1989

An O(h^6) Quintic Spline Collocation Method for Second Order Two-Point Boundary Value Problems

M. Irodotou-Ellina

Elias N. Houstis
Purdue University, enh@cs.purdue.edu

S. B. Kim

Report Number:

89-904

AN O(H6) QUINTIC SPLINE COLLOCATION METHOD FOR SECOND ORDER TWO-POINT BOUNDARY VALUE PROBLEMS

M. Irodotou-Ellina
E. H. Houstis
S. B. Kim

CSD-TR-904
August 1989

AN $O(h^6)$ QUINTIC SPLINE COLLOCATION METHOD FOR SECOND ORDER TWO-POINT BOUNDARY VALUE PROBLEMS

M. Irodotou-Ellina
University of Thessaloniki
Department of Mathematics
Thessaloniki, Greece

and

E.N. Houstis and S.B. Kim*
Department of Computer Science
Purdue University
W. Lafayette, IN 47907
U.S.A.

CSD-TR 904

Abstract

Collocation methods based on quintic splines are formulated and analyzed for the second-order two-point boundary value problems with mixed boundary conditions. These methods determine quintic spline approximation to the solution of the boundary value problem, by forcing the approximating solution to satisfy the given operator equation, or a perturbed one at the nodes, the boundary conditions and auxiliary end conditions. The methods that are based on the initial operator equation produce not optimal approximation, as compared to the corresponding interpolation procedures. This paper derives appropriate perturbations of the initial differential equation, such that the application of the collocation procedure leads to optimal approximating schemes. The theoretical behavior of the method has been verified numerically on a variety of benchmark problems found in the literature.

1. INTRODUCTION

In this paper we consider a collocation of the solution u of the second order two-point boundary problem

$$Lu \equiv D^2u(s) + p(s)\,Du(s) + q(s)\,u(s) = f(s)\,, \quad a \le s \le b\,, \tag{1.1a}$$

subject to boundary conditions

$$Bu \equiv \sum_{j=0}^{1} (a_{ij}\,D^j u(a) + b_{ij}\,D^j u(b)) = g_i\,, \quad i = 0, 1 \tag{1.1b}$$

Specifically, quintic spline approximation of u is determined using the method of collocation. The standard formulation of the method leads to a fourth order convergence which is not optimal as compared to implementation with quintic splines. In this paper we formulate optimal methods by applying the collocation method to a perturbed L operator or right hand side f. Several authors [1,2,4,6,9] have considered spline collocation methods for approximating the problem (1). Optimal spline collocation schemes are introduced in [2,4] for cubic splines and [6] for quadratic splines. The method considered here can be formulated for nonlinear problems and two dimensional elliptic problems [5]. The organization of the paper is as follows. Section 2 presents some preliminary interpolating results used to formulate and analyze the convergence of the methods presented. Section 3 presents the formulation of optimal quintic spline collocation methods. The convergence analysis of the method is discussed in Section 4. Finally Section 5 presents the results of some numerical experiments used to verify the theoretical convergence of the method.

2. QUINTIC SPLINE INTERPOLATION RESULTS

In this section we present the error analysis of a special quintic spline interpolant w and derive several asymptotic relations to be used for the formulation of the optimal quintic spline collocation method. For this we consider w to be an element of $Sp_5(\Delta_N) \equiv \{u \mid u \in C^4[a, b]$ and u is a polynomial of degree at most 5 on each subinterval of the partition $\Delta_N\}$ where $\Delta_N \equiv \{a = s_0 \le s_1 \le \cdots \le s_N = b,$ $h = s_i - s_{i-1},\ 1 \le i \le N\}$ is the uniform partition of the interval $[a, b]$. Throughout, we denote by $\{B_k\}$ the set of B-splines [1] for $Sp_5(\Delta_n)$ and define $w(s) = \sum_{k=i}^{i+5} a_k\, B_k(s)$, for $s \in [s_i, s_{i+1}]$ to be the quintic spline interpolant of u in $C^{10}[a, b]$, satisfying

(a) the interpolation conditions:

$$w(s_i) = u(s_i) \ \text{ for } \ 0 \le i \le N \tag{2.1a}$$

and

(b) for $i = 0, 1, N-1, N$ the end conditions:

$$w''''(s_i) = u''''(s_i) - u^{(6)}(s_i)\,\frac{h^2}{12} + u^{(8)}(s_i)\,\frac{h^4}{240}\,, \tag{2.1b}$$

$$w'''(s_i) = u'''(s_i) - u^{(7)}(s_i)\,\frac{h^4}{240}\,, \tag{2.1c}$$

$$w''(s_i) = u''(s_i) + u^{(6)}(s_i)\,\frac{h^4}{720}\,. \tag{2.1d}$$

Let's denote $w_i \equiv w(s_i)$ and $w_i^{(p)} \equiv w^{(p)}(s_i)$ for all p where $g^{(p)} \equiv D^p g$. Further we define Λ by $\Lambda g_i \equiv g_{i-2} + 26g_{i-1} + 66g_i + 26g_{i+1} + g_{i+2}$ for any function g evaluated at the nodes of partition Δ_N. Then we have the following recursive relations connecting w and its derivatives [3]:

$$\Lambda w' = \frac{120}{24h}\,(-\,w_{i-2} - 10w_{i-1} + 10w_{i+1} + w_{i+2})\,, \tag{2.2a}$$

$$\Lambda w'' = \frac{120}{6h^2}\,(w_{i-2} + 2w_{i-1} - 6w_i + 2w_{i+1} + w_{i+2})\,, \tag{2.2b}$$

$$\Lambda w''' = \frac{120}{2h^3}\,(-\,w_{i-2} + 2w_{i-1} - 2w_{i+1} + w_{i+2})\,, \tag{2.2c}$$

$$\Lambda w'''' = \frac{120}{h^4}\,(w_{i-2} - 4w_{i-1} + 6w_i - 4w_{i+1} + w_{i+2})\,, \tag{2.2d}$$

for $i = 2(1)\,N-2$. Since w interpolates u, after expanding in Taylor series, we obtain the following relations.

