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GRADUATE SCHOOL
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By Lucia Rut Capdevila

Entitled

A TRANSFER NETWORK LINKING EARTH, MOON, AND THE TRIANGULAR LIBRATION POINT REGIONS IN THE EARTH-MOON SYSTEM

For the degree of Doctor of Philosophy

Is approved by the final examining committee:

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4/19/2016

Date

A TRANSFER NETWORK LINKING EARTH, MOON,
AND THE TRIANGULAR LIBRATION POINT REGIONS
IN THE EARTH-MOON SYSTEM

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Lucía R. Capdevila

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

May 2016

Purdue University

West Lafayette, Indiana

For my family.

Fiat lux.

ACKNOWLEDGMENTS

Throughout my time as a Ph.D student, I have learned far more than what I can synthesize in this dissertation, and I owe that to everyone around me during the past 8 years.

Growing up, my parents always said “there is always money for books”. I would like to thank my parents, Ana and César, for making education a priority, and for their continued re-assurance that I would not get kicked out of Purdue. I would also like to thank my sister, Caro, for always defending me from “the bad guys” and for listening to me. Perhaps, I would have never finished this degree if it wasn’t for my dear Doctor Husband, Masaki. I would like to thank him for the seemingly endless support he has given me since we first became friends.

I am incredibly lucky that Professor Kathleen C. Howell agreed to become my advisor. I would like to thank her for her patience and understanding throughout all the years I have been her student, for the freedom to pursue my interests, and for not giving up on me. I would also like to thank my committee members, Professor Martin J. Corless, Professor William A. Crossley, and Professor James M. Longuski for their support since my time as an undergraduate student in their classes. I would like to express my gratitude to Professor Hiroshi Yamakawa for the amazing opportunity to work with him at Kyoto University, and for the hospitality that him and Mrs. Norie Yamakawa showed me during my time in Kyoto. I am also forever indebted to Professor Mai Bando for her kindness and friendship. I would like to thank my mentor, Nathan J. Strange, for the great opportunity to work with him and learn from him during my time at the Jet Propulsion Laboratory in Pasadena, California.

Dr. Navindran Davendralingam, Dr. Amanda J. Knutson, Dr. Aurélie Héritier, and Mrs. Linda Flack helped me through some of the darkest times during my Ph.D. I can never thank them enough for that. Through the years, I have been the beneficiary

of many enlightening discussions with Dr. Daniel J. Grebow and Dr. Thomas A. Pavlak. I am grateful to them for their help and, most importantly, for their friendship.

Throughout my time in Professor Howell's research group, I have had the honor to meet an incredible group of people: her students. I am thankful to each and everyone of them for all the things they taught me, for their support as we navigated through graduate school together, and for their friendship.

I would also like to thank the School of Aeronautics & Astronautics and the School of Engineering Education at Purdue for the financial support that made my graduate education possible. I would also like to express my gratitude to Mr. Eric Holloway and Professor Heidi Diefes-Dux for their support during my time as a teaching assistant in First-Year Engineering and for the tremendous development they personally coached me through.

As a doctoral student, I have had opportunities that influenced the course of my work and fostered immense personal growth. These experiences were made possible by the financial support of the National Science Foundation's (NSF) and Japan Society for the Promotion of Science's (JSPS) East Asia and Pacific Summer Institutes (EAPSI) fellowship, and the Jet Propulsion Laboratory Summer Internship Program (JPLSIP). I am grateful to both of these programs.

TABLE OF CONTENTS

	Page
LIST OF TABLES	vii
LIST OF FIGURES	viii
ABBREVIATIONS	xii
ABSTRACT	xiii
1 Introduction	1
1.1 Problem Introduction and Motivation	1
1.2 Previous Contributions	2
1.2.1 The Circular Restricted Three-Body Problem and Mission Trajectory Design	2
1.2.2 Transfers into Lunar Halo Orbits	5
1.2.3 Transfers into Lunar Distant Retrograde Orbits	5
1.2.4 Transfers to/from the Vicinity of L_4/L_5	6
1.3 Goal, Objectives, and Scope of this Investigation	6
2 Circular Restricted Three-Body Problem	9
2.1 Equations of Motion	9
2.2 Jacobi Constant	11
2.3 Equilibria	12
2.4 State Transition Matrix	13
2.5 Invariant Manifold Theory	15
2.5.1 Invariant Manifolds for Fixed Points	15
2.5.2 Maps	19
2.5.3 Invariant Manifolds Associated with Periodic Orbits	21
2.5.4 Computation of Invariant Manifolds	23
3 Numerical Methods	26
3.1 Differential Corrections	26
3.2 Continuation	27
3.2.1 Natural Parameter Continuation	28
3.2.2 Pseudo-Arclength Continuation	28
4 Regions of Stability in the Earth-Moon CR3BP	30
4.1 Distant Retrograde Orbits near Moon	30
4.2 Near-Rectilinear Halo Orbits near the Moon	39
4.3 Motion near the Triangular Libration Points	43

	Page
4.4 Selected Orbits	46
5 Transfer Network	48
5.1 Transfer Options between LEO and Lunar DRO	50
5.1.1 Methodology	50
5.1.2 Prograde Earth Departures	51
5.1.3 Retrograde Earth Departures	57
5.2 Transfer Options between Lunar DRO and $L_{4/5}$ SPOs	62
5.2.1 Methodology	62
5.2.2 Results for L_4 SPO	66
5.2.3 Results for L_5 SPO	74
5.3 Transfer Options between L_2 NRHO and Lunar DRO	80
5.3.1 Methodology	80
5.3.2 Results	81
5.4 Single Transfer Family from LEO to L_4 SPO	86
6 Summary, Concluding Remarks and Recommendations	91
6.1 Summary and Concluding Remarks	91
6.2 Recommendations	93
REFERENCES	94
Appendix: Transfer Data	101
VITA	111

LIST OF TABLES

Table	Page
2.1 Jacobi constant for libration points of the Earth-Moon CR3BP	13
4.1 Departure and arrival motions energy level, orbital period, and initial conditions	47
A.1 Data for select transfer options from LEO to lunar DRO with prograde Earth departures	102
A.2 Data for select transfer options from LEO to lunar DRO with retrograde Earth departures	104
A.3 Data for select transfer options from lunar DRO to L_4 SPO	105
A.4 Data for select transfer options from lunar DRO to L_5 SPO	107
A.5 Data for select transfer options from L_2 NRHO to lunar DRO	109
A.6 Data for select transfer options from LEO to L_4 SPO	110

LIST OF FIGURES

Figure	Page
2.1 Earth-Moon CR3BP rotating and inertial frames	10
2.2 Earth-Moon CR3BP libration points	12
2.3 Stable and unstable manifolds associated with a fixed point	18
2.4 Poincaré Map	20
2.5 Planar projection of stable and unstable manifolds associated with an L_2 halo orbit in the Earth-Moon system	25
4.1 Lunar DRO properties in the Earth-Moon CR3BP, (a) geometry in the Earth-Moon rotating frame for varying energy levels (low values of Jacobi constant, C , indicated by blue-like colors, are associated with high energy levels), (b) lunar DRO orbital period and resonances with the lunar orbital period	32
4.2 Lunar DRO planar, ν_1 , and out-of-plane, ν_2 , stability indices, (a) full ν range and planar period tripling bifurcations at $\nu_1 = -0.5$ (dashed green) (b) close-up view near bifurcation to 3D DRO family, $\nu_2 = 1$	33
4.3 Lunar DRO (black) and period-3 DRO (green) properties in the Earth-Moon CR3BP, (a) Earth-Moon rotating frame geometry at $C = 3$, (b) Earth-Moon rotating frame geometry at $C = 2.895$, (c) Earth-Moon rotating frame geometry at $C = 2.4$, (d) orbital period for various energy levels	35
4.4 Lunar period-3 DRO planar, ν_1 , and out-of-plane, ν_2 , stability indices, (a) full ν range, (b) close-up view near stability boundaries $\nu = \pm 1$	36
4.5 The lunar DRO stability region in the Earth-Moon CR3BP, (a) lunar DRO, period-3 DRO, and quasi-periodic DRO at $C = 2.91$ in the Earth-Moon rotating frame, (b) $x - \dot{x}$ surface of section at $y = 0, C = 2.91$, (c) period-3 DRO stable (blue) and unstable (red) manifold trajectories at $C = 2.91$, (d) size of DRO stability region at various Jacobi constant values	37
4.6 Lunar L_1 halo orbits in rotating barycentered Earth-Moon frame view .	40
4.7 Lunar L_2 halo orbits in rotating barycentered Earth-Moon frame view .	40

Figure	Page
4.8 Lunar halo orbits planar, ν_1 , and out-of-plane, ν_2 , stability indices (a) full ν range for L_1 halo orbits, (b) close-up near $\nu = \pm 1$ for L_1 halo orbits, (c) full ν range for L_2 halo orbits, (d) close-up near $\nu = \pm 1$ for L_2 halo orbits	41
4.9 Lunar halo orbital period (top figures) and energy level (bottom figures) for various perilune altitudes, (a) for L_1 halo orbits, (b) for L_2 halo orbits	42
4.10 Periodic orbits near $L_{4,5}$ in the Earth-Moon CR3BP, (a) short-period orbits in the Earth-Moon barycentered rotating frame, (b) short-period orbits stability (top) and orbital period (bottom) at various energy levels, (c) long-period orbits in the Earth-Moon barycentered rotating frame, (d) long-period orbits stability (top) and orbital period (bottom) at various energy levels	45
4.11 Departure and arrival motions near the Earth, Moon, and triangular libration points in the barycentered rotating Earth-Moon frame in the Earth-Moon CR3BP	46
5.1 Diagram of transfer network linking (black arrows) the regions near the Earth (dashed blue), Moon (dashed grey), and triangular libration points (dashed green) in the Earth-Moon system	48
5.2 Transfer options into lunar DRO from LEO with prograde Earth departures in terms of Time of Flight (TOF), lunar DRO insertion cost, ΔV_2 , and energy level in terms of Jacobi constant, C , saturated at $C = 1.6535$. Blue colors indicate low-energy transfer options, and red colors indicate high-energy solutions	52
5.3 Select transfer options from LEO into lunar DRO with prograde Earth departures in the rotating barycentered Earth-Moon frame view	53
5.4 Select transfer options from LEO into lunar DRO with prograde Earth departures in the inertial Earth-centered frame view	54
5.5 Period-3 DRO stable manifold trajectory arc at $C = 2.4069$ in the Earth-Moon rotating barycentered view	56
5.6 Period-3 DRO stable manifold trajectory arc at $C = 2.4667$ in the Earth-Moon rotating barycentered view	57
5.7 L_1 Lyapunov orbit at $C \approx 2.40629$ in the Earth-Moon rotating barycentered view	58
5.8 Transfer options into lunar DRO from LEO with retrograde Earth departures in terms of Time of Flight (TOF), lunar DRO insertion cost, ΔV_2 , and energy level in terms of Jacobi constant, C . Blue colors indicate low-energy transfer options, and red colors indicate high-energy solutions	59

Figure	Page
5.9 Select transfer options from LEO into lunar DRO with retrograde Earth departures in the rotating barycentered Earth-Moon frame view	60
5.10 Select transfer options from LEO into lunar DRO with retrograde Earth departures in the inertial Earth-centered frame view	61
5.11 Retrograde orbits about the Earth by R.A. Broucke in barycentered rotating Earth-Moon frame view, (a) Family G, member 70, $C = 1.0381$, (b) Family BD, member 71, $C = 0.8329$	63
5.12 Schematic depicting surface of section $\Sigma(\theta)$ and an initial guess for a transfer trajectory between lunar DRO and L_4 SPO	64
5.13 Transfer options from lunar DRO to L_4 SPO in terms of Time of Flight (TOF), total maneuver capability requirements, $\Sigma\Delta V$, and energy level in terms of Jacobi constant, C . Blue colors indicate low-energy transfer options, and red colors indicate high-energy solutions	67
5.14 Select transfer options from lunar DRO to L_4 SPO in the rotating barycentered Earth-Moon frame view	68
5.15 Select transfer options from lunar DRO to L_4 SPO in the rotating barycentered Earth-Moon frame view (continued)	69
5.16 Select transfer options from lunar DRO to L_4 SPO in the inertial Earth-centered frame view	70
5.17 Select transfer options from lunar DRO to L_4 SPO in the inertial Earth-centered frame view (continued)	71
5.18 Dynamical structures associated with transfer options from lunar DRO to L_4 SPO in the barycentered rotating Earth-Moon frame view, (a) 3:4 resonant orbit at $C = 2.8240$, (b) 3:4 resonant orbit at $C = 2.6310$, (c) 3:4 and 2:3 resonant orbits at $C = 2.9499$, (d) period-3 DRO and associated manifold trajectory arc at $C = 2.9036$, (e) period-3 DRO and associated manifold trajectory arc at $C = 2.9407$, (f) L_4 long-period orbit at $C = 2.8404$	73
5.19 Transfer options from lunar DRO to L_5 SPO in terms of Time of Flight (TOF), total maneuver capability requirements, $\Sigma\Delta V$, and energy level in terms of Jacobi constant, C . Blue colors indicate low-energy transfer options, and red colors indicate high-energy solutions	75
5.20 Select transfer options from lunar DRO to L_5 SPO in the rotating barycentered Earth-Moon frame view	76
5.21 Select transfer options from lunar DRO to L_5 SPO in the rotating barycentered Earth-Moon frame view (continued)	77

Figure	Page
5.22 Select transfer options from lunar DRO to L_5 SPO in the inertial Earth-centered frame view	78
5.23 Select transfer options from lunar DRO to L_5 SPO in the inertial Earth-centered frame view (continued)	79
5.24 Dynamical structures associated with transfer options from lunar DRO to L_5 SPO in the rotating barycentered frame view, (a) period-3 DRO unstable manifold trajectory arc at $C = 2.8846$ (b) long-period orbit near L_5 at $C = 2.8644$	80
5.25 Transfer options from L_2 NRHO to lunar DRO in terms of Time of Flight (TOF), total maneuver capability requirements, $\Sigma\Delta V$, and energy level in terms of Jacobi constant, C . Blue colors indicate low-energy transfer options, and red colors indicate high-energy solutions	82
5.26 Select transfer options from L_2 NRHO to lunar DRO in the rotating barycentered Earth-Moon frame view	83
5.27 Select transfer options from L_2 NRHO to lunar DRO in the rotating barycentered Earth-Moon frame view (continued)	84
5.28 Select transfer options from L_2 NRHO to lunar DRO in the inertial Moon-centered frame view	85
5.29 L_2 Axial orbit and associated manifold trajectory arc at $C = 3.0111$ in the rotating barycentered Earth-Moon frame view	86
5.30 L_2 NRHO and associated manifold trajectory arc at $C = 3.0244$ in the rotating barycentered Earth-Moon frame view	87
5.31 Vertical orbits and associated manifold trajectories at $C = 3.0445$ in the rotating barycentered Earth-Moon frame view. L_1 vertical orbit: Unstable manifolds (red), stable manifolds (blue). L_2 vertical orbit: Unstable manifolds (magenta), stable manifolds (light blue)	88
5.32 Direct transfer from LEO to L_4 SPO, (a) rotating barycentered frame view, (b) inertial Earth-centered frame view	89
5.33 Family of transfer options from LEO to L_4 SPO in terms of Time of Flight (TOF), L_4 SPO insertion cost, ΔV_2 , and energy level in terms of Jacobi constant, C . Blue colors indicate low-energy transfer options, and red colors indicate high-energy solutions	89
5.34 Transfer from LEO to L_4 SPO with lunar flyby	90

ABBREVIATIONS

BCP	Bi-Circular Problem
CR3BP	Circular Restricted Three-Body Problem
DPO	Distant Prograde Orbit
DRO	Distant Retrograde Orbit
DST	Dynamical Systems Theory
GTO	Geostationary Orbit
LEO	Low Earth Orbit
LLO	Low Lunar Orbit
LPO	Long-Period Orbit
ND	Nondimensional
NRHO	Near-Rectilinear Halo Orbit (also labeled ‘NRO’ in recent literature)
SEP	Solar Electric Propulsion
SPO	Short-Period Orbit
SRP	Solar Radiation Pressure
STM	State Transition Matrix
TOF	Time of Flight

ABSTRACT

Capdevila, Lucía R. PhD, Purdue University, May 2016. A Transfer Network Linking Earth, Moon, and the Triangular Libration Point Regions in the Earth-Moon System. Major Professor: Kathleen C. Howell.

