma 3.3 ♠

Note again as before, that in the above theorem, the quadric surfaces $S_1, \ldots S_k$ need not be in the form given (as constructed via Lemma 3.1), but may instead be an m parameter family of solutions, obtained by G^1 interpolation of input curves with possibly "normal" data, as explained in the previous section 2. Also note, that given k conics in space, in general, k quadrics above, may not form a G^1 continuous mesh (no non-trivial solution for the generated system (1) of polynomial equations). In

[3]This nonlinear system was solved with the aid of MACSYMA, on a Symbolics 3650

this case one may try increasing the number of quadric surface patches between any two of the given curves. This yields the theorem below, a variation of theorem 3.4.

Theorem 3.5 *Let $C_1 : (f_1 = 0, q_1 = 0)$, and $C_2 : (f_2 = 0, q_2 = 0)$ be two conics in space. These curves can be smoothly contained by two quadrics $S_1 : g_1 = a_1 f_1 + b_1 q_1^2 = 0$, $S_2 : g_2 = a_2 f_2 + b_2 q_2^2$ which together with k other quadrics $T_1 : h_1 = 0, \ldots, T_k : h_k = 0$ form a G^1 continuous mesh if and only if there exist nonzero constants a_1, a_2, b_1, b_2, $c_{i0}, \ldots c_{i9}$ (the coefficients of the quadric $T_i : h_i = 0$), $i = 1 \ldots k$, and $\alpha_1, \ldots, \alpha_{k+1}$, $\beta_1, \ldots, \beta_{k+1}$, and planes $R_1 : r_1(x, y, z) = 0, \ldots, R_{k+1} : r_{k+1}(x, y, z) = 0$ such that*

$$\begin{aligned} a_1 f_1 + b_1 q_1^2 - \alpha_1 h_1 - \beta_1 r_1^2 &= 0 \\ a_2 f_2 + b_2 q_2^2 - \alpha_{k+1} h_k - \beta_{k+1} r_{k+1}^2 &= 0 \\ h_i = \alpha_i h_{i-1} + \beta_i r_i^2, \quad i &= 2, \ldots, k \end{aligned} \tag{2}$$

.

Necessarily the complexity of the nonlinear system of equations also goes up.

If the above generated systems (1),(2) of polynomial equations, do not yield a satisfactory G^1 solution, one may instead try intermixing cubic surfaces with quadrics. To do this one first considers the lemma below similar to lemma 3.1 and a corollary of theorem 3.2

Lemma 3.4 *Let $S : f(x, y, z) = 0$ be an irreducible quadric surface, and $Q : q(x, y, z) = 0$ be a plane which intersects S in a conic C. Then, a cubic surface $T_1 : f_1(x, y, z)$ is tangent to S along C if and only if there exists nonzero constants $a_1, \ldots, a_4$, and $b_1, \ldots, b_4$ such that $f_1 = (a_1 x + a_2 y + a_3 z + a_4) f + (b_1 x + b_2 y + b_3 z + b_4) q^2$.*

Similar to lemma 3.3 one obtains

Lemma 3.5 *Let $C_1 : (f_1 = 0, q_1 = 0)$ and $C_2 : (f_2 = 0, q_2 = 0)$ be two conics in space. These two curves can be smoothly contained by two quadrics $S_1 : g_1 = a_1 f_1 + b_1 q_1^2 = 0$ and $S_2 : g_2 = a_2 f_2 + b_2 q_2^2$ both of which meet a cubic surface $T_1 : h_1 = 0$ if there exist nonzero constants a_1, a_2, b_1, b_2, $\alpha_{11}, \ldots, \alpha_{14}$, $\alpha_{21}, \ldots, \alpha_{24}$ $\beta_{11}, \ldots, \beta_{14}$, $\beta_{21}, \ldots, \beta_{24}$ and planes $R_1 : r_1(x, y, z) = 0$, $R_2 : r_2(x, y, z) = 0$ such that $h_1 = (\alpha_{11} x + \alpha_{12} y + \alpha_{13} z + \alpha_{14}) g_1 + (\beta_{11} x + \beta_{12} y + \beta_{13} z + \beta_{14}) r_1^2 = (\alpha_{21} x + \alpha_{22} y + \alpha_{23} z + \alpha_{24}) g_2$ - $(\beta_{21} x + \beta_{22} y + \beta_{23} z + \beta_{24}) r_2^2$*

Proof: It follows from Bezout's theorem 3.1 for surface intersection, that the a quadrics S_1 and a cubic surface T_1 must meet in either a space cubic, a plane cubic, an irreducible conic or straight lines. Consider only the plane intersection curves and assume they lie on an unknown plane Q, then just apply Lemma 3.4 . ♠

In both the above lemmas, T_1 need not be in the above form but may instead be a l-parameter family of solutions, obtained by G^1 interpolation of input curves with possibly "normal" data, as explained in the previous section 2. These parameterized cubic surfaces may be intermixed with the quadric surfaces in theorems 3.4 and 3.5 to form a G^1 continuous mesh of alternating quadric and cubic surfaces in the obvious manner. I'll skip the details here.

