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Abstract

The functional equation relating the eigenvalues of the block Symmetric Successive Overrelaxation (SSOR) iteration matrix with those of the block Jacobi iteration matrix found by Chong and Cai is used in order to obtain precise domains of convergence of the block SSOR iteration method associated with a class of generalized consistently ordered (GCO) $(k, p-k)-$ matrices A ($p \geq 2$, $k = 1(1)p-1$). We show that the domain of dependence depends only on the relaxation parameter ω, the spectral radius of the block Jacobi iteration matrix and the value of $\ell = k/p$. *Unlike the case* $(k, p-k) = (1, p-1)$, $p \geq 3$, *which was studied in an earlier paper of the authors, beyond certain critical values of* ℓ, *the domain of dependence does not grow monotonically with* ℓ *and manifests various sorts of behavior*. However, as in the case $(k, p-k) = (1, p-1)$, it is shown that the intersection of the convergence domains taken over all pairs (p, k) still remains the exact domain of convergence of the point SSOR iteration method associated with nonsingular H-matrices A.

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1. INTRODUCTION

In this paper we shall be concerned with the region of convergence of the block Symmetric Successive Overrelaxation (SSOR) method for solving linear systems of equations $Ax = b$, where A is a block nonsingular matrix

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 & A_{1,p-k+1} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 & 0 & A_{2,p-k+2} & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & & 0 & 0 & \cdots & A_{kp} \\ A_{k+1,1} & 0 & \cdots & & & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & A_{p,p-k} & 0 & 0 & \cdots & A_{pp} \end{bmatrix}, \qquad (1.1)$$

where each diagonal block is square and nonsingular. Let D be the block diagonal matrix given by $D := \text{diag}\,(A_{11}, A_{22}, \ldots, A_{pp})$. Then the *block Jacobi iteration matrix associated with* A is given by $J_B^A := I - D^{-1} A$ and has the form

$$J_B^A = \begin{bmatrix} 0 & 0 & \cdots & 0 & C_{1,p-k+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & C_{2,p-k+2} & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & & 0 & 0 & \cdots & C_{kp} \\ C_{k+1,p} & 0 & \cdots & & & 0 & \cdots & \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & C_{p,p-k} & 0 & 0 & \cdots & 0 \end{bmatrix}. \qquad (1.2)$$

In the language of Varga [14] we say that J_B^A is a *weakly Cyclic matrix of index* p and A

is p-cyclic. In the language of Young [16] A is a *generalized consistently ordered* (GCO) $(k, p-k)$-matrix.

Let L and U be strictly lower and strictly upper triangular matrices respectively such that

$$J_B^A = L + U \quad . \tag{1.3}$$

Then the block SSOR iteration method for solving the system $Ax = b$ has the *block* SSOR *iteration matrix associated with A* given by

$$S_\omega^A = (I - \omega\, U)^{-1}\, [(1-\omega)\, I + \omega\, L]\, (I - \omega\, L)^{-1}\, [(1-\omega) I + \omega\, U],\ \omega \neq 0, 2\ . \tag{1.4}$$

Since a necessary condition for the SSOR method to converge (see e.g., Varga [14] and Young [16]) is that $\omega \in (0, 2)$ we shall restrict ourselves to considering values of ω in this interval only.

Recently Chong and Cai [3] have extended a previous result due to Varga, Niethammer and Cai [15] about the functional relationship between the eigenvalues of the block Jacobi and the block SSOR matrices associated with p-cyclic matrices of the form (1.1). Upon inspection the Chong and Cai result can be slightly strengthened to read:

Lemma 1.1: Consider the equation

$$\left[\lambda - (\omega - 1)^2\right]^p = \lambda^k \left[\lambda - (\omega - 1)\right]^{p-2k} (2-\omega)^{2k}\, \omega^p\, \mu^p \tag{1.5}$$

and assume that $\omega \neq 0, 2$. *Then*

i) $0 = \lambda \in \sigma(S_\omega^A)$ if and only if $\omega = 1$ and,

ii) $0 \neq \lambda \in \sigma(S_\omega^A)$ if and only if $\mu \in \sigma(J_B^A)$, where $\sigma(\cdot)$ denotes the spectrum of a matrix. □

Let $\rho(\cdot)$ denote the spectral radius of a matrix and put $\nu := \rho\,(J_B^A)$. We raise here the following question:

For which points (ν, ω) *in the* (ν, ω) *– plane do all roots* λ *of (1.5) lie in the interior of the unit circle thereby guaranteeing that*

$$p\ (S_\omega^A) < 1 \quad ? \tag{1.6}$$

In passing we comment as follows: i) For the case $(p, k) = (2, 1)$ it can be deduced from the works of D'Sylva and Miles [4] and Lynn [9] that (1.6) holds for all points (ν, ω) satisfying $0 \le \nu < 1$ and $0 < \omega < 2$. ii) For the case $(p, k) = (p, 1)$, $p \ge 3$, (1.6) holds provided that $(\nu, \omega) \in R(\ell)$, where $R(\ell)$ is the region in the (ν, ω)-plane given by (1.7) – (1.10) below with $\ell := k/p$ and $k = 1$. The result (1.7) – (1.10) for this case was obtained by the present authors in [8].

The first main result of this paper can be summarized in the following statement:

Theorem 1.1: Let A be a nonsingular GCO *$(k, p-k)$-matrix, $p \ge 3$, $k \ne p/2$, whose diagonal blocks are square and nonsingular. Let J_B^A and S_ω^A be the block Jacobi and the block SSOR iteration matrices associated with A and given in (1.2) – (1.3) and (1.4), respectively. Suppose that $\rho\ (J_B^A) = \nu$. Then $\rho\ (S_\omega^A) < 1$, provided that $(\nu, \omega) \in R(\ell)$, where $R(\ell)$ is the region in the (ν, ω)-plane defined by:*

$$R(\ell) = \begin{cases} 0 < \omega \le 1, & 0 \le \nu < 1 \ , \qquad\qquad (1.7a) \\ 1 \le \omega \le \omega_\ell^*, & 0 \le \nu < \dfrac{1 + (\omega - 1)^2}{(2 - \omega)^{2\ell}\ \omega^{2 - 2\ell}} =: \nu_{1,\ell}(\omega), \qquad (1.7b) \\ \omega_\ell^* \le \omega < 2, & 0 \le \nu < \dfrac{(\omega - 1)^{1/2}\ (\phi(\omega) + 1)^\ell}{\omega\ (1 - 2\ell)^{(1-2\ell)/2}\ (2\ell)^\ell} =: \nu_{2,\ell}(\omega), \qquad (1.7c) \end{cases}$$

where

$$\ell = k/p \ , \tag{1.8}$$

$$\omega^* := \omega_\ell^* := \frac{2(\phi_\ell^* + 2)^{1/2}}{(\phi_\ell^* + 2)^{1/2} + (\phi_\ell^* - 2)^{1/2}}\ , \quad \phi^* := \phi_\ell^* := \frac{1 + (9 - 16\ell)^{1/2}}{2(1 - 2\ell)} \tag{1.9}$$

and

$$\phi := \phi(\omega) := \omega - 1 + \frac{1}{\omega - 1} \quad . \ \Box \tag{1.10}$$