In the near future, several space applications in the Earth-Moon system may require a spacecraft to hold a stable motion, but the transfer trajectory infrastructure to access such stable motions has not been fully investigated yet. The triangular libration points, L_4 and L_5 , in the Earth-Moon system have long been thought of as ideal locations for a communications satellite. Recently, Distant Retrograde Orbits (DROs) and Near-Rectilinear Halo Orbits (NRHOs) near the Moon have been identified as motion of interest for manned and unmanned missions with a focus on operations in cislunar space. The triangular libration points, as well as lunar DROs and NRHOs describe special types of possible motion for a spacecraft/satellite that is influenced solely by the gravitational fields of the Earth and the Moon. What is common to the three types of solutions is that they are practically stable, that is, a spacecraft/satellite can naturally follow the solution for extended periods of time without requiring significant course adjustment maneuvers. This investigation contributes to the infrastructure of a network of transfer trajectories connecting regions of stability located near the Earth, Moon, and the triangular libration points in the Earth-Moon system. Several transfer options between regions of stability are presented and discussed, including transfer options between Low Earth Orbit (LEO) and lunar DRO, lunar DRO and periodic orbits near L_4 and L_5 , as well as lunar DRO and L_2 NRHOs.

1. Introduction

1.1 Problem Introduction and Motivation

According to the 2013 Global Exploration Roadmap the Moon and lunar vicinity could be used as a stepping stone to furthering the exploration of deep space by better pairing manned and unmanned capabilities. [1] NASA's Evolvable Mars Campaign (EMC) includes a series of Proving Grounds Missions (PGMs) that will utilize cislunar space as a test bed to validate the technology required to, eventually, lead a manned expedition beyond the Earth-Moon system. [2] NASA's Asteroid Redirect Mission (ARM), including the Asteroid Redirect Robotic Mission (ARRM) that will return an asteroid boulder to the vicinity of the Moon, and the Asteroid Redirect Crewed Mission (ARCM) that will fly astronauts to rendezvous with the boulder in lunar orbit to bring samples back to Earth, is considering a lunar DRO as the asteroid boulder's lunar destination. [2] Lunar DROs have also been suggested as potential orbits for resource aggregation in cislunar space as part of the Evolvable Mars Campaign (EMC). [3] The lunar vicinity would become a resource and supply allocation and aggregation point, as well as staging location for missions in and beyond cislunar space. [4] Lunar 'halo' orbits have been considered ideal locations for a translunar space station due to the unobstructed line-of-sight from Earth. [5] Recently, however, Near-Rectilinear Halo Orbits (NRHOs, also termed *NROs* in the recent literature) are being sought after for their added ease of access to the lunar surface, and like lunar DROs, also being considered as potential resource aggregation orbits in cislunar space. [4] An increased human presence in deep space may place higher demands on the current communications infrastructure. In the future, deep space vehicles may benefit from the placement of relay communication satellites at the triangular Lagrange points in the Earth-Moon system, L_4 and L_5 , offering almost

continuous line-of-sight from Earth and the Moon, while remaining distant from the noise near the Earth. [5–7] While the current space exploration directions may shift, they highlight a larger, and more permanent need for space exploration: a transfer network between regions of interest in the Earth-Moon system. The goal of this investigation is to contribute to the development of the transfer network infrastructure in the Earth-Moon system by providing new transfer options linking regions of stability near the Earth, Moon and triangular libration points. New transfer options provide more flexibility to meet mission requirements in the trajectory design process for a variety of future missions in the Earth-Moon system.

1.2 Previous Contributions

1.2.1 The Circular Restricted Three-Body Problem and Mission Trajectory Design

The solution to the n -Body Problem (n BP) in celestial mechanics has been a focus of study for centuries. As early as 1687, Sir Isaac Newton published his Law of Gravitation in *Philosophiae Naturalis Principia Mathematica*. Previously, Newton had proven that the center of mass of n bodies moves with uniform speed and in a straight line. He had also geometrically solved the Two-Body Problem (2BP) by considering the relative motion of one body with respect to the other and proved that the solution is an ellipse. However, Newton’s 1687 publication caused a shift in the search approach for a solution to the n BP. The investigation moved away from geometry and toward analytical mechanics.

The formulation of the Circular Restricted Three-Body Problem (CR3BP) originated as an intermediate step in the attempt to solve a larger problem. The Three-Body Problem (3BP) was first formulated in the rotating frame by Leonhard Euler in his second lunar theory, published in 1772. That same year, Joseph Louis Lagrange identified the five equilibrium solutions in the CR3BP in his memoir. Lagrange’s solutions do predict the Trojan asteroids in the Sun-Jupiter system. However, the

predictions from Lagrange were not verified until 1906, when 588 Achilles was first discovered in orbit near the triangular libration point L_4 , i.e., ahead of Jupiter in its path around the Sun. In 1836, Carl Gustav Jacob Jacobi reformulated the problem and demonstrated that an integral of the motion exists. This integral leads to the constant of integration, Jacobi's Constant, that is associated with the equations of motion as formulated in a rotating frame. George William Hill used Jacobi's Constant to support the statement that the Moon can never leave its orbit about the Earth as a result of the zero velocity curves, bounding curves that exist as a result of Jacobi's Constant. In his publications from 1877 and 1878, Hill considered a modified version of the Sun-Earth restricted three-body problem to solve for the motion of the Moon. The modifying assumptions included the following: (i) zero solar parallax, (ii) zero solar eccentricity, and (iii) zero lunar inclination. Furthermore, Hill determined a family of periodic orbits around the Earth, one of which possessed the same period as that of the Moon. [8] [9]

Influenced by Hill, one of the most important contributors to an understanding of the 3BP was Jules Henri Poincaré. Between 1892 and 1899, Poincaré published the three volumes that comprise *Les Méthodes Nouvelles de la Mécanique Céleste*. Throughout these three volumes, Poincaré detailed new tools to study the 3BP. He identified various types of periodic orbits and proved the existence of an infinite number of such perfectly repeatable motions. He also proved that additional transcendental integrals necessary to reduce the order of the equations of motion and, thus, solve the 3BP, do not exist. [9]

Finally, in 1912, Karl Sundman published a solution to the problem in terms of a complete convergent series, but only for $n \leq 3$. [10] Then, in 1991, Qui-Dong Wang published a more general convergent series solution to the n -body problem. [11] However, the solutions presented by Sundman and Wang possess such a slow rate of convergence that both are currently impractical for any applications in mission design. Hence, the series solutions presently offer no additional insight into the n BP. Fortunately, other means of investigation are available. [10]

One of the first numerical approaches to investigate the 3BP was accomplished by Sir George Howard Darwin. Between 1897 and 1910, Darwin numerically determined several types of periodic orbits in the vicinity of the two libration points closest to the second primary, as well as periodic orbits about the primaries themselves. [8] Between 1913 and 1939, the members of the Copenhagen Observatory identified several types of orbits near the collinear points under the direction of Elis Strömgren. This group of researchers also identified families of analytical solutions that asymptotically approached and departed the triangular libration points. [8] In 1920, Forest Ray Moulton published a series of papers, compiled into one volume, under the name *Periodic Orbits*. This volume included analytical and numerical studies of periodic orbits with applications to the CR3BP. In particular, Moulton outlined the proof for the existence of three types of orbits near the collinear points. [12] This compilation also includes the construction of two- and three-dimensional orbits in the vicinity of the equilateral points by Thomas Buck. [13] Additionally, Moulton identified a family of retrograde periodic orbits that enclose L_4 . These retrograde orbits may be related to the analytical solutions that asymptotically approach and depart the triangular libration points originally studied by Strömgren. [8] As research into the CR3BP became more popular, different research groups focused on specific areas of the problem.

While early trajectory design for missions beyond the Earth were greatly influenced by two-body motion, [14] Farquhar's concept of a 'halo' orbit [15] and the success of International Sun-Earth Explorer-3 (ISEE-3) spacecraft [16] changed this forever. The ISEE-3 spacecraft was the first venture of a manmade vehicle to the region of space near a collinear libration point. [12] With a successful insertion into a Sun-Earth L_1 halo orbit on November 20th, 1978, and the achievement of its scientific goals, ISEE-3 paved the way for several subsequent libration point missions. [16] The CR3BP has since been successfully used for mission trajectory design. In 2010, the Acceleration, Reconnection, Turbulence, and Electrodynamics of the Moons Interaction with the Sun (ARTEMIS) mission spacecraft P1 was the first artificial satellite to

insert into orbit about the translunar Earth-Moon libration point, L_2 . [17] The success of ARTEMIS, perhaps, is a significant contributing factor to the recent increased interest in cislunar space.

1.2.2 Transfers into Lunar Halo Orbits

The three-dimensional, collinear libration point orbiting, periodic solution known as the ‘halo’ orbit is perhaps the most popular Circular Restricted Three-Body solution available. These solutions were first found by Farquhar and Kamel, [18] and multitudes of researchers have contributed the understanding of halo orbits and their properties since. [19–22] Transfers into lunar halo orbits from Earth have been the subject of numerous investigations including direct, low-energy and low-thrust transfers. [23–26] A great deal of effort has also been vested in the computation low-cost transfers, low-thrust, and homoclinic and heteroclinic connections between lunar halos and other lunar orbits that possess manifolds. [27–29] Very recently, Parrish et al. have presented low-thrust transfers from a 5.5 day lunar DRO to 14 day L_2 Halo orbit. [30] However, transfers between stable lunar orbits, like DROs, and stable lunar halo orbits, like NRHOs, has not been investigated yet.

1.2.3 Transfers into Lunar Distant Retrograde Orbits

Distant retrograde orbits are well-known trajectory types. [31–35] Such orbits have also been the focus of a number of investigations concerning transfers. However, most of these transfers have involved DROs about the Earth [36,37] or Jupiter’s moon Europa. [38–40]. The investigation of transfers into Lunar DROs is more recent. In 2009, Ming and Shijie presented the first two transfer trajectories from LEO into a lunar DRO: a low-energy transfer, and a high energy arc and Lyapunov manifold combination type transfer. [41] The announcement of ARM, however, led to an increased interest in transfers into lunar DROs. Direct [42–48] and powered lunar flyby [43,46,48,49] type transfers have been heavily investigated in the recent years

as potential transfer options for ARCM, the crewed mission to rendezvous with an asteroid in lunar DRO. [50] Longer TOF transfer options into lunar DRO have also been investigated for cargo type missions that may serve in the process of resource aggregation in cislunar space. Transfer options with longer TOF before lunar DRO insertion include low-energy transfers [51,52] and combination type transfers, where the spacecraft uses a high energy, Hohmann-like, transfer arc to leave the Earth’s vicinity and inserts into a Lyapunov orbit [43] or Lyapunov manifold arc [51] as a means to reach lunar DRO insertion. Few indirect type transfers have been explored. [52]

1.2.4 Transfers to/from the Vicinity of L_4/L_5

The first transfers to/from the triangular libration points in the Earth-Moon system first appeared in 1979 when Broucke investigated direct transfers connecting the libration points of the Earth-Moon system and the Moon in support of the placement of large manned space stations near the CR3BP equilibrium points. [53] Since Broucke’s results, other investigations have focused on direct transfers to the vicinity of Earth-Moon L_4 and L_5 , however, originating at Earth. [54–56] Indirect transfers from Earth to the Earth-Moon triangular points have also been investigated, both interior [55–59] and exterior [56,60] type transfers. Few investigations have focused on transfers to the Earth-Moon triangular points from the Moon since Broucke’s results. [61]

1.3 Goal, Objectives, and Scope of this Investigation

The goal of this investigation is to contribute to the development of a transfer network connecting regions of stability near the Earth, Moon, and triangular libration points in the Earth-Moon system. Therefore, to address the goal of this investigation, the following supporting objectives have been identified:

- Contribute transfer options from Earth to regions of stability near the Moon

- Contribute transfer options between regions of stability near the Moon and the triangular libration points in the Earth-Moon system
- Contribute transfer options between regions of stability near the Moon
- Contribute insight about Dynamical Systems Theory (DST) mechanisms associated with transfer options identified

In particular, this investigation focuses on the regions of stability along the Distant Retrograde Orbit (DRO) family, the L_2 Near-Rectilinear Halo Orbits (NRHOs) near the Moon, and Short-Period Orbits (SPOs) near the triangular libration points in the Earth-Moon system. Thus, in support of this investigations objectives, the following tasks were identified:

1. Compute direct and indirect transfers between LEO and a lunar DRO
2. Compute transfers between a lunar DRO and an L_2 NRHO
3. Compute transfers between a lunar DRO and and L_4 and an L_5 SPO
4. Identify underlying dynamical structures guiding transfer options between regions of interest

Representative solutions are selected from the lunar DRO, L_2 NRHO, and triangular libration point SPO families and only two maneuver transfer options are considered here. Furthermore, this investigation is organized as follows:

- **Chapter 2:** Background information about the Circular Restricted Three-Body Problem (CR3BP) is provided in this chapter, including equations of motion, invariant manifold theory, and definitions of stability used in this investigation.
- **Chapter 3:** Numerical methods employed in this investigation include differential corrections and continuation schemes. A brief description of both is provided in this chapter.

- **Chapter 4:** Each stability region of interest to this investigation is discussed in detail in this chapter, and representative selected orbits are presented.
- **Chapter 5:** All transfer options found in this investigation are presented and discussed in this chapter.
- **Chapter 6:** Concluding remarks and recommendations for future work are offered in this chapter.

2. Circular Restricted Three-Body Problem

In this investigation, the motion of the spacecraft in the Earth-Moon environment is modeled in terms of the Circular Restricted Three-Body Problem (CR3BP). The CR3BP model is complex enough to retain important characteristics of the real Earth-Moon system, yet simple enough to retain a great deal of dynamical structure that can be used in trajectory design.

2.1 Equations of Motion

In the Earth-Moon CR3BP, the primaries move along circular orbits about their common barycenter, and the third particle, that is, the spacecraft is assumed to be massless. The equations of motion (EOMs) formulated in the rotating frame can be expressed as follows

$$\begin{aligned}\ddot{x} - 2\dot{y} &= \frac{\partial U}{\partial x} \\ \ddot{y} + 2\dot{x} &= \frac{\partial U}{\partial y} \\ \ddot{z} &= \frac{\partial U}{\partial z}\end{aligned}\tag{2.1}$$

where $U_a = \frac{\partial U}{\partial a}$ and U is the pseudo-potential function that can be computed as follows

$$U = \frac{1}{2}(x^2 + y^2) + \frac{(1 - \mu)}{r_{13}} + \frac{\mu}{r_{23}}\tag{2.2}$$

where the scalar quantities in the denominator, r_{13} and r_{23} , are the distances to the spacecraft measured from the Earth and the Moon, respectively. These equations describe the position of the spacecraft, $\vec{R} = [x, y, z]^T$, and its velocity, $\dot{\vec{R}} = \vec{V} = [\dot{x}, \dot{y}, \dot{z}]^T$, in terms of components in the rotating reference frame. The rotating reference frame is centered at the barycenter of the Earth-Moon system,

B , and rotates with the primaries. The rotating $\hat{x}$ axis is directed from the Earth to the Moon, $\hat{y}$ is perpendicular to $\hat{x}$ and in the plane of the primaries, and $\hat{z} = \hat{x} \times \hat{y}$ completes the right handed triad. The rotating reference frame is related to the inertial frame through a simple rotation, θ , about the $\hat{z} = \hat{Z}$ axis as seen in figure 2.1. The rotating frame coordinates are nondimensional based on the characteristic

Figure 2.1. Earth-Moon CR3BP rotating and inertial frames

length, $L^* \cong 384,388$ km (the distance between the Earth and the Moon), and time, $T^* = \sqrt{\frac{L^{*3}}{G(m_{\oplus} + m_{\lrcorner})}}$, where G is the universal gravitational constant, and $m_{\oplus}$ and $m_{\lrcorner}$ are the masses of the Earth and the Moon, respectively. The characteristic mass parameter, μ , is defined as $\mu = \frac{m_{\lrcorner}}{m_{\oplus} + m_{\lrcorner}}$, and for this investigation is $\mu = 0.012151$. Furthermore, a useful property of the CR3BP EOMs is given by the *Mirror Theorem*

Theorem 2.1 (The Mirror Theorem) *If n-point masses are acted upon by their mutual gravitational forces only, and at a certain epoch the radius vector from the (assumed stationary) center of mass of the system is perpendicular to every velocity vector, then the orbit of each mass after that epoch is a mirror image of its orbit prior to that epoch.*

Moreover, if any trajectory satisfies the CR3BP EOMs, the reflection of this trajectory across the $\hat{x}$ axis is also a solution to the equations in negative time. [62]

2.2 Jacobi Constant

The CR3BP EOMs admit an integral of the motion known as the Jacobi constant, C . The scalar value depends of the pseudo-potential function, U , and the speed of the spacecraft such that

$$C = 2U - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad (2.3)$$

where the pseudo-potential function given in equation (2.2). The Jacobi constant is a measure of energy that combines information about a trajectory's geometry and speed, that is, a comprehensive parameter to characterize a trajectory in the CR3BP. Therefore, the Jacobi constant proves useful in distinguishing regimes of motion in the CR3BP. [8]

The minimum energy (or maximum Jacobi constant value) associated with a particular position in the CR3BP is very useful in the design of transfer trajectories. Rearranging equation (2.3) to solve for the velocity magnitude, v , yields $v = \sqrt{2U - C}$. Thus, if $v \in \Re$, then $2U - C \geq 0$. Thus, if a spacecraft is at position $\vec{R}_1$ with pseudo-potential function U_1 , for v_1 to be real, the bounds on Jacobi constant are such that $C \leq 2U_1$. The maximum Jacobi constant associated with position $\vec{R}_1$ is

$$C_{1\max} = 2U_1 \quad (2.4)$$

and the range of possible energies at position $\vec{R}_1$, that is, $C_1 : C \in [C_{1\max}, -\infty)$.