4 Conclusion

We have implemented the G^1 interpolation and G^1 continuous meshing algorithms as presented in sections 2 and 3, as part of our geometric modeling system [1]. The program takes as input any collection of geometric data points, curves, with and without associated "normals". Both implicit and rational parametric representations of the space curves and their derivatives are allowed. The program solves the linear system of equations using a variant of Gaussian elimination with scaled partial pivoting. The rank computation is done implicitly during the solution steps. The result, when nontrivial solutions exist, are expressed in terms of symbolic coefficients and represent a family of interpolation surfaces. Values are specified for these coefficients by means of either the least-squares approximation approach [5] or using Bezier control weights [4]. The system of polynomial equations of section 3, is currently solved by linking to Gröbner basis routines in Macsyma. We are currently improving our software implementation to include:

1. a linking to our own algebraic geometry package [6], optimized for solving systems of polynomial equations
2. the development of a more user-friendly method of inputting geometric data and of selecting the appropriate interpolated solutions
3. incorporating a way of automatically satisfying nonsingular and irreducibility constraints of interpolating and meshing surfaces

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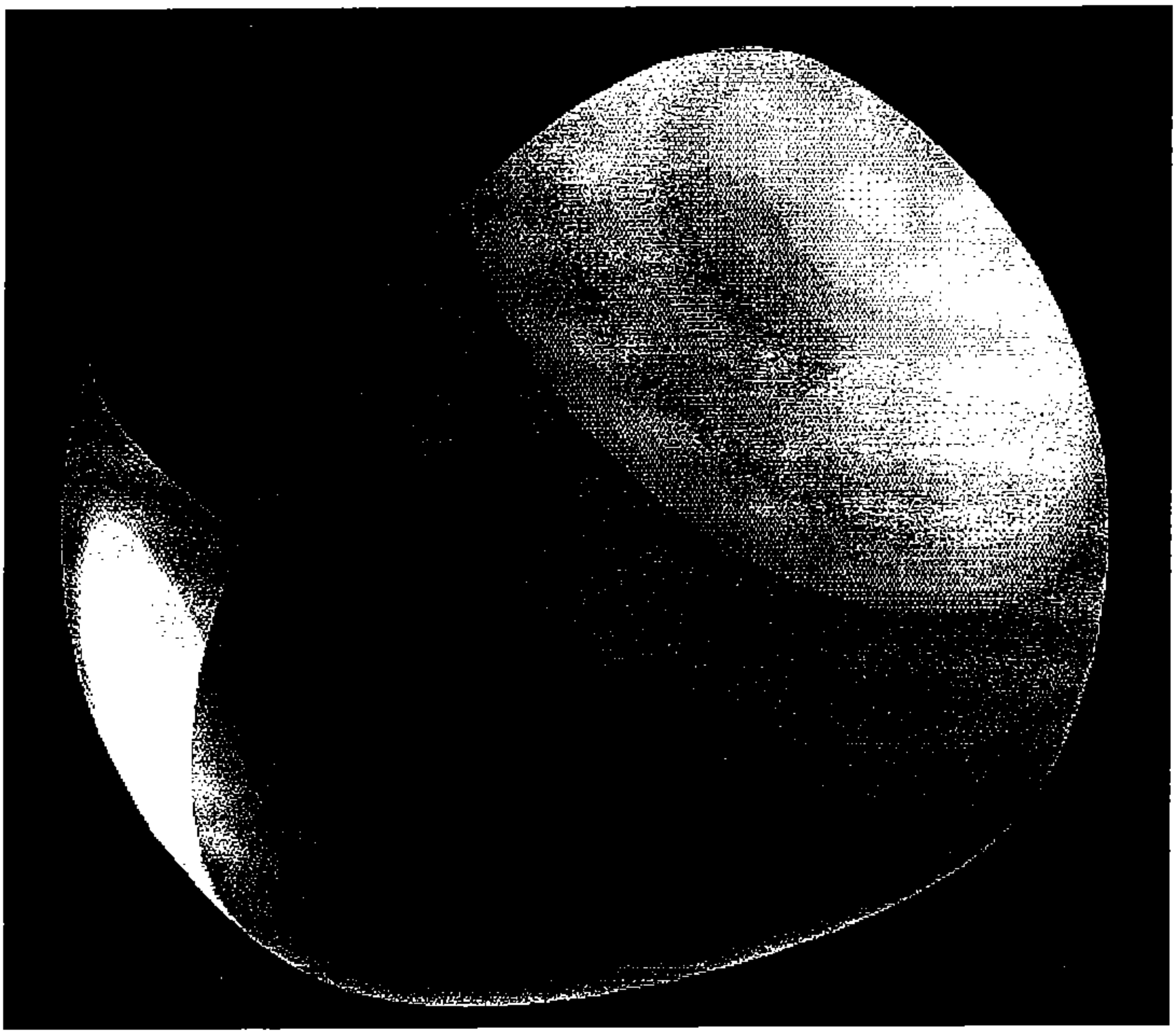

