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Let N^t be queue length at the t-th departure epoches. Then

$$N^{t+1} - N^t = A^t - D^t \tag{1}$$

where A^t and D^t represent the number of arrivals and departures in the k-th epoches. Then (1) implies

$$EN^{t+1} - EN^t = EA^t - ED^t = EA^t - Pr\{N^t > 0\}. \tag{2}$$

since $ED^t = Pr\{N^t > 0\}$.

To study properties of (2) we need the following lemma.

Lemma. Let a_n be nonnegative sequence, i.e., $a_n \geq 0$. Then, the following can be proved.

(i) If $\lim\limits_{n\to\infty} \sup a_n = \infty$, then $\lim\sup(a_{n+1} - a_n) \geq 0$.

(ii) If $\lim\limits_{n\to\infty} \sup a_n < \infty$, then $\lim\inf(a_{n+1} - a_n) \leq 0$.

(iii) For any $a_n \geq 0$ and $b_n \geq 0$

$$\lim\sup(a_n - b_n) \leq \lim\sup a_n - \lim\inf b_n \tag{3a}$$

$$\lim_{n\to\infty}\inf(a_n - b_n) \geq \lim\inf a_n - \lim\sup b_n \tag{3b}$$

Proof. We prove first (i). Since $\limsup a_n = \infty$, then there exists an increasing subsequence a_{n_k} tending to infinity. Hence $a_{n_k} - a_{n_{k-1}}$. But then

$$0 \leq a_{n_k} - a_{n_{k-1}} = (a_{n_k} - a_{n_k-1}) + (a_{n_k-1} - a_{n_k-2}) + \cdots + (a_{n_{k-1}+1} - a_{n_{k-1}}) \tag{4}$$

so (4) implies that at least one of the terms in the parenthesis of the RHS of (4) must be positive. So, we can pick up in each set of integers (n_k, n_{k-1}), $k = 1,2,\ldots,$ such an index $n_j \in (n_k, n_{k-1})$ that $a_{n_j} - a_{n_{j-1}} \geq 0$. This implies that lim sup $(a_{n+1} - a_n) \geq 0$.

The proof for (ii) goes in the same manner as above. In particular, lim sup $a_k < \infty$, i.e., lim sup $a_n = \inf \sup (a_n, a_{n+1}, a_{n+2}, \ldots)$. So $\sup (a_n, a_{n+1})$ is nonincreasing, hence $a_{n_k} - a_{n_{k-1}} \leq 0$. This can be expressed as in (4), so at least one term in (4) is respective, and this implies lim inf $(a_{n+1} - a_n) \leq 0$.

Finally, part (iii) is standard and follows from the following known fact about limsup, namely $\limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n$; for (3b) we note that $\limsup c \cdot a_n = c \liminf a_n$. ■

Now we are ready to formulate our main result.

Theorem. For a general G|G|1 queue the following holds.

(i) If $\limsup EN^t < \infty$, then $\liminf EA^t \leq \limsup Pr\{N^t > 0\}$

(ii) If $\limsup EN^t = \infty$, then $\limsup EA^t \geq \liminf Pr\{N^t > 0\}$.

Proof. We prove first (i). From (2), Lemma (ii) and (3b) we have,

$$0 \geq \liminf (EN^{t+1} - EN^t) = \liminf (EA^t - Pr\{N^t > 0\}) \geq \liminf EA^t - \limsup Pr\{N^t > 0\}$$

which implies (i).

The second thesis comes from (2), Lemma (i) and (3b), i.e.,

$$0 \leq \limsup (EN^{t+1} - EN^t) = \limsup (EA^t - Pr\{N^t > 0\}) \geq \limsup EA^t - \liminf Pr\{N^t > 0\}$$

and this immediately proves (ii). ■