Lemma 2.1. If $u \in C^{10}[a,\ b]$ and w is the quintic-spline interpolant of u, defined by (2.1) then we have

$$\Lambda w' = 120u_i' + 30u_i^{(3)}\,h^2 + \frac{7}{2}\,u_i^{(5)}\,h^4 + O(h^6)\,, \tag{2.3a}$$

$$\Lambda w'' = 120u_i'' + 30u_i^{(4)}\,h^2 + \frac{11}{3}\,u_i^{(6)}\,h^4 + O(h^6)\,, \tag{2.3b}$$

$$\Lambda\, w''' = 120u_i''' + 30u_i^{(5)}\,h^2 + 3u_i^{(7)}\,h^4 + O(h^6)\,, \tag{2.3c}$$

$$\Lambda w'''' = 120u_i'''' + 20u_i^{(6)}\,h^2 + \frac{3}{2}\,u_i^{(8)}\,h^4 + O(h^6)\,. \tag{2.3d}$$

With the above relations as a basis, we prove the following.

Theorem 2.1. Let w be the unique quintic spline satisfying equations (2.1) for a given function $u \in C^{10}[a,\ b]$. Then for uniform partitions we have for $i = 0(1)N$

$$w_i' = u_i' + O(h^6)\,, \tag{2.4a}$$

$$w_i'' = u_i'' + \frac{h^4}{720}\, u_i^{(6)} + O(h^6)\,, \tag{2.4b}$$

$$w_i''' = u_i''' - \frac{h^4}{240}\, u_i^{(7)} + O(h^6)\,, \tag{2.4c}$$

$$w_i'''' = w_i'''' - \frac{h^2}{12}\, u_i^{(6)} + \frac{h^4}{240}\, u_i^{(8)} + O(h^6)\,. \tag{2.4d}$$

Furthermore, the following interpolating error estimates hold

$$||(w-u)^{(m)}||_\infty = O(h^{6-m}), \text{ for } m = 0(1)4\,.$$

Proof: First we prove the relation (2.4d). For this we consider any function $g \in C^6[a, b]$ and easily show that $\Lambda g_i = 120 g_i + 30 g_i''\, h^2 + \frac{7}{2}\, g_i''''\, h^4 + O(h^6)$. Letting $g = u'''' - \frac{h^2}{12}\, u^{(6)} + \frac{h^4}{240}\, u^{(8)}$ we find

$$\Lambda\left[\, u_i'''' - \frac{h^2}{12}\, u_i^{(6)} + \frac{h^4}{240}\, u_i^{(8)} \right] = 120 u_i^{(4)} + 20 u_i^{(6)}\, h^2 + \frac{3}{2}\, u_i^{(8)}\, h^4 + O(h^6). \tag{2.5}$$

If we define $d_i \equiv w_i'''' - u_i'''' + \frac{h^2}{12}\, u_i^{(6)} - \frac{h^4}{240}\, u_i^{(8)}$ and subtract equation (2.5) from equation (2.3d) we conclude that

$$\Lambda d_i = O(h^6 ||u^{(10)}||_\infty) \text{ for } 2 \le i \le N-2 \text{ and } d_0 = d_1 = d_{N-1} = d_N = 0\,. \tag{2.6}$$

Since the coefficient matrix of the equations (2.6) is diagonally dominant and its inverse has L_∞-norm bounded by 1/12, we have that $d_i = O(h^6)$ uniformly in i. This proves relation (2.4d). Equations (2.4b), (2.4c) can be proved following similar arguments. It remains to verify equation (2.4a). For this we observe that w_i' can be written in terms of w, w'' and w''' according to its definition. Specifically, for $i = 0(1)\, N-1$, we have

$$w_i' = (w_{i+1} - w_i)/h - h(42 w_i'' + 18 w_{i+1}'')/120 + h^2(2 w_{i+1}''' - 3 w_i''')/60\,,$$

and for $i = 1(1)N$

$$w_i' = (w_i - w_{i-1})/h + h(3w_{i-1}'' + 7w_i'')/20 + h^2(2w_{i-1}''' - 3w_i''')/60 .$$

After using the relations (2.1a), (2.4b) and (2.4c) for $w_i - u_i$, $w_i'' - u_i''$, $w_i''' - u_i'''$ and expanding in Taylor series, we obtain $w_i' = u_i' + O(h^6)$ for both of the above equations. It is known that the piecewise linear interpolant yields $O(h^2)$ accuracy and since w'''' is the piecewise linear interpolant of an $O(h^2)$ perturbation of u'''', it is clear that $\| w'''' - u'''' \|_\infty = O(h^2)$. Consequently $\| w''' - u''' \|_\infty = O(h^3)$, after integrating $w'''' - u''''$ from s_i to s for $s_i \le s \le s_{i+1}$ and taking the norms. Similarly, we conclude that $\| w'' - u'' \|_\infty = O(h^4)$, $\| w' - w' \|_\infty = O(h^5)$, and $\| w - u \|_\infty = O(h^6)$. This completes the proof of the Theorem. ■

For later use, we derive the approximation of $u^{(6)}$ by a linear combination of values of w''. We first define the difference operator δ such that $\delta w_i \equiv w_{i-1} - 2w_i + w_{i+1}$ and $\delta^2 w_i \equiv w_{i-2} - 4w_{i-1} + 6w_i - 4w_{i+1} + w_{i+2}$.

Corollary 2.1. Under the hypotheses of Theorem 2.1, we have

$$u_i^{(6)} = \delta^2 w_i''/h^4 + O(h^2) \text{ for } 2 \le i \le N-2 . \tag{2.7}$$

Proof: From the asymptotic relation (2.4b), we have

$$\delta^2 w_i''/h^4 = \delta^2 u_i''/h^4 + \frac{1}{720}\delta^2 u_i^{(6)} + O(h^2)$$

$$= u_i^{(6)} + \frac{1}{6} u_i^{(8)} h^2 + O(h^4) + \frac{1}{720} u_i^{(10)} h^4 + O(h^6) + O(h^2) = u_i^{(6)} + O(h^2) .$$

Corollary 2.2. If $u \in C^{10}[a, b]$, then the following approximations hold at the knots s_i,

$$\begin{aligned} u_i' &= w_i' + O(h^6) \text{ for } i = 1(1)N-1 \text{, and} \\ u_i'' &= w_i'' - \delta^2 w_i''/720 + O(h^6) \text{ for } 2 \le i \le N-2 . \end{aligned} \tag{2.8}$$

At the end points now, we have similar relations to (2.8).

