

1992

# Calculation of the Displacement of a Wankel Rotary Compressor

J. E. Beard

*Michigan Technological University*

G. R. Pennock

*Purdue University*

Follow this and additional works at: <http://docs.lib.purdue.edu/icec>

---

Beard, J. E. and Pennock, G. R., "Calculation of the Displacement of a Wankel Rotary Compressor" (1992). *International Compressor Engineering Conference*. Paper 788.

<http://docs.lib.purdue.edu/icec/788>

This document has been made available through Purdue e-Pubs, a service of the Purdue University Libraries. Please contact [epubs@purdue.edu](mailto:epubs@purdue.edu) for additional information.

Complete proceedings may be acquired in print and on CD-ROM directly from the Ray W. Herrick Laboratories at <https://engineering.purdue.edu/Herrick/Events/orderlit.html>

# CALCULATION OF THE DISPLACEMENT OF A WANKEL ROTARY COMPRESSOR

J. E. Beard  
Associate Professor  
ME-EM Department  
Michigan Technological University  
Houghton, Michigan 49931

G. R. Pennock  
Associate Professor  
School of Mechanical Engineering  
Purdue University  
West Lafayette, Indiana 47907

## ABSTRACT

The volumetric displacement of a Wankel rotary compressor is a function of the trochoid ratio and the pin size ratio, assuming that the number of lobes is specified. The mathematical expression which defines the displacement contains a function which can be evaluated directly and a normal elliptic integral of the second type which does not have an explicit solution. This paper focuses on the contribution of the elliptic integral to the total displacement of the compressor. The paper shows that the influence of the elliptic integral can account for as much as 20% of the total displacement, depending on the trochoid ratio and pin size ratio. The paper also shows that the numerical method used to evaluate the elliptic integral has a minimal effect on the accuracy of the calculated displacement (for a practical number of integration steps). The numerical integration technique that is used in this paper is the trapezoidal rule. The bounds on the error are included in the paper. For illustrative purposes, the paper includes a numerical example of the common three lobed Wankel rotary compressor.

## NOMENCLATURE

The following notation is used consistently throughout the paper:

$X_1 O_1 Y_1$  = Cartesian reference frame attached to the smaller pitch circle  
(regarded as stationary, or fixed, in this paper)

$X_2 O_2 Y_2$  = Cartesian reference frame attached to the larger pitch circle  
(the pitch circle that contains the generating pins)

$O_1$  = center of the smaller pitch circle ,  $O_2$  = center of the larger pitch circle

$r_1$  = radius of the smaller pitch circle ,  $r_2$  = radius of the larger pitch circle

$T$  = number of generating lobes on the larger pitch circle

$T - 1$  = number of generated lobes on the smaller pitch circle

$C$  = center of the generating pin ,  $r$  = radius of the generating pin

$r_C$  = radius  $O_2 C$  of the epitrochoidal path of point  $C$

$e$  = trochoid eccentricity = length of the crank  $O_1 O_2$

$Q$  = internal point of contact between the generating pin and the generated shape

$H$  = external point of contact between the generating pin and the generated shape

$\phi$  = input angle (position of the crank relative to the  $X_1$ -axis)

$\alpha$  = crank angle (position of the crank relative to the  $X_2$ -axis)

$\psi$  = angle between the  $X_1$ -axis and the  $X_2$ -axis

$\mu$  = trochoid ratio ,  $\lambda$  = pin size ratio and  $\Delta A$  = cross-sectional area of a pocket

## INTRODUCTION

The Wankel rotary compressor is a gerotor with three generating lobes on the larger pitch circle; i.e.,  $T = 3$  [Wankel, 1965; Ansdale, 1969; Yamamoto, 1981]. A gerotor is a planar mechanism consisting of a pair of pitch circles one of which encloses the other. The number of generating lobes on the larger pitch circle is one more than the number of generated lobes on the smaller pitch circle. The basic geometry of the gerotor mechanism is, for the most part, well-known and can be found in references such as; Hall [1968], Schell [1969], Colbourne [1975], Leemhius and Soedel [1978], Sadler and Nelle [1979], and Beard et al. [1991a]. The influence of the design parameters on the curvature of the generated shape, the displacement, and the compression ratio has also been well documented.

The lobe generating pins generate two shapes, commonly referred to as the inner envelope and the outer envelope [Ansdale, 1969; Wydra, 1986]. The inner-most envelope is produced by the internal point of contact between the pin and the generated shape, and the outer-most envelope is produced by the external point of contact. The size and the placement of the generating pins has a significant effect on the volumetric displacement of the Wankel rotary compressor [Beard and Pennock, 1990]. The analytical equation for the displacement can be expressed in terms of the trochoid ratio and the pin size ratio [Beard et al. 1991b]. The equation contains two distinct terms: (i) an explicit function which can be evaluated directly, and (ii) a normal elliptic integral of the second type which does not have an explicit solution. This paper focuses on the contribution of the elliptic integral to the total displacement of the compressor. The accuracy of the calculated displacement, depending on the numerical method that is adopted to evaluate the elliptic integral, has not been investigated previously. For illustrative purposes, the numerical method that is presented in the following section is the trapezoidal rule and the local and global errors associated with this method are also investigated.

### DEVELOPMENT OF THE THEORY

Since a Wankel rotary compressor has a constant cross-section (not including the pocket in the rotor), the volume contained in a pocket is the cross-sectional area of that pocket multiplied by the depth of the pocket. Several authors; e.g., Colbourne [1974; 1975] and Beard et al. [1989], have shown that the cross-sectional area of a gerotor pocket can be expressed as

$$\Delta A = \left( \frac{4 r_2^2 \mu}{T-1} \right) \sin \left( \frac{\pi}{T} \right) \pm$$

$$r_2^2 \mu \frac{\lambda}{T} \int_{\frac{\pi}{T-1}}^{\frac{\pi(T+1)}{T-1}} \left\{ \sqrt{1 + \mu^2 - 2 \mu \cos \phi \left( \frac{T-1}{T} \right)} - \sqrt{1 + \mu^2 - 2 \mu \cos \left[ \frac{2\pi - \phi(T-1)}{T} \right]} \right\} d\phi \quad (1)$$

where the "+" sign of the  $\pm$  is used for the external contact gerotor and the "-" sign is used for the internal contact gerotor. The integral in Eq. (1) is a normal elliptic integral of the second type and does not have an explicit solution [Byrd and Friedman, 1954]. Therefore, a numerical integration scheme must be used to approximate the integral. The error introduced by the numerical method is an important consideration in the design of a rotary machine, in general, and a Wankel rotary compressor, in particular. The most common approaches for numerical integration are: (i) the Newton-Coates formulas (e.g., the trapezoidal rule, Simpson's 1/3 rule, and Simpson's 3/8 rule), and (ii) Romberg integration and Gauss quadrature. The trapezoidal rule is adopted in this presentation since it is regarded to be the simplest numerical technique and is generally considered to provide good, but not the best, accuracy.

The angle between the crank  $O_1 O_2$  and the moving X-axis, see Figure 1, can be expressed as

$$\alpha = \phi \left( \frac{T-1}{T} \right) \quad (2)$$

Substituting Eq. (2) into Eq. (1) and changing the limits of integration, the cross-sectional area of a pocket may be written as

$$\Delta A = \frac{r_2^2 \mu}{T-1} \left\{ 4 \sin \pi/T \pm \lambda \int_{\beta_1}^{\beta_2} (f_1 - f_2) d\alpha \right\} \quad (3)$$

where  $\beta_1 = \pi/T$  ,  $\beta_2 = \pi(T+1)/T$  (4a)

$$f_1 = \sqrt{1 + \mu^2 - 2\mu \cos \alpha} \quad \text{and} \quad f_2 = \sqrt{1 + \mu^2 - 2\mu \cos(2\pi/T - \alpha)} \quad (4b)$$

The major goal of this paper is to present the results in a manner that will allow the designer to make a direct comparison between gerotors with different trochoid ratios and pin size ratios and the associated errors. Therefore, the equation for the displacement will be normalized by scaling the gerotor to fit inside a unit circle. The containment radii for internal and external contact, see Figures 2a and 2b, respectively, are

$$r_I = \frac{2r_2}{T} + r_C - r \quad \text{and} \quad r_E = \frac{r_2}{T} + r_C + r \quad (5)$$

Therefore, the radius of the generating pitch circle can be written in terms of the internal containment radius as

$$r_2 = \frac{T r_I}{2 + \mu T (1 - \lambda)} \quad (6a)$$

or in terms of the external containment radius as

$$r_2 = \frac{T r_E}{1 + \mu T (1 + \lambda)} \quad (6b)$$

Equations (6) provide the designer with the freedom to choose the parameters  $T$ ,  $\mu$ ,  $\lambda$ , and  $r_I$  (or  $r_E$ ) and directly solve for the radius of the generating pitch circle.

For convenience, the cross-sectional area of a pocket will be expressed in dimensionless form. For illustrative purposes we will consider only external contact, however, a similar expression can also be written for internal contact. The area contained by the circle of radius  $r_E$  is

$$A_E = \pi r_E^2 \quad (7a)$$

Rearranging Eq. (6b) in terms of  $r_E$  and substituting into Eq. (7a), the contained area is

$$A_E = \frac{\pi r_2^2 [1 + \mu T (1 + \lambda)]^2}{T^2} \quad (7b)$$

Finally, dividing Eq. (3) by Eq. (7b), the cross-sectional area of a pocket (for external contact) in dimensionless form is

$$(\Delta \bar{A})_E = \frac{\mu T^2}{\pi (T-1) [1 + \mu T (1 + \lambda)]^2} \left\{ 4 \sin \pi/T + \lambda \int_{\beta_1}^{\beta_2} (f_1 - f_2) d\alpha \right\} \quad (8)$$

The integral in Eq. (8); i.e.,

$$I = \int_{\beta_1}^{\beta_2} (f_1 - f_2) d\alpha \quad (9)$$

is a normal elliptic integral of the second type. As noted earlier, the integral does not have an explicit solution [Byrd and Friedman, 1954]. Using the trapezoidal rule [Chapra and Canale, 1985], the integral of a continuously differentiable function, say  $g(x)$ , can be expressed as

$$\int_{x_1}^{x_2} g(x) dx = \frac{h}{2} [g_1 + 2g_2 + 2g_3 + \dots + g_{n+1}] \quad (10)$$

where  $n$  is the number of steps and  $h$  is the step size. The local error, which is the error associated with a single step [Chapra and Canale, 1985], can be written as

$$E_L = -\frac{1}{12} h^3 g''(\xi) \quad \text{where} \quad x_0 \leq \xi \leq x_1 \quad (11)$$

and the global error can be written as

$$E_G = -\frac{1}{12} h^3 [f''(\xi_1) + f''(\xi_2) + f''(\xi_3) + \dots + f''(\xi_n)] \quad (12)$$

From the mean value theorem, the global error for a continuous function can be written as

$$E_G = -\frac{1}{12} h^3 n f''(\xi) \quad \text{where} \quad x_0 \leq \xi \leq x_n \quad (13)$$

The number of steps can be expressed in terms of the limits of integration and the step size; namely

$$n = \frac{\beta_2 - \beta_1}{h} \quad (14a)$$

Therefore, substituting Eqs. (4a) into Eq. (14a), the number of steps can be written as

$$n = \frac{\pi}{h} \quad (14b)$$

Finally, substituting Eq. (14b) into Eq. (13), the global error can be written as

$$E_G = -\frac{\pi}{12} h^2 (f''_1 - f''_2) \quad (15)$$

where the second-order derivatives, from Eq. (4b), are

$$f''_1 = \frac{(\mu^2 \cos \alpha - \mu)(\mu - \cos \alpha)}{(\mu^2 - 2\mu \cos \alpha + 1)^{3/2}} \quad (16a)$$

$$\text{and} \quad f''_2 = \frac{[\mu^2 \cos(2\pi/T - \alpha) - \mu][\mu - \cos(2\pi/T - \alpha)]}{[\mu^2 - 2\mu \cos(2\pi/T - \alpha) + 1]^{3/2}} \quad (16b)$$

Note that if the number of segments is doubled then the truncation error will be quartered. The global error can be bounded by calculating the minimum and maximum error over the interval  $\beta_1$  to  $\beta_2$ . The local and global errors for a given number of lobes  $T$  and arbitrary values of  $\mu$  and  $\lambda$  are given by Eqs. (11) and (15), respectively. The bounds of the error can be determined because the function consists of the two functions  $f_1$  and  $f_2$  which are continuously differentiable over the region of interest.

## RESULTS AND CONCLUSIONS

The following results and conclusions are for a Wankel rotary compressor with  $T=3$  and for external contact. First, the total pocket displacement is plotted against  $\mu$  for various values of  $\lambda$ , see Figure 3. As noted by Beard et al. [1991b], the displacement decreases for: (i) an increasing value of  $\mu$  and a specified value of  $\lambda$ ; and (ii) an increasing value of  $\lambda$  and a specified value of  $\mu$ . The contribution of the elliptic integral to the total pocket displacement is shown in Figure 4. Remember that both the displacement and the integral have been scaled according to Eq. (8) and are dimensionless values.

For a specified value of  $\lambda$  and an increasing value of  $\mu$ , the percentage of the displacement attributed to the elliptic integral may be regarded as constant, see Figure 5. However, the figure also shows that for a specified value of  $\mu$  and an increasing value of  $\lambda$ , the percentage of the displacement attributed to the integral is significant. The percentage variation in the displacement is between 5% and

20% as  $\lambda$  increases from 0.05 to 0.25. Therefore, the numerical method that is selected to determine the displacement can have a significant effect on the accuracy of the volumetric displacement. It should be noted that 100 steps was used in the trapezoidal rule for approximating the elliptic integral.

The upper bound on the percentage integration error versus  $\mu$ ; i.e., the maximum percentage error that may exist in the numerical method used to evaluate the elliptic integral, is shown in Figure 6a. Remember that this value is the upper limit (or bound) on the error and that the actual error may be less. Since the percentage error of the integrand is independent of  $\lambda$ , see Eq. (8), only one value of  $\lambda$  ( $= 0.05$ ) is plotted against  $\mu$ . The maximum percentage error for the Wankel rotary compressor considered here is 0.017%. For the sake of completeness, the integration error for the total displacement was also considered. The total percentage error in the total displacement varies from  $0.07 \times 10^{-2}\%$  to  $0.34 \times 10^{-2}\%$ , see Figure 6b. Since this percentage error is very small it can be neglected for all practical purposes and the calculated displacement can be regarded as exact. Future research will include the effects of the manufacturing tolerances on the total pocket displacement.

#### REFERENCES

1. R. F. Ansdale with a special contribution by D.J. Lockley (1969), *The Wankel RC Engine: Design and Performance*, A.S. Barnes and Company, Inc., Cranbury, New Jersey.
2. J.E. Beard, G.R. Pennock, and M.M. Stanisic (1989), "The Effects of the Design Parameters on the Generated Curvature and Displacement of Epitrochoidal Gerotor Pumps", *Trans. SAE, Journal of Commercial Vehicles*, Section 2, Vol. 98, pp. 217-228, SAE Technical Paper No. 891831.
3. J.E. Beard, and G.R. Pennock (1990), "The Effects of the Design Parameters on the Displacement and Compression Ratio of the Wankel Rotary Compressor", *Proc. of 10th International Compressor Engineering Conference*, Purdue University, West Lafayette, Indiana, July 17-20, Vol. 1, pp. 24-33.
4. J.E. Beard, A.S. Hall, Jr., and W. Soedel (1991a), "Comparison of Hypotrochoidal and Epitrochoidal Gerotors", *Trans. ASME, Journal of Mechanical Design*, June, Vol. 113, No. 2, pp. 133-141.
5. J.E. Beard, D.W. Yannitell, and G.R. Pennock (1991b), "The Effects of the Generating Pin Size and Placement on the Generated Curvature and Displacement of Internal Gear Pumps", *Proc. of 8th World Congress on the Theory of Machines and Mechanisms*, Prague, Czechoslovakia, August 26-31, Vol. 5, pp. 1411-1414.
6. P.F. Byrd, and M.D. Friedman (1954), *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer-Verlag, Berlin, p. 8.
7. S.C. Chapra, and R.P. Canale (1985), *Numerical Methods for Engineers with Personal Computer Applications*, Chapter 13, McGraw-Hill Book Company, Inc., New York.
8. J.R. Colbourne (1974), "The Geometry of Trochoidal Envelopes and Their Application in Rotary Pumps", *Mechanism and Machine Theory*, Pergamon Press, Vol. 9, pp. 421-435.
9. J.R. Colbourne (1975), "Gear Shape and Theoretical Flow Rate in Internal Gear Pumps", *Trans. of Canadian Society of Mechanical Engineering*, Vol. 3, No. 4, pp. 215-223.
10. A.S. Hall, Jr. (1968), "Pin Gearing", *ASME Paper 68-MECH-58*, *Proc. of ASME Mechanisms Conference*, Atlanta, Georgia, October 5-9.
11. R.S. Leemhuis, and W. Soedel (1978), "Vector Loop Analysis of Wankel Compressors and Engines", *ASME Paper 78-DET-78*.
12. J.P. Sadler, and D.E. Nelle (1979), "Design Analysis of an Epicyclic Rotary Pump Mechanism", *Trans. of ASME, Journal of Mechanical Design*, January, Vol. 101, page 99-107.
13. C.P. Schell (1969), "Define the Geometry of the Inner and Outer Tooth Contours of a Gear Type Oil Pump", *General Motors Engineering Journal*, July-August-September 1969, pages 53 and 54.
14. F. Wankel (1965), *Rotary Piston Engines*, translated by R.F. Ansdale, Iiffe Books, Ltd., London.

15. L. Wydra (1986), "Development of Outer-Envelope Trochoidal Compressors", Proc. of 8th International Compressor Engineering Conference, Purdue University, West Lafayette, Indiana, July 16-19, Vol. 1, pp. 282-291.
16. K. Yamamoto (1981), *Rotary Engine*, Toyo Kogyo Co., Ltd., Japan.

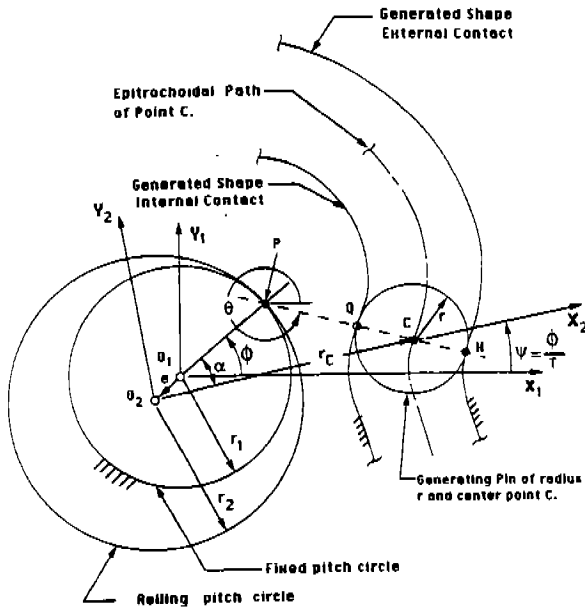


Fig. 1. Epitrochoidal path of the center of the generating pin  $C$ . A portion of the generated shape for internal contact and external contact is also shown.

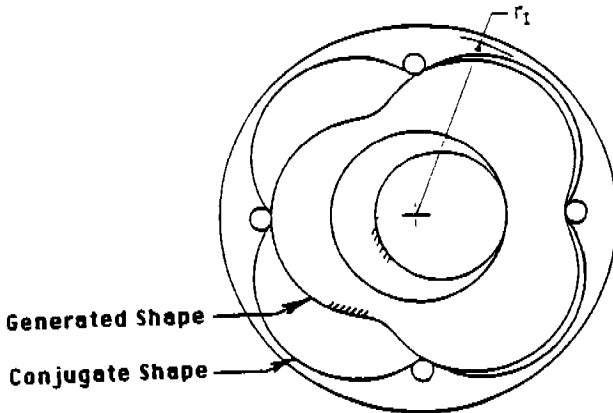


Fig. 2a. Containment Radius for Internal Contact.

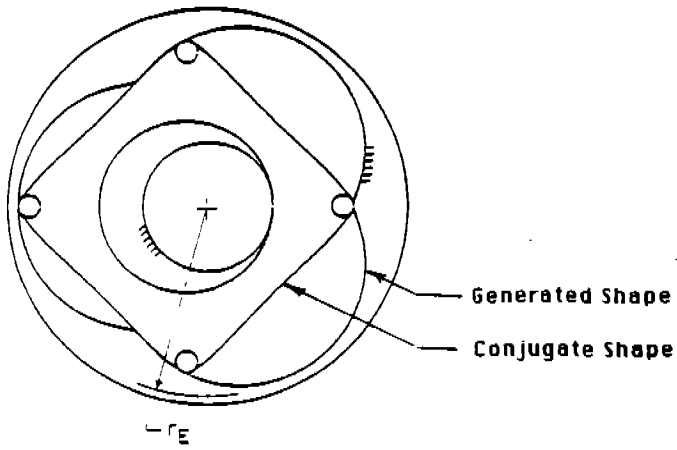


Fig. 2b. Containment Radius for External Contact.

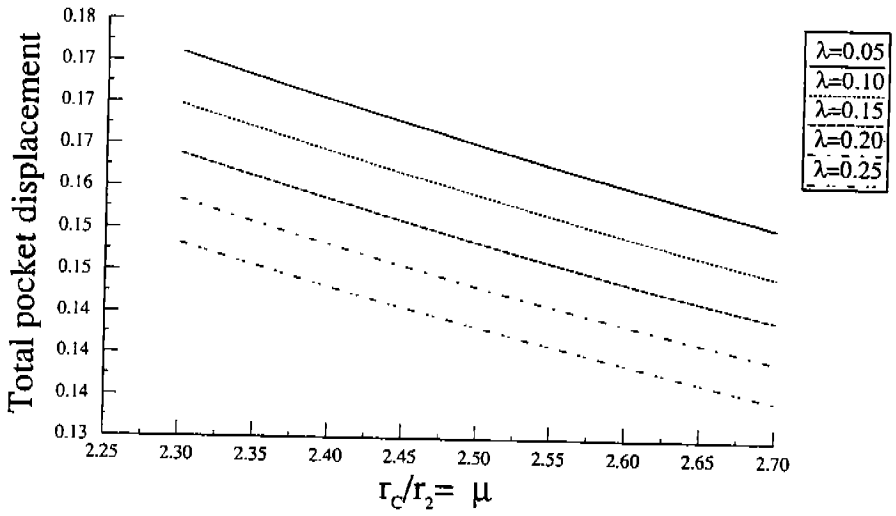


Fig. 3. Total relative pocket displacement for  $T = 3$ , external contact.



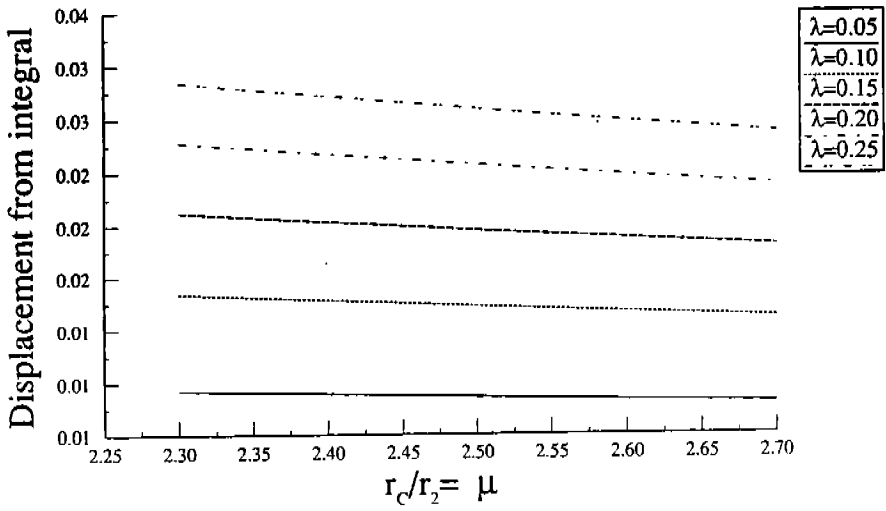


Fig. 4. The relative pocket displacement attributed to  $\lambda$  times the elliptical integral.

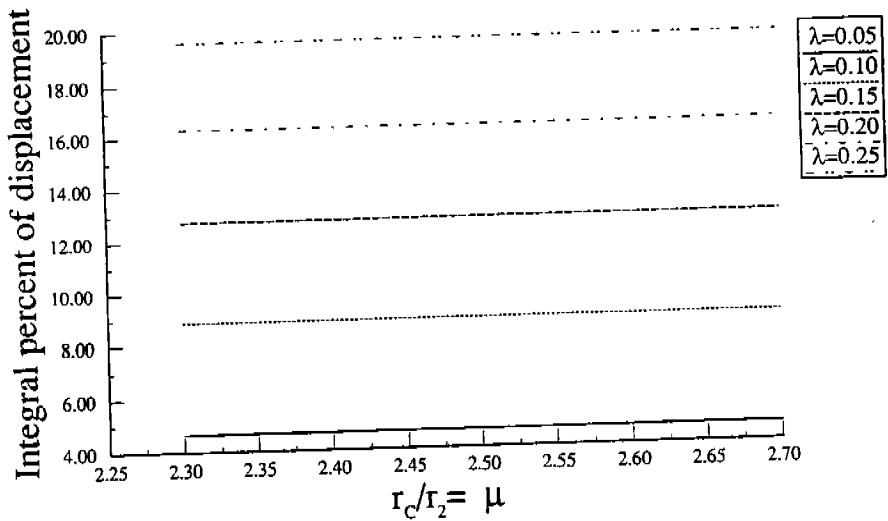


Fig. 5. The percent of the total displacement attributed to  $\lambda$  times the elliptical integral.

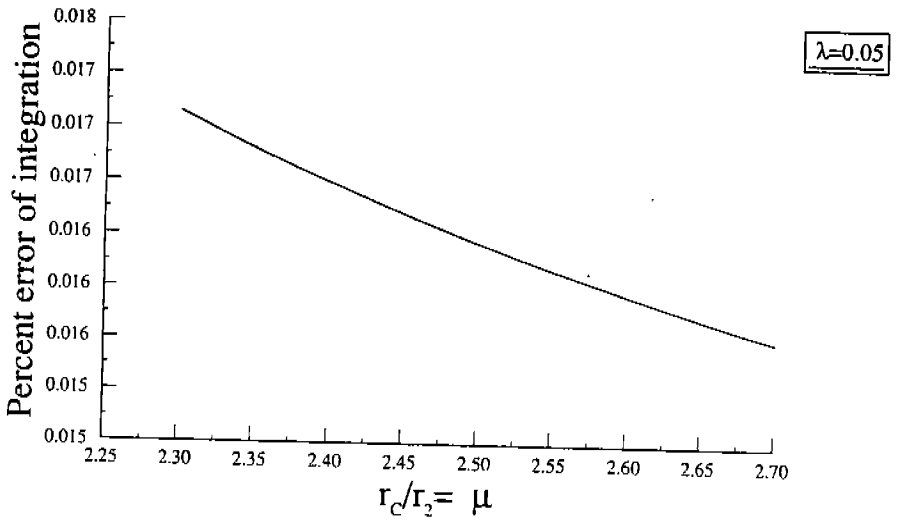


Fig. 6a. The upper bounds of the percent error associated with the numerical method used to evaluate the integral.

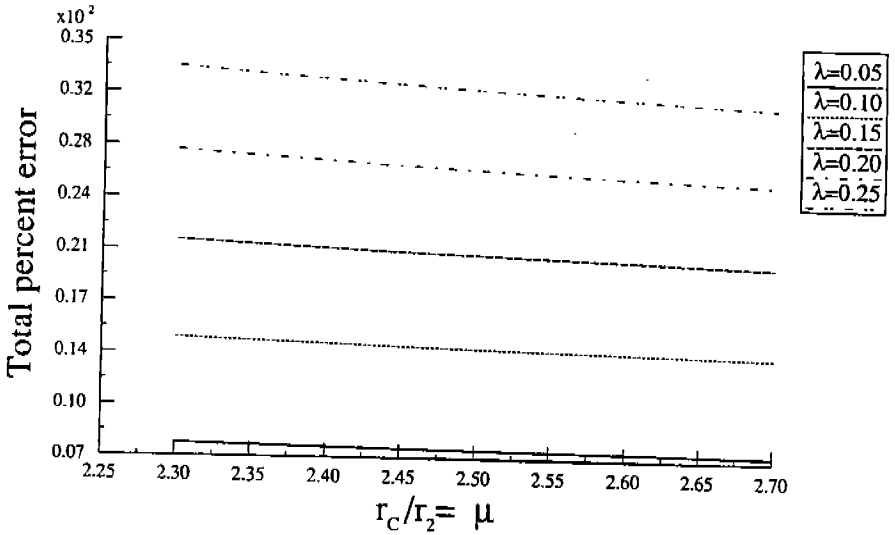


Fig. 6b. The total percentage error in the displacement calculations attributed to the numerical technique used to evaluate the integral.