We prove Theorem (1.1) in Section 2. The main idea in the proof is to apply Rouché's theorem, as in [8], for the location of the zeros of analytic functions to various factors of the equation in (1.5). In Section 3 we study in more details the properties of the boundary of the region $R(\ell)$ given in (1.7). The analysis there yields the second main result of this paper:

Theorem 1.2: Under the conditions and notations of Theorem 1.1 and for any numbers $\ell_1, \ell_2 \in (0, 1/2)$ *we have,*

$$R(\ell_1) \subset R(\ell_2) \tag{1.11}$$

if and only if

$$(i) \quad 0 < \ell_1 < \ell_2 \leq 3/8 \tag{1.12}$$

or

$$(ii) \quad 0 < \ell_1 \leq \ell^* < 3/8 < \ell_2 < 1/2\,, \tag{1.13}$$

where ℓ^* *is the unique real root in* $(0, 3/8)$ *of the equation*

$$v_{2,\ell^*}(2) = v_{2,1/2}(2) = \lim_{\ell \to (1/2)^-} v_{2,\ell}(2) = \sqrt{3/2} \tag{1.14}$$

or

$$(iii) \quad 0 < \ell^* < \ell_1 < 3/8 < \ell_2 \leq \ell_2^*(\ell_1) < 1/2\,, \tag{1.15}$$

where $\ell_2^*(\ell_1)$ *is the unique real root in* $(3/8, 1/2)$ *of the equation*

$$v_{2,\ell_2^*(\ell_1)}(2) = v_{2,\ell_1}(2) \tag{1.16}$$

or

$$(iv) \quad 0 < \ell^* < \ell_1^*(\ell_2) \leq \ell_1 < 3/8 < \ell_2 < 1/2\,, \tag{1.17}$$

where $\ell_1^*(\ell_2)$ *is the unique real root in* $(\ell^*, 3/8)$ *of the equation*

$$\nu_{2,\ell_1^*(\ell_2)}(2) = \nu_{2,\ell_2}(2) \quad . \tag{1.18}$$

For all other pairs of distinct numbers $\ell_1, \ell_2 \in (0, 1/2)$ *neither of* $R(\ell_1)$, $R(\ell_2)$ *is a proper subset of the other nor do they coincide. Moreover*

$$R := \bigcap_{\ell \in (0,\, 1/2)} R(\ell) := \begin{cases} 0 < \omega < 2,\ 0 \le \nu \le 1/2 \\ 0 < \omega \le \dfrac{2}{1 + (2\nu - 1)^{1/2}},\ 1/2 < \nu < 1 \end{cases} \quad . \ \square \tag{1.19}$$

We remark the following:

(i) The analysis in Sections 2 and 3 will show that the region of convergence $R(\ell)$ of the SSOR method depends only on ν, ω and on the ratio $\ell = k/p$.

(ii) For $\ell \in (0, 3/8]$ and $\ell \in (3/8, 1/2)$ the region $R(\ell)$ is the (ν, ω)-plane which is specified by (1.7) – (1.10) is illustrated in the shaded region in Figures 1 and 2, respectively. Figs 1,2

(iii) The region of intersection which (1.19) specifies is illustrated in Figure 3 and is precisely the supremum of the convergence domain of the point SSOR method associated with the nonsingular H-matrices A of all orders which satisfy the $\rho(|J_p^A|) = \nu$. Here J_p^A denotes the point Jacobi matrix associated with A and $|B|$ denotes the matrix whose elements are the moduli of the corresponding elements of B. The latter region was essentially obtained by Neumaier and Varga [11] and an open question concerning convergence along its curved boundary was settled by Hadjidimos and Neumann [7]. Fig. 3

(iv) We shall indeed show that as $\ell \to 0^+$ the point of intersection of the curves $\nu_1(\omega) := \nu_{1,\ell}(\omega)$ and $\nu_2(\omega) := \nu_{2,\ell}(\omega)$, which turns out to be a point of tangency between them, tends to the point $(1/2, 2)$.

(v) For $\ell = 1/2$ the region $R(1/2)$ is the whole rectangle shown in Figures 1, 2, and 3 excluding its upper, lower and right boundaries. As was commented by Varga, Niethammier and Cai [15], in the special case $(p,k) = (2, 1)$, this result can be concluded from earlier results of D'Sylva and Miles [4] and Lynn [9]. For $(p, k) = (2k, k)$, $k > 1$, the result is new. Clearly $R(\ell) \subset R(1/2)$, $\forall\, \ell \in (0, 1/2)$, so that we can strengthen (1.19) as follows:

$$\bigcap_{\ell \in (0,\, 1/2]} R(\ell) = \bigcap_{\ell \in (0,1/2)} R(\ell) =: R \quad . \tag{1.20}$$

(vi) The case $(p, k) = (p, 1)$, $p \geq 3$, was studied in Hadjidimos and Neumann [8]. It is a special case of the result in this paper since there $\ell = 1/p \leq 1/3 < 3/8 < 1/2$.

(vii) In this paper we restrict ourselves to considering rational values of the parameter $\ell = k/p$, although our analysis covers all real values ℓ, from the interval $(0, 1/2)$. Actually our analysis covers values of ℓ from the entire interval $(0, 1)$ as we can readily ascertain that

$$\sigma\,(S_\omega^A) = \sigma\,(S_\omega^{A^T}) \quad , \tag{1.21}$$

where A^T is the transpose of A. Thus, if A is a GCO $(k,\, p-k)$-matrix with $\ell = k/p \in (1/2, 1)$, then A^T is a GCO $(p-k,\, k)$-matrix with $\ell' = k'/p' = (p-k)/p = 1 - \ell \in (0, 1/2)$.

(viii) Finally we comment that in the case $(p,k) = (2, 1)$, with A hermitian positive definite matrix, the use of the SSOR splitting as a means for the preconditioning of the system $Ax = b$ is well known (see e.g., the survey paper by Axelsson [1]). For non hermitian matrices the case $(p, k) = (3, 1)$ has arisen in an iterative method for finding the least squares solution to a system with an $m \times n$ matrix coefficient of full column rank (see Chen [2] and also Niethammer, de Pillis and Varga [12], Markham, Neumann and Plemmons [10] and Freund [5]).

2. PROOF OF THEOREM 1.1

Recall the statement of Rouché's theorem (e.g., Tall [13]): *Suppose g and f are analytic functions in a domain containing the track and interior of a closed Jordan contour γ described anticlockwise. If*

$$|f(\lambda) - g(\lambda)| < |g(\lambda)|$$

on the track γ, then $f(\lambda)$ and $g(\lambda)$ have the same number of zeros inside γ.

Consider the equation (1.5). We wish to determine the locations of its roots λ as functions of $\nu = \max_{\mu \in \sigma(J_B^A)} |\mu| = \rho(J_B^A)$. Our idea is this: Put

$$f(\lambda) = \Big[\lambda - (\omega - 1)^2\Big]^p - \lambda^k \Big[\lambda - (\omega - 1)\Big]^{p-2k} (2-\omega)^{2k} \omega^p \mu^p \qquad (2.1)$$

and

$$g(\lambda) = \Big[\lambda - (\omega - 1)^2\Big]^p \quad . \qquad (2.2)$$

Since $g(\lambda)$ has for any $\omega \in (0, 2)$ all its root in the interior of the unit circle and in view of Rouché's theorem we ask: *Given an* $\omega \in (0, 2)$, *for which* $\mu \in \mathbb{C}$ *does it hold that*

$$|\lambda - (\omega-1)^2|^{p-2k} (2-\omega)^{2k} \omega^p |\mu|^p = |f(\lambda) - g(\lambda)| < |g(\lambda)| \qquad (2.3)$$

$$= |\lambda - (\omega - 1)^2|^p , \ \forall \lambda \in \partial\Omega \quad ,$$

where Ω *denotes the unit disc*?