Lemma 2.2. If $u \in C^{10}[a, b]$ then we have the following approximations to $u^{(6)}$ and $u^{(7)}$ at the end points. For $k = 6,7$

$$u_j^{(k)} = [(3-j)\delta^2 w_2^{(k-4)} - (2-j)\delta^2 w_3^{(k-4)}]/h^4 + O(h^2), \quad j = 0, 1 \text{ and}$$
$$u_\lambda^{(k)} = [(3-m)\delta^2 w_{N-2}^{(k-4)} - (2-m)\delta^2 w_{N-3}^{(k-4)}]/h^4 + O(h^2), (\lambda, m) = (N-1, 1), (N, 0) . \tag{2.9}$$

Proof: We approximate first $u_1^{(6)}$, $u_0^{(6)}$, $u_{N-1}^{(6)}$, $u_N^{(6)}$ by $2u_2^{(6)} - u_3^{(6)}$, $2u_1^{(6)} - u_2^{(6)}$, $2u_{N-2}^{(6)} - u_{N-3}^{(6)}$, $2u_{N-1}^{(6)} - u_{N-2}^{(6)}$ respectively and then use Corollary 2.1.

3. AN OPTIMAL QUINTIC SPLINE COLLOCATION METHOD

We consider now the linear second order equation

$$Lu = f \tag{3.1a}$$

subject to homogeneous boundary conditions

$$Bu = 0 . \tag{3.1b}$$

From now on, we assume that $u \in C^{10}[a, b]$ is the solution to the problem (3.1) and w is the quintic spline interpolant of u defined by (2.1). Based on relations (2.8) and (2.9), we observe that w satisfies,

$$w_0'' - (3\delta^2 w_2'' - 2\delta^2 w_3'')/720 + p_0 w_0' + q_0 w_0 = f_0 + O(h^6) \tag{3.2a}$$

$$w_1'' - (2\delta^2 w_2'' - \delta^2 w_3'')/720 + p_1 w_1' + q_1 w_1 = f_1 + O(h^6) \tag{3.2b}$$

$$w_i'' - \delta^2 w_i''/720 + p_i w_i' + q_i w_i = f_i + O(h^6) , \quad i = 2(1)N-2 \tag{3.2c}$$

$$w_{N-1}'' - (2\delta^2 w_{N-2}'' - \delta^2 w_{N-3}'')/720 + p_{N-1} w_{N-1}' + q_{N-1} w_{N-1} = f_{N-1} + O(h^6) \tag{3.2d}$$

$$w_N'' - (3\delta^2 w_{N-2}'' - 2\delta^2 w_{N-3}'')/720 + p_N w_N' + q_N w_N = f_N + O(h^6) \tag{3.2e}$$

$Bw \equiv a_{i0} w_0 + a_{i1} w_0' + b_{i0} w_N + b_{i1} w_N' = 0 \; i = 0, 1.$

For later use we need the following auxiliary boundary conditions:

$$Au \equiv h \frac{d}{ds}(Lu) = h \frac{df}{ds} \text{ at } s_i, \quad i = 0, N ,$$

and we observe that w satisfies the relations

$$A'w_0 = h[\, w_0''' + (3\delta^2 w_2''' - 2\delta^2 w_3''')/240 +$$
$$p_0[\, w_0'' - (3\delta^2 w_2'' - 2\delta^2 w_3'')/720] + (p_0' + q_0)\, w_0' + q_0'\, w_0] = hf_0' + O(h^6) , \tag{3.4}$$

and

$$A'w_N = h[\, w_N''' + (3\delta^2 w_{N-2}''' - 2\delta^2 w_{N-3}''')/240 +$$
$$p_N[\, w_N'' - (3\delta^2 w_{N-2}'' - 2\delta^2 w_{N-3}'')/720] + (p_N' + q_N)\, w_N' + q_N'\, w_N] = hf_N' + O(h^6) .$$

Let L' denote the perturbation of L defined by the left side of equations (3.2), specifically L' is defined by

$$L'g_i = Lg_i - \delta^2 g_i''/720 \text{ for } i = 2(1)N-2 .$$

So we have w satisfying the relations

$$L'\, w_i = f_i + O(h^6) \text{ for } 0 \le i \le N , \quad Bw = O(h^6) \tag{3.5a}$$

$$[A'\, w]_{s_i} = hf_{s_i}' + O(h^6) , \quad i = 0, N .$$

The above observations can be summarized in the following lemma.

Lemma 3.1. Let w be the quintic spline interpolant of the solution u to problem (3.1). If $u \in C^{10}[a, b]$, then w satisfies the relations

$$[Lw - f]_{s_i} = O(h^4) , \qquad Bw = O(h^6) , \tag{3.6a}$$

$$[Aw - hf']_{0,N} = O(h^4), \tag{3.6b}$$

and

$$[L'w - f]_{s_i} = O(h^6) , \quad Bw = O(h^6), \tag{3.6c}$$

$$[A'w - hf']_{0,N} = O(h^6) . \tag{3.6d}$$

3.1 Formulation of the quintic spline collocation method

In this method we determine a quintic spline $z \in Sp_5(\Delta_N)$ which satisfies the following equations,

$$[L'z - f]_{s_i} = 0 \ \text{for} \ 0 \le i \le N \ \text{and} \ Bz = 0 \tag{3.7a}$$

and

$$[A'z - hf']_{0,N} = 0 . \tag{3.7b}$$

Before we proceed to the main analysis of this method, we must verify the solvability of its equations (3.2) in the case that $p(s) \equiv q(s) \equiv 0$. For this we denote by Q the coefficient matrix of the system (3.2), where $p(s) \equiv q(s) \equiv 0$, and easily prove Lemma 3.2 and 3.3.

Lemma 3.2. If $p(s) \equiv q(s) \equiv 0$, then the system of equations (3.2) is uniquely solvable for $\{z_i''\}_0^N$, and the solution vector satisfies $\max\{|z_0''|, \dots, |z_N''|\} \le 1.022 \max\{|f_0|, \dots, |f_N|\}$. That is the matrix Q satisfies $||Q^{-1}|| \le 1.022$.