Now suppose, the spacecraft is initially at position $\vec{R}_1$ (with v_1 and U_1) and it is desired to perform a tangential maneuver that will put the spacecraft in a trajectory with Jacobi constant C_2 , where C_2 is bounded such that $C_2 \leq 2U_1$. The speed after the burn will be given by

$$v_2 = \sqrt{2U_1 - C_2} \quad (2.5)$$

and the required maneuver magnitude is

$$\Delta v = \sqrt{2U_1 - C_2} - v_1 \quad (2.6)$$

The calculation of this tangential maneuver to achieve a specific post maneuver energy level can be used to construct a trajectory that is tangent to a periodic orbit, but exists at a different energy level.

2.3 Equilibria

The CR3BP EOMs admit an infinity of non-trivial solutions. Five particular constant solutions are the equilibrium points, $L_i, i = 1, 2, \dots, 5$, also known as the libration points, and illustrated in Figure 2.2 as black dots. The collinear libration

Figure 2.2. Earth-Moon CR3BP libration points

points, L_1, L_2 , and L_3 , are linearly unstable, and the triangular or equilateral points, L_4 and L_5 , are linearly stable. Other, non-constant solutions are discussed in later chapters. Furthermore, each libration point exists at a different energy level. Table 2.1 lists the Jacobi constant corresponding to each libration point in the Earth-Moon CR3BP. It is apparent that $C_{L_1} > C_{L_2} > C_{L_3} > C_{L_{4,5}}$. The lower the Jacobi constant is, the higher the energy associated with the equilibria state. Furthermore, linearized motion about the libration points is an important building block in the understanding

Table 2.1. Jacobi constant for libration points of the Earth-Moon CR3BP

L_i	1	2	3	4,5
C_{L_i}	3.1883	3.1722	3.0121	2.9880

of nonlinear dynamics in the CR3BP; many families of periodic orbits in the CR3BP begin at a libration point. [8]

2.4 State Transition Matrix

Given a set of initial conditions, a numerical integration process will propagate the state forward in time. For various applications, the path may require modification, e.g., targeting, rendezvous, determination of periodic orbits, and guidance algorithms. To adjust the path correctly and efficiently requires information available from the State Transition Matrix (STM).

The linearized equations of motion can be obtained via a first-order Taylor expansion relative to the reference solution. If the reference solution is some nominal path, $\vec{x}_n(\tau) = \begin{bmatrix} x_n(\tau) & y_n(\tau) & z_n(\tau) & \dot{x}_n(\tau) & \dot{y}_n(\tau) & \dot{z}_n(\tau) \end{bmatrix}^T$, that varies with time, the variation $\delta\vec{x}(\tau)$ is defined as

$$\delta\vec{x}(\tau) = \vec{x}(\tau) - \vec{x}_n(\tau) \quad (2.7)$$

and the linear variational equations are

$$\delta\ddot{x} - 2\delta\dot{y} = U_{xx}|_{\vec{x}_n(\tau)}\delta x + U_{xy}|_{\vec{x}_n(\tau)}\delta y + U_{xz}|_{\vec{x}_n(\tau)}\delta z \quad (2.8)$$

$$\delta\ddot{y} + 2\delta\dot{x} = U_{yx}|_{\vec{x}_n(\tau)}\delta x + U_{yy}|_{\vec{x}_n(\tau)}\delta y + U_{yz}|_{\vec{x}_n(\tau)}\delta z \quad (2.9)$$

$$\delta\ddot{z} = U_{xz}|_{\vec{x}_n(\tau)}\delta x + U_{yz}|_{\vec{x}_n(\tau)}\delta y + U_{zz}|_{\vec{x}_n(\tau)}\delta z \quad (2.10)$$

where $U_{ab} = \frac{\partial U}{\partial a \partial b}$ is the second order partial of U with respect to a and b , and can be computed as follows

$$U_{xx} = 1 - \frac{(1-\mu)}{r_{13}^3} - \frac{\mu}{r_{23}^3} + \frac{3(1-\mu)(x+\mu)^2}{r_{13}^5} + \frac{3\mu(x-1+\mu)^2}{r_{23}^5} \quad (2.11)$$

$$U_{xy} = \frac{3(1-\mu)(x+\mu)y}{r_{13}^5} + \frac{3\mu(x-1+\mu)y}{r_{23}^5} = U_{yx} \quad (2.12)$$

$$U_{xz} = \frac{3(1-\mu)(x+\mu)z}{r_{13}^5} + \frac{3\mu(x-1+\mu)z}{r_{23}^5} = U_{zx} \quad (2.13)$$

$$U_{yy} = 1 - \frac{(1-\mu)}{r_{13}^3} - \frac{\mu}{r_{23}^3} + \frac{3(1-\mu)y^2}{r_{13}^5} + \frac{3\mu y^2}{r_{23}^5} \quad (2.14)$$

$$U_{yz} = \frac{3(1-\mu)yz}{r_{13}^5} + \frac{3\mu yz}{r_{23}^5} = U_{zy} \quad (2.15)$$

$$U_{zz} = -\frac{(1-\mu)}{r_{13}^3} - \frac{\mu}{r_{23}^3} + \frac{3(1-\mu)z^2}{r_{13}^5} + \frac{3\mu z^2}{r_{23}^5} \quad (2.16)$$

where variations are defined with respect to the nominal and the partials are evaluated along the path. Thus, equations (2.8)–(2.10) can be rewritten in matrix form as follows

$$\delta \dot{\vec{x}}(\tau) = A(\tau) \delta \vec{x}(\tau) \quad (2.17)$$

where $A(\tau)$ is

$$A(\tau) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ U_{xx}|_{\vec{x}_n(\tau)} & U_{xy}|_{\vec{x}_n(\tau)} & U_{xz}|_{\vec{x}_n(\tau)} & 0 & 2 & 0 \\ U_{yx}|_{\vec{x}_n(\tau)} & U_{yy}|_{\vec{x}_n(\tau)} & U_{yz}|_{\vec{x}_n(\tau)} & -2 & 0 & 0 \\ U_{zx}|_{\vec{x}_n(\tau)} & U_{zy}|_{\vec{x}_n(\tau)} & U_{zz}|_{\vec{x}_n(\tau)} & 0 & 0 & 0 \end{bmatrix} \quad (2.18)$$

Since the variational equations are linear with time-varying coefficients, the general solution to equation (2.17) is known to be

$$\delta \vec{x}(\tau) = \Phi(\tau, \tau_0) \delta \vec{x}(\tau_0) \quad (2.19)$$

where $\delta\vec{x}(\tau_0)$ is the initial vector variation in the state with respect to the nominal. Then, $\delta\vec{x}(\tau)$ is the variation downstream from the initial state at some time $\tau > \tau_0$. The STM is represented as $\Phi(\tau, \tau_0)$ and is defined as the matrix of partials

$$\Phi(\tau, \tau_0) = \frac{\partial \vec{x}(\tau)}{\partial \vec{x}(\tau_0)} \quad (2.20)$$

Furthermore, from equation (2.19), it is apparent that at time $\tau = \tau_0$, $\delta\vec{x}(\tau) = \delta\vec{x}(\tau_0)$ and

$$\Phi(\tau_0, \tau_0) = I \quad (2.21)$$

If equation (2.19) is substituted into equation (2.17), the following system of differential equations is obtained for the elements of $\Phi(\tau, \tau_0)$

$$\dot{\Phi}(\tau, \tau_0) = A(\tau)\Phi(\tau, \tau_0) \quad (2.22)$$

Thus, the elements of the STM can be integrated numerically for all time, τ .

2.5 Invariant Manifold Theory

An understanding of invariant manifold theory is critical for trajectory design in this regime. Manifolds offer a basis for the computation of transfers between different regions in the CR3BP. Invariant manifolds associated with periodic orbits are of particular interest. Therefore, relevant concepts about invariant manifolds for fixed points are initially introduced. Maps are then used to relate fixed points to periodic orbits. Finally, invariant manifolds for periodic orbits and the process for their numerical computation is detailed.

2.5.1 Invariant Manifolds for Fixed Points

Investigation of nonlinear systems includes the determination of equilibrium points and the linearization relative to such solutions whenever possible. Such a step usually offers the first insight into the behavior of the system. Nonlinear systems possess other properties as well.

Consider the following system of nonlinear differential equations

$$\dot{\vec{x}}(\tau) = \vec{f}(\vec{x}(\tau)) \quad (2.23)$$

where $\vec{x}(\tau)$ is an m -dimensional state vector. Recall that the system in equation (2.23) can be linearized about some reference or nominal solution, $\vec{x}_n(\tau)$, such that

$$\delta\dot{\vec{x}}(\tau) = A(\tau)\delta\vec{x}(\tau) \quad (2.24)$$

where $A(\tau) = D\vec{f}(\vec{x}_n(\tau))$ is the Jacobian matrix of first partial derivatives of $\vec{f}$, evaluated on the reference solution. If $\vec{x}_n(\tau) = \vec{x}_n$ is a fixed point, i.e., an equilibrium or stationary point of the system in equation (2.23), then, $A(\tau) = A$ is constant. In the case of a constant A , the general solution to equation (2.24) is

$$\delta\vec{x}(\tau) = e^{A(\tau-\tau_0)}\delta\vec{x}(\tau_0) \quad (2.25)$$

Equation (2.25) can be written in the form

$$\delta\vec{x}(\tau) = \sum_{j=1}^m C_j e^{\lambda_j(\tau-\tau_0)} \vec{v}_j \quad (2.26)$$

where C_j are coefficients to be determined from the initial conditions; $\lambda_1, \dots, \lambda_m$ are the eigenvalues of A and $\vec{v}_1, \dots, \vec{v}_m$ are the eigenvectors of A that span $\mathfrak{R}^m$. [63] [64]

To gain insight into the behavior of the solution in equations (2.25) and (2.26), it is necessary to examine the vector subspace(s) containing the solution at the initial time, τ_0 . Let s eigenvalues possess a negative real part, u eigenvalues have a positive real part and, then, c eigenvalues are purely imaginary, such that $m = s + u + c$. Thus, $\mathfrak{R}^m$ can be decomposed into three vector subspaces:

$$\begin{aligned} \text{(i) the stable subspace, } E^s &= \text{span} \{ \vec{v}_1, \dots, \vec{v}_s \} \\ \text{(ii) the unstable subspace, } E^u &= \text{span} \{ \vec{v}_{s+1}, \dots, \vec{v}_{s+u} \} \\ \text{(iii) the center subspace, } E^c &= \text{span} \{ \vec{v}_{s+u+1}, \dots, \vec{v}_{s+u+c} \} \end{aligned} \quad (2.27)$$

Note that equation (2.26) depends on the initial conditions, $\delta\vec{x}(\tau_0)$, that dictate which C_j will be nonzero. Thus, $\delta\vec{x}(\tau_0)$ also specifies the vector subspace(s) that define the

solution at τ_0 . Since E^s , E^u , and E^c are invariant, if $\delta\vec{x}(\tau)$ is contained in a vector subspace(s) at τ_0 , then, $\delta\vec{x}(\tau)$ will remain in the same vector subspace(s) as the system evolves. Thus, from equation (2.26), it is apparent that for any $\delta\vec{x}(\tau_0) \in E^s$, $\delta\vec{x}(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$, for any $\delta\vec{x}(\tau_0) \in E^u$, $\delta\vec{x}(\tau) \rightarrow 0$ as $\tau \rightarrow -\infty$, and for any $\delta\vec{x}(\tau_0) \in E^c$, $\delta\vec{x}(\tau)$ will remain in the vicinity of $\vec{x}_n$ as $\tau \rightarrow \pm\infty$. [64]

The flow in the vicinity of the fixed point, $\vec{x}_n$, can be linked to the stable and unstable invariant manifolds corresponding to that fixed point through the following definition by Guckenheimer and Holmes [64]:

Definition 2.1 *The local stable and unstable manifolds of $\vec{x}_n$, $W_{loc}^s(\vec{x}_n)$, and $W_{loc}^u(\vec{x}_n)$, are defined as follows*

$$\begin{aligned} W_{loc}^s(\vec{x}_n) &= \{ \vec{x} \in U \mid \phi_\tau(\vec{x}) \rightarrow \vec{x}_n \text{ as } \tau \rightarrow \infty, \text{ and } \phi_\tau(\vec{x}) \in U \text{ for all } \tau \geq 0 \} \\ W_{loc}^u(\vec{x}_n) &= \{ \vec{x} \in U \mid \phi_\tau(\vec{x}) \rightarrow \vec{x}_n \text{ as } \tau \rightarrow -\infty, \text{ and } \phi_\tau(\vec{x}) \in U \text{ for all } \tau \leq 0 \} \end{aligned}$$

where U is a neighborhood of $\vec{x}_n$ and $\phi_\tau(\vec{x})$ is a mapping from $\mathbb{R}^m$ to $\mathbb{R}^m$. The invariant manifolds, $W_{loc}^s(\vec{x}_n)$ and $W_{loc}^u(\vec{x}_n)$, are nonlinear analogues of the eigenvector subspaces of the linear system, E^s and E^u .

The following theorem, as stated by Guckenheimer and Holmes, is particularly useful to relate the stable and unstable invariant manifolds to the stable and unstable vector subspaces at the fixed point. [64]

Theorem 2.2 (Center Manifold Theorem for Flows) *Let $\vec{f}$ be a C^r vector field on $\mathbb{R}^m$ vanishing at the origin ($\vec{f}(\vec{0}) = \vec{0}$) and let $A = D\vec{f}(\vec{0})$. Divide the spectrum of A into three parts, σ_s , σ_c , σ_u with*

$$\Re(\lambda) \begin{cases} < 0 & \text{if } \lambda \in \sigma_s, \\ = 0 & \text{if } \lambda \in \sigma_c, \\ > 0 & \text{if } \lambda \in \sigma_u. \end{cases}$$

Let the (generalized) eigenspaces of σ_s , σ_c , and σ_u be E^s , E^c , and E^u , respectively. Then there exist C^r stable and unstable invariant manifolds W^u and W^s tangent to E^u and E^s at 0 and a C^{r-1} center manifold W^c tangent to E^c at 0. The manifolds W^u ,

W^s , and W^c are all invariant for the flow of $\vec{f}$. The stable and unstable manifolds are unique, but W^c need not be.

The invariant manifolds, W^s and W^u , mentioned in the *Center Manifold Theorem for Flows*, are the global analogues to the local invariant manifolds, W_{loc}^s and W_{loc}^u . The global invariant manifolds can be obtained by propagating points in W_{loc}^s backwards in time and by propagating points in W_{loc}^u forwards in time. [64] A planar projection of the stable and unstable manifolds at $\vec{x}_n$ appears in Figure 2.3. For the example in Figure 2.3, the stable and unstable eigenvector subspaces are each comprised of one eigenvector, i.e., $E^s = \vec{v}_s$ and $E^u = \vec{v}_u$. The invariant manifold branch W^{s+} corresponds to the propagation of $\vec{v}_s$ and, W^{s-} corresponds to the propagation of $-\vec{v}_s$. Similarly, W^{u+} corresponds to the propagation of $\vec{v}_u$ and W^{u-} corresponds to the propagation of $-\vec{v}_u$.

