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Abstract

In this paper we introduce the Modified Accelerated Overrelaxation (MAOR) method, a generalization of the AOR one, for the iterative solution of the nonsingular linear system $Ax = b$. We assume that A is in a $p \times p$ partitioned form and belongs to a subclass of the p -cyclic consistently ordered matrices. It is pointed out that for specific choices of the "acceleration" and "relaxation" matrices the MAOR method reduces to an extrapolation of the Jacobi or the Modified (M)SOR method with different parameters corresponding to the row blocks of A . First, an eigenvalue relationship connecting the spectra of the block Jacobi and MAOR matrices associated with A is derived from which many well-known eigenvalue relationships can be recovered. Then, by considering the particular case $p = 2$ it is shown that a matrix analogue of the aforementioned eigenvalue relationship holds and the MAOR method is equivalent to a certain 2-step one. Finally, the precise domains of convergence of the MAOR method are derived, when the spectrum of the Jacobi matrix is real or pure imaginary and a brief discussion follows which concludes the present study.

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1. Introduction and Preliminaries

Suppose we are given the nonsingular linear system

$$Ax = b \quad , \quad (1.1)$$

with $A \in \mathbb{C}^{n,n}$ and $b \in \mathbb{C}^n$. Suppose also that A is in a $p \times p$ block partitioned form, where the diagonal blocks A_{jj} are square and nonsingular of order n_j , $j = 1(1)p$, $\sum_{j=1}^p n_j = n$. Let $D := \text{diag}(A_{11}, A_{22}, \dots, A_{pp})$ and let

$$T := I - D^{-1} A =: L + U \quad (1.2)$$

be the Jacobi matrix associated with A , where L and U are strictly lower and strictly upper triangular matrices respectively. For the iterative solution of (1.1) many methods can be used (see e.g., [17], [21], [9] or [6]). In this paper we introduce and propose the following generalized version of the Accelerated Overrelaxation (AOR) method ([5], [14])

$$x^{(m+1)} = \mathcal{L}_{R,\Omega} x^{(m)} + (I - RL)^{-1} \Omega D^{-1} b \quad , \quad m = 0, 1, 2, \dots, \quad (1.3)$$

with

$$\mathcal{L}_{R,\Omega} := (I - RL)^{-1} [I - \Omega + (\Omega - R)L + \Omega U] \quad . \quad (1.4)$$

In (1.3) R and Ω may be any matrices of order n provided $I - RL$ and Ω are invertible and $(I - RL)^{-1}$ is easy to compute. It is noted that for $R = rI$ and $\Omega = \omega I$ the classical AOR method is recovered.

Our main objective is to study some of the basic properties of the method introduced above for matrices A which belong to the class of generalized consistently ordered (GCO) $(q, p - q)$ -matrices (cf. [21]) when T is of the form

$$T := \begin{bmatrix} 0 & 0 & \dots & 0 & T_{1,p-q+1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & T_{2,p-q+2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & T_q \\ T_{q+1,1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & T_{q+2,2} & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & T_{p,p-q} & 0 & 0 & \dots & 0 \end{bmatrix} \quad , \quad (1.5)$$

with $\gcd(p, q) = 1$. For reasons which will become clear later we restrict ourselves to considering diagonal matrices R and Ω such that

$$R = \text{diag}(r_1 I_1, r_2 I_2, \dots, r_p I_p), \quad \Omega = \text{diag}(\omega_1 I_1, \omega_2 I_2, \dots, \omega_p I_p) \quad , \quad (1.6)$$

with I_j unit matrices of order n_j , $j = 1(1)p$ and $r_j, \omega_j (\neq 0)$ scalars. A careful study of (1.3) – (1.4), having in mind (1.2), (1.5) and (1.6), reveals that : i) For $R = 0$, the MAOR method reduces to an Extrapolated Jacobi method with extrapolation (diagonal) matrix Ω , since then

$$\mathcal{L}_{0,\Omega} := I - \Omega + \Omega T, \quad (I - RL)^{-1} \Omega D^{-1} b = \Omega D^{-1} b \quad , \quad (1.7)$$

and ii) For $\det(R) \neq 0$ and $R \Omega^{-1} = sI$, with s being a scalar, it reduces to an Extrapolated Modified Successive Overrelaxation (EMSOR) method (cf. [7], [8] and [2]) with extrapolation matrix $R^{-1} \Omega$ ($:= \frac{1}{s} I$) and overrelaxation one R , because

$$\begin{aligned} \mathcal{L}_{R,\Omega} &:= I - R^{-1} \Omega + R^{-1} \Omega \mathcal{L}_{R,R}, \\ (I - RL)^{-1} \Omega D^{-1} b &= R^{-1} \Omega (I - RL)^{-1} R D^{-1} b \quad . \end{aligned} \quad (1.8)$$

Obviously (1.7) and (1.8) reduce to the classical Extrapolated Jacobi and Extrapolated SOR methods for $\Omega = \omega I$ and $R = rI$, $\Omega = \omega I$ respectively. It is for this reason that the new method is called Modified Accelerated Overrelaxation (MAOR) method. It is also worth noticing that because of the structure of T in (1.5) and the fact that R appears always as a coefficient to L the first q diagonal element of R , r_j , $j = 1(1)q$, may be considered as any arbitrary scalars unless otherwise specified.