As $\lambda - (\omega - 1) \neq 0$ for $\lambda \in \partial\Omega$, to answer our question it suffices to determine for a given $\omega \in (0, 2)$ those $\mu \in \mathbb{C}$ for which

$$\min_{\lambda \in \partial\Omega} \frac{|\lambda - (\omega-1)^2|}{|\lambda - (\omega - 1)|^{1-2\ell}} > (2-\omega)^{2\ell} \omega |\mu| \qquad (2.4)$$

with $\ell = k/p$.

Let $\lambda \in \partial\Omega$ and represent λ as $\lambda = x + iy$, $x, y \in \mathbb{R}$. Then the ratio appearing in (2.4) admits the following expression

$$\frac{|\lambda - (\omega-1)^2|}{|\lambda - (\omega-1)|^{1-2\ell}} = \frac{[1 + (\omega-1)^4 - 2(\omega-1)^2 x]^{1/2}}{[1 + (\omega-1)^2 - 2(\omega-1)x]^{1/2-\ell}} =: h(x, \omega) \quad . \qquad (2.5)$$

Clearly the inequality in (2.4) holds if and only if

$$\min_{x \in [-1,\, 1]} h(x,\ \omega) > (2-\omega)^{2\ell}\, \omega |\mu| \quad . \tag{2.6}$$

Thus for a fixed $\omega \in (0, 2)$ we shall investigate the behavior of $h(x, \omega)$ as a function of x in the interval $[-1, 1]$. It will be convenient to use the notation " ~ " to denote equality of sign between two expressions.

Before beginning our investigation in earnest, we make the following simple observation.

Observation 2.1: For $\omega = 1$ a necessary and sufficient condition for (2.3) to hold is that $|\mu| < 1$. Moreover, when $|\mu| = 1$, a necessary and sufficient condition for $\lambda \in \partial\Omega$ to be a root of (2.1) is that $\lambda^k = \mu^p$.

Proof: Because $h(x, 1) = 1$ for all $x \in [-1, 1]$, we easily see that (2.6) holds when, and only when, $|\mu| < 1$. Next observe that when $\omega = 1$, $f(\lambda) = \lambda^{p-k}(\lambda^k - \mu^p)$, from which the remainder of the statement follows trivially. □

In view of Observation 2.1 from now on we shall concern ourselves with $1 \neq \omega \in (0, 2)$. For such ω define the functions

$$h_1 := h_1(x,\ \omega) := 1 + (\omega-1)^4 - 2(\omega-1)^2 x \tag{2.7}$$

and

$$h_2 := h_2(x,\ \omega) := 1 + (\omega-1)^2 - 2(\omega-1)\, x \tag{2.8}$$

and note that by (2.5), $h(x) = h_1^{1/2} / h_2^{1/2-\ell}$ and that both functions admit only positive values. But then we have the following

$$\begin{aligned} \partial h(x,\ \omega)/\partial x \ &\sim -(\omega-1)^2 h_2^{1/2-\ell}\, h_1^{-1/2} + (\omega-1)(1-2\ell) h_2^{-1/2-\ell}\, h_1^{1/2} \\ &\sim (\omega-1)(x-\psi(\omega)) \quad , \end{aligned} \tag{2.9}$$

where

$$\psi(\omega) := \frac{1}{4\ell}\left(\omega - 1 + \frac{1}{\omega - 1}\right) - \frac{(1-2\ell)}{4\ell}\left[(\omega-1)^2 + \frac{1}{(\omega-1)^2}\right] . \qquad (2.10)$$

Now the function $h(x, \omega)$ is well defined and differentiable in an interval which strictly contains $[-1, 1]$ and therefore in considering its critical points in the (closed) interval $[-1, 1]$ it suffices to consider those points in $[-1, 1]$ at which the left expression on the right hand side of (2.9) vanishes. Recall that for the moment we are assuming that $\omega \in (0, 2)\backslash\{1\}$ to be fixed. However, our analysis of the behavior of $h(x, \omega)$ requires that we consider two possibilities.

Case 1: $0 < \omega < 1$. In this case $\partial h(x, \omega) / \partial x \sim -(x - \psi(\omega))$. Moreover $(\omega - 1) + 1/(\omega - 1) < -2$ and $(\omega - 1)^2 + 1/(\omega - 1)^2 > 2$ and so, by (2.10), $\psi(\omega) < -1/2\ell - (1 - 2\ell)/2\ell = -1/\ell + 1 < -2 + 1 = -1$ (as $\ell < 1/2$). Whence $-(x - \psi(\omega)) < 0$ as $x \in [-1, 1]$, showing that $\partial h(x, \omega) / \partial x < 0$ for all such x. This proves that

$$\min_{x \in [-1, 1]} h(x, \omega) = h(1, \omega) = (2 - \omega)^{2\ell}\, \omega . \qquad (2.11)$$

Combining (2.11) with (2.10) leads to the following conclusions:

Lemma 2.1: For any $\omega \in (0, 1)$:

(i) *A necessary and sufficient condition for (2.3) to hold is* $|\mu| < 1$.

(ii) *When* $|\mu| = 1 = \nu_0(\omega)$, *a necessary and sufficient condition for* $\lambda \in \partial\Omega$ *to be a root of* $f(\lambda)$ *is that* $\lambda = 1$ *and* $\mu^p = 1$.

Proof: The proof of (i) has been done in the arguments leading to the statement, so we only need to prove (ii). The sufficiency of the condition is immediate on the examination of (2.1). Conversely, suppose that $\lambda = x + iy \in \partial\Omega$ is a root of $f(\lambda)$. Taking into account that $|\mu| = 1$ we obtain from (2.1) and (2.5) that $h(x, \omega) = (2 - \omega)^{2\ell}\omega$. But then, as (2.6) holds for all $|\mu| < 1$, we must have that $x = 1$ so that $\lambda = 1$. The remaining part of the proof now follows by setting $f(1) = 0$ in (2.1). □

Case 2: $1 < \omega < 2$. Observe now that by (2.9), $\partial h(x, \omega) / \partial x \sim (x - \psi(\omega))$. Moreover on letting

$$\phi(\omega) := \omega - 1 + \frac{1}{\omega - 1} \qquad (2.12)$$

we see that

$$\phi(\omega) > 2 \tag{2.13}$$

and

$$(\omega - 1)^2 + \frac{1}{(\omega - 1)^2} = \phi^2(\omega) - 2 \quad . \tag{2.14}$$

Substituting (2.12) and (2.14) in (2.10) we have that

$$\psi(\omega) = \frac{1}{4\ell}\,[-(1-2\ell)\,\phi^2(\omega) + \phi(\omega) + 2(1-2\ell)] \quad . \tag{2.15}$$

We proceed to investigate the sign of $x - \psi(\omega)$ as x varies between -1 and 1. For this purpose we shall first consider the function $\psi(\omega)$.