Figure 2.3. Stable and unstable manifolds associated with a fixed point

2.5.2 Maps

The techniques employed to study the behavior of the flow in the vicinity of an equilibrium point are no longer useful when the reference solution is a function of time. However, if the continuous-time system can be reduced to a discrete-time system via a mapping, then the time-varying nominal solution can be reduced to a fixed point. Congruously, methods similar to those previously presented for a stationary point can be developed to study the behavior of the flow in the vicinity of a time-varying reference solution. A periodic orbit is an example of such a solution.

Let $\vec{x}_n^*$ identify a fixed point. Consider the periodic orbit Γ and let it be a solution to equation (2.23) through point $\vec{x}_n^*$ and with period T . Let Σ be the $(m - 1)$ -dimensional hypersurface transversal to Γ at $\vec{x}_n^*$. Then, for any point $\vec{x} \in \Sigma$ sufficiently close to $\vec{x}_n^*$, the solution to equation (2.23) through $\vec{x}$, will intersect Σ again at point $P(\vec{x})$, near $\vec{x}_n^*$. The mapping $P : \vec{x} \rightarrow P(\vec{x})$ is a first return or Poincaré map. [65] [66] Furthermore, P is smooth and possesses a smooth inverse, i.e., it is diffeomorphic. [64] Figure 2.4 illustrates the concept of a Poincaré map for a low-dimensional system.

A stroboscopic map is a special type of Poincaré map that samples the flow at periodic intervals. When the reference is the periodic orbit, the sampling period is that of the periodic orbit. This type of mapping effectively reduces Γ to the fixed point, $\vec{x}_n^* \in \Sigma$, since $\vec{x}^* \rightarrow P(\vec{x}^*) = \vec{x}^*$. Recall that equation (2.23) can be linearized relative to the nominal solution, in this case Γ , to obtain equation (2.24); the general solution to equation (2.24) is equation (2.19). Furthermore, equation (2.19) can be rewritten for the discrete-time system with period T as follows

$$\delta\vec{x}(\tau_0 + kT) = \Phi(\tau_0 + kT, \tau_0 + (k - 1)T) \delta\vec{x}(\tau_0 + (k - 1)T) \quad (2.28)$$

where $\delta\vec{x}(\tau_0 + kT)$ and $\delta\vec{x}(\tau_0 + (k - 1)T)$ are the perturbations from the periodic orbit at the k^{th} and $(k - 1)^{th}$ crossing of Σ . Note that the STM is the first derivative of P ,

Figure 2.4. Poincaré Map

i.e., $\Phi(\tau_0 + kT, \tau_0 + (k-1)T) = DP$. Using the STM for each crossing, equation (2.28) can be rewritten in the following manner

$$\delta\vec{x}(\tau_0 + kT) = \prod_{j=1}^k \Phi(\tau_0 + (k-j+1)T, \tau_0 + (k-j)T) \delta\vec{x}(\tau_0) \quad (2.29)$$

where $\delta\vec{x}(\tau_0)$ is some initial perturbation with respect to Γ . Because the following property is true for all integers $k \geq 2$,

$$\Phi(\tau_0 + (k-1)T, \tau_0 + (k-2)T) = \Phi(\tau_0 + kT, \tau_0 + (k-1)T) \quad (2.30)$$

equation (2.29) reduces to

$$\delta\vec{x}(\tau_0 + kT) = \Phi(\tau_0 + T, \tau_0)^k \delta\vec{x}(\tau_0) \quad (2.31)$$

where $\Phi(\tau_0 + T, \tau_0)$ is the STM for one period of Γ , termed the monodromy matrix. If $\Phi(\tau_0 + T, \tau_0)$ possesses m distinct eigenvalues, $\lambda_1, \dots, \lambda_m$, with corresponding eigenvectors, $\vec{v}_1, \dots, \vec{v}_m$, then, equation (2.31) is expressed as

$$\delta\vec{x}(\tau_0 + kT) = \sum_{j=1}^m C_j \lambda_j^k \vec{v}_j \quad (2.32)$$

where C_j are coefficients determined from the initial perturbation, $\delta\vec{x}(\tau_0 + kT)$. As is apparent from equation (2.32), the magnitude of the eigenvalues $|\lambda_j|$ will dictate the expansion or contraction of the perturbation after k iterations of the map. If $|\lambda_j| > 1$, $|\delta\vec{x}(\tau_0 + kT)|$ will expand as $k \rightarrow \infty$, if $|\lambda_j| < 1$, $|\delta\vec{x}(\tau_0 + kT)|$ will contract as $k \rightarrow \infty$ and if $|\lambda_j| = 1$, $|\delta\vec{x}(\tau_0 + kT)|$ remain unchanged as $k \rightarrow \infty$. [66]

2.5.3 Invariant Manifolds Associated with Periodic Orbits

For a periodic orbit represented as a fixed point on a map, it is possible to define the invariant stable and unstable manifolds associated with the periodic orbit using the map. Once again, consider the periodic orbit Γ , represented by fixed point $\vec{x}^*$ on the hypersurface Σ , and the stroboscopic map $P : \vec{x} \rightarrow P(\vec{x})$. Parker and Chua [66] give the following definitions for stable and unstable invariant manifolds of Γ at $\vec{x}^*$:

Definition 2.2 *The stable manifold of Γ , denoted by $W^s(\Gamma)$, is defined as the set of all points $\vec{x}$ such that $P^k(\vec{x})$ approaches Γ as $k \rightarrow \infty$.*

Definition 2.3 *The unstable manifold of Γ , denoted by $W^u(\Gamma)$, is defined as the set of all points $\vec{x}$ such that $P^k(\vec{x})$ approaches Γ as $k \rightarrow -\infty$.*

Let $DP = \Phi(\tau_0 + T, \tau_0)$ be evaluated as the monodromy matrix associated with a periodic orbit of period T . The m matrix eigenvalues are computed where the number of eigenvalues that can be identified such that $|\lambda_j| < 1$ is s , u eigenvalues possess magnitudes $|\lambda_j| > 1$, and c eigenvalues are computed with unit magnitude

such that $|\lambda_j| = 1$, such that $s + u + c = m$. Then, the stable, unstable, and center vector subspaces associated with the monodromy matrix at $\vec{x}^*$ are [64]

$$\begin{aligned} E^s &= \text{span} \{ \vec{v}_j \mid |\lambda_j| < 1 \} \\ E^u &= \text{span} \{ \vec{v}_j \mid |\lambda_j| > 1 \} \\ E^c &= \text{span} \{ \vec{v}_j \mid |\lambda_j| = 1 \} \end{aligned} \tag{2.33}$$

Then, $W^s(\Gamma)$ and $W^u(\Gamma)$ are locally tangent to E^s and E^u at $\vec{x}^*$, respectively, and have the same dimension as the associated vector subspace. [66] Furthermore, the monodromy matrix possesses a very specific eigenstructure. Lyapunov's theorem states [67, 68]

Theorem 2.3 (Lyapunov's Theorem) *If λ_j is an eigenvalue of the monodromy matrix, $\Phi(\tau_0 + T, \tau_0)$, of a t -invariant system, then λ_j^{-1} is also an eigenvalue of $\Phi(\tau_0 + T, \tau_0)$, with the same structure of elementary divisors.*

In the CR3BP, the second-order system possesses three degrees of freedom and, thus, the monodromy matrix is defined in terms of six eigenvalues. Because Γ is a periodic orbit, one pair of eigenvalues is always equal to one; let $\lambda_1 = \lambda_2 = 1$. If some $\lambda_3 \in \Re$, then there exists a $\lambda_4 \in \Re$ such that $\lambda_4 = \frac{1}{\lambda_3}$. Of course λ_3 and λ_4 then correspond to the local stable and unstable modes. Also, if some λ_5 is complex, then there exists a complex conjugate $\lambda_6 = \lambda_5^*$. However, periodic solutions do not always include a reciprocal pair of real eigenvalues, e.g., λ_3 and λ_4 . Thus, periodic solutions are not always associated with one-dimensional stable and unstable invariant manifolds.

Any family of periodic orbits in the CR3BP may include some members that possess stable and unstable invariant manifolds and some that do not. Therefore, it is of interest to identify those members that do possess stable and unstable manifolds for potential transfers. Two stability indices, ν_1 and ν_2 , can be defined for each member in a family of periodic orbits to qualify the member's stability, that is,

$$\nu_1 = \frac{1}{2}(\lambda_3 + \lambda_4) \tag{2.34}$$

$$\nu_2 = \frac{1}{2}(\lambda_5 + \lambda_6) \tag{2.35}$$

where the pairs of eigenvalues, $\lambda_{3,4}$ and $\lambda_{5,6}$ may be real or complex. A periodic orbit is said to be stable when $|\nu_i| \leq 1, i = 1, 2$ and ν_i is real. When a periodic orbit is stable, there are no stable and unstable manifolds associated with it, only the center manifold exists. Otherwise, if any $|\nu_i| > 1, i = 1, 2$, the periodic orbit is considered unstable, and there exist a stable and an unstable manifold associated with it. Furthermore, the larger the stability index value gets, the faster the manifolds approach or depart the periodic orbit. [20, 69, 70] The eigenvalues of the monodromy matrix associated with a periodic orbit in the CR3BP can also provide information about other types of neighboring flow. Certain values of ν_i indicate the presence and type of nearby families of periodic orbits. The explanations by Grebow and Bosanac are very helpful in understanding and applying this concept. [71, 72]

2.5.4 Computation of Invariant Manifolds

To use invariant manifolds for transfers, their computation is, of course, required. Let the monodromy matrix associated with a periodic orbit include the eigenvalue λ_u associated with eigenvector $\vec{v}_u$, such that $|\lambda_u| > 1$. The invariant manifolds associated with the periodic orbit Γ intersect at the fixed point $\vec{x}^*$, that is, the point that represents the orbit on hyperplane Σ . Thus, if $\vec{x}^*$ is removed from $W^u(\vec{x}^*)$, two half-manifolds, $W^{u+}(\vec{x}^*)$ and $W^{u-}(\vec{x}^*)$, result. Note that, similar to an equilibrium point, $W^{u+}(\vec{x}^*)$ and $W^{u-}(\vec{x}^*)$ diverge from $\vec{x}^*$ along the directions $\vec{v}_u$ and $-\vec{v}_u$, respectively. [66, 68] Let $\vec{x}^{W^{u+}} \in W^{u+}(\vec{x}^*)$ be a point on the half-manifold, and let $\vec{W} \subset W^{u+}(\vec{x}^*)$ be the set of points on $W^{u+}(\vec{x}^*)$ between $\vec{x}^{W^{u+}}$ and the first iterate of the map, $P(\vec{x}^{W^{u+}})$. Then,

$$W^{u+}(\vec{x}^*) = \bigcup_{j=-\infty}^{\infty} P^j(\vec{W}) \quad (2.36)$$

The set $\vec{W}$ can be determined by selecting some $\vec{x}^{W^{u+}}$ that lies on $\vec{v}_u$ and is close to $\vec{x}^*$

$$\vec{x}^{W^{u+}} = \vec{x}^* + \alpha \vec{v}_u \quad (2.37)$$

where $\alpha > 0$. [66] Traditionally, the eigenvector $\vec{v}_u$ is normalized in position, since position is usually much larger than the velocity components. Thus, equation (2.37) can be written such that

$$\vec{x}^{W^{u+}} = \vec{x}^* + \left(\frac{d}{v_u^{(R)}} \right) \vec{v}_u \quad (2.38)$$

where $v_u^{(R)}$ is the norm of the position components in the eigenvector $\vec{v}_u$ and $d > 0$. The displacement d is sufficiently small to avoid violating the linear approximation but still large enough to allow the unstable manifold to actually depart the vicinity of the periodic orbit within a reasonable time interval. For example, a typical value for the Earth-Moon system is $d \cong 50$ km. [68] Thus, the initial conditions from equation (2.38) are numerically integrated forwards in time to obtain $W^{u+}(\vec{x}^*)$. Similarly, initial conditions are obtained for $W^{u-}(\vec{x}^*)$, $W^{s+}(\vec{x}^*)$ and $W^{s-}(\vec{x}^*)$ as follows

$$\vec{x}^{W^{u\pm}} = \vec{x}^* \pm d \frac{\vec{v}_u}{v_u^{(R)}} \quad (2.39)$$

$$\vec{x}^{W^{s\pm}} = \vec{x}^* \pm d \frac{\vec{v}_s}{v_s^{(R)}} \quad (2.40)$$

Like $\vec{x}^{W^{u+}}$, $\vec{x}^{W^{u-}}$ is integrated numerically forward in time to obtain $W^{u-}(\vec{x}^*)$. However, $\vec{x}^{W^{s+}}$ and $\vec{x}^{W^{s-}}$ are integrated backwards in time to obtain $W^{s+}(\vec{x}^*)$ and $W^{s-}(\vec{x}^*)$, respectively. The planar projections of the stable and unstable manifolds for an L_2 halo orbit in the Earth-Moon system appear in Figure 2.5.

Figure 2.5. Planar projection of stable and unstable manifolds associated with an L_2 halo orbit in the Earth-Moon system

3. Numerical Methods

3.1 Differential Corrections

Finding trajectories with specific characteristics can be done in a variety of ways. In this investigation, an initial guess that approximates the desired solution is first produced, and then refined until a more precise approximation is found. In particular, a multidimensional Newton-Raphson's method, or Newton's method, is employed. [73] Usually, desired trajectory characteristics such as specific altitude from a body, time of flight, periodicity, etc, are known and can be expressed as a set of m linear or nonlinear scalar equations, $f_j, j = 1, 2, \dots, m$, the elements of the constraints vector, $\vec{F}(\vec{X})$, in the form given by equation (3.1)

$$\vec{F}(\vec{X}) = \begin{Bmatrix} f_1(\vec{X}) \\ f_2(\vec{X}) \\ \dots \\ f_m(\vec{X}) \end{Bmatrix} = \vec{0} \quad (3.1)$$

The constraints are functions on the problem's n free-variables, $x_k, k = 1, 2, \dots, n$, expressed in vector form, $\vec{X}$, as shown in (3.2)

$$\vec{X} = [x_1, x_2, \dots, x_n]^T \quad (3.2)$$

where the free-variables themselves could be the state variables in (2.1) and/or functions of the state variables. According to the Newton-Raphson Method, the vector of free-variables that satisfies (3.1), that is $\vec{X}^*$, can be found iteratively to within a specified tolerance, ϵ , by applying the recursive formula in (3.3) when the number of free-variables is the same as the number of constraints, that is, $m = n$.

$$\vec{X}_{i+1} = \vec{X}_i - D\vec{F}(\vec{X}_i)^{-1}\vec{F}(\vec{X}_i) \quad (3.3)$$

Note that $D\vec{F}(\vec{X}_i)$ is the Jacobian, $\frac{\partial \vec{F}(\vec{X}_i)}{\partial \vec{X}_i}$. When the number of free-variables is larger than the number of constraints, that is, $m < n$, formula in (3.4) should be used instead to update the free-variables vector,

$$\vec{X}_{i+1} = \vec{X}_i - D\vec{F}(\vec{X}_i)^+ \vec{F}(\vec{X}_i) \quad (3.4)$$

where $D\vec{F}(\vec{X}_i)^+$ is the Moore-Penrose pseudo-inverse, in this case defined by the right matrix inverse of $D\vec{F}(\vec{X}_i)$ that yields the minimum norm solution to the linear systems of equation in (3.5)

$$D\vec{F}(\vec{X}_i)(\vec{X}_{i+1} - \vec{X}_i) = \vec{F}(\vec{X}_i) \quad (3.5)$$

and can be computed by (3.6)

$$D\vec{F}(\vec{X}_i)^+ = D\vec{F}(\vec{X}_i)^T \left(D\vec{F}(\vec{X}_i) D\vec{F}(\vec{X}_i)^T \right)^{-1} \quad (3.6)$$

The iterative method begins with the initial guess, $\vec{X}_0$. During each iteration both, $\vec{F}(\vec{X}_i)$ and $D\vec{F}(\vec{X}_i)$ must be computed, and doing so will require the numerical propagation of the CR3BP equations of motion in (2.1). Finally, the solution is deemed found when the norm of the constraints function is within the specified tolerance, that is $\|\vec{F}(\vec{X}^*)\| \leq \epsilon$.