The outline of this paper is as follows. In Section 2 we derive the new functional relationship

$$\prod_{j=1}^p (\lambda + \omega_j - 1) = \prod_{j=1}^q \omega_j \prod_{j=q+1}^p (\omega_j - r_j + r_j \lambda) \mu^p \quad , \quad (1.9)$$

which connects the sets of eigenvalues $\mu \in \sigma(T)$ and $\lambda \in \sigma(\mathcal{L}_{R,\Omega})$. From (1.9) all the similar known equations can be readily recovered. E.g., the one by Saridakis [15] for $r_j = r$ and $\omega_j = \omega$, $j = 1(1)p$, and the special case of it for $(p, q) = (2, 1)$ [5]. Taylor's equation [16] for $r_j = \omega_j$, $j = 1(1)p$, and its special case for $(p, q) = (2, 1)$ [20]. Also, Verner-Bernal's equation [19] for $r_j = \omega_j = \omega$, $j = 1(1)p$, and from the latter the famous equations of Young, $(p, q) = (2, 1)$, and of Varga, $(p, q) = (p, 1)$, etc. Sections 3 – 5 are devoted to the case $p = 2$. More specifically, in Section 3 we derive the matrix analogue of (1.9) and prove the equivalence of the MAOR method to a certain 2-step one. In Section 4 the precise convergence domains of the MAOR method when $\sigma(T)$ is real or pure imaginary are determined extending in this way previous results by Hadjidimos [5], Niethammer [12], Missirlis [10], Moussavi [11], Yeyios and Psimarni [22],

Hadjidimos and Yeyios [8] and others. Finally in Section 5 special cases of the results obtained in Section 4 are discussed and some concluding remarks are made.

2. The Functional Equation (1.9)

In order to derive (1.9) we follow an approach due to Varga, Niethammer and Cai [18], which was already used successfully in [7] and [8]. Also, to simplify the analysis we work out the case $(p, q) = (5, 2)$. The generalization to any (p, q) becomes obvious. In our particular case we have

$$T := \begin{bmatrix} 0 & 0 & 0 & T_{14} & 0 \\ 0 & 0 & 0 & 0 & T_{25} \\ T_{31} & 0 & 0 & 0 & 0 \\ 0 & T_{42} & 0 & 0 & 0 \\ 0 & 0 & T_{53} & 0 & 0 \end{bmatrix}, \quad (2.1)$$

$$L := \begin{bmatrix} & 0 & & 0 \\ T_{31} & 0 & 0 & \\ 0 & T_{42} & 0 & 0 \\ 0 & 0 & T_{53} & \end{bmatrix}, \quad U := \begin{bmatrix} 0 & T_{14} & 0 \\ 0 & 0 & T_{25} \\ 0 & 0 & \end{bmatrix}.$$

Let $u = [u_1^T, \dots, u_5^T]^T \neq 0$, partitioned in accordance with A in (1.1), be an eigenvector of $\mathcal{L}_{R, \Omega}$ with corresponding eigenvalue λ . From the equation $\mathcal{L}_{R, \Omega} u = \lambda u$ and in view of (1.4), (1.2) (1.6) and (2.1) we have

$$\begin{aligned} i) \quad & \omega_1 T_{14} u_4 = (\lambda + \omega_1 - 1)u_1 \\ ii) \quad & \omega_2 T_{25} u_5 = (\lambda + \omega_2 - 1)u_2 \\ iii) \quad & (\omega_3 - r_3 + r_3 \lambda) T_{31} u_1 = (\lambda + \omega_3 - 1)u_3 \\ iv) \quad & (\omega_4 - r_4 + r_4 \lambda) T_{42} u_2 = (\lambda + \omega_4 - 1)u_4 \\ v) \quad & (\omega_5 - r_5 + r_5 \lambda) T_{53} u_3 = (\lambda + \omega_5 - 1)u_5 \end{aligned} \quad (2.2)$$

In (2.2), at least one of the u_j 's, $j = 1(1)5$, will be different from zero, since $u \neq 0$. Let it be u_3 . We multiply (2.2iii) by $\prod_{j=1, j \neq 3}^5 (\lambda + \omega_j - 1)$ and in the left hand side obtained we replace the factors $(\lambda + \omega_1 - 1)u_1$, then $(\lambda + \omega_4 - 1)u_4$, and so on, by using the equations in (2.2) in a cyclic manner. Thus we have

$$\prod_{j=1}^2 \omega_j \prod_{j=3}^5 (\omega_j - r_j + r_j \lambda) T_{31} T_{14} T_{42} T_{25} T_{53} u_3 = \prod_{j=1}^5 (\lambda + \omega_j - 1) u_3 \quad (2.3)$$