Lemma 2.2: For $\omega \in (1, 2)$,

$$(i) \quad \psi(\omega) \le 1 \quad . \tag{2.16}$$

$$(ii) \quad \psi(\omega) \le -1 \tag{2.17}$$

when

$$\phi^* = \frac{1 + (3 - 16\ell)^{1/2}}{2(1 - 2\ell)} \le \phi(\omega) < \infty \tag{2.18}$$

and

$$\psi(\omega) \ge -1 \tag{2.19}$$

when

$$2 < \phi(\omega) \le \phi^* \quad . \tag{2.20}$$

Proof: (i) The inequality (2.16) is equivalent, in view of (2.15), to

$$(1-2\ell)\,\phi^2(\omega)-\phi(\omega)-2(1-4\ell)\geq 0 \quad , \tag{2.21}$$

which always holds when $\phi(\omega)>2$. Moreover, we see that equality holds in (2.21) for $\ell=3/8$.

(ii) The inequality (2.17) holds if and only if

$$(1-2\ell)\,\phi^2(\omega)-\phi(\omega)-2\leq 0$$

and this inequality can be shown to hold, by arguments involving roots of quadratic polynomials, whenever $\phi(\omega)$ satisfies the conditions of (2.18). The validity of (2.19) subject to the conditions in (2.20) follows along similar arguments. □

Let us note from (2.12) that for $\omega\in(1,2)$, the inverse function to $\phi:=\phi(\omega)$ is given by

$$\omega(\phi)=\frac{2(\phi+2)^{1/2}}{(\phi+2)^{1/2}+(\phi-2)^{1/2}} \quad . \tag{2.22}$$

In what follows for the value of ϕ^* in (2.18) we shall let

$$\omega^*=\omega(\phi^*) \quad . \tag{2.23}$$

Notice that $\phi(\omega)$ given in (2.12) is a strictly decreasing function of ω in (1, 2) and so $\omega(\phi)$ is strictly decreasing in the interval $(2,\infty)$. Thus when

$$1<\omega\leq\omega^* \tag{2.24}$$

we have from (2.17) that for $x\in(-1,1]$, $x-\psi(\omega)>0$, so that in this interval $\partial h(x,\omega)/\partial x>0$ by (2.9). Hence $h(x,\omega)$ is strictly increasing in the entire interval $[-1,1]$ and

$$\min_{x\in[-1,1]} h(x,\omega)=h(-1,\omega)=\frac{1+(\omega-1)^2}{\omega^{1-2\ell}} \quad . \tag{2.25}$$

Combining (2.25) with (2.6) shows that for any $\omega \in (1, \omega^*]$, (2.3) holds if and only if

$$|\mu| < \frac{1 + (\omega - 1)^2}{(2 - \omega)^{2\ell}} \, \omega^{2-2\ell} =: \nu_{1,\ell}(\omega) =: \nu_1(\omega) \quad . \tag{2.26}$$

Next, when

$$\omega^* \leq \omega < 2 \tag{2.27}$$

we see from (2.16), (2.19) and (2.9) that for $x \in [-1, 1]$

$$\partial h(x, \omega) / \partial x \leq 0 \quad \text{when} \quad x \leq \psi(\omega) \tag{2.28}$$

with equality if and only if $x = \psi(\omega)$ (< 1), while

$$\partial h(x, \omega) / \partial x > 0 \quad \text{when} \quad x > \psi(\omega) \quad . \tag{2.29}$$

This means that

$$\min_{x \in [-1, 1]} h(x, \omega) = h(\psi(\omega), \omega) \quad , \tag{2.30}$$

where, to remind ourselves, $\psi(\omega)$ is given in (2.10). Combing (2.30) with (2.6) yields that for any $\omega \in [\omega^*, 2)$, (2.3) holds if and only if

$$|\mu| < \frac{[1 + (\omega - 1)^4 - 2(\omega - 1)^2 x]^{1/2}}{\omega(2 - \omega)^{2\ell} [1 + (\omega - 1)^2 - 2(\omega - 1)x]^{1/2-\ell}} =: \nu_{2,\ell}(\omega) =: \nu_2(\omega) \tag{2.31a}$$

or by making use of (2.15), (2.12) and (2.14), we obtain the alternative expressions

$$|\mu| < \frac{(\omega - 1)^{1/2}}{\omega} \; \frac{(\phi(\omega) + 1)^{\ell}}{(1 - 2\ell)^{(1-2\ell)/2} \, (2\ell)^{\ell}} =: \nu_{2,\ell}(\omega) =: \nu_2(\omega) \quad . \tag{2.31b}$$

We summarize the foregoing results in the following expanded statement:

Lemma 2.3: Let $\omega \in (1, 2)$ and consider ω^* of (2.23) obtained from (2.18) using (2.22). Then:

(i) *For $\omega \in (1, \omega^*]$, (2.3) holds if and only if $|\mu| < \nu_1(\omega)$. Moreover for any ω in this interval,*

$$\nu_1(\omega) < 1 \quad . \tag{2.32}$$

(ii) *For $\omega \in [\omega^*, 2)$, (2.3) holds if and only if $|\mu| < \nu_2(\omega)$. Moreover for any ω in this interval,*

$$\nu_2(\omega) < 1 \quad . \tag{2.33}$$

(iii) *For $\omega \in (1, \omega^*]$ and $|\mu| = \nu_1(\omega)$, $\lambda \in \partial\Omega$ is a root of $f(\lambda)$ if and only if $\lambda = -1$ and $\mu^p = (-1)^k [1 + (\omega - 1)^2]^p / (2 - \omega)^{2k} \omega^{2p-2k}$.*

(iv) For $\omega \in (\omega^*, 2)$ and $|\mu| = \nu_2(\omega)$, if λ is a root of $f(\lambda)$ then λ is not real.

Proof: (i) The initial part of the statement is our result (2.26). To show (2.32) recall that for any $\omega \in (1, \omega^*]$, the function $h(x, \omega)$ is strictly increasing in $[-1, 1]$. Thus from (2.5) we find that

$$\max_{x \in [-1, 1]} h(x, \omega) = h(1, \omega) = (2 - \omega)^{2l} \omega \quad .$$

Hence, from (2.25)

$$(2 - \omega)^{2l} \omega > \frac{1 + (\omega - 1)^2}{\omega^{1-2l}} \quad .$$

Dividing both sides of this inequality by $(2 - \omega)^{2l} \omega$, (2.32) obtains.

(ii) The initial part of the statement is simply (2.31). To show (2.33) recall that in $[\omega^*, 2)$, the function $h(x, \omega)$ first strictly decreases until $x = \psi(\omega)$ and then strictly increases. Therefore on utilizing (2.5) and (2.30) we see that

$$h(1, \omega) = (2-\omega)^{2\ell}\,\omega > h(\psi(\omega), \omega) \quad .$$

Dividing both sides of this inequality by $(2-\omega)^{2\ell}\,\omega$, (2.33) obtains.

(iii) Assume that $\omega \in (1, \omega^*]$. The "if" part of the claim is trivial so we prove the "only if" part. Suppose then that $\lambda \in \partial\Omega$ is a root of $f(\lambda)$. Then as $\lambda = x + iy$ and $|\mu| = \nu_1(\omega)$, we see from (2.1) and (2.5) that $h(x, \omega) = [1 + (\omega - 1)^2] / \omega^{1-2\ell}$. But then as (2.6) holds for all $|\mu| < \nu_1(\omega)$, we must have that $\lambda = -1$ (see (2.25)). The remainder of the proof now follows by setting $f(-1) = 0$ in (2.1).

(iv) Assume that $\omega \in (\omega^*, 2)$ and that $\lambda \in \partial\Omega$ is a root of (2.1). Then as $\lambda = x + iy$ and $|\mu| = \nu_2(\omega)$, we see from (2.1) and (2.5) that $h(x, \omega) = h(\psi(\omega), \omega)$. But then because (2.6) holds for all $|\mu| < \nu_2(\omega)$, we must have (see (2.30)) that x lies strictly between -1 and 1 showing that $\lambda = x + iy \in \partial\Omega$ can not be real. □

The validity of Theorem 1.1 is now a consequence of Observation 2.1 and Lemmas 2.1, 2.3 and 2.4. As we mentioned in the introduction, we devote the next section to an investigation of some of the boundaries of the region $R(\ell)$.

3. THE GEOMETRY OF THE CURVES $\nu_1(\omega)$ AND $\nu_2(\omega)$

Consider Figures 1 and 2. They illustrate that the curves $\nu_1(\omega)$ and $\nu_2(\omega)$ are tangential at $\omega = \omega^*$, a fact we shall prove momentarily. We comment that from (2.26) one readily sees that $\nu_1(\omega)$ is well defined in the entire interval (0, 2) and not only in the interval $(1, \omega^*]$ in which it was defined and used in Section 2. As for $\nu_2(\omega)$ it too is well defined in $(1, \infty)$ and not only in the interval $[\omega^*, 2)$. This is readily seen from (2.31b).

Lemma 3.1: At the point $\omega = \omega^*$ given in (1.9) the curves $\nu_1(\omega)$ and $\nu_2(\omega)$ are tangential.

Proof: Since $\nu_1(\omega^*) = \nu_2(\omega^*)$ we see that

$$\frac{\partial}{\partial\omega}\left[\nu_1^p(\omega) - \nu_2^p(\omega)\right]\Big|_{\omega=\omega^*} = p\,\nu_1^{p-1}(\omega^*)\left[\nu'_1(\omega^*) - \nu'_2(\omega^*)\right] \quad . \tag{3.1}$$

But then, because $\nu_1(\omega^*) \neq 0$, to prove our claim it suffices to show that (3.1) vanishes. Now

$$\Delta(\omega) := \frac{\partial}{\partial \omega} [\nu_1^p(\omega) - \nu_2^p(\omega)]$$

$$= \frac{1}{(2-\omega)^{4k}\,\omega^{4p-4k}} \Big\{ (2-\omega)^{2k}\,\omega^{2p-2k}\, p\, [1+(\omega-1)^2]^{p-1}\, 2(\omega-1)$$

$$- [1+(\omega-1)^2]^p\, [-2k(2-\omega)^{2k-1}\,\omega^{2p-2k} + (2p-2k)\,(2-\omega)^{2k}\,\omega^{2p-2k-1} \Big\}$$

$$- \frac{1}{\omega^{2p}\,(2-\omega)^{4k}\,[1+(\omega-1)^2 - 2(\omega-1)\,\psi(\omega)]^{p-2k}} \cdot$$

$$\cdot \Big\{ \omega^p\,(2-\omega)^{2k}\,[1+(\omega-1)^2 - 2(\omega-1)\,\psi(\omega)]^{p/2-k}\,\frac{p}{2} \cdot \tag{3.2}$$

$$\cdot [1+(\omega-1)^4 - 2(\omega-1)^2\,\psi(\omega)]^{p/2-1}\,[4(\omega-1)^3 - 4(\omega-1)\,\psi(\omega) - 2(\omega-1)^2\,\psi'(\omega)]$$

$$- [1+(\omega-1)^4 - 2(\omega-1)^2\,\psi(\omega)]^{p/2} \Big\{ p\omega^{p-1}\,(2-\omega)^{2k}\,[1+(\omega-1)^2 - 2(\omega-1)\,\psi(\omega)]^{p/2-k}$$

$$+ (\frac{p}{2} - k)\,\omega^p(2-\omega)^{2k}\,[1+(\omega-1)^2 - 2(\omega-1)\,\psi(\omega)]^{p/2-k-1} \cdot$$

$$\cdot [2(\omega-1) - 2\psi(\omega) - 2(\omega-1)\,\psi'(\omega)] \Big\} \Big\} .$$

Now according to the analysis leading to (2.25), for each $\omega \in (1, \omega^*]$, $h(x, \omega)$ has an absolute minimum which it attains uniquely at $x = -1$. On the other hand according to the analysis leading to (2.30), in $[\omega^*, 2)$, $h(x, \omega)$ has an absolute minimum which it attains uniquely at $x = \psi(\omega)$. Thus we must have $\psi(\omega^*) = -1$. But then a simple inspection of the two fractional multipliers which appear in $\Delta(\omega)$ shows that, at $\omega = \omega^*$, their denominators become equal, their common value being $(2-\omega^*)^{4k}\,(\omega^*)^{4p-4k} \neq 0$. Moreover, since

$$[1+(\omega^*-1)^2 - 2(\omega^*-1)\,\psi(\omega^*)] = (\omega^*)^2$$

and

$$[1+(\omega^*-1)^4 - 2(\omega^*-1)^2\,\psi(\omega^*)] = [1+(\omega^*-1)^2]^2 ,$$

one can obtain by inspection that

$$\Delta(\omega^*) = \frac{[1+(\omega^*-1)^2]^{p-2}}{(2-\omega^*)^{2k+1}\,(\omega^*)^{2p-2k+2}}\,\Delta_1(\omega^*) ,$$

where

$$
\begin{aligned}
\Delta_1(\omega^*) &= 2p(2-\omega^*)(\omega^*-1)(\omega^*)^2\,[1+(\omega^*-1)^2] \\
&\quad - [1+(\omega^*-1)^2]^2\,[-2k(\omega^*)^2+(2p-2k)(2-\omega^*)\,\omega^*] \\
&\quad - p(\omega^*)^2\,(2-\omega^*)(\omega^*-1)\,[2(\omega^*-1)^2+2-(\omega^*-1)\,\psi'(\omega^*)] \\
&\quad + [1+(\omega^*-1)^2]^2\left\{ p\omega^*\,(2-\omega^*)-2k(\omega^*)^2+(p-2k)(2-\omega^*)[\omega^*-(\omega^*-1)\,\psi'(\omega^*)]\right\} \\
&= (2-\omega^*)(\omega^*-1)\left\{ p(\omega^*)^2\,(\omega^*-1)-(p-2k)\,[1+(\omega^*-1)^2]^2 \right\}\psi'(\omega^*) \quad .
\end{aligned}
\tag{3.3}
$$

Consider next the rightmost expression in (3.3). Since $\omega^* \neq 1$, it follows from (2.10) that $d\psi(\omega^*)\,/\,d\omega$ is bounded. We shall finally show that the expression in the braces of the extreme right hand side of (3.3) vanishes. Now

$$
\begin{aligned}
& p(\omega^*)^2\,(\omega^*-1)-(p-2k)\,[1+(\omega^*-1)^2]^2 \\
&= -(p-2k)\,[1+(\omega^*-1)^4+2(\omega^*-1)^2]+p(\omega^*-1)\,[(\omega^*-1)^2+2(\omega^*-1)+1] \\
&= (\omega^*-1)^2\left\{-(p-2k)\,[\frac{1}{(\omega^*-1)^2}+(\omega^*-1)^2+2]+p\ [(\omega^*-1)+\frac{1}{(\omega^*-1)}+2]\right\} \\
&= p(\omega^*-1)^2\left\{-(1-2\ell)\,[(\phi^*)^2-2+2]+(\phi^*+2)\right\} \\
&= p(\omega^*-1)\left\{-(1-2\ell)\,(\phi^*)^2+\phi^*+2\ell\right\} = 0
\end{aligned}
$$

(see (2.12), (2.14) and (2.18)). □

We now focus our attention on the behavior of the curve $\nu_1(\omega)$. First we claim that $\nu_1(\omega)$ has a unique turning point in (1,2). Note that

$$
\begin{aligned}
\nu'_1(\omega) &= (2-\omega)^{2\ell}\,\omega^{2-2\ell}\,2(\omega-1)-[1+(\omega-1)^2] \quad \cdot \\
&\quad \cdot[-2\ell(2-\omega)^{2\ell-1}\,\omega^{2-2\ell}+(2-\omega)^{2\ell}\,(2-2\ell)\,\omega^{1-2\ell}] \\
&= -[(1-2\ell)\,\omega^2-4(1-\ell)\,\omega+4(1-\ell)] \quad .
\end{aligned}
\tag{3.4}
$$

Now the zeros $\tilde{\omega}_1$ and $\tilde{\omega}_2$ of the quadratic which appears on the extreme right hand side of (3.4) are given by

$$\tilde{\omega}_1 = \frac{2(1-\ell)^{1/2}}{(1-\ell)^{1/2}+\ell^{1/2}}\ , \quad \tilde{\omega}_2 = \frac{2(1-\ell)^{1/2}}{(1-\ell)^{1/2}-\ell^{1/2}} \tag{3.5}$$

and hence they satisfy: $1 < \tilde{\omega}_1 < 2 < \tilde{\omega}_2$. The following table thus shows where $v_1(\omega)$ increases and decreases:

ω	0		1		$\tilde{\omega}_1$		2
$v_1(\omega)$	∞	$\searrow$		$\searrow$	min	$\nearrow$	∞

(3.6)

It is easily concluded that $\tilde{\omega}_1$ is the unique turning point of $v_1(\omega)$ in the interval (1, 2). We can now further deduce that:

Lemma 3.2: The turning point $(v_1(\tilde{\omega}_1), \tilde{\omega}_1)$ *of the curve* $v_1(\omega)$ *lies beneath the point* $(v_1(\omega^*), \omega^*)$ *of tangency of* $v_1(\omega)$ *and* $v_2(\omega)$. *That is* $\tilde{\omega}_1 < \tilde{\omega}_2$.

Proof: Recall the expression for ω^* given in (1.9). We require to show that

$$\frac{2(1-\ell)^{1/2}}{(1-\ell)^{1/2}+\ell^{1/2}} < \frac{2(\phi^*+2)^{1/2}}{(\phi^*+2)^{1/2}+(\phi^*-2)^{1/2}}$$

or, equivalently, that

$$\frac{1-\ell}{\ell} < \frac{\phi^*+2}{\phi^*-2} \ .$$

But that this holds follows readily from the substitution $\phi^* = [1+(9-16\ell)^{1/2}]/2(1-2\ell)$. □

Until now we have essentially considered behavior of $v_1(\omega)$ (and $v_2(\omega)$) as a function of ω only. Since, according to (2.26), $v_1(\omega) = v_{1,\ell}(\omega)$ depends also on ℓ, let us now investigate the dependency of $v_{1,\ell}(\omega)$ on ℓ.

Lemma 3.3: For $0 < \ell_1 < \ell_2 < 1/2$ *the following hold:*

$$\nu_{1,\ell_1}(\omega) > \nu_{1,\ell_2}(\omega), \quad \omega \in (0,1) \quad , \tag{3.7a}$$

$$\nu_{1,\ell_1}(1) = \nu_{1,\ell_2}(1) \quad , \tag{3.7b}$$

$$\nu_{1,\ell_1}(\omega < \nu_{1,\ell_2}(\omega), \quad \omega \in (1,2) \quad . \tag{3.7c}$$

Moreover, for all $\omega \in (1,2)$,

$$\lim_{\ell \to 0^+} \nu_{1,\ell}(\omega) = \frac{1 + (\omega - 1)^2}{\omega^2} \quad . \tag{3.8}$$

Proof: From (2.26) we see that

$$\frac{\nu_{1,\ell_1}(\omega)}{\nu_{1,\ell_2}(\omega)} = \left[\frac{2-\omega}{\omega} \right]^{2(\ell_2 - \ell_1)}$$

from which (3.7a) – (3.7c) follow. (3.8) follows by letting $\ell \to 0^+$ in (2.26). □

Let us denote the turning point ϖ_1 for $\nu_{1,\ell}(\omega)$ ($= \nu_1(\omega)$) in (3.5) by $\varpi_{1,\ell}$ and let us denote by ω_ℓ^* the point of tangency of $\nu_{1,\ell}(\omega)$ and $\nu_{2,\ell}(\omega)$ ($= \nu_2(\omega)$) given in (1.9). By (3.5),

$$\lim_{\ell \to 0^+} \varpi_{1,\ell} = \lim_{\ell \to 0^+} \frac{2(1-\ell)^{1/2}}{(1-\ell)^{1/2} + \ell^{1/2}} = 2$$

and so, as $\nu_{1,\ell}$ is a continuous function of both ω and $\ell \in (0, 1/2)$, we have, by (3.8), that

$$\lim_{\ell \to 0^+} \nu_{1,\ell}(\varpi_{1,\ell}) = 1/2$$

showing that, as $\ell \to 0^+$, the turning points satisfy

$$\lim_{\ell \to 0^+} (\nu_{1,\ell}(\tilde{\omega}_{1,\ell}),\ \tilde{\omega}_{1,\ell}) = (1/2,2) \quad . \tag{3.9}$$

Consider now the behavior of the tangency points ω_ℓ^* as $\ell \to 0^+$. First we claim that for $0 < \ell_1 < \ell_2 < 1/2$,

$$\omega_{\ell_1}^* > \omega_{\ell_2}^* \quad . \tag{3.10}$$

For this purpose we use (1.9) to show that $d\omega_\ell^* \,/\, d\phi_\ell^* < 0$ and $d\phi_\ell^* \,/\, d\ell > 0$. But then $d\omega_\ell^* \,/\, d\ell < 0$ from which (3.10) follows. Secondly, it can be ascertained from (1.9) that $\lim_{\ell \to 0^+} \omega_\ell^* = 2$ so that, on appealing once again to (3.8), we conclude that

$$\lim_{\ell \to 0^+} (\nu_{1,\ell}\,(\omega_\ell^*),\ \omega_\ell^*) = (1/2,2) \quad . \tag{3.11}$$

Having considered in detail the behavior of the function $\nu_{1,\ell}(\omega)$, we now analyze the function $\nu_{2,\ell}(\omega)$ ($= \nu_2(\omega)$) first as a function of ω for an ℓ fixed in $(0,1/2)$ and then as a function of ℓ for an ω fixed in $(1,2]$.

Lemma 3.4: Let $\ell \in (0,1/2)$. Then

(i) *For any $\ell \in (0,3/8]$,*

$$\nu_2(\omega_1) < \nu_2(\omega_2),\quad 1 < \omega_1 < \omega_2 \le 2 \quad . \tag{3.12}$$

(ii) *For any $\ell \in [3/8,1/2)$ the following hold:*

$$\nu_2(\omega_1) < \nu_2(\omega_2),\quad 1 < \omega_1 < \omega_2 \le \tilde{\omega}'_1 \tag{3.13a}$$

and

$$\nu_2(\omega_1) > \nu_2(\omega_2),\quad \tilde{\omega}'_1 < \omega_1 < \omega_2 \le 2 \quad , \tag{3.13b}$$

where

$$\tilde{\omega}'_1 = \frac{2}{1 + (-3 + 8\ell)^{1/2}} \quad . \tag{3.14}$$

Proof: (i) Because $\nu_2(\omega)$ is defined for all $\omega > 1$, it follows by (2.31b) that

$$\begin{aligned} \nu'_2(\omega) &= \omega[\frac{1}{2}(\omega - 1)^{-1/2} (\phi + 1)^{\ell} + (\omega - 1)^{1/2} \ell(\phi + 1)^{\ell-1} \phi'] \\ &\quad - (\omega - 1)^{1/2} (\phi + 1)^{\ell} \\ &= \omega(\phi + 1) + 2\ell\omega(\omega - 1) \phi' - 2(\omega - 1)(\phi + 1) \\ &= (\phi + 1)(2 - \omega) + 2\ell\omega(\omega - 1) \phi' \\ &= (\omega - 1 + \frac{1}{\omega - 1} + 1)(2 - \omega) + 2\ell\omega(\omega - 1)[1 - \frac{1}{(\omega - 1)^2}] \\ &= (1 - 2\ell)\omega^2 - \omega + 1 \quad . \end{aligned} \tag{3.15}$$

For $\ell \in (0,3/8]$ the discriminant $D = 8\ell - 3$ of the quadratic in (3.15) is negative or zero, hence our conclusion.

(ii) For $\ell \in [3/8,1/2)$, the real distinct zeros $\tilde{\omega}'_1$ and $\tilde{\omega}'_2$ of the quadratic in (3.15) are given by

$$\tilde{\omega}'_1 = \frac{2}{1 + (-3 + 8\ell)^{1/2}}, \quad \tilde{\omega}'_2 = \frac{2}{1 - (-3 + 8\ell)^{1/2}}$$

and satisfy: $1 < \tilde{\omega}'_1 < 2 < \tilde{\omega}'_2$. Consequently the following table shows where $\nu_2(\omega)$ increases and decreases

(3.16)

ω	1		$\tilde{\omega}'_1$		2		$\tilde{\omega}'_2$		∞
$\nu_2(\omega)$	$1 - 2\ell$	↗		↘		↘		↗	∞
			max						

A careful study of the table in (3.16) yields our claims. □

We remark that $\tilde{\omega}'_1$ is the unique turning point of $\nu_2(\omega)$ in the interval $(1,2]$.

It can be further deduced that:

Lemma 3.5: The turning point $(\nu_2(\bar{\omega}'_1), \bar{\omega}'_1)$ *of the curve* $\nu_2(\omega)$ *lies above the point* $(\nu_2(\omega^*), \omega^*)$ *of tangency of* $\nu_1(\omega)$ *and* $\nu_2(\omega)$. *That is, for any fixed* $\ell \in (3/8, 1/2)$, $\omega^* < \bar{\omega}'_1$.

Proof: Using the expression for ω^* given in (1.9) we require to show that

$$\frac{2(\phi^* + 2)^{1/2}}{(\phi^* + 2)^{1/2} + (\phi^* - 2)^{1/2}} < \frac{2}{1 + (-3 + 8\ell)^{1/2}} \quad .$$

This inequality is equivalent to

$$\frac{\phi^* + 2}{\phi^* - 2} < \frac{1}{-3 + 8\ell}$$

which follows from the substitution $\phi^* = [1 + (9 - 16\ell)^{1/2}] / 2(1 - 2\ell)$. □

Before we investigate the dependency of $\nu_2(\omega) = \nu_{2,\ell}(\omega)$ on ℓ we examine the behavior of the limiting curve $\lim_{\ell \to (1/2)^-} \nu_{2,\ell}(\omega)$ for all $\omega \in (1,2]$. From the expression (2.31b) and recalling that $\lim_{\ell \to (1/2)^-} (1 - 2\ell)^{1-2\ell} = 1$, we obtain that

$$\lim_{\ell \to (1/2)^-} \nu_{2,\ell}(\omega) = \frac{(\omega^2 - \omega + 1)^{1/2}}{\omega}, \quad \forall\, \omega \in (1,2] \quad . \tag{3.17}$$

Hence

$$\nu'_{2,1/2}(\omega) \sim \omega - 2 \le 0 \quad .$$

This implies that $\nu_{2,1/2}(\omega)$ strictly decreases in (1,2] which concurs with the results of Lemma 3.4 for the case $\ell \in (3/8, 1/2)$. It is also commented that as $\ell \to (1/2)^-$ all three points $(\nu_{1,1/2}(\bar{\omega}_1), \bar{\omega}_1)$, $(\nu_{1,1/2}(\omega^*), \omega^*)$ and $(\nu_{1,1/2}(\bar{\omega}'_1), \bar{\omega}'_1)$ coincide with the point $(\nu, \omega) = (1,1)$. However, note that $\lim_{\ell \to (1/2)^-} \lim_{\omega \to 1^+} \nu_{2,\ell}(\omega) = 0 \ne 1 = \lim_{\omega \to 1^+} \lim_{\ell \to (1/2)^-} \nu_{2,\ell}(\omega)$!

In the sequel two lemmas are given and proved which show very clearly the dependency of $\nu_2(\omega)$ on ℓ.

Lemma 3.6: For $0 < \ell_1 < \ell_2 \leq 3/8$ *the following hold*:

$$\nu_{2,\ell_1}(\omega) < \nu_{2,\ell_2}(\omega), \quad 1 < \omega \leq 2 \quad . \tag{3.18}$$

Proof: From the expression (2.31b) for $\nu_{2,\ell}(\omega)$ we have that each of its factors is positive. Then $\nu_{2,\ell}(\omega)$ is positive for any pair (ℓ, ω). Now set

$$\begin{aligned} z &:= z(\ell,\omega) := \ell n \nu_{2,\ell}(\omega) \\ &= \ell n \frac{(\omega-1)^{1/2}}{\omega} + \ell\, \ell n(\phi+1) - \frac{1}{2}\, \ell n(1-2\ell) + \ell\, \ell n(1-2\ell) - \ell\, \ell n(2\ell) \quad . \end{aligned} \tag{3.19}$$

Thus on differentiating with respect to ℓ we have that

$$\begin{aligned} \partial z / \partial \ell &= \ell n(\phi+1) - \frac{(-2)}{2(1-2\ell)} + \ell n(1-2\ell) + \ell \cdot \frac{(-2)}{(1-2\ell)} - \ell n(2\ell) - \ell \cdot \frac{2}{2\ell} \\ &= \ell n[(1-2\ell)(\phi+1)/2\ell] \quad . \end{aligned} \tag{3.20}$$

As can be readily checked $\phi(\omega)$ strictly decreases in the interval (1,2] and since $\phi(2) = 2$, it follows that

$$\frac{(1-2\ell)}{2\ell}(\phi+1) \geq \frac{3(1-2\ell)}{2\ell} \quad . \tag{3.21}$$

But for all $\ell \in (0, 3/8]$, $\frac{3(1-2\ell)}{2\ell} \geq 1$. This result, in view of (3.20) and (3.19), implies that for a fixed $\omega \in (1,2]$, $\nu_{2,\ell}(\omega)$ strictly increases with ℓ in the interval $(0, 3/8]$. □

Lemma 3.7: As a function of ℓ, $\nu_{2,\ell}(2)$ *strictly increases in the interval* $[0, 3/8]$ *and strictly decreases in the interval* $[3/8, 1/2]$.

Proof: From the proof of the previous lemma and by virtue of (3.19), (3.20) and (3.21) the behavior of the function $\nu_{2,\ell}(2)$ in the intervals (0,3/8] and [3/8,1/2) follows immediately on considering the position of the magnitude $\dfrac{3(1-2\ell)}{2\ell}$ with respect to 1. The endpoints 0 and $(1/2)^-$ of the two intervals can be included by continuity arguments, by taking limits as ℓ tends to 0^+ and (1/2), respectively, and recalling that $\lim_{\ell \to 0^+} \ell^{\ell} = 1$ and $\lim_{\ell \to (1/2)^-} (1-2\ell)^{1-2\ell} = 1$. □

A careful examination of the results of this section, particularly in Lemmas 3.1 – 3.7, proves our claims in Theorem 1.2. Finally (1.19) readily follows from (3.8), (3.9), (3.11), and the fact that for all $1 < \omega < 2$ the limiting point $\lim_{\ell \to 0^+} (\nu_{1,\ell}(\omega), \omega)$ is the point $\left[\dfrac{1+(\omega-1)^2}{\omega^2}, \omega \right]$, where $\nu = \dfrac{1+(\omega-1)^2}{\omega^2}$ is the inverse function of $\omega = \dfrac{2}{1+(2\nu-1)^{1/2}}$ for $1/2 < \nu < 1$.

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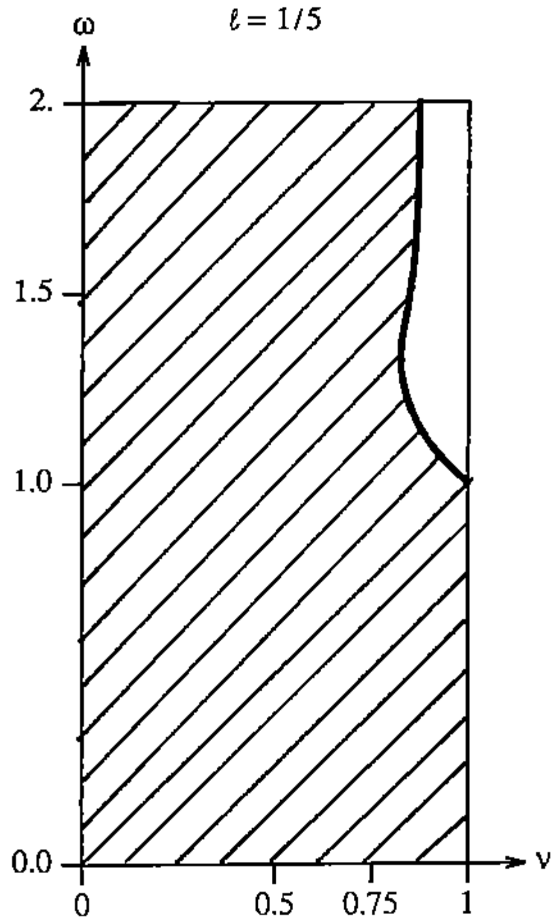

Figure 1

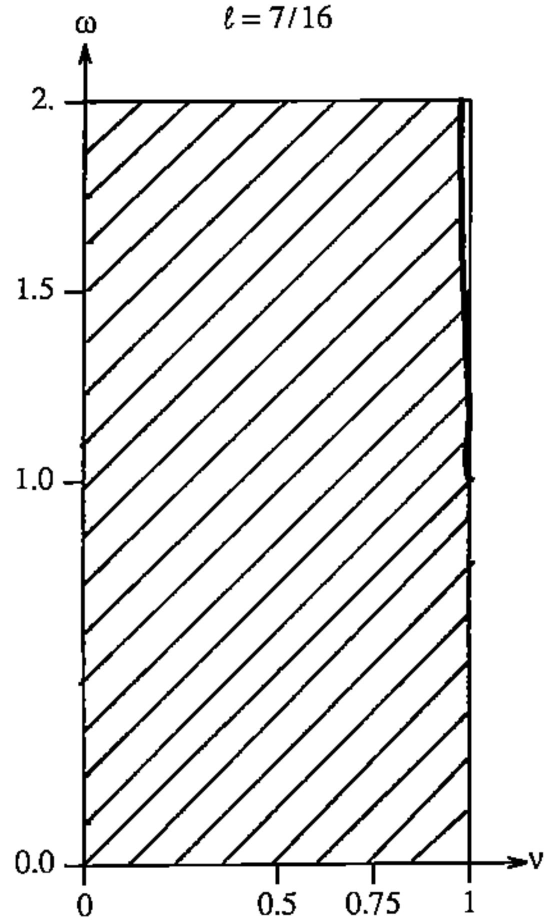

Figure 2

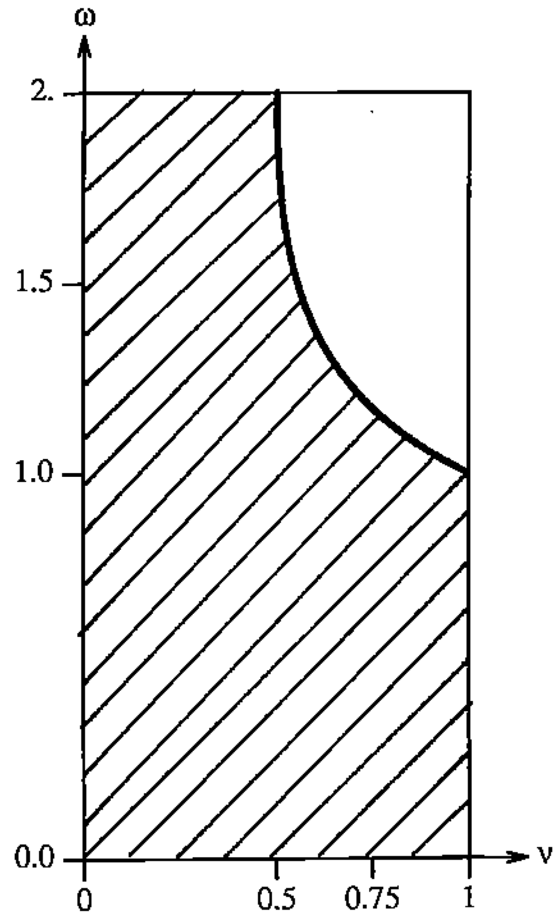

Figure 3