Frequently, trajectories of interest traverse sensitive regions of the CR3BP, and it may be difficult for a Newton-Raphson solver to find a solution when all the sensitivity of the trajectory is concentrated at the initial state. Thus, the trajectory is usually broken up into smaller trajectory arc segments, by adding ‘nodes’ along the trajectory and enforcing full state continuity between the arcs at these nodes.

3.2 Continuation

Having generated and corrected an initial guess using the Newton-Raphson method described in section 3.1 to obtain a trajectory with the desired characteristics, it may be of interest to find the pertaining family curve that the initial solution belongs to. In this investigation, a family curve is found by natural parameter or pseudo-arclength

continuation. Depending on the problem, a particular method of continuation may be better suited to obtain the family of solutions sought.

3.2.1 Natural Parameter Continuation

Natural parameter continuation can be achieved by letting one of the constraints in equation (3.1) enforce the desired value of a natural parameter throughout the corrections process, and then taking “small” steps in that parameter to obtain a new solution along the same family. For example, if the natural parameter is the initial x -position, x_0 , and it is desired to set it to the specific value x_d , then the constraint can be written as $x_0 - x_d = 0$. After an initial solution, $\vec{X}_1^*$, is obtained by applying the Newton-Raphson iterative method, an initial guess for the next solution, $\vec{X}_2^{*'}$, can be obtained by taking a sufficiently small step, Δx_0 , in x_0 , that is, $x_{0_1}^* + \Delta x_0$, while other free-variables are unchanged. Then, the new solution, $\vec{X}_2^*$, can be found by applying the Newton-Raphson method again, where this time $x_d = x_{0_1}^* + \Delta x_0$. This process can be repeated to obtain the family of solutions.

3.2.2 Pseudo-Arclength Continuation

Pseudo-arclength continuation predicts a new solution in the same family curve of solutions using the family tangent at a solution already known. Suppose initial solution, $\vec{X}_1^*$ has already been computed using a Newton-Raphson method, where the Jacobian, $D\vec{F}(\vec{X}_1^*)$ has one more column than rows, that is, there is one more free-variable than there are constraints. The family tangent, $\vec{\eta}_1^*$ is the nullspace of the Jacobian at $\vec{X}_1^*$, that is, $\mathcal{N}(D\vec{F}(\vec{X}_1^*))$. Thus, the prediction, $\vec{X}_2^{*'}$, for the new solution can be calculated by (3.7)

$$\vec{X}_2^{*'} = \vec{X}_1^* + ds\vec{\eta}_1^* \quad (3.7)$$

where s is the step-size of the pseudo-arclength continuation method, and d sets the direction of the continuation, that is, $+1$ or -1 . For the first prediction in either

direction, d must be set arbitrarily to one of the two possible values. Then, to find $\vec{X}_2^*$, the Newton-Raphson's constraints vector function is augmented by one constraint as expressed in (3.8)

$$\tilde{\vec{F}}(\vec{X}) = \begin{bmatrix} \vec{F}(\vec{X}) \\ (\vec{X} - \vec{X}_1^*)^T \vec{\eta}_1^* - s \end{bmatrix} \quad (3.8)$$

This constraint ensures that solutions are evenly spaced apart, but is not absolutely necessary. If included, though, the Jacobian matrix must also be augmented by one row as given in (3.9)

$$D\tilde{\vec{F}}(\vec{X}) = \begin{bmatrix} D\vec{F}(\vec{X}) \\ \vec{\eta}_1^{*T} \end{bmatrix} \quad (3.9)$$

so as to become square. Then, the Newton-Raphson method can be applied to $\vec{X}_2^{*/}$ using $\tilde{\vec{F}}(\vec{X})$ and $D\tilde{\vec{F}}(\vec{X})$ instead of the original constraints vector function and Jacobian. Once, the first new solution is found, $\vec{X}_2^*$, the process can be repeated to predict a second new solution on the same family curve. In this second prediction, d can be computed by (3.10)

$$d = \text{sign}(\vec{\eta}_1^* \cdot \vec{\eta}_2^*) \quad (3.10)$$

where $\vec{\eta}_2^*$ is the family tangent direction at the corrected new solution, $\mathcal{N}(D\vec{F}(\vec{X}_2^*))$.

4. Regions of Stability in the Earth-Moon CR3BP

The objective of this investigation is to contribute to the development of a transfer network to connect regions of stability near the Earth, Moon, and triangular libration points in the Earth-Moon system. In particular, it is of interest to connect the vicinity of the Earth with stability regions defined by the lunar Distant Retrograde Orbit (DRO) family, the L_2 Near Rectilinear Halo Orbits (NRHOs), and the Short-Period Orbits (SPOs) near L_4 and L_5 in the Earth-Moon system. In this chapter, each stability region of interest is introduced. Representative orbits in each region are selected and presented at the end.

4.1 Distant Retrograde Orbits near Moon

The emergence of DROs in the literature is not recent. As early as 1968, Broucke's work offers evidence of the existence of DROs in the Earth-Moon system. Broucke's comprehensive investigation on x -axis symmetric periodic orbits in the planar Earth-Moon CR3BP mentions a family of retrograde lunar orbits labeled family "C". Members with single loops around the Moon, as viewed in the rotating reference frame, are observed as stable. Broucke also notes that family C corresponds to Stromgren's class "f". [74] Several authors have since contributed to the understanding of the dynamical properties of DROs and the surrounding neighborhood in different systems. [31,32,34,35] In Hill's limiting case of the planar restricted three-body problem, these orbits were investigated by Hénon. The 1969 publication by Hénon refers to these orbits as family "f". [31]

While there are many ways to compute DROs, here, their computation begins with a set of initial conditions provided by Hénon in the Hill's problem. [31] Hénon's initial conditions are transitioned to the Earth-Moon CR3BP using a differential corrections

method that adjusts the initial conditions to obtain a periodic orbit in the Earth-Moon CR3BP. The DRO initial conditions have the form $[x_0, 0, 0, 0, \dot{y}_0, 0]^T$. During the corrections process, x_0 remains fixed, $\dot{y}_0$ is adjusted to achieve a perpendicular crossing of the x -axis on the other side of the Moon, while time to the next crossing is free to vary. The Earth-Moon DRO is then used in a natural parameter continuation process in x_0 to obtain other members in the family. The corrected DRO initial conditions are used as an initial guess for the next member in the family, with the only difference between the two being a small step in x_0 . Then, the new initial guess is re-corrected to produce a new DRO. A large portion of the DRO family appears in Figure 4.1(a). The color of each orbit indicates the Jacobi constant value, specified by the colorbar on the right. It is apparent that the path of some members extends very far from the Moon, hence the name Distant Retrograde Orbits (DROs). It is also evident that DROs orbiting close to the Moon possess high Jacobi constant values, close to the Jacobi constant values for L_1 ($C_{L_1} = 3.18834$) and L_2 ($C_{L_2} = 3.17216$), while DROs with close passes of the Earth are characterized by much lower Jacobi constant values.

The period of an orbit complements the geometry in terms of the speed of the spacecraft along a particular path in comparison to others; speed is also indicative of the specific three-body system. The DRO orbital periods in figure 4.1(b), are plotted as a function of Jacobi constant. DROs close to the Moon (high C) are characterized by very short periods, but DROs grow in size (decreasing C) as the orbital period passes through the 1:3 and 2:3 lunar resonances, and asymptotically approaches a 1:1 resonance with the Moon. From Broucke’s 1968 report, a family of Earth retrograde orbits also exists in the Earth-Moon planar CR3BP, i.e., family “ A_1 ”. As Earth DROs evolve from orbits in the close vicinity of the Earth to orbits with close passages of the Moon, they appear as a mirror image of the lunar DROs across the line $x \approx 0.5$. Similar to the family of lunar DROs, as Earth DROs approach the collision orbit with the Moon, the period of these orbits approaches the period of the primary system. Also, notable is the fact that not all Earth DROs are stable. [74]

(a)

(b)

Figure 4.1. Lunar DRO properties in the Earth-Moon CR3BP, (a) geometry in the Earth-Moon rotating frame for varying energy levels (low values of Jacobi constant, C , indicated by blue-like colors, are associated with high energy levels), (b) lunar DRO orbital period and resonances with the lunar orbital period

The stability of a periodic orbit reveals much information concerning the behavior of the associated neighboring flow, and a change in stability can signal bifurcations with other families of periodic orbits. The stability indices, ν_1 and ν_2 , for the lunar DRO family appear in figure 4.2(a) as a function of energy level, that is, Jacobi constant, C . In this case, since this family of lunar DROs is planar, the in-plane and

Figure 4.2. Lunar DRO planar, ν_1 , and out-of-plane, ν_2 , stability indices, (a) full ν range and planar period tripling bifurcations at $\nu_1 = -0.5$ (dashed green) (b) close-up view near bifurcation to 3D DRO family, $\nu_2 = 1$

out-of-plane motions are de-coupled, and ν_1 is associated with the in-plane flow, while ν_2 is associated with the out-of-plane flow. Lunar DROs are stable for $C \geq 2.36871$, as indicated by $|\nu_i| \leq 1, i = 1, 2$ in figure 4.2(a). Also, several period-multiplying bifurcations can be detected. The lowest multiplicity occurs at $C \cong 2.95494$ and $C \cong 2.84913$, where $\nu_1 = -0.5$, indicating a period-tripling bifurcation. Hénon labeled this family as “g3”. [32] Here, however, such three-revolution periodic orbits are denoted as period-3 DROs (P3DROs). Higher multiplicity period-multiplying bifurcations also exist, as introduced by Markellos in 1974 in the Sun-Jupiter CR3BP, and in 2007 by Douskos, Kalantonis and Markellos in the Earth-Moon system. [33, 35] However, when out-of-plane motion is considered, a bifurcation with a three-dimensional family

is detected at $C \cong 2.36871 < C_{L_2} < C_{L_1}$, indicated by $|\nu_2| > 1$ in figure 4.2(b). Vaquero and Howell note this bifurcation and show the three-dimensional family. [42] The three-dimensional family of orbits is stable, while DROs that exist at $C < 2.36871$ are slightly sensitive to perturbations in the out-of-plane direction.

Although P3DROs are only one of numerous period-multiplying families that bifurcate from the planar DRO family, they play an important role in characterizing the neighboring DRO dynamics. The computation of period-3 DROs begins with the identification of the period-tripling bifurcating orbit in the DRO family, that is, a DRO whose monodromy matrix has a pair of eigenvalues at the 3rd root of unity, $\sqrt[3]{1}$. This first P3DRO is then used to compute other members in the same family using a multiple-shooting differential corrections method and natural parameter continuation in Jacobi constant in a similar fashion to Bosanac’s “variable-time multiple shooting algorithm”. [72] Given that a DRO at a particular energy level exists, a corresponding P3DRO also exists. The rotating frame geometry of both, P3DRO and DRO families, at select energy levels appear in figure 4.3, where the DRO appears in black, and the P3DRO is colored in green. For the sample trajectories, it is evident that the path along the P3DRO remains loosely in the vicinity of the DRO. The time along one revolution across the family of P3DROs evolves similarly to the DRO orbital period, as demonstrated in figure 4.3(d). However, in contrast with DROs, P3DROs are linearly unstable. The stability indices for the P3DRO family appear in figure 4.4. Since P3DROs are a planar family of orbits, the in-plane and out-of-plane motions are decoupled. In this case, ν_1 is associated with in-plane motion, while ν_2 is associated with out-of-plane motion. Figure 4.4(a) shows $\nu_1 > 1$ for the entire range of lunar P3DROs shown, reflecting the family’s linear instability. Figure 4.4(b) shows a closer look near $-1 \geq \nu_i \geq 1$ range revealing that $\nu_1 = 1$ at $C \cong 2.95494$ and $C \cong 2.84913$, corresponding to the bifurcation with the DRO family. It is also apparent from figure 4.4(b) that $\nu_2 = 1$ at $C \cong 2.6048, C \cong 2.6466$ signaling two bifurcations to families of periodic orbits, and three bifurcations to period-doubling families at $C \cong 2.3471, C \cong 2.8795, C \cong 2.9102$, when $\nu_2 = -1$. Since, in this case ν_2

Figure 4.3. Lunar DRO (black) and period-3 DRO (green) properties in the Earth-Moon CR3BP, (a) Earth-Moon rotating frame geometry at $C = 3$, (b) Earth-Moon rotating frame geometry at $C = 2.895$, (c) Earth-Moon rotating frame geometry at $C = 2.4$, (d) orbital period for various energy levels

is associated with out-of-plane frequencies, the period-doubling bifurcations indicated by ν_2 point to three-dimensional families of periodic orbits.

The ‘DRO stability region’ is comprised of motion associated with the stable, unstable and center manifold subspaces of DROs and period-3 DROs. As unstable orbits, P3DROs have associated stable and unstable manifolds, that is, flow that asymptotically approaches or departs the orbit, respectively. Stable DROs, how-

Figure 4.4. Lunar period-3 DRO planar, ν_1 , and out-of-plane, ν_2 , stability indices, (a) full ν range, (b) close-up view near stability boundaries $\nu = \pm 1$

ever, only possess an associated center manifold subspace, that is, flow that neither departs nor approaches the orbit. The ‘DRO stability region’ is comprised of the various period-multiplying families previously mentioned as well as quasi-periodic DROs (QPDROs), that is, solutions on the center manifold subspace associated with the DROs. [35] The P3DRO stable and unstable manifolds bound the DRO stability region at all energy levels. [39] However, the stability region is destroyed at the each of the period-tripling bifurcations previously highlighted. This phenomenon was first observed by Hénon in Hill’s problem; [32] Markellos, then, confirmed it in the Sun-Jupiter CR3B problem. [33] Winter, in 2000, and then Douskos, Kalantonis and Markellos, in 2007, corroborated the same dynamical behavior in the Earth-Moon CR3BP. [34, 35]

Understanding the DRO stability region is useful in the identification of dynamical mechanisms that enable transfer options into/out of this region. To better understand and visualize this region, consider the energy level $C = 2.91$. The lunar DRO that exists at this Jacobi constant value is plotted in black in figure 4.5(a). The green path is the P3DRO, and a planar QPDRO at the same energy level appears in gray.