If for some $j = 3(1)5$, $\omega_j - r_j + r_j \lambda = 0$, then from (2.3) $\lambda + \omega_k - 1 = 0$ for some $k = 1(1)5$ and (1.9) is satisfied for any μ . A necessary condition for this is to have $\omega_j - r_j + r_j(1 - \omega_k) = 0$ or equivalently, for a pair of indices indicated previously, $\omega_j = r_j \omega_k$. (Obviously such a condition can not be true for the Extrapolated Jacobi matrix since then $R = 0$.) On the other hand if one of the factors on the right hand side of (2.3) is zero that is for some $k = 1(1)5$ we have $\lambda = 1 - \omega_k$ then from (2.3) either for some $j = 3(1)5$, $\omega_j = r_j \omega_k$ and (1.9) is satisfied for any μ or if $\omega_j \neq r_j \omega_k$ for all $j = 3(1)5$ and $k = 1(1)5$ then 0 will be an eigenvalue of $T_{31} T_{14} T_{42} T_{25} T_{53}$. Since this matrix is one of the diagonal submatrices of T^5 it is implied that $0 = \mu \in \sigma(T)$, which satisfies (1.9). Also, if a number λ , such that $\omega_j - r_j + r_j \lambda \neq 0$, $j = 3(1)5$, satisfies (1.9) with $0 \neq \mu \in \sigma(T)$ then there exists $u'_3 \neq 0$ for which $T_{31} T_{14} T_{42} T_{25} T_{53} u'_3 = \mu^5 u'_3$, with $\mu^5 = \prod_{j=1}^5 (\lambda + \omega_j - 1) / (\prod_{j=1}^2 \omega_j \prod_{j=3}^5 (\omega_j - r_j + r_j \lambda))$. Defining u'_5, u'_2, u'_4 and u'_1 via (2.2v), (2.2ii), (2.2iv) and (2.2i) and using (1.9) we find out that u'_3 satisfies (2.2iii). This implies that $u' = [u'_1, \dots, u'_5]^T \neq 0$ is an eigenvector of $\mathcal{L}_{R,\Omega}$ with eigenvalue λ . For $0 = \mu \in \sigma(T)$ a limiting process argument leads to the same conclusion. Thus we have just proved:

Theorem 1: Given the square nonsingular matrix A partitioned in a $p \times p$ block form with block diagonal submatrices square and nonsingular. Assume that A is a GCO $(q, p - q)$ -matrix, with $\gcd(p, q) = 1$, and its associated block Jacobi matrix T is defined by (1.2) and has the form (1.5). Let $\mathcal{L}_{R,\Omega}$ in (1.4) be the associated MAOR matrix with R and Ω , $\det(\Omega) \neq 0$, being defined in (1.6). If λ and μ satisfy the relationship

$$\prod_{j=1}^p (\lambda + \omega_j - 1) = \prod_{j=1}^q \omega_j \prod_{j=q+1}^p (\omega_j - r_j + r_j \lambda) \mu^p \quad (2.4)$$

then the following statements are true:

- i) If $\lambda \in \sigma(\mathcal{L}_{R,\Omega})$ and $\omega_j - r_j + r_j \lambda = 0$ for some $j = q + 1(1)p$ then $\omega_j = r_j \omega_k$ (equivalently $\lambda = 1 - \omega_k$) for some $k = 1(1)p$.
- ii) If $\omega_k - 1 = \lambda \in \sigma(\mathcal{L}_{R,\Omega})$ for some $k = 1(1)p$ then either $\omega_j - r_j + r_j \lambda = 0$ (equivalently $\omega_j = r_j \omega_k$) for some $j = q + 1(1)p$ or $0 = \mu \in \sigma(T)$.
- iii) If $\omega_j - r_j + r_j \lambda \neq 0$ for all $j = q + 1(1)p$ then $\lambda \in \sigma(\mathcal{L}_{R,\Omega})$ if and only if $\mu \in \sigma(T)$. $\square$

Remarks: i) If one follows very closely the analysis in Taylor [16], provided Ω , and therefore R , is considered as in [16], one can show that the theorem above is true for the entire class of GCO $(q, p - q)$ matrices. ii) The present theorem and all the results in it are new. Furthermore, under appropriate additional assumptions all the

similar well-known eigenvalue relationships can be recovered as special cases of it as this was pointed out in the introductory section. However, there seems that for $r_j = 0$, $j = q + 1(1)p$, (the Extrapolated Jacobi case given in (1.7)) the result obtained from Thm 1 is new. $\square$

3. Equivalence of the 2-Cyclic MAOR and a Certain 2-Step Method.

An equivalence of the 2-cyclic MAOR and a certain 2-step method will allow one to study the convergence properties of either method via the other. Despite the fact that in the cases of real and pure imaginary spectra $\sigma(T)$ we are concerned with in the sequel the study is made by using the MAOR (the EMSOR, to be precise,) method, for more general spectra it would be more convenient if we used the 2-step method. For such an equivalence to be established a matrix analogue of the eigenvalue relationship (1.9) will provide us with the key point needed (see, e.g., [2] – [4] and [8]). For this the following theorem is stated and proved:

Theorem 2: Let $p = 2$; then under the assumptions of Theorem 1, and the additional assumption that $R \Omega^{-1} = sI$, with $s (\neq 0)$ being a scalar, the following matrix identities hold

$$\begin{aligned} & \left[\mathcal{L}_{R,\Omega} + (\omega_1 - 1)I \right] \left[\mathcal{L}_{R,\Omega} + (\omega_2 - 1)I \right] \\ &= \left[\left(1 - \frac{r_2}{\omega_2}\right)I + \frac{r_2}{\omega_2} \mathcal{L}_{R,\Omega} \right] (\Omega T)^2 = (\Omega T)^2 \left[\left(1 - \frac{r_2}{\omega_2}\right)I + \frac{r_2}{\omega_2} \mathcal{L}_{R,\Omega} \right] . \end{aligned} \quad (3.1)$$