Lemma 3.3. If $p(s) \equiv q(s) \equiv 0$ and $Bz = z_0 = z_N = 0$, then the quintic spline $z \in Sp_5(\Delta_N)$ which satisfies equations (3.7), exists uniquely.

Proof: From Lemma 3.2 and equation (3.7), we observe that z satisfies the following collocation equations,

$$\mathbf{z} = (Q^{-1}\,\mathbf{f}) = \mathbf{g} \ \text{at} \ s_i, \quad i = 0(1)N$$

and specifically at the points s_0 and s_1, we have respectively

$$(a_0 + 2a_1 - 6a_2 + 2a_3 + a_4)/6h^2 = g_0 \tag{3.8a}$$

and

$$(a_1 + 2a_2 - 6a_3 + 2a_4 + a_5)/6h^2 = g_1 . \tag{3.8b}$$

The auxiliary equation (3.7b) estimated at s_0 becomes

$$(-243a_0 + 500a_1 - 54a_2 - 410a_3 + 212a_4 - 42a_5 + 70a_6 - 46a_7 + 15a_8 - 2a_9)/480h^2 = hf_0' . \tag{3.9}$$

Also the boundary condition at s_0 becomes

$$a_0 + 26a_1 + 66a_2 + 26a_3 + a_4 = 0$$

So we have that

$$a_0 = -26a_1 - 66a_2 - 26a_3 - a_4 . \tag{3.10}$$

Now if we replace a_0 in (3.9) and (3.8a) by the right-hand side of (3.10) we get the following new equations, respectively,

$$(6818a_1 + 15984a_2 + 5908a_3 + 455a_4 - 42a_5 + 70a_6 - 46a_7 + 15a_8 - 2a_9)/480h^2 = hf_0' \tag{3.11a}$$

and

$$(-24a_1 - 72a_2 - 24a_3)/6h^2 = g_0 . \tag{3.11b}$$

With the same way as we have done before, if we replace a_1 estimated from (3.11a) in (3.8b) and (3.11b), we observe that a_2 and a_3 respectively, dominate others. We can do the same with coefficients a_{N+3}, a_{N+4}. Thus, the system of equations (3.7) has a unique solution by the diagonal dominance property. ■

3.2 Formulation of a deferred correction quintic spline collocation method

Here we have an alternative formulation of the method, that is, we determine the z through a two step collocation method. We need equations (3.2) to be written in the following form:

$$Lw_0 = f_0 + (3\delta^2 w_2'' - 2\delta^2 w_3'')/720 + O(h^6) , \tag{3.12a}$$

$$Lw_1 = f_1 + (2\delta^2 w_2'' - \delta^2 w_3'')/720 + O(h^6) , \tag{3.12b}$$

$$Lw_i = f_i + \delta^2 w_i''/720 + O(h^6), \text{ for } 2 \le i \le N-2 , \tag{3.12c}$$

$$Lw_{N-1} = f_{N-1} + (2\delta^2 w_{N-2}'' - \delta^2 w_{N-3}'')/720 + O(h^6) , \tag{3.12d}$$

$$Lw_N = f_N + (3\delta^2 w_{N-2}'' - 2\delta^2 w_{N-3}'')/720 + O(h^6) . \tag{3.12e}$$

The Two-Step Collocation Method is defined as follows:

Step 1: Determine a $\bar{z} \in Sp_5(\Delta_N)$ such that it satisfies

$$[L\bar{z} - f]_i = 0 \text{ for } i = 0(1)N, \quad B\bar{z} = 0 \tag{3.13a}$$

and

$$[A\bar{z} - hf']_{0,N} = 0 . \tag{3.13b}$$

Step 2: Find a $\bar{\bar{z}} \in Sp_5(\Delta_N)$ such that it satisfies

$$[L\bar{\bar{z}} - \bar{f}] = 0 \text{ for } i = 0(1)N, \quad B\bar{\bar{z}} = 0 \tag{3.14a}$$

$$[A\bar{\bar{z}} - h\bar{f}']_{0,N} = 0 \tag{3.14b}$$

where $\bar{f}_i$, $i = 0(1)N$ and $\bar{f}_j'$, $j = 0, N$ are defined as follows:

$$\begin{aligned}
\bar{f}_0 &= f_0 + (3\delta^2 \bar{z}_2'' - 2\delta^2 \bar{z}_3'')/720 , \\
\bar{f}_1 &= f_1 + (2\delta^2 \bar{z}_2'' - \delta^2 \bar{z}_3'')/720 , \\
\bar{f}_i &= f_i + \delta^2 \bar{z}_i''/720 , \text{ for } i = 2(1)N-2 , \\
\bar{f}_{N-1} &= f_{N-1} + (2\delta^2 \bar{z}_{N-2}'' - \delta^2 \bar{z}_{N-3}'')/720 , \\
\bar{f}_N &= f_N + (3\delta^2 \bar{z}_{N-2}'' - 2\delta^2 \bar{z}_{N-3}'')/720 , \\
\bar{f}_0' &= f_0' - [3\delta^2 \bar{z}_2''' - 2\delta^2 \bar{z}_3''']/240 + [3\delta^2 \bar{z}_2'' - 2\delta^2 \bar{z}_3'']/720 , \\
\bar{f}_N' &= f_N' - [3\delta^2 \bar{z}_{N-2}''' - 2\delta^2 \bar{z}_{N-3}''']/240 + [3\delta^2 \bar{z}_{N-2}'' - 2\delta^2 \bar{z}_{N-3}'']/720 .
\end{aligned} \tag{3.15}$$

In the above formulation, we have assumed that the $u^{(6)}$ and $u^{(7)}$ can be estimated at $\{s_i\}_0^N$ by

$$u_i^{(6)} = \delta^2 \bar{z}_i''/h^4 + O(h^2) \text{ for } i = 2(1)N-2 \tag{3.16}$$

and for $k = 6, 7$

$$u_j^{(k)} = [(3-j)\delta^2\, \overline{z}_2^{(k-4)} - (2-j)\delta^2\, \overline{z}_3^{(k-4)}]/h^4 + O(h^2), \quad j = 0, 1$$

$$u_\lambda^{(k)} = [(3-m)\delta^2\, \overline{z}_{N-2}^{(k-4)} - (2-m)\delta^2\, \overline{z}_{N-3}^{(k-4)}]/h^4 + O(h^2), \quad (\lambda, m) = (N-1, 1), (N, 0) .$$

Lemma 3.4. The systems of equations (3.13) and (3.14) have a unique solution if $q(s) < 0$ and $\overline{z}$, $\overline{\overline{z}}$ satisfy Dirichlet boundary conditions.