Figure 4.5. The lunar DRO stability region in the Earth-Moon CR3BP, (a) lunar DRO, period-3 DRO, and quasi-periodic DRO at $C = 2.91$ in the Earth-Moon rotating frame, (b) $x - \dot{x}$ surface of section at $y = 0$, $C = 2.91$, (c) period-3 DRO stable (blue) and unstable (red) manifold trajectories at $C = 2.91$, (d) size of DRO stability region at various Jacobi constant values

However, there exists an infinity of QPDROs at this same energy level, but, due to the high number of lunar revolutions, it is not useful to visualize them in the $x - y$ space. Instead, the dynamical structures in the region surrounding the DRO are more easily visualized on a surface of section. Consider a two-sided surface of section along the x -axis at a specific Jacobi constant value, C^* , that is, $y = 0$ and $C = C^*$. The surface associated with $C^* = 2.91$ yields the map that appears in figure 4.5(b). It is apparent that similar structures emerge for positive (left) and negative (right) y -velocities, that is, on both DRO crossings of the x -axis. Focusing only on the left half with $\dot{y} > 0$, the DRO is represented as a single black dot, several QPDROs appear in gray, and the P3DRO appears as three hyperbolic points on the section. As previously mentioned, the stable (blue) and unstable (red) P3DRO manifolds bound the stability region, hence, they appear at the edge of all the quasi-periodic motion about the DRO. A spacecraft moving along any path within the stability region, will remain there indefinitely unless it suffers a change in energy. The P3DRO manifolds provide a natural means of transport to/from the edges of the DRO stability region. Figure 4.5(c) shows the stable (blue) and unstable (red) manifolds associated with the P3DRO (green) that exists at $C = 2.91$ as they appear in the $x - y$ plane of the rotating frame, and the DRO at the same energy level appears in black. The $x - y$ view of the P3DRO manifolds shows how they reach from the vicinity of the DRO (black) to the vicinity of the Earth, as well as the triangular libration points, L_4 and L_5 . Note, however, that only larger P3DROs (P3DROs that pass close to Earth) possess manifolds that reach LEO. This information about P3DRO manifolds is useful in understanding the dynamical structures that enable transfer options to/from the DRO region. The distance between the DRO fixed point, and the perpendicular crossing of the P3DRO can be used to represent the ‘size’ of the stability region. If these fixed points are computed at different energy levels, it is possible to generate a graphical representation that indicates the ‘size’ of the stability region at different energy levels. A limited range of information is plotted in figure 4.5(d) across the DRO, P3DRO and period-4 DRO families in the (x_0, C) plane. In the figure, x_0 is the

x -position of the periodic orbit as it crosses the x -axis perpendicularly ($\dot{x} = 0$) with positive y -velocity ($\dot{y} > 0$), and C is the orbit's Jacobi constant. As expected, DROs and P3DROs intersect when $C \cong 2.95494$ and $C \cong 2.84913$ at the period-tripling bifurcations. The quasi-periodic region surrounding the DRO disappears at these Jacobi constant values. The period-4 DROs originate from a period-quadrupling bifurcation with the DRO family, at $C \approx 2.99572$, and exist within the stability region. [35]

4.2 Near-Rectilinear Halo Orbits near the Moon

The renowned L_1 and L_2 ‘halo’ orbit families accompany the DRO in the lunar region. Farquhar first formulated the concept of a ‘halo orbit’ when looking for a translunar coverage orbit without obstruction by the Moon. [15] In 1973, Farquhar and Kamel used the Lindstedt-Poincaré method to approximate three-dimensional quasi-periodic solutions near translunar libration point L_2 and found “a ‘halo orbit’ of the Moon”, for large amplitude solutions. [18] In 1979, Breakwell and Brown continued the family of L_2 halo orbits, and computed the L_1 halo orbit family, as they evolve from the vicinity of the libration point towards the Moon reaching almost-rectilinear orbits. [19] Howell extended these families, and the L_3 halo orbit family to other systems. [20] [21]

Here, L_1 and L_2 halo orbits are computed using a differential corrections process and a natural parameter continuation scheme. [61] Part of the L_1 and L_2 halo orbit family appears in figures 4.6 and 4.7, respectively, where each closed curve represents a periodic orbit as they appear in the rotating frame. The $y - z$ projection in figures 4.6 and 4.7 reveals the characteristic ‘halo’ around the Moon shape that gives them their name. This geometry allows for continuous line of sight with the Earth. Note that the halos shown here are typically referred to as ‘northern halos’ because they extend above the xy plane in the rotating frame. ‘Southern halos’ can be obtained by applying the *Mirror Theorem*. [62] The orbits colored in green represent unstable

Figure 4.6. Lunar L_1 halo orbits in rotating barycentered Earth-Moon frame view

Figure 4.7. Lunar L_2 halo orbits in rotating barycentered Earth-Moon frame view

periodic orbits, while those colored in black represent stable orbits. Recall that a periodic orbit is unstable when at least one of the stability indices is greater than 1 in magnitude, and when both indices are within ± 1 , $|\nu_i| \leq 1, i = 1, 2$, the periodic orbit is stable. Figures 4.8(a) and 4.8(c) show the value of both stability indices for each of the L_1 and L_2 halo orbit families as a function of perilune altitude. It is

Figure 4.8. Lunar halo orbits planar, ν_1 , and out-of-plane, ν_2 , stability indices (a) full ν range for L_1 halo orbits, (b) close-up near $\nu = \pm 1$ for L_1 halo orbits, (c) full ν range for L_2 halo orbits, (d) close-up near $\nu = \pm 1$ for L_2 halo orbits

apparent that as the halo orbits approach the Moon, the ν_1 decreases exponentially. A closer view at the $-1 \geq \nu_i \geq 1$ range shown in figures 4.8(b) and 4.8(d) reveals ν_2 as remaining within or close to $-1 \geq \nu_i \geq 1$ for the entire range of both the L_1 and L_2 halo orbit families. It is apparent that there is a range of L_1 halo orbits over

which $|\nu_i| \leq 1$ that corresponds to the halo orbits colored in black in figure 4.6. For the L_2 halo orbits, the stability indices are within ± 1 at the same time over two separate ranges corresponding to the two bands of black orbits in figure 4.7. While technically unstable, the lower perilune altitude halo orbits have stability indices that of such small magnitude, that the unstable manifolds require extremely long times to leave the vicinity of the orbit. Therefore, there is a practical stability region near the Moon along the L_1 and L_2 halo orbit families. Furthermore, the halo orbit stability indices also indicate various types of nearby families of periodic orbits that are not discussed here. Apparent from the $x - z$ view in figures 4.6 and 4.7, halos near the Moon are near-rectilinear, thus, they are termed Near-Rectilinear Halo Orbits (NRHOs). The low lunar passage associated with NRHOs may be advantageous for missions requiring access to the lunar surface. The stability of NRHOs also makes them suitable for crewed missions in cis-lunar space, and/or as a resource aggregation orbit in the lunar region. [4] Figures 4.9(a) and 4.9(b) show how the orbital period and Jacobi constant decrease dramatically as halos approach the Moon. However, note

Figure 4.9. Lunar halo orbital period (top figures) and energy level (bottom figures) for various perilune altitudes, (a) for L_1 halo orbits, (b) for L_2 halo orbits

that while L_2 halo orbits terminate at the Moon, the L_1 halo orbit family continues beyond the range shown here to wrap around the Earth, and cycle back onto itself

where it begins as shown by Grebow in his dissertation. [71] Furthermore, many other 3-body orbits accompany the halo families in the lunar region, including families of vertical and axial orbits. [70]

4.3 Motion near the Triangular Libration Points

Among destinations beyond the Moon, the triangular Lagrange points, L_4 and L_5 , in the Earth-Moon system have long been regarded as ideal locations to place communications relay satellites to support deep space missions. [5] Also, in 1961, Kordylewski reported cloud-like satellites near the Earth-Moon triangular points. [75] Although there have been observations to support [76], as well as those that could not confirm Kordylewski's report [77], the notion of cloud-like satellites near the Earth-Moon triangular libration points has motivated countless investigations.

In 1970, Schutz and Tapley found initial conditions for a particle initially at Earth-Moon L_5 that would yield a 14-year period librating orbit when propagated with the ephemeris Sun, Earth, and Moon. However, the deviations from the libration point did not coincide with the observed Kordylewski clouds. [78] In 1975, Katz's simulations, incorporating the true eccentricities of the Sun, Earth and Moon, as well the effect of Solar Radiation Pressure (SRP), indicate that a particle in the vicinity of the triangular points does not possess long term stability. [79] In 2000, Castella and Jorba demonstrated that when the Sun's gravitational perturbation is added to the Earth-Moon CR3BP motion, the point mass is no longer in equilibrium at the triangular points. In the Bi-Circular Problem (BCP), where the Sun moves in a co-planar circular orbit about the Earth-Moon CR3BP, L_4 and L_5 are replaced by dynamical equivalents: three distinct planar periodic orbits, each with an orbital period that matches that of the Sun around the Earth-Moon system. Also, associated with each of the planar orbits, there exists a family of three-dimensional quasi-periodic orbits. [80] Two of the planar orbits, as well as the associated vertical family of quasi-periodic orbits are linearly stable, and the region of stability around them persists in

the real ephemeris model. [81] More recently, Jorba et al. added SRP to the BCP and showed that the regions of stability near the triangular points off the Earth-Moon plane may still be present for objects larger than dust-like particles and/or small asteroids. [82] In this investigation, however, the motion of the spacecraft is modeled according to the Earth-Moon CR3BP to reduce the complexity in this initial look at transfer options into/out of the region of stability near the triangular libration points.

In chapter IX of Moulton's 1920 "Periodic Orbits" Thomas Buck provides analytical approximations for planar short-period orbits (SPOs) and long-period orbits (LPOs), as well as a three-dimensional vertical orbit near the triangular points. [13] In this investigation, the focus is on planar transfers to/from the triangular region, therefore, only SPOs and LPOs will be discussed further. In 1935, Pedersen contributes the first 3^{rd} order approximation of equilateral SPOs. [83] In his 1961 publication, Rabe includes a survey of planar periodic orbits of the Trojan type, that is, orbits in the vicinity of the triangular points in the Sun-Jupiter system, including equilateral LPOs. [84] The following year, 1962, Rabe and Schanzle reproduce the investigation in the Earth-Moon system. [85] Later, in 1966, Goodrich uses Rabe's approach to compute SPOs near L_4 and L_5 . [86] In 1967, Deprit et al. provide the full genealogy of both, the short- and long-period families at L_4 and L_5 . [87]

The computation of triangular libration point SPOs and LPOs begins with an initial guess from the linearized motion in the vicinity of the equilibria as described by Szebehely [8] Then, through a predictor-corrector algorithm developed by Markellos and Halioulas for asymmetric periodic solutions [88], it is possible to generate numerous members in both families of SPOs [70] and LPOs. [61]. Figures 4.10(a) and 4.10(c) show part of the SPO and LPO families, respectively, as they appear in the rotating frame. The color of each orbit indicates its energy in terms of Jacobi constant, C . Note that while the colorbar looks the same, the Jacobi constant range is not the same in both figures. It is apparent that SPOs become more energetic as they move away from the immediate vicinity of the libration point, while LPOs do the opposite. The bottom plot in figures 4.10(b) and 4.10(d) show that orbital

periods increase with increasing Jacobi constant, meaning, as orbits move away from the libration point SPOs have increasingly shorter periods, while LPOs have increasingly longer periods. The upper plot in figures 4.10(b) and 4.10(d) shows that SPOs are all stable for the range shown since both stability indices are within the stability envelope of ± 1 , while LPOs become slightly unstable at $C = 2.995961$. Beyond the

Figure 4.10. Periodic orbits near $L_{4,5}$ in the Earth-Moon CR3BP, (a) short-period orbits in the Earth-Moon barycentered rotating frame, (b) short-period orbits stability (top) and orbital period (bottom) at various energy levels, (c) long-period orbits in the Earth-Moon barycentered rotating frame, (d) long-period orbits stability (top) and orbital period (bottom) at various energy levels

ranges shown here, the families of SPOs and LPOs evolve as explained by Deprit et al. The LPO family evolves away from the equilateral point, developing three loops that grow to match a member of the SPO family at a period-tripling bifurcation. The

SPO family evolves past the period-tripling bifurcation orbit with the LPO family to a bifurcation orbit where the SPO families from L_4 and L_5 meet the L_3 Lyapunov family. While the family of LPOs has many stable and unstable members, all SPO family members are stable in the planar problem. [87] Furthermore, LPOs are useful in understanding transfer options into/out of the triangular libration point region.

4.4 Selected Orbits

Prior to the study of transfer options, representative orbits are selected from each stability region of interest. All selected orbits appear in black in figure 4.11. Table

Figure 4.11. Departure and arrival motions near the Earth, Moon, and triangular libration points in the barycentered rotating Earth-Moon frame in the Earth-Moon CR3BP

4.1 give the characteristics, that is, the Jacobi constant, C , the period, and the initial state for each of the selected orbits. In the vicinity of the Earth, the orbit selection

Table 4.1. Departure and arrival motions energy level, orbital period, and initial conditions

Orbit Name	C	Period [days]	x_0 [nd]	y_0 [nd]	z_0 [nd]	$\dot{x}_0$ [nd]	$\dot{y}_0$ [nd]	$\dot{z}_0$ [nd]
DRO	2.9604	10.0225	0.8495	0	0	0	0.4794	0
L_2 NRHO	3.0277	7.9414	1.0455	0	0.1946	0	-0.1490	0
L_4 SPO	2.9132	28.3488	-0.2255	0.8660	0	-0.2384	0.2494	0
L_5 SPO	2.9132	28.3488	-0.2255	-0.8660	0	0.2384	0.2494	0

is limited to a circular, 200 km altitude LEO. In the lunar region, a 10-day DRO and at almost 8-day NRHO have been selected. At the triangular points, a rather large, 28.8 day SPO has been selected.

5. Transfer Network

Having identified representative departure/arrival motions in each stability region of interest, it is possible to explore transfer options, and thus, establish a transfer network between these regions. The selected lunar DRO, L_2 NRHO, L_4 and L_5 SPOs serve as departure and arrival motions to be connected by various transfer options. Each pair of orbits considered in this investigation is represented in figure 5.1 by the black bidirectional arrow connecting the two orbits. Each black arrow also represents

Figure 5.1. Diagram of transfer network linking (black arrows) the regions near the Earth (dashed blue), Moon (dashed grey), and triangular libration points (dashed green) in the Earth-Moon system

the various transfers options that may exist between the pair of orbits it connects. The dual direction on each arrow symbolizes transfers to and from each orbit. For example, transfers from LEO to lunar DRO, and from lunar DRO to LEO are both considered in this investigation.

It is readily apparent from figure 5.1 that not all possible pair of orbits are considered in this investigation. Some of the possible pairs are not considered in this investigation because they have been investigated before, while others are recom-

mended for future work. In particular, transfer options between LEO and L_4 and L_5 SPO are generally not considered here because the literature shows that transfers that pass by the lunar region can significantly reduce the transfer ΔV cost. [55,57,58]. Instead, it is assumed that a transfer between the Earth and triangular libration point regions can be effectuated by combining two transfers that meet at the lunar region, one from LEO to the lunar region and a second one from the lunar region to the triangular libration point region. However, one family of direct transfers from LEO to L_4 SPO is computed to demonstrate that the incorporation of a lunar flyby does, in fact, reduce the total transfer ΔV cost.

All transfer options considered in this investigation are two-impulse transfers. The first maneuver, $\|\Delta\vec{V}\|_1 = \Delta V_1$, is allowed to insert onto the transfer trajectory from the departure orbit. The following maneuver, $\|\Delta\vec{V}\|_2 = \Delta V_2$, is allowed to insert into the arrival orbit. Therefore, each transfer trajectory can be characterized by the transfer Time of Flight (TOF), and the total onboard maneuver capability required. In the case of transfers from LEO, ΔV_1 can be delivered by the launch vehicle, so only ΔV_2 is reported. However, for transfers between other orbits, the onboard maneuver capability required is $\sum \Delta V = \Delta V_1 + \Delta V_2$. Furthermore, the general methodology employed to identify two-maneuver transfer options between a pair of orbits is as follows:

1. Generate a set of initial guesses
2. Employ a differential corrections scheme to correct each initial guess and obtain a set of initial solutions
3. Use continuation seeded by the set of initial solutions to populate the transfer solution space

This general methodology varies slightly for each orbit pair.

5.1 Transfer Options between LEO and Lunar DRO

5.1.1 Methodology

Topputo's 2013 extensive investigation on transfer trajectories between a LEO and Low lunar Orbit (LLO) uses a simple, yet powerful methodology. [89] In this investigation, a similar procedure is adapted to compute transfer trajectories from an LEO into lunar DRO.

The computation of transfer options from LEO to lunar DRO begins with the generation of an initial guess. Such initial guess can be produced by propagating trajectories from lunar DRO backwards in time, and selecting those that come within the vicinity of Earth. Trajectories are constructed with a final position on the DRO, and a final velocity that is tangent to that of the DRO. Final trajectory positions are selected along the entire DRO, and velocities are selected at various energy levels. Once an initial guess is available, it is discretized into segments that are differentially corrected in a Newton-Raphson scheme to meet the following constraints:

- At initial time, the transfer trajectory is be at a 200 km altitude Earth apse.
- At final time, the transfer trajectory matches lunar DRO in position, and velocity direction.
- Between initial and final time, the transfer trajectory is continuous in position and velocity.

Using, the corrected set of initial guesses, that is, the set of initial solutions, a pseudo-arclength continuation scheme is employed to compute families of transfers. However, the continuation is not allowed to go through primaries, that is, the continuation process will be forced to terminate if a transfer has a primary pass below a certain altitude. This procedure is also limited by a maximum TOF of 100 days, and a maximum lunar DRO insertion cost of 1.6 km/s. This means that there could be solutions with longer TOF, or higher insertion cost, but they are not considered in this investigation.

Using the methodology described it is possible to generate various families of transfer trajectories from LEO to lunar DRO. Some transfer families exhibit prograde Earth departures, while others are retrograde in the vicinity of the Earth. Furthermore, return transfers, that is, transfers from lunar DRO to LEO can be obtained by making use of the *Mirror Theorem*. [62] Recall that the lunar DRO is symmetric about the $\hat{x}$ axis. Therefore, all transfers into lunar DRO can be reflected across the $\hat{x}$ axis to obtain a transfer trajectory from lunar DRO back to LEO.