Proof: We distinguish the two cases $R = 0$ and $\det(R) \neq 0$. i) For $R = 0$, $\mathcal{L}_{R,\Omega} = \mathcal{L}_{0,R}$ is given in (1.7). So, using (1.5) the left hand side of (3.1) becomes

$$\begin{aligned} & \begin{bmatrix} I - \Omega + \Omega T + (\omega_1 - 1)I \\ 0 & \omega_1 T_{12} \\ \omega_2 T_{21} & (\omega_1 - \omega_2)I_2 \end{bmatrix} \begin{bmatrix} I - \Omega + \Omega T + (\omega_2 - 1)I \\ (\omega_2 - \omega_1)I_1 & \omega_1 T_{12} \\ \omega_2 T_{21} & 0 \end{bmatrix} = \\ & \begin{bmatrix} \omega_1 \omega_2 T_{12} T_{21} & 0 \\ 0 & \omega_1 \omega_2 T_{21} T_{12} \end{bmatrix} = (\Omega T)^2 . \end{aligned}$$

However, the expression $(\Omega T)^2$ above is nothing but the middle and rightmost expression in (3.1), since for $R = 0$, $r_2 = 0$ and $\left(1 - \frac{r_2}{\omega_2}\right)I + \frac{r_2}{\omega_2} \mathcal{L}_{0,\Omega} = I$. ii) For $\det(R) \neq 0$, and in view of the fact that $R \Omega^{-1} = sI$, from (1.8) we have that

$$\mathcal{L}_{R,\Omega} = (1 - \frac{1}{s})I + \frac{1}{s} \mathcal{L}_{R,R} . \quad (3.2)$$

On the other hand it is known that (3.2) is true for $s = 1$ (see [3] and also [8]), therefore

$$\left[\mathcal{L}_{R,R} + (r_1 - 1)I \right] \left[\mathcal{L}_{R,R} + (r_2 - 1)I \right] = \quad (3.3)$$

$$\mathcal{L}_{R,R}(RT)^2 = (RT)^2 \mathcal{L}_{R,R} .$$

Solving (3.1) for $\mathcal{L}_{R,R}$ and substituting into the leftmost and the middle expressions of (3.3) we obtain

$$\begin{aligned} & \left[s \mathcal{L}_{R,\Omega} - (s - 1)I + (r_1 - 1)I \right] \left[s \mathcal{L}_{R,\Omega} - (s - 1)I + (r_2 - 1)I \right] = \\ & s^2 \left[\mathcal{L}_{R,\Omega} + (\omega_1 - 1)I \right] \left[\mathcal{L}_{R,\Omega} + (\omega_2 - 1)I \right] \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} & \left[s \mathcal{L}_{R,\Omega} - (s - 1)I \right] \left[s \Omega T \right]^2 = \\ & s^2 \left[\left(1 - \frac{r_2}{\omega_2}\right)I + \frac{r_2}{\omega_2} \mathcal{L}_{R,\Omega} \right] \left[\Omega T \right]^2 \end{aligned} \quad (3.5)$$

respectively. Equating (3.4) and (3.5) we obtain the first equality in (3.1). The second equality in (3.1) is now trivially proved. $\square$

Let $x^{(m)}$, $m = 0, 1, 2, \dots$, be the MAOR iterates from (1.3) – (1.4) and let us apply the leftmost and rightmost operators in (3.1) to $x^{(m)}$ to obtain

$$\begin{aligned} & \left[\mathcal{L}_{R,\Omega} + (\omega_1 - 1)I \right] \left[\mathcal{L}_{R,\Omega} + (\omega_2 - 1)I \right] x^{(m)} = \\ & (\Omega T)^2 \left[\left(1 - \frac{r_2}{\omega_2}\right)I + \frac{r_2}{\omega_2} \mathcal{L}_{R,\Omega} \right] x^{(m)} . \end{aligned}$$

Equivalently we have

$$\begin{aligned} \mathcal{L}_{R,\Omega}^2 x^{(m)} + (\omega_1 + \omega_2 - 2) \mathcal{L}_{R,\Omega} x^{(m)} + (\omega_1 - 1)(\omega_2 - 1) x^{(m)} = \\ (1 - \frac{r_2}{\omega_2})(\Omega T)^2 x^{(m)} + \frac{r_2}{\omega_2} (\Omega T)^2 \mathcal{L}_{R,\Omega} x^{(m)} \end{aligned}$$

or by using (1.3) – (1.4)

$$\begin{aligned} (x^{(m+2)} - (\mathcal{L}_{R,\Omega} + I)(I - RL)^{-1} \Omega D^{-1} b) + \\ (\omega_1 + \omega_2 - 2)(x^{(m+1)} - (I - RL)^{-1} \Omega D^{-1} b) + (\omega_1 - 1)(\omega_2 - 1) x^{(m)} = \\ \omega_1(\omega_2 - r_2)T^2 x^{(m)} + \omega_1 r_2 T^2 (x^{(m+1)} - (I - RL)^{-1} \Omega D^{-1} b) . \end{aligned}$$

Solving for $x^{(m+2)}$, rearranging terms and simplifying we finally obtain

$$\begin{aligned} x^{(m+2)} = & \left[(2 - \omega_1 - \omega_2)I + \omega_1 r_2 T^2 \right] x^{(m+1)} - \\ & \left[(\omega_1 - 1)(\omega_2 - 1)I - \omega_1 (\omega_2 - r_2)T^2 \right] x^{(m)} \\ & + \left[(\mathcal{L}_{R,\Omega} + I) + (\omega_1 + \omega_2 - 2)I - \omega_1 r_2 T^2 \right] (I - RL)^{-1} \Omega D^{-1} b . \end{aligned} \quad (3.6)$$

Due to the nonsingularity of A we have $1 \notin \sigma(T^2)$, which, in turn, implies that $1 \notin \sigma(\mathcal{L}_{R,\Omega})$, and the constant vector term in (3.6) can be simplified (see e.g., [8] for a similar simplification). So, one finally has

$$\begin{aligned} x^{(m+2)} = & \left[(2 - \omega_1 - \omega_2)I + \omega_1 r_2 T^2 \right] x^{(m+1)} - \\ & \left[(\omega_1 - 1)(\omega_2 - 1)I - \omega_1 (\omega_2 - r_2)T^2 \right] x^{(m)} + \omega_1 \omega_2 (I + T) D^{-1} b . \end{aligned} \quad (3.7)$$

In the 2-step method above $x^{(0)}$ and $x^{(1)}$ can be taken arbitrarily [13]. It can be checked that (3.6) and (3.7) coincide with the schemes (4.4) and (4.5) in [8] respectively when in the former ones $r_2 = \omega_2$ (MSOR case) and in the latter ones $p = 2$. So, it has just been proved that:

Theorem 3: Under the assumptions of Theorem 2 the MAOR method (1.3) – (1.4) and the 2-step method (3.7) are equivalent in the sense that they have the same asymptotic convergence rates. $\square$

4. Convergence Analysis

For $p = 2$, (1.9) becomes

$$(\lambda + \omega_1 - 1)(\lambda + \omega_2 - 1) = \omega_1 (\omega_2 - r_2 + r_2 \lambda) \mu^2 \quad (4.1)$$

or equivalently, if we drop the indices to simplify the notation,

$$\lambda^2 - b\lambda + c = 0 \quad , \quad (4.2)$$

where

$$b := 2 - \omega - \omega' + \omega' r \mu^2 \quad , \quad c := (\omega - 1)(\omega' - 1) + \omega' (r - \omega) \mu^2 \quad (4.3a)$$

and

$$\omega' := \omega_1, \quad \omega := \omega_2, \quad r := r_2 \quad . \quad (4.3b)$$

We assume that $\omega, \omega', r \in \mathbb{R}$, $\omega\omega' \neq 0$, and $\sigma(T)$ is real or pure imaginary. So, we distinguish two cases:

Case I: $\sigma(T) \in \mathbb{R}$.

Let $\sigma(T) \subset [-\bar{\mu}, \bar{\mu}]$, $\bar{\mu} = \rho(T)$, and $\sigma(T^2) \subset [\underline{\mu}^2, \bar{\mu}^2] =: M$, where $0 \leq \underline{\mu}^2 \leq \bar{\mu}^2$. Then the following theorem can be stated and proved.

Theorem 4: Under the assumptions of Thms 2 and 3 and the assumption $\sigma(T) \in \mathbb{R}$ the precise domains of convergence of the MAOR method defined in (1.3) – (1.4) is the union of all the subdomains defined in Table 1 for $0 < \underline{\mu} \leq \bar{\mu} < 1$, in (4.10) for $0 = \underline{\mu} \leq \bar{\mu} < 1$, and in Table 2 for $1 < \underline{\mu} \leq \bar{\mu}$.

Proof: The MAOR method converges (strongly in the sense of Young [21]) if and only if for all $\mu^2 \in M$ the two roots of (4.2) are less than one in modulus. This, because of Lemma 2.1 of [21, p. 171], holds if and only if

$$|c| < 1, \quad |b| < 1 + c, \quad \text{for all } \mu^2 \in M \quad . \quad (4.4)$$

In view of (4.3a), (4.4) are equivalent to

$$\begin{aligned} \omega\omega'(1 - \mu^2) &> 0 \quad , \\ -2 &< \omega\omega'(1 - \mu^2) - \omega - \omega' + \omega'r\mu^2 < 0 \quad , \end{aligned}$$

$$4 + \omega\omega'(1 - \mu^2) - 2\omega - 2\omega' + 2\omega'r\mu^2 > 0 ,$$

for all $\mu^2 \in M$

or to

$$0 < \omega\omega'(1 - \mu^2) < 4 ,$$

$$\omega + \omega' - 2 - \frac{1}{2} \omega\omega' (1 - \mu^2) < \omega'r\mu^2 < \omega + \omega' - \omega\omega' (1 - \mu^2) , \quad (4.5)$$

for all $\mu^2 \in M$.

Obviously, for $1 \in M$ the leftmost inequality in the first set of inequalities (4.5) can not be satisfied. So, to have convergence it must be either a) $\bar{\mu} < 1$ or b) $\underline{\mu} > 1$.