Figure 1: G^1 interpolation with quadric and quartic surface patches.

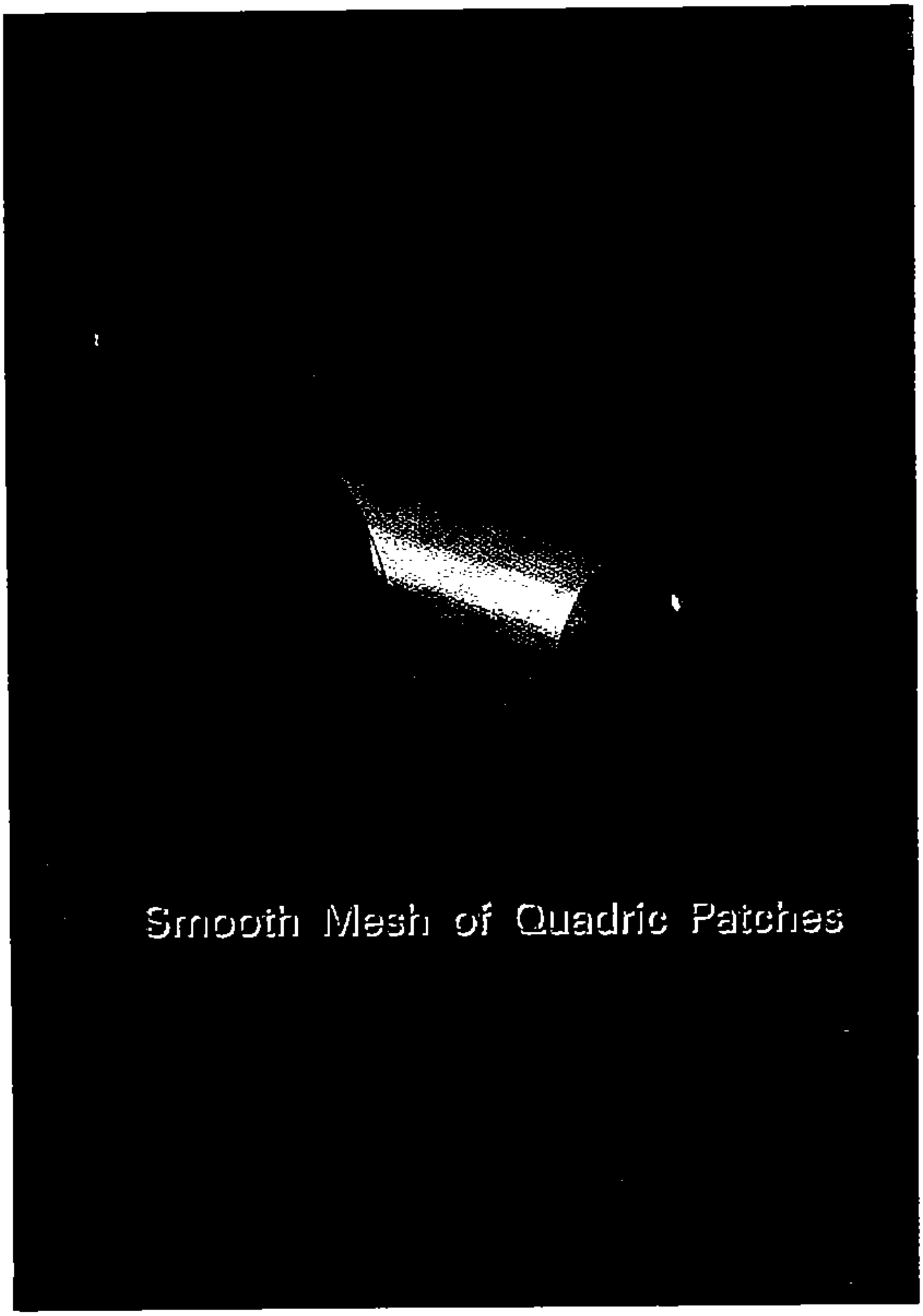

Figure 2: G^1 mesh of quadric surface patches.