5.1.2 Prograde Earth Departures

The totality of the solutions found with prograde Earth departures appear in figure 5.2. Each hollow dot represents a two maneuver transfer between LEO and lunar DRO in terms of the transfer TOF in days, the DRO insertion cost in km/s, and colored by the transfer Jacobi constant, that is, the transfer energy level. While the minimum Jacobi constant value for the set of all solutions found is $C = -0.7863$, the color in figure 5.2 is saturated at a Jacobi constant value of 1.6535, that is, transfers with Jacobi constants of 1.65 or lower are represented by the same maroon-like color. The maximum transfer Jacobi constant is $C = 2.4667$. The transfer energy range for solutions found is related to the bounds on TOF and insertion cost imposed on the continuation process, and well above the minimum energy (maximum Jacobi constant) a spacecraft can possess when holding a position matching that of the lunar DRO selected here. It is not surprising that transfers with Jacobi constants closest to $C = 2.96$ (the Jacobi constant value associated with the 10 day DRO selected for this investigation) are associated with lower DRO insertion costs, since the energy difference between two tangential trajectories is proportional to the difference in speed. There also exist higher energy two-maneuver transfer options, that is, transfers with a Jacobi constant below $C = -0.7863$ that are not shown in figure 5.2. However, these higher energy transfers options are retrograde about the Earth. Transfers with retrograde Earth departures are presented and discussed later.

Figure 5.2. Transfer options into lunar DRO from LEO with prograde Earth departures in terms of Time of Flight (TOF), lunar DRO insertion cost, ΔV_2 , and energy level in terms of Jacobi constant, C , saturated at $C = 1.6535$. Blue colors indicate low-energy transfer options, and red colors indicate high-energy solutions

Transfer options appear to be grouped along family curves that experience local minima in ΔV_2 . Local minima in insertion cost are of interest because they represent the lowest insertion cost option of a along a particular family curve within a given TOF range. Therefore, each insertion cost local minima is numbered in figure 5.2, and is plotted as it appears in the rotating frame in figures 5.3(a)-5.3(w). The inertial frame view of local minima appears in figures 5.4(a)-5.4(w). If a family curve has multiple local minima, the curve is given a letter. Each transfer's departure from LEO is indicated with a green dot, and the DRO insertion is marked by a red dot in both, the rotating and inertial views. The transfer appears as a solid line colored

Figure 5.3. Select transfer options from LEO into lunar DRO with prograde Earth departures in the rotating barycentered Earth-Moon frame view

Figure 5.4. Select transfer options from LEO into lunar DRO with prograde Earth departures in the inertial Earth-centered frame view

according to energy level (Jacobi constant) in agreement with figure 5.2, while the DRO is plotted in solid black. If the transfer comes within close proximity of the Moon, the inertial view indicates the location of closest approach with the * symbol along the transfer and the Moon is depicted as a grey dot in its orbit (dashed grey) at the time of closest approach. Furthermore, transfer data for numbered local minima is provided in table A.1 in the appendix.

Examination of local minima provides insight into the transfer options into lunar DRO from LEO. It is apparent from figure 5.2, that a single curve may have multiple local minima. Inspection of local minima along a single curve reveals that the minima with longer TOF incorporates a new or closer lunar flyby. For example, compare transfers on curve A in figures 5.3(a) and 5.3(h). In comparing the inertial view of transfers 1 and 8 in figures 5.4(a) and 5.4(h) it seems the lunar flyby and extra TOF available before DRO insertion along transfer 8 serves to better align the inbound DRO transfer trajectory with the DRO, thus, reducing the DRO insertion cost. Similar observations are made for transfers 6 and 19 on curve B, transfers 10 and 11 on curve C, transfers 13 and 14 on curve D, and transfers 16 and 17 on curve E. Interestingly, family curve E is a closed curve in a shape resembling an infinity sign. Transfer numbered 15, with rotating and inertial views in figures 5.3(o) and 5.4(o), respectively, possesses the lowest insertion cost among all transfers into lunar DRO with prograde Earth departures found in this investigation. This transfer takes approximately 66 days to reach lunar DRO, and 337 m/s for insertion.

While the methodology employed here is independent of auxiliary periodic orbits and/or associated manifold structures, P3DROs and associated manifold trajectories seem to enable transfer options from LEO into lunar DRO with prograde departures. For example, transfer 7 (figure 5.3(g)) exists at a Jacobi constant of $C = 2.4069$. At this same energy level, there exists a P3DRO shown in a dashed grey line in figure 5.5. An associated P3DRO stable manifold trajectory is shown in black. The P3DRO stable manifold portion highlighted in a thicker black line resembles transfer 7 very closely in geometry. Similarly, transfer 15 (figure 5.3(o)) exists at $C = 2.4667$, like

Figure 5.5. Period-3 DRO stable manifold trajectory arc at $C = 2.4069$ in the Earth-Moon rotating barycentered view

the P3DRO shown in a dashed grey line in figure 5.6. An associated P3DRO stable manifold trajectory (black), has a portion (arc highlighted in thicker black) that exhibits a geometry very similar to that of transfer 15. Similarly, other transfers from LEO into lunar DRO with prograde departures closely resemble P3DRO manifold trajectories at the transfer energy level.

The geometry of P3DRO manifolds, however, also shares similarities with various periodic orbits in the Earth-Moon CR3BP, and thus, transfers may also look like a variety of other periodic orbits. For example, the geometry of transfer 3 in figure 5.3(c) resembles that of the family of “2:1 resonant cyclers” presented by Vaquero. [90], and transfer 3 exists at an energy level that is within the energy range given for Vaquero’s “2:1 resonant cyclers”. Therefore, it seems that transfer 3 may be associated with a 2:1 resonant dynamical structure. Similar observations can be made for transfers 4, 5, 9-12, 18, 20 and 21. The geometry for transfer 7 in figure 5.3(g) also resembles the L_1

Figure 5.6. Period-3 DRO stable manifold trajectory arc at $C = 2.4667$ in the Earth-Moon rotating barycentered view

Lyapunov orbit in figure 5.7 (extracted from the Earth-Moon Catalog module within Adaptive Trajectory Design (ATD) environment developed by Purdue University and NASA Goddard Space Flight Center [46, 91]) exists at the same energy level. Thus, it appears that the L_1 Lyapunov orbit dynamical structure is also associated with transfer 7. Similar observations can be made about transfers 13-17 and 22.

5.1.3 Retrograde Earth Departures

The transfer options into lunar DRO with retrograde Earth departures appear in figure 5.8. Each hollow dot represents a two impulse transfer from LEO into lunar DRO in terms of the transfer TOF in days, the DRO insertion cost in km/s, and colored by the transfer Jacobi constant, that is, the transfer energy level, where the Jacobi constant value ranges from $C = -0.9291$ to $C = 1.6226$. From figure 5.8 it is apparent that transfer family curves experience local minima in insertion cost. As

Figure 5.7. L_1 Lyapunov orbit at $C \approx 2.40629$ in the Earth-Moon rotating barycentered view

previously mentioned, local minima are key transfer options to consider, thus, each local minima is labeled with a number. Also, transfer family curves with multiple minima are given a letter. The rotating views of local minima in figures 5.9(a)-5.9(o) exhibit tangential lunar DRO insertions. However, figures 5.10(a)-5.10(o) show that lunar DRO insertion is not necessarily tangent in the inertial frame. A lunar flyby enables transfers 7 and 13 to switch the transfer direction and achieve a close to tangential lunar DRO insertion in both the rotating and inertial frames, incurring much lower DRO insertion cost than all other transfers with retrograde Earth departures found here. In fact, transfer 7 has the lowest insertion cost of all retrograde Earth departure transfer options found here, requiring 854 m/s to insert into lunar DRO after an approximately 8 days TOF.

While the methodology employed to calculate transfers with retrograde Earth departure into lunar DRO is independent of specific periodic orbits, it seems that

Figure 5.8. Transfer options into lunar DRO from LEO with retrograde Earth departures in terms of Time of Flight (TOF), lunar DRO insertion cost, ΔV_2 , and energy level in terms of Jacobi constant, C . Blue colors indicate low-energy transfer options, and red colors indicate high-energy solutions

Figure 5.9. Select transfer options from LEO into lunar DRO with retrograde Earth departures in the rotating barycentered Earth-Moon frame view

Figure 5.10. Select transfer options from LEO into lunar DRO with retrograde Earth departures in the inertial Earth-centered frame view

retrograde orbits about the Earth may be linked to these transfers. Broucke’s 1968 publication on periodic orbits in the Earth-Moon system contains several examples of retrograde Earth orbits. [74] Although transfer 4 in figure 5.9(d) passes much closer to the Earth than Broucke’s family G member 70, shown here in figure 5.11(a), the transfer and the retrograde periodic orbit possess similar geometry. Additionally, family G member 70 and transfer 4 also exist at comparable Jacobi constant values, $C = 1.0381$ and $C = 0.90313$, respectively. Similarly, transfer 10 in figure 5.9(j) appears similar in geometry to Broucke’s family BD member 71, shown here in figure 5.11(b). Transfer 10 and family BD member 71 are also close in Jacobi constant, where transfer 10 has a Jacobi constant of $C = 0.98772$, and family BD member 71 exists at $C = 0.8329$.

5.2 Transfer Options between Lunar DRO and $L_{4/5}$ SPOs

5.2.1 Methodology

The computation of orbit-to-orbit transfers in the case where both the departure and arrival orbits are stable and non-Keplerian, as is the case with the lunar DRO and $L_{4/5}$ SPOs selected for this investigation, can be challenging. Stable orbits lack stable/unstable manifolds to be exploited in an initial guess strategy. Also, a periapse along the transfer trajectory may not indicate an auspicious orbit departure or insertion location.

In this investigation, a transfer trajectory between lunar DRO and $L_{4/5}$ SPO is generated by connecting a trajectory arc that comes from the lunar DRO to a trajectory that goes to the $L_{4/5}$ SPO at some midway point between the two orbits. Let the surface of section, Σ be defined by the angle θ that is measured from the $+\hat{x}$ direction in the rotating frame, in the counter clock-wise direction as shown in figure 5.12. Surface of section $\Sigma(\theta)$ locates the midway point between lunar DRO and $L_{4/5}$ SPO. In this investigation, a value of $\theta = 20^\circ$ is employed for transfers to L_4 SPO, and $\theta = -20^\circ$ for transfers to L_5 SPO, but there may be other values of θ that

(a)

(b)

Figure 5.11. Retrograde orbits about the Earth by R.A. Broucke in barycentered rotating Earth-Moon frame view, (a) Family G, member 70, $C = 1.0381$, (b) Family BD, member 71, $C = 0.8329$

produce similar results. Trajectories arriving at $\Sigma(\theta)$ from lunar DRO are constructed having an initial position on the lunar DRO and an initial velocity that is tangent to the periodic orbit, but may differ in velocity magnitude, that is, the trajectory may exist at a different energy level than lunar DRO. Similarly, trajectories on $\Sigma(\theta)$

Figure 5.12. Schematic depicting surface of section $\Sigma(\theta)$ and an initial guess for a transfer trajectory between lunar DRO and L_4 SPO

outbound towards $L_{4/5}$ SPO are constructed having a final position on the $L_{4/5}$ SPO and a final velocity that is tangent to the periodic orbit, but may differ in velocity magnitude. For a specific energy level (Jacobi constant value), trajectories from lunar DRO are propagated forwards in time, while trajectories from $L_{4/5}$ SPO are propagated backwards in time towards $\Sigma(\theta)$. Upon arrival to the surface of section, only the first intersection of each trajectory is recorded. This process is repeated for trajectories starting/ending at different locations along the lunar DRO/ $L_{4/5}$ SPO. Then, pairs of trajectories (one propagated from lunar DRO, and one propagated from $L_{4/5}$ SPO) that are sufficiently “close” on $\Sigma(\theta)$ are selected as an initial guess. Initial guesses are sought at all feasible energy levels. Each initial guess is then differentially corrected using a Newton-Raphson algorithm to generate a two-maneuver transfer such that it meets the following constraints:

- At initial time, the transfer trajectory matches the lunar DRO in position, and the transfer velocity is tangent to that of the DRO.

- At final time, the transfer trajectory matches the $L_{4/5}$ SPO in position, and the transfer velocity is tangent to the $L_{4/5}$ SPO.
- Between initial and final time, the transfer trajectory is continuous in position and velocity.

Applying the differential corrections process to all initial guesses found, yields a set of initial solutions.

The set of initial transfer trajectories is used to seed a continuation process to further populate the solution space. Each initial solution is discretized into 5 day trajectory arcs, and a pseudo-arclength continuation process bounded by a maximum TOF of 350 days, and a maximum $\sum \Delta V = 1$ km/s is utilized compute other solutions in the family curve associated with each initial solution. The continuation process is not designed to continue families of solutions through primaries. Therefore, if a solution comes within certain distance of either primary, the continuation process is forced to stop.

The methodology employed here recovers numerous families of transfers from lunar DRO to $L_{4/5}$ SPO, but there exist important implicit limitations associated with it that must be noted. The very definition of $\Sigma(\theta)$, and considering first intersections only may limit the geometry of transfers found here. For example, this methodology may not be able to recover initial (seed) transfer trajectories that loop around the Earth before reaching $L_{4/5}$ SPO because that would require multiple intersections with $\Sigma(\theta)$. Then, since the continuation process does not allow continuation through primaries, an Earth loop may not develop along a family curve.

Furthermore, return transfer from $L_{4/5}$ SPO to lunar DRO are readily available from the transfers computed here by means of the *Mirror Theorem*. [62] Each transfer trajectory from lunar DRO to L_4 SPO can be reflected across the $\hat{x}$ axis to yield a transfer trajectory from L_5 SPO (the reflection across the $\hat{x}$ axis of the L_4 orbit) to lunar DRO. Similarly, transfer trajectories from a lunar DRO to L_5 SPO can be

reflected across the $\hat{x}$ axis to yield a transfer trajectory from L_4 SPO (the reflection across the $\hat{x}$ axis of the L_5 orbit) back to lunar DRO.

5.2.2 Results for L_4 SPO

The totality of the resulting transfer options from lunar DRO to L_4 SPO obtained here appear in figure 5.13, where each hollow dot represents a single transfer in terms the transfer time, TOF, total transfer cost, $\Sigma\Delta V$, (transfer injection at lunar DRO, and $L_{4/5}$ SPO insertion maneuvers) and the color indicates the transfer Jacobi constant, that is, the energy. Visual inspection of figure 5.13 reveals that transfer options lie on family curves that experience local minima in $\Sigma\Delta V$. Each curve is labeled by a letter, and each local minima is labeled with a number assigned from top left to bottom right of the figure. Each local minima is shown as it appears in the rotating frame in figures 5.14-5.15. The Earth-centered inertial views appear in figures 5.16-5.17. Note that in figures 5.14-5.15, and 5.16-5.17, local minima are ordered as they appear along family curves, starting with the lowest TOF local minima in each curve, and since some family curves experience turns in the TOF dimension, the transfer labels do not appear in numerical order. For example, family curve C containing local minima 4 evolves in the increasing TOF direction to produce local minima 8, and then evolves in the decreasing TOF direction to produce local minima 6. Also, some transfer families are closed curves, that is the case with curve F, G, H, J, K and L. Figure 5.13 also reveals gaps for certain ranges of TOF. It is possible that these gaps are the product of the methodology employed here that only considers the first intersection with surface of section $\Sigma(20^\circ)$ in the early stages of of the initial guess generation process. Furthermore, among all DRO to L_4 SPO transfers recovered here, local minima 18 (on family curve I) incurs the lowest total transfer cost, approximately 42 m/s, and requires 247 days of transfer time, at a Jacobi constant of $C = 2.9503$.

The lunar DRO to L_4 SPO transfer options presented here are enabled by various dynamical mechanisms of the Earth-Moon CR3BP. One of those dynamical mecha-

Figure 5.13. Transfer options from lunar DRO to L_4 SPO in terms of Time of Flight (TOF), total maneuver capability requirements, $\Sigma\Delta V$, and energy level in terms of Jacobi constant, C . Blue colors indicate low-energy transfer options, and red colors indicate high-energy solutions

Figure 5.14. Select transfer options from lunar DRO to L_4 SPO in the rotating barycentered Earth-Moon frame view

Figure 5.15. Select transfer options from lunar DRO to L_4 SPO in the rotating barycentered Earth-Moon frame view (continued)

Figure 5.16. Select transfer options from lunar DRO to L_4 SPO in the inertial Earth-centered frame view

Figure 5.17. Select transfer options from lunar DRO to L_4 SPO in the inertial Earth-centered frame view (continued)

nisms is resonant orbits. For example, transfer 9 in figure 5.14(i) seems to be exploiting the dynamics of a 3:4 resonant orbit like that one shown in figure 5.18(a) (extracted from the Earth-Moon Catalog module within Adaptive Trajectory Design (ATD) environment developed by Purdue University and NASA Goddard Space Flight Center [46,91]) Transfer 11, in figure 5.14(k) appears to be utilizing the dynamical structures associated with a family of 3:4 resonant orbits like the one shown in figure 5.18(b) computed by Vaquero. [90] Note that there are multiple families of 3:4 resonant orbits, and the orbit in figure 5.18(a) is in a different family than the one shown in figure 5.18(b). The dynamical behavior exhibited by transfer 18 resembles a combination of periodic orbits in 3:4 and 2:3 resonance with the Moon like the ones shown in figure 5.18(c), thus, indicating a possible link between the two resonant orbits and the transfer dynamics.