Case Ia: Let $0 \leq \underline{\mu} \leq \bar{\mu} < 1$. Assume first that $\underline{\mu} > 0$. Then (4.5) are equivalent to either

$$\omega > 0, \quad 0 < \omega' < 4 / \left[\omega(1 - \underline{\mu}^2) \right] ,$$

$$\max_{\mu^2 \in M} A(\mu^2) < r < \min_{\mu^2 \in M} B(\mu^2) \quad (4.6)$$

or

$$\omega < 0, \quad 4 / \left[\omega(1 - \underline{\mu}^2) \right] < \omega' < 0 ,$$

$$\max_{\mu^2 \in M} B(\mu^2) < r < \min_{\mu^2 \in M} A(\mu^2) , \quad (4.7)$$

where

$$A(\mu^2) := \frac{1}{\omega'\mu^2} \left[\omega + \omega' - 2 - \frac{1}{2} \omega\omega'(1 - \mu^2) \right]$$

$$B(\mu^2) := \frac{1}{\omega'\mu^2} \left[\omega + \omega' - \omega\omega'(1 - \mu^2) \right] . \quad (4.8)$$

It can be found out that

$$\frac{\partial A(\mu^2)}{\partial \mu^2} = \frac{(2 - \omega)(2 - \omega')}{2\omega'\mu^4} , \quad \frac{\partial B(\mu^2)}{\partial \mu^2} = \frac{\omega\omega' - \omega - \omega'}{\omega'\mu^4} . \quad (4.9)$$

In order to determine the extreme values for A and B in (4.6) and (4.7) we have to

determine the sign of the expressions in (4.9). For these signs we must take into account not only the intervals for ω and ω' defined in (4.6) and (4.7) but also the relative positions of ω with respect to (wrt) 2, that of ω' wrt 2 and $\frac{\omega}{\omega-1}$ and therefore the relative position of ω wrt 1 and that of $\frac{\omega}{\omega-1}$ wrt 2. Considering in the (ω, ω') -plane all possible subdomains defined in this way one can very easily study the behavior of $A(\mu^2)$ and $B(\mu^2)$ and therefore define the corresponding ranges for r . After a long and tedious elementary analysis one ends up with the following Table 1, where the results obtained are given in the most compact form.

Table 1
 $0 < \underline{\mu} \leq \bar{\mu} < 1$
 (Increasing (I), Decreasing (D))

Case	Range of ω	Sub-case	Range of ω'	Behavior of $A(\underline{\mu}^2)$	Behavior of $B(\underline{\mu}^2)$	Range of r
1	$0 < \omega \leq \frac{2}{1+\underline{\mu}}$	(i)	$0 < \omega' \leq 2$	I	D	$A(\bar{\mu}^2) < r < B(\bar{\mu}^2)$
		(ii)	$2 \leq \omega' < \frac{4}{\omega(1-\underline{\mu}^2)}$	D	D	$A(\underline{\mu}^2) < r < B(\bar{\mu}^2)$
2	$\frac{2}{1+\underline{\mu}} \leq \omega \leq 2$	(i)	$0 < \omega' \leq 2$	I	D	$A(\bar{\mu}^2) < r < B(\bar{\mu}^2)$
		(ii)	$2 \leq \omega' \leq \frac{\omega}{\omega-1}$	D	D	$A(\underline{\mu}^2) < r < B(\bar{\mu}^2)$
		(iii)	$\frac{\omega}{\omega-1} \leq \omega' < \frac{4}{\omega(1-\underline{\mu}^2)}$	D	I	$A(\underline{\mu}^2) < r < B(\underline{\mu}^2)$
3	$2 \leq \omega \leq \frac{2}{1-\underline{\mu}^2}$	(i)	$0 < \omega' \leq \frac{\omega}{\omega-1}$	D	D	$A(\underline{\mu}^2) < r < B(\bar{\mu}^2)$
		(ii)	$\frac{\omega}{\omega-1} \leq \omega' \leq 2$	D	I	$A(\underline{\mu}^2) < r < B(\underline{\mu}^2)$
		(iii)	$2 \leq \omega' < \frac{4}{\omega(1-\underline{\mu}^2)}$	I	I	$A(\bar{\mu}^2) < r < B(\underline{\mu}^2)$
4	$\frac{2}{1-\underline{\mu}^2} \leq \omega \leq \frac{2}{1-\underline{\mu}}$	(i)	$0 < \omega' \leq \frac{\omega}{\omega-1}$	D	D	$A(\underline{\mu}^2) < r < B(\bar{\mu}^2)$
		(ii)	$\frac{\omega}{\omega-1} \leq \omega' < \frac{4}{\omega(1-\underline{\mu}^2)}$	D	I	$A(\underline{\mu}^2) < r < B(\underline{\mu}^2)$
5	$\frac{2}{1-\underline{\mu}} \leq \omega < \infty$		$0 < \omega' < \frac{4}{\omega(1-\underline{\mu}^2)}$	D	D	$A(\underline{\mu}^2) < r < B(\bar{\mu}^2)$
6	$-\infty < \omega < 0$		$\frac{4}{\omega(1-\underline{\mu}^2)} < \omega' < 0$	D	D	$B(\underline{\mu}^2) < r < A(\bar{\mu}^2)$

Consequently, the union of all the domains, defined by all the subcases of Table 1, which are cylindrical ones bounded from above and below (r -direction) by the surfaces of two hyperboloids, gives the precise region of convergence in the particular case examined so far. It has to be remarked, however, that in the subcases 1ii, 2ii, 3i, 3iii, 4i

5 and 6 the corresponding cylinders may be empty sets or may not extend over the entire subregion of the (ω, ω') -plane defined by the corresponding subcase. This depends on whether $A(\underline{\mu}^2) < B(\bar{\mu}^2)$, for subcases 1ii, 2ii, 3i, 4i and 5, $A(\bar{\mu}^2) < B(\underline{\mu}^2)$, for subcase 3iii and $B(\underline{\mu}^2) < A(\bar{\mu}^2)$, for case 6, hold for none or some of the corresponding pairs (ω, ω') . This remark also holds in all the other cases which are examined in the sequel. Assume now that $\underline{\mu} = 0$. Then from (4.5) and Table 1 one can very easily find out that the domain of convergence in the (ω, ω', r) -space is defined by

$$0 < \omega < 2, \quad 0 < \omega' < 2, \quad A(\bar{\mu}^2) < r < B(\bar{\mu}^2) \quad . \quad (4.10)$$

We simply note that r is arbitrary in the special case $\bar{\mu} = 0$.

Case Ib: Let $1 < \underline{\mu} \leq \bar{\mu}$. Then (4.5) are equivalent to either

$$\begin{aligned} \omega > 0, \quad 4 / \left[\omega(1 - \bar{\mu}^2) \right] < \omega' < 0 \quad , \\ \max_{\mu^2 \in M} B(\mu^2) < r < \min_{\mu^2 \in M} A(\mu^2) \end{aligned} \quad (4.11)$$

or

$$\begin{aligned} \omega < 0, \quad 0 < \omega' < 4 / \left[\omega(1 - \bar{\mu}^2) \right] \quad , \\ \max_{\mu^2 \in M} A(\mu^2) < r < \min_{\mu^2 \in M} B(\mu^2) \quad , \end{aligned} \quad (4.12)$$

where $A(\mu^2)$ and $B(\mu^2)$ are given in (4.8). Following an analysis similar to the one in Case Ia we can determine the precise domain of convergence as the union of all nonempty subdomains of Table 2.

Table 2
 $(1 < \underline{\mu} \leq \bar{\mu})$
 (Increasing (I), Decreasing (D))

Case	Range of ω	Sub-case	Range of ω'	Behavior of $A(\mu^2)$	Behavior of $B(\mu^2)$	Range of r
1	$0 < \omega \leq \frac{2}{1 + \bar{\mu}}$	(i)	$\frac{\omega}{\omega - 1} \leq \omega' < 0$	D	I	$B(\bar{\mu}^2) < r < A(\bar{\mu}^2)$
		(ii)	$\frac{4}{\omega(1 - \bar{\mu}^2)} < \omega' \leq \frac{\omega}{\omega - 1}$	D	D	$B(\underline{\mu}^2) < r < A(\bar{\mu}^2)$
2	$\frac{2}{1 + \bar{\mu}} \leq \omega \leq 2$		$\frac{4}{\omega(1 - \bar{\mu}^2)} < \omega' < 0$	D	I	$B(\bar{\mu}^2) < r < A(\bar{\mu}^2)$
3	$2 \leq \omega < \infty$		$\frac{4}{\omega(1 - \bar{\mu}^2)} < \omega' < 0$	I	I	$B(\bar{\mu}^2) < r < A(\bar{\mu}^2)$
4	$\frac{2}{1 - \bar{\mu}^2} \leq \omega < 0$	(i)	$0 < \omega' \leq \frac{\omega}{\omega - 1}$	I	I	$A(\bar{\mu}^2) < r < B(\bar{\mu}^2)$
		(ii)	$\frac{\omega}{\omega - 1} \leq \omega' < 2$	I	D	$A(\bar{\mu}^2) < r < B(\bar{\mu}^2)$
		(iii)	$2 \leq \omega' < \frac{4}{\omega(1 - \bar{\mu}^2)}$	D	D	$A(\underline{\mu}^2) < r < B(\bar{\mu}^2)$
5	$\frac{2}{1 - \bar{\mu}} \leq \omega < \frac{2}{1 - \bar{\mu}^2}$	(i)	$0 < \omega' \leq \frac{\omega}{\omega - 1}$	I	I	$A(\bar{\mu}^2) < r < B(\bar{\mu}^2)$
		(ii)	$\frac{\omega}{\omega - 1} \leq \omega' < \frac{4}{\omega(1 - \bar{\mu}^2)}$	I	D	$A(\bar{\mu}^2) < r < B(\bar{\mu}^2)$
6	$-\infty < \omega \leq \frac{2}{1 - \bar{\mu}}$		$0 < \omega' < \frac{4}{\omega(1 - \bar{\mu}^2)}$	I	I	$A(\bar{\mu}^2) < r < B(\bar{\mu}^2)$

The derivation of the Tables 1 and 2 and the result (4.10) complete the proof of the present theorem. $\square$

Case II: $\sigma(T) \in i\mathbb{R}$

Let $\sigma(T) \subset [-i\bar{\mu}, i\bar{\mu}]$, $i^2 = -1$, $\rho(T) = \bar{\mu}$, and $\sigma(T^2) \subset [-\bar{\mu}^2, -\underline{\mu}^2] =: -M$, where $0 \leq \underline{\mu} \leq \bar{\mu}$. Then a theorem, similar to Theorem 4, can be stated and proved. For this

we must bear in mind that the expressions in (4.3a) are a little different due to the pure imaginary nature of $\sigma(T)$. The corresponding expressions are obtained from (4.3a) when μ^2 is replaced by $-\mu^2$. More specifically

$$b := 2 - \omega - \omega' - \omega' r \mu^2, \quad c := (\omega - 1)(\omega' - 1) - \omega'(r - \omega)\mu^2 \quad . \quad (4.3a)'$$

Thus we have:

Theorem 5: Under the assumptions of Thms 2 and 3 and the assumption $\sigma(T) \in i\mathbb{R}$ the precise domain of convergence of the MAOR method defined in (1.3) – (1.4) is the union of all the subdomains defined in Table 3, for $0 < \underline{\mu} \leq \bar{\mu}$ and in (4.13), for $0 = \underline{\mu} \leq \bar{\mu}$.

Proof: The MAOR method converges if and only if the two roots of (4.2), with b, c being defined in (4.3a)' for all $\mu^2 \in M$, are less than one in modules. This is equivalent to

$$|c| < 1, \quad |b| < 1 + c, \quad \text{for all } \mu^2 \in M \quad . \quad (4.4)'$$

Assuming that $\underline{\mu} > 0$ and working in a way similar to that for the real case we end up with Table 3, where again the results obtained are given in the most compact form. The expressions for A and B are the ones in (4.8).

Table 3

($\underline{\mu} > 0$)

(Increasing (I), Decreasing (D))

Case	Range of ω	Sub-case	Range of ω'	Behavior of $A(\underline{\mu}^2)$	Behavior of $B(\underline{\mu}^2)$	Range of r
1	$0 < \omega \leq \frac{2}{1+\underline{\mu}^2}$	(i)	$0 < \omega' \leq 2$	I	D	$B(-\underline{\mu}^2) < r < A(-\underline{\mu}^2)$
		(ii)	$2 \leq \omega' < \frac{4}{\omega(1+\underline{\mu}^2)}$	I	I	$B(-\underline{\mu}^2) < r < A(-\underline{\mu}^2)$
2	$\frac{2}{1+\underline{\mu}^2} \leq \omega \leq 2$		$0 < \omega' < \frac{4}{\omega(1+\underline{\mu}^2)}$	I	D	$B(-\underline{\mu}^2) < r < A(-\underline{\mu}^2)$
3	$2 \leq \omega < \infty$		$0 < \omega' < \frac{4}{\omega(1+\underline{\mu}^2)}$	I	I	$B(-\underline{\mu}^2) < r < A(-\underline{\mu}^2)$
4	$-\infty < \omega < 0$		$\frac{4}{\omega(1+\underline{\mu}^2)} < \omega' < 0$	I	I	$A(-\underline{\mu}^2) < r < B(-\underline{\mu}^2)$

For $\underline{\mu} = 0$, we can find the following domains of convergence in the (ω, ω', r) -space.

$$\begin{aligned}
 i) \quad & 0 < \omega \leq \frac{2}{1+\underline{\mu}^2}, \quad 0 < \omega' < 2, \quad B(-\underline{\mu}^2) < r < A(-\underline{\mu}^2), \\
 ii) \quad & \frac{2}{1+\underline{\mu}^2} \leq \omega < 2, \quad 0 < \omega' < \frac{4}{\omega(1+\underline{\mu}^2)}, \quad B(-\underline{\mu}^2) < r < A(-\underline{\mu}^2).
 \end{aligned} \tag{4.13}$$

For $\underline{\mu} = 0$ then $0 < \omega < 2$, $0 < \omega' < 2$ and r is arbitrary. $\square$

5. Concluding Remarks and Discussion

The precise domains of convergence of the 2-cyclic consistently ordered MAOR method with $\sigma(T)$ real or pure imaginary, determined and presented in the previous section, are new. We remark, however, that in very special cases, where the three parameters involved in the MAOR method are somehow reduced to two, the corresponding domains can be directly obtained from the ones in Section 4 and most of them have been known. Thus: i) For $r \neq 0$ and $\omega = \omega'$ (ESOR method) for $\sigma(T)$ real or pure imaginary with $\underline{\mu} = 0$ they can be found in [12], while for $\underline{\mu} > 0$ and $\sigma(T)$ real and $\sigma(T)$ pure imaginary in [1] and [10] respectively. ii) For $r = \omega \neq \omega'$ (MSOR method) for

$\sigma(T)$ real with $\mu = 0$ they can be found in [21], while in all other cases in [22] or they can be directly obtained from the analysis in [8]. iii) For $r = 0$, $\omega \neq \omega'$ (two-parameter Extrapolated Jacobi method) the convergence domains obtained in the previous section, at least in some cases, must be new.

An interesting open problem, which is under investigation, is the one concerned with the determination of the optimum parameters in the various cases of Section 4. Again it is noted that the optimum parameters of the special cases which we referred to in the end of the previous paragraph can be found in the literature and some of them in the references given previously.

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