Proof: The proof is given in [1].

4. CONVERGENCE ANALYSIS AND ERROR BOUNDS

Before we proceed to analyze the quintic spline collocation method, we had to introduce some notation in order to represent the equations (3.1), (3.7a) and (3.14a) in an integral form. First we assume that the boundary value problem $u'' = 0$, $Bu = 0$ is uniquely solvable. This means that there is a Green's function $G(s, t)$ for this problem. If we denote $u'' \equiv \nu$, $z'' \equiv \nu_N$, $\overline{z}'' \equiv \mu_N$ and $\overline{\overline{z}}'' \equiv \xi_N$ and assume that ν, ν_N, μ_N and ξ_N satisfy the boundary conditions, then u, z, $\overline{z}$ and $\overline{\overline{z}}$ can be obtained via the Green's function. We have that

$$u^{(m)}(s) = \int_a^b \frac{\partial^m G(s,t)}{\partial s^m}\, \nu(t)\, dt\ , \quad z^{(m)}s = \int_a^b \frac{\partial^m G(s,t)}{\partial s^m}\, \nu_N(t)\, dt\ ,$$

$$\overline{z}^{(m)}(s) = \int_a^b \frac{\partial^m G(s,t)}{\partial s^m}\, \mu_N(t)\, dt\ , \quad \overline{\overline{z}}^{(m)}(s) = \int_a^b \frac{\partial^m G(s,t)}{\partial s^m}\, \xi_N(t)\, dt\ ,$$

for $m = 0, 1$.

Now we define the following operators:

D_N: $C[a, b] \to R^{N+1}$, $(D_N g)_i = g(s_i)$ for $0 \le i \le N$,

M_N: $R^{N+1} \to C[a, b]$ via piecewise linear interpolation at $\{s_i\}_0^N$,

K: $C[a, b] \to C[a, b]$ such that $Kg(s) = p(s)\int_a^b \frac{\partial G(s,t)}{\partial s}\, g(t)dt + q(s)\int_a^b G(s, t)g(t)\, dt.$

4.1 Convergence analysis of quintic spline collocation method

With the notations we discussed in the previous section, we observe that equations (3.1) and (3.7a) can be written in the form

$$(I + K)v = f \tag{4.1a}$$

and

$$QD_N\, v_N + D_N\, Kv_N = D_N f \tag{4.1b}$$

respectively, recalling that Q is the matrix of the system (3.2) where $p(s) \equiv q(s) \equiv 0$. Because of that Q is non-singular then equation (4.1b) can be written equivalently as

$$(I + P_N K)v_N = P_N f\,, \tag{4.2}$$

where $P_N \equiv M_N\, Q^{-1}\, D_N$ is an operator that maps $C[a,\, b]$ onto the continuous piecewise linear functions with breakpoints $\{s_i\}_0^N$, and we have from [2] that the sequence of operator P_N converges strongly to the identity operator I: $C[a,\, b] \rightarrow C[a,\, b]$, that is $||P_N g - g||_\infty \rightarrow 0$ for each fixed $g \in C[a,\, b]$.

We present now the main convergence theorem.

Theorem 4.1. If we assume that

(a1) the coefficients p, q and f are continuous in $I = [a,\, b]$,
(a2) the boundary value problem (3.1) has a unique solution u in $C^{10}[a,\, b]$,
(a3) the problem $u'' = 0$, $Bu = 0$ is uniquely solvable,

then

(r1) the collocation approximation $z \in Sp_5(\Delta_N)$ defined by equation (3.7a) exists,
(r2) we have the global error estimates $||(u - z)^{(m)}||_\infty = O(h^{6-m})$, $m = 0,\, 1,\, 2$,
(r3) we have the local estimates $|(u - z)''_{s_i}| = O(h^4)$, $|(u - z)'_{s_i}| = O(h^6)$ and $|(u - z)_{s_i}| = O(h^6)$.

Proof: From the assumptions (a3) and (a2), we have that $(I + K)^{-1}$ exists and it is a bounded linear operator. So using that $P_N K \rightarrow K$, we conclude that $(I + P_N K)^{-1}$ exists and it is bounded. This proves (r1). Now we consider the problem $w'' = r_N$, $Bw = O(h^6)$. From (a3) it follows that there is a linear function c such that

$$Bc = Bw = O(h^6)\,, \quad ||c^{(k)}||_\infty = O(h^6)\,, \quad k = 0,\, 1\,. \tag{4.3}$$

From the solvability of $(w - c)'' = r_N$, $B(w - c) = 0$, from (a3) and equations (3.6c), we have that

$$(I + P_N K)(w'' - c'') = P_N f + O(h^6)\,. \tag{4.4}$$

Subtracting (4.4) and (4.2), we have

$$(I + P_N K)(w'' - c'' - z'') = O(h^6)\,. \tag{4.5}$$

Because of the boundedness of $(I + P_N K)^{-1}$ and from (4.5) we conclude that

$$||w'' - c'' - z''||_\infty = O(h^6)\,. \tag{4.6}$$

Since $(w - c - z)'' = \theta_N$, $B(w - c - z) = 0$ is uniquely solvable (a3), we have

$$(w - c - z)(s) = \int_a^b G(s,\, t)(w'' - c'' - z'')(t)\, dt\,. \tag{4.7}$$

Using equations (4.6) and (4.7) we obtain

$$||(w - c - z)'||_\infty = O(h^6) \text{ and } ||(w - c - z)||_\infty = O(h^6)\,, \tag{4.8}$$

and thus the proof is completed from relations (4.3), (4.7), (4.8) and Theorem 2.1. ∎

4.2 Convergence analysis and error bounds for the two-step collocation method

This method was defined in Section 3.2. With the notations of Section 4 as a basis, we can write the equations (3.13a) and (3.14a) in the form

$$(I + P_N K)\mu_N = P_N f \tag{4.9}$$

$$(I + P_N K)\xi_N = P_N \bar{f} \tag{4.10}$$

respectively.

First we proceed to present the convergence of Step 1.

Theorem 4.2. Under the assumptions of Theorem 4.1, we have

(r1) the collocation approximation $\bar{z} \in Sp_5(\Delta_N)$ defined by equation (3.13a) exists,
(r2) we have the global error estimates $\| (u - \bar{z})^{(m)} \|_\infty = O(h^4) \quad m = 0, 1, 2,$
(r3) we have the local error estimates $| (u - \bar{z})_i^{(m)} | = O(h^4) \quad m = 0, 1, 2.$

Proof: The result (r1) is a direct consequence of (a3) and the uniform boundedness of $(I + P_N K)^{-1}$. Considering the problem $w'' = r_N\ Bw = O(h^6)$, from (a3) it follows that there is a linear function g such that

$$Bg = Bw = O(h^6)\,, \quad \| g \|_\infty = O(h^6) \text{ and } \| g' \|_\infty = O(h^6)\,. \tag{4.11}$$

From the solvability of $(w - g)'' = r_N, B(w - g) = 0$ and the equation (3.6a), we conclude that

$$(I + P_N K)(w'' - g'') = P_N f + O(h^4)\,. \tag{4.12}$$

Subtracting (4.12) and (4.9), we have

$$(I + P_N K)(w'' - g'' - \bar{z}'') = O(h^4)\,. \tag{4.13}$$

This implies that

$$\| w'' - g'' - \bar{z}'' \|_\infty = O(h^4)$$

Since $(w - g - \bar{z})'' = \sigma_N, B(w - g - \bar{z}) = 0$ is uniquely solvable, we have

$$(w - g - \bar{z})(s) = \int_a^b G(s, t)(w'' - g'' - \bar{z}'')(t)\, dt$$

so

$$\| w' - g' - \bar{z}' \|_\infty = O(h^4)\,, \quad \| w - g - \bar{z} \|_\infty = O(h^4)\,. \tag{4.14}$$

Thus, (r2) and (r3) follow from Theorem 2.1 and equations (4.11). This completes the proof of the theorem. ■

From equations (2.7), (2.9) and (3.16), we have $\delta^2 w_i'' = \delta^2 \bar{z}_i'' + O(h^6)$. So equations (3.6a) become

$$[Lw - \bar{f}]_{s_i} = O(h^6)\,,\ \ 0 \le i \le N \ \text{ and } \ Bw = O(h^6)\,. \tag{4.15}$$

If we subtract relations (3.14a) and (4.15), we obtain

$$L(w - \bar{\bar{z}})_{s_i} = O(h^6)\,,\ \ 0 \le i \le N \ \text{ and } \ B(w - \bar{\bar{z}}) = O(h^6)\,. \tag{4.16}$$

Now we consider the problem $(w - \bar{\bar{z}})'' = \theta_N$, $B(w - \bar{\bar{z}}) = O(h^6)$. Notice that there is a linear function g such that

$$B(w - \bar{\bar{z}}) = Bg = O(h^6)\,,\ \ ||g||_\infty = O(h^6) \ \text{ and } \ ||g'||_\infty = O(h^6)\,. \tag{4.17}$$

From (a3) and $B(w - \bar{\bar{z}} - g) = 0$ we can write equations (4.16) as $(I + P_N K)(w'' - \bar{\bar{z}}'' - g'') = O(h^6)$ and this implies

$$||w'' - \bar{\bar{z}}'' - g''||_\infty = O(h^6)\,.$$

Continuing as in Theorem 4.2, we have the following results.

Theorem 4.3. Under the hypotheses of Theorem 4.1, we conclude that

(r1) the collocation approximation $\bar{\bar{z}} \in Sp_5(\Delta_N)$ exists,
(r2) $||(u - \bar{\bar{z}})^{(m)}||_\infty = O(h^{6-m}) \quad m = 0, 1, 2,$
(r3) $|(u - \bar{\bar{z}})''_{s_i}| = O(h^4)$, $\ |(u - \bar{\bar{z}})'_{s_i}| = O(h^6)$ and $|(u - \bar{\bar{z}}_{s_i}| = O(h^6)$.

5. Numerical Results

In this section, we present a number of numerical results to verify the theoretical convergence of the quantic-spline collocation method introduced in Section 3. The implementation of the one step formulation of the method in FORTRAN is referred as P5C4COL. The fourth order method based on the straightforward application of collocation criterion is referred by P5C4COL (order=4) and the sixth order one that corresponds to extrapolated formulation is denoted by P5C4COL (order=6). We have chosen the same problems as in [Hous 88] to verify the theoretical behavior of the method. Some of these examples are used in References X and Y in order to allow comparison with other collocation methods. All computations were carried out on a SEQUENT-SYMMETRY system in double precision. For problem 2, we present some data for the Galerkin and collocation methods, based on quadratic splines as implemented in the program (P2C1GAL and P2C1COL). The data indicate complete agreement between the analytical and numerical behavior of the method.

Problem 1

This example is chosen to test convergence of P5C4COL (order=6) for various smoothness assumptions on u.

$$u''(x) + \left[\frac{16x}{1+4x^2} \right] u'(x) + \frac{8}{1+4x^2}\, u(x) = f \quad \text{for } 0 < x < 1$$

subject to boundary conditions

$$A_0 u(a) + B_0 u'(a) = g_0 \quad \text{and} \quad A_1 u(b) + B_1 u'(b) = g_1$$

The functions f, g_0 and g_1 are chosen so that $u(x) = x^{\alpha/2}$. Three values of α, 13, 11 and 9 are used which put $u(x)$ in $C^{6.5}$, $C^{5.5}$, $C^{4.5}$, respectively. We present tables of the norms of the observed errors for $n = 8$ to 256 kpoints in the partition Δ (see Tables I, III and V). From these we derive estimates of the orders of convergence which are shown in Tables II, IV and VI.

Table I. Errors of P4C4COL (order=6) for the case $\alpha = 13$ with $A_0 = A_1 = 1$ and $B_0 = B_1 = 0$.

n	$\|\|u - u_\Delta\|\|_\infty$	$\|\|u' - u'_\Delta\|\|_\infty$	$\|\|u'' - u''_\Delta\|\|_\infty$
8	.12D–05	.14D–04	.73D–03
16	.23D–07	.43D–06	.45D–04
32	.39D–09	.13D–07	.28D–05
64	.64D–11	.41D–09	.17D–06
128	.93D–13	.13D–10	.11D–07
256	.40D–13	.31D–12	.68D–09

Table II. Estimated orders of convergence of the P5C4COL (order=6) for $\alpha = 13$ (Table I).

n	u_Δ global convergence rate	u'_Δ global convergence rate	u''_Δ global convergence rate
8			
	5.73	5.00	4.02
16			
	5.87	5.02	4.00
32			
	5.92	5.01	4.00
64			
	6.11	5.02	4.00
128			
	1.21	5.32	4.00
256			

Table III. Errors of P5C4COL (order=6) for the case $\alpha = 11$.

n	$\|\|u - u_\Delta\|\|_\infty$	$\|\|u' - u'_\Delta\|\|_\infty$	$\|\|u'' - u''_\Delta\|\|_\infty$
8	.44D–06	.92D–05	.33D–03
16	.11D–07	.43D–06	.29D–04
32	.24D–09	.20D–07	.26D–05
64	.54D–11	.84D–09	.23D–06
128	.12D–12	.37D–10	.20D–07
256	.51D–13	.12D–11	.18D–08

Table IV. Estimated orders of convergence of P5C4COL (order=6) for $\alpha = 11$ (Table III).

n	u_Δ global convergence rate	u'_Δ global convergence rate	u''_Δ global convergence rate
8			
	5.38	4.41	3.49
16			
	5.45	4.46	3.50
32			
	5.50	4.53	3.50
64			
	5.42	4.52	3.50
128			
	1.28	4.90	3.50
256			

Table V. Errors of P5C4COL (order=6) for the case $\alpha = 9$ with $A_0 = A_1 = 1, B_0 = B_1 = 0$.

n	$\|\|u - u_\Delta\|\|_\infty$	$\|\|u' - u'_\Delta\|\|_\infty$	$\|\|u'' - u''_\Delta\|\|_\infty$
8	.21D–05	.44D–04	.10D–02
16	.10D–06	.40D–05	.18D–03
32	.45D–08	.36D–06	.32D–04
64	.20D–09	.32D–07	.57D–05
128	.87D–11	.28D–08	.10D–05
256	.38D–12	.25D–09	.18D–06

Table VI. Estimated orders of convergence of P5C4COL (order=6) for $\alpha = 9$ (Table V).

n	u_Δ global convergence rate	u'_Δ global convergence rate	u''_Δ global convergence rate
8			
	4.39	3.43	2.50
16			
	4.47	3.48	2.50
32			
	4.49	3.51	2.50
64			
	4.54	3.50	2.50
128			
	4.50	3.50	2.50
256			

We now consider another version of Problem 1 where f, g_0 and g_1 are chosen to make $u(x) = 1/(1+4x^2)$. This should give the highest possible order of convergence and, as the results of Table VII show, the observed errors are smaller. The estimated orders of convergence seen in Table VIII are as predicted by Theorem 4.1 and essentially the same as in Table II.

Table VII. Errors of P5C4COL (order=6) for Problem 1 with Dirichlet boundary conditions ($A_0 = A_1 = 1, B_0 = B_1 = 0$), and $u(x) = 1/(1+4x^2)$.

n	$\|\|u - u_\Delta\|\|_\infty$	$\|\|u' - u'_\Delta\|\|_\infty$	$\|\|u'' - u''_\Delta\|\|_\infty$
8	.11D–03	.23D–02	.40D–01
16	.12D–05	.21D–04	.92D–03
32	.33D–07	.58D–06	.80D–04
64	.47D–09	.11D–07	.43D–05
128	.68D–11	.29D–09	.25D–06
256	.17D–12	.12D–11	.15D–07

Table VIII. Estimated orders of convergence of P5C4COL (order=6) for the case of $u = 1/(1+4x^2)$ (Table VII).

n	u_Δ global convergence rate	u'_Δ global convergence rate	u''_Δ global convergence rate
8			
	6.49	6.79	5.44
16			
	5.19	5.16	3.53
32			
	6.15	5.70	4.20
64			
	6.10	5.25	4.13
128			
	5.30	7.95	4.04
256			

Problem 2

This is a trivial second order problem used very often for verifying the convergence of various numerical methods, the equation is

$$u'' - 4u = 4\cosh(1)$$

subject to boundary conditions $u(0) = u(1) = 0$. It has the true solution

$$u(x) = \cosh(2x - 1) - \cosh(1)$$

The computational results (Tables IX,X) show almost exact agreement with the orders of convergence predicted by Theorem 4.1.

We also solved this problem using the program P2C1GAL which implements the Galerkin method using quadratic-spline basis functions. The results shown in Tables XIII and XIV indicate the rates of convergence expected for such a method. Comparing with Table IX and X for P2C1COL (order=4), we see that the quadratic spline collocation method is slightly more accurate and they both exhibit the same order of convergence. The collocation method is more general as it does not require a self-adjoint operator.

Table IX. Errors of P5C4COL (order=6) for Problem 2.

n	$\|\|u - u_\Delta\|\|_\infty$	$\|\|u' - u'_\Delta\|\|_\infty$	$\|\|u'' - u''_\Delta\|\|_\infty$
8	.28D–07	.66D–06	.29D–04
16	.68D–09	.20D–07	.20D–05
32	.12D–10	.63D–09	.13D–06
64	.20D–12	.19D–10	.82D–08
128	.62D–13	.80D–12	.51D–09
256	.19D–12	.73D–12	.34D–10

Table X. Estimated orders of convergence for P5C4COL (order=6) for Problem 2 (Table IX).

n	u_Δ global convergence rate	u'_Δ global convergence rate	u''_Δ global convergence rate
8			
	5.39	5.02	3.87
16			
	5.86	5.03	3.96
32			
	5.87	5.02	3.99
64			
	1.68	4.59	4.00
128			
	−1.60	.13	4.90
256			

Table XI. Errors of P2C1COL (order=4) for Problem 2.

n	$\|\|u - u_\Delta\|\|_\infty$	$\|\|u' - u'_\Delta\|\|_\infty$	$\|\|u'' - u''_\Delta\|\|_\infty$
8	.15D–03	.71D–02	.46D–00
16	.18D–04	.18D–02	.23D–00
32	.21D–05	.50D–03	.12D–00
64	.27D–06	.12D–03	.55D–01
128	.34D–07	.29D–01	.28D–01
256	.38D–08	.70D–05	.15D–01

Table XII. Estimated orders of convergence of P2C1COL (order=4) for Problem 2 (Table XI).

n	u_Δ global convergence rate	u'_Δ global convergence rate	u''_Δ global convergence rate
8			
	3.13	1.97	1.01
16			
	3.10	1.86	0.90
32			
	2.96	2.11	1.18
64			
	2.98	1.99	0.95
128			
	3.13	2.07	0.92
256			

Table XIII. Errors for quadratic spline galerkin (P2C1GAL) for Problem 2.

n	$\|\|u - u_\Delta\|\|_\infty$	$\|\|u' - u'_\Delta\|\|_\infty$	$\|\|u'' - u''_\Delta\|\|_\infty$
8	.14D–03	.12D–01	.49D–00
16	.18D–04	.30D–02	.26D–00
32	.23D–05	.76D–03	.13D–00
64	.29D–06	.19D–03	.66D–00
128	.36D–07	.48D–04	.33D–01

Table XIV. Estimated orders of convergence for P2C1GAL for Problem 2 (Table XIII).

n	u_Δ global convergence rate	u_Δ' global convergence rate	u_Δ'' global convergence rate
8			
	2.98	1.96	0.95
16			
	2.99	1.98	0.97
32			
	3.00	1.99	0.99
64			
	3.00	2.00	0.99
128			

Problem 3

This example was considered in Reference [10]. The equation and boundary conditions are

$$u'' + xu'(x) - u(x) = xe^x + |x|(3x^3 - 2x^2 + 12x - 6)$$
$$u(-1) = e^{-1} - 2, \quad u(1) = e$$

which has the unique solution

$$u(x) = \begin{cases} e^x - x^3 + x^4 & x \geq 0 \\ e^x + x^3 - x^4 & x \leq 0 \end{cases}$$

The derivatives of order three and four of $u(x)$ have jump discontinuities at the origin. Results are shown in Tables (XV–XIII).

Table XV. Errors of P5C4COL (order=6) for Problem 3. The mesh includes the discontinuity point.

n	$\|\|u - u_\Delta\|\|_\infty$	$\|\|u' - u'_\Delta\|\|_\infty$	$\|\|u'' - u''_\Delta\|\|_\infty$
8	.20D–01	.38D–01	.31D+00
16	.51D–02	.98D–02	.14D+00
32	.13D–02	.25D–02	.68D–01
64	.33D–03	.62D–03	.33D–01
128	.83D–04	.16D–03	.15D–01
256	.21D–04	.39D–04	.76D–02

Table XVI. Estimated orders of convergence of P5C4COL (order=6) for Problem 3 (Table XV).

n	u_Δ global convergence rate	u'_Δ global convergence rate	u''_Δ global convergence rate
8			
	1.97	1.96	1.12
16			
	1.97	1.98	1.07
32			
	1.99	1.99	1.03
64			
	1.99	1.99	1.13
128			
	2.00	2.00	1.01
256			

Table XVII. Errors of P5C4COL (order=6) for Problem 3. The mesh *does not* include the discontinuity point.

n	$\|\|u - u_\Delta\|\|_\infty$	$\|\|u' - u'_\Delta\|\|_\infty$	$\|\|u'' - u''_\Delta\|\|_\infty$
7	.15D–01	.35D–01	.56D+00
15	.32D–02	.77D–02	.27D+00
31	.73D–03	.18D–02	.13D+00
63	.17D–03	.43D–03	.64D–01
127	.43D–04	.11D–03	.32D–01
255	.11D–04	.26D–04	.16D–01

Table XVIII. Estimated orders of convergence of P5C4COL (order=6) for Problem 3 (Table XVII).

n	u_Δ global convergence rate	u'_Δ global convergence rate	u''_Δ global convergence rate
7			
	2.29	2.18	1.07
15			
	2.12	2.10	1.04
31			
	2.06	2.05	1.02
63			
	2.03	2.03	1.01
127			
	2.02	2.01	1.00
255			

REFERENCES

1. Ahlberg, J.H. and Ito T., A Collocation Method for Two-Point Boundary Value Problems, Mathem. of Computation, Vol. 29, pp. 761–776 (1975).

2. Daniel, J.W. and Swartz, B.K., Extrapolated Collocation For Two-Point Boundary Value Problems Using Cubic Splines, J. Inst. Math. Applics. 16, pp. 161–174 (1975).

3. Fyfe, D.J., Linear Dependence Relations Connecting Equal Interval *N*th Degree Splines and Their Derivatives, J. Inst. Math. Appl., 7, pp. 398–406 (1971).

4. Fyfe, D.J., The Use of Cubic Splines in the Solution of Certain Fourth Order Boundary Value Problems, Computer Journal 13, pp. 204–205 (1970).

5. Irodotou-Ellina M., $O(h^6)$ Spline Collocation Methods for High Order Elliptic Boundary Value Problems, University of Thessaloniki, Greece, Ph.D. thesis (1987).

6. Houstis, E.N., C.C. Christara and J.R. Rice, Quadratic Spline Collocation Methods for Two-Point Boundary Value Problems, Int. J. for Num. Methods in Engineering, Vol. 26, pp 935–952 (1988).

7. Lucas, T.R., Asymptotic Expansions for Interpolating Periodic Splines, SIAM J. Numer. Anal., 19, pp. 1051–1066 (1982).

8. Naimack, M.A., Linear Differential Equations, Part I, English translation by E.R. Dawson, ed. by W.N. Everett, New York, Unger (1967).

9. Russell, R.D. and Shampine, L.F., A Collocation Method for Boundary Value Problems, Numer. Math. 19, pp. 1–28 (1972).