Period-3 DROs and associated manifolds also play an important role in the DRO departure phase of transfers options from lunar DRO to L_4 SPO. For example, at the same energy level that transfer 29 exists, there also exists a period-3 DRO shown in dashed grey in figure 5.18(d). Portion of a period-3 DRO unstable manifold trajectory is depicted in solid black. The similarity between the manifold trajectory and the transfer during the DRO departure phase suggests that the period-3 DRO at this energy level as well as the associated manifolds may be dynamical mechanism that enables transfer 29. Similar observations can be made for transfers 14-17, 20-25, and 28 since they exist at similar energy levels as transfer 29 and exhibit similar geometry during the DRO departure phase. Similarly, figure 5.18(e) shows another period-3 DRO in grey and a portion of an unstable manifold arc in black that looks very similar to transfer 30 in figure 5.15(m) during the first half of the transfer. Similar observations can be made for transfers 19, and 26-27 because they exist at similar energy levels as transfer 30 and exhibit similar geometry during the DRO departure phase.

Transfer 13, in figure 5.14(m), exhibits some complex dynamics in the neighborhood of L_4 that resembles the family of long-period period orbits at L_4 . Figure 5.18(f)

Figure 5.18. Dynamical structures associated with transfer options from lunar DRO to L_4 SPO in the barycentered rotating Earth-Moon frame view, (a) 3:4 resonant orbit at $C = 2.8240$, (b) 3:4 resonant orbit at $C = 2.6310$, (c) 3:4 and 2:3 resonant orbits at $C = 2.9499$, (d) period-3 DRO and associated manifold trajectory arc at $C = 2.9036$, (e) period-3 DRO and associated manifold trajectory arc at $C = 2.9407$, (f) L_4 long-period orbit at $C = 2.8404$

shows a planar long-period L_4 orbit that exists at the same energy level as transfer 13. [46] The similarity between the L_4 long-period orbit and transfer 13 suggests that the transfer maybe enabled by the dynamical structures associated with the periodic L_4 long-period orbit. Furthermore, transfers 15-17, 19-30 exhibit similar type of dynamics.

5.2.3 Results for L_5 SPO

Transfer options from lunar DRO to L_5 SPO found appear in figure 5.19, where each hollow dot represents a single transfer in terms the transfer time, TOF, total transfer cost, $\Sigma\Delta V$, and the color indicates the transfer Jacobi constant, that is, the energy level. Family curves are labeled by a letter, and each local minima in $\Sigma\Delta V$ is labeled with a number assigned from top left to bottom right of figure 5.19. Each local minima is shown as it appears in the rotating frame in figures 5.20-5.21, and the Earth-centered inertial views appear in figures 5.22-5.23. Rotating and inertial views are organized keeping local minima that occur on the same family curve together, and appear in the order they occur along the curve starting with the local minima with lowest TOF and highest $\Sigma\Delta V$. Closed family curves include G, H, I, N, O, and R. Much like in transfers from lunar DRO to L_4 SPO, figure 5.19 also reveals gaps for certain ranges of TOF. It is possible that these gaps are also the product of the methodology employed here that only considers the first intersection with surface of section $\Sigma(-20^\circ)$ in the early stages of of the initial guess generation process. Furthermore, among all DRO to L_5 SPO transfers recovered here, local minima 34 (on family curve T) incurs the lowest total transfer cost, approximately 47.5 m/s, and requires 317 days of transfer time, at a Jacobi constant of $C = 2.9383$.

As it is the case with transfer options from lunar DRO to L_4 SPO, transfer options to L_5 SPO also seem to be enabled by P3DRO manifold dynamics, as well long-period orbits near the triangular points, in this case, L_5 . For example, the DRO departure phase along transfer 4 in figure 5.20(d) resembles very much the P3DRO manifold

Figure 5.19. Transfer options from lunar DRO to L_5 SPO in terms of Time of Flight (TOF), total maneuver capability requirements, $\Sigma\Delta V$, and energy level in terms of Jacobi constant, C . Blue colors indicate low-energy transfer options, and red colors indicate high-energy solutions

Figure 5.20. Select transfer options from lunar DRO to L_5 SPO in the rotating barycentered Earth-Moon frame view

Figure 5.21. Select transfer options from lunar DRO to L_5 SPO in the rotating barycentered Earth-Moon frame view (continued)

Figure 5.22. Select transfer options from lunar DRO to L_5 SPO in the inertial Earth-centered frame view

Figure 5.23. Select transfer options from lunar DRO to L_5 SPO in the inertial Earth-centered frame view (continued)

trajectory arc shown in black in figure 5.24(a) that exists at the same energy level as transfer 4. Also, the arrival phase along transfer 11 in figure 5.20(o) looks like the L_5 long-period orbit shown in figure 5.24(b) that exist at an energy level close to that of transfer 11. [46]

Figure 5.24. Dynamical structures associated with transfer options from lunar DRO to L_5 SPO in the rotating barycentered frame view, (a) period-3 DRO unstable manifold trajectory arc at $C = 2.8846$ (b) long-period orbit near L_5 at $C = 2.8644$

5.3 Transfer Options between L_2 NRHO and Lunar DRO

5.3.1 Methodology

The calculation of transfer options between lunar DRO and the L_2 NRHO begins with the search for suitable initial guesses. To generate the set of initial guesses for transfer trajectories between NRHO and DRO, trajectories departing tangentially from the NRHO are propagated starting at different locations around the NRHO, and at a range of Jacobi constant values (energy levels) in search of trajectories that may come close to the DRO within a maximum TOF of 50 days. Each initial guess is differentially corrected and locally optimized using a constrained optimization scheme where the objective function to be minimized is the total transfer cost, $\Sigma\Delta V$, and

the transfer is constrained to match the NRHO in position at initial time, and to match the lunar DRO in position at final time, while the TOF is fixed. The resulting transfers are the initial set of solutions. The set of initial solutions is used to seed a continuation process to populate the transfer solution space. Here, a natural parameter continuation process is employed in combination with the constrained optimization scheme. Each initial solution is discretized into 3 day trajectory arc segments, and used in a TOF continuation scheme. This particular methodology was selected here due to the difficulty in finding tangential transfers with TOFs within 50 days. Furthermore, for every transfer option generated to travel from NRHO to lunar DRO, a return transfer from the DRO to the NRHO can be generated using the *Mirror Theorem*. The reflection in this case is across the $\hat{x}$ and $\hat{z}$ axes. [62]

5.3.2 Results

Applying the methodology described, it is possible to obtain various transfer options from NRHO to lunar DRO. All solutions recovered are represented in figure 5.25, where each hollow dot represents a distinct transfer in terms of the transfer time of flight, TOF, and total transfer cost, $\Sigma\Delta V$, and colored in agreement with the colorbar that indicates the transfer Jacobi constant, that is, the energy level. From figure 5.25 it is apparent that transfer options lie on family curves, and that these family curves experience local minima in $\Sigma\Delta V$. Each local minima is numbered, and shown as it appears in the rotating frame in figure 5.3.2-5.3.2. Additionally, family curves that experience more than one local minima are labeled with a letter. The Moon-centered inertial view of each local minima appears in figure 5.3.2. The lowest cost minima is transfer 16, requiring 402.62 m/s of onboard maneuvering capability, and takes approximately 30 days to reach lunar DRO.

The vicinity of the Moon is inhabited by an infinite variety of dynamical structures, and thus, it is not surprising to find this fact reflected in the variety of transfer options available between L_2 NRHO and lunar DRO. Some of the dynamical struc-

Figure 5.25. Transfer options from L_2 NRHO to lunar DRO in terms of Time of Flight (TOF), total maneuver capability requirements, $\Sigma\Delta V$, and energy level in terms of Jacobi constant, C . Blue colors indicate low-energy transfer options, and red colors indicate high-energy solutions

tures in the lunar region, in addition to the lunar orbits already discussed, include the axial and vertical orbit families. For example, transfer 6 in figure 5.26(f) resembles the manifold (thick black) associated with the axial orbit in dashed grey (extracted from the Earth-Moon Catalog module within Adaptive Trajectory Design (ATD) environment developed by Purdue University and NASA Goddard Space Flight Center [46, 91]) shown in figure 5.29 that exists at the same energy level as transfer 6. Transfer 12, in figure 5.27(d) seems to follow the dynamical pattern of manifolds associated with nearby L_2 halo orbits like the one shown in thick black in figure 5.30 that exists at the same energy level as transfer 12. The flow governing transfer 15 in figure 5.27(g) seems to be guided by the manifolds associated with the

(a) 1

(b) 2

(c) A:3

(d) A:4

(e) A:5

(f) 6

(g) 7

(h) 8

Figure 5.26. Select transfer options from L_2 NRHO to lunar DRO in the rotating barycentered Earth-Moon frame view

(a) 9

(b) B:10

(c) 11

(d) 12

(e) 13

(f) B:14

(g) 15

(h) 16

Figure 5.27. Select transfer options from L_2 NRHO to lunar DRO in the rotating barycentered Earth-Moon frame view (continued)

Figure 5.28. Select transfer options from L_2 NRHO to lunar DRO in the inertial Moon-centered frame view

Figure 5.29. L_2 Axial orbit and associated manifold trajectory arc at $C = 3.0111$ in the rotating barycentered Earth-Moon frame view

two vertical orbits (extracted from the Earth-Moon Catalog module within Adaptive Trajectory Design (ATD) environment developed by Purdue University and NASA Goddard Space Flight Center [46, 91]) shown in figure 5.31 that exist at the same energy level as the transfer. [46] The complex geometry of transfer 15 may be dictated by a combination of stable and unstable manifolds associated with both vertical orbits in the lunar vicinity.

5.4 Single Transfer Family from LEO to L_4 SPO

In the literature it has been demonstrated that it is possible to lower the total ΔV cost associated with a transfer from Earth to the vicinity of the triangular libration points by including a passage through the lunar region in the design of the transfer trajectory, while incurring a longer TOF. [55, 57, 58] For example, consider a direct

Figure 5.30. L_2 NRHO and associated manifold trajectory arc at $C = 3.0244$ in the rotating barycentered Earth-Moon frame view

transfer from a prograde LEO with 200 km altitude to the working L_4 SPO ($C = 2.91$) shown in figure 5.4. This transfer from LEO to L_4 SPO requires an insertion $\Delta V_2 = 0.61314$ km/s, and a TOF = 7.1734 days. Using this transfer in a pseudo-arc-length continuation scheme yields the family of transfer shown in figure 5.33, where each dot represents a transfer from LEO to L_4 SPO in terms of insertion cost, ΔV_2 and TOF, and the original transfer appears denoted as “1”. It is apparent from 5.33, that transfer “1” is a local minima in ΔV_2 , and that there is another local minima, labeled “2”, occurring at $\Delta V_2 = 0.50627$ km/s, and TOF = 37.8185 days. In agreement with the literature, the geometry of the second local minima shown in figure 5.4, reveals that the lower insertion cost and longer TOF transfer incorporates a lunar flyby. Furthermore, transfer “2” can be seen as the composition of a transfer from LEO to the lunar region, and a transfer from the lunar region to the triangular

Figure 5.31. Vertical orbits and associated manifold trajectories at $C = 3.0445$ in the rotating barycentered Earth-Moon frame view. L_1 vertical orbit: Unstable manifolds (red), stable manifolds (blue). L_2 vertical orbit: Unstable manifolds (magenta), stable manifolds (light blue)

region. Since this investigation already considers transfers from the Earth to the lunar region, and from the lunar region to the triangular libration point regions, it did not seem necessary to consider transfers from the Earth to the triangular libration points region explicitly.

Figure 5.32. Direct transfer from LEO to L_4 SPO, (a) rotating barycentered frame view, (b) inertial Earth-centered frame view

Figure 5.33. Family of transfer options from LEO to L_4 SPO in terms of Time of Flight (TOF), L_4 SPO insertion cost, ΔV_2 , and energy level in terms of Jacobi constant, C . Blue colors indicate low-energy transfer options, and red colors indicate high-energy solutions

Figure 5.34. Transfer from LEO to L_4 SPO with lunar flyby

(a) rotating barycentered frame view, (b) inertial Earth-centered frame view

6. Summary, Concluding Remarks and Recommendations

6.1 Summary and Concluding Remarks

The goal of this investigation is to contribute to the development of the transfer network infrastructure in the Earth-Moon system by providing new transfer options connecting regions of stability near the Earth, Moon and triangular libration points, with particular focus on the regions spanned by stable lunar DROs, L_2 NRHOs and $L_{4/5}$ SPOs. An initial look at the problem restricts the investigation to two-impulse transfer options between departure and arrival motions defined by periodic orbits in each region of interest. Various solutions have been identified offering transfer options with a variety of geometries, orbit departure/arrival locations, energy levels, flight times, and maneuvering capability requirements. Additionally, various underlying dynamical structures, including low energy manifold trajectories, have been found to guide the various transfer options computed.

New transfer options linking the regions near the Earth and the Moon are identified as two-impulse direct and indirect type transfer options between LEO and lunar DRO. The lowest insertion cost transfer option into lunar DRO with prograde Earth departure, transfer 15 (see figures 5.3(o) and 5.4(o)), requires approximately 66 days to reach lunar DRO, and nearly 340 m/s of maneuver capability for lunar DRO insertion. The lowest insertion cost transfer option into lunar DRO with retrograde Earth departure, transfer 7 (see figures 5.9(g) and 5.10(g)), requires approximately 850 m/s to insert into lunar DRO after an 8 days TOF. Transfer options into lunar DRO with prograde Earth departures seem to be guided by manifolds associated with P3DROs at various energy levels, as well as L_1 Lyapunov orbits and 2:1 resonant orbits, while transfer options into lunar DRO with retrograde Earth departures seem to be guided

by retrograde orbits about the Earth. In general, retrograde Earth departure transfer options require more ΔV capability than transfers with prograde Earth departures.

Novel transfer options connecting the regions near the Moon and the neighborhood of the triangular libration points are identified here as two-impulse transfer trajectories between lunar DRO and $L_{4/5}$ SPO. Triangular libration point LPOs and manifold trajectories associated with P3DROs seem to enable the various transfer options between lunar DROs and $L_{4/5}$ SPOs. The lowest total ΔV cost transfer option from lunar DRO into L_4 SPO corresponds to transfer 18 (see figures 5.15(a) and 5.17(a)), and requires approximately 42 m/s of total ΔV and around 250 days of transfer time to reach the L_4 SPO. The lowest total ΔV cost transfer option from lunar DRO into L_5 SPO corresponds to transfer 34 (see figures 5.21(p) and 5.23(p)) that requires approximately 48 m/s of maneuver capability and reaches the L_5 SPO in about 320 days. However, in general, for similar transfer times, transfers between lunar DRO and L_5 SPO require lower total maneuvering capability than transfers between lunar DRO and L_4 SPO. Furthermore, a single family of transfers from LEO to L_4 SPO demonstrates that incorporating a lunar flyby to transfers from LEO to the triangular region can yield insertion ΔV cost savings on the order of 100 m/s.

Connecting stable periodic orbits in the lunar region, various two-impulse transfer options between lunar L_2 NRHO and DRO are identified. Various families of lunar periodic orbits seem to enable transport between NRHO and lunar DRO, including Axial, Vertical and other Near-rectilinear Halo Orbits. The lowest total ΔV cost transfer option found here is transfer 16 (see figures 5.27(h) and 5.28(o)), requires approximately 403 m/s of onboard maneuvering capability, and takes about 30 days to reach lunar DRO without leaving the lunar region.

The various transfer options connecting the regions near the Earth, Moon, and triangular libration points identified in this investigation can be combined to produce different itineraries traversing the Earth-Moon system. For example, a round-trip trajectory from Earth to the vicinity of L_4 can be designed by combining transfer options between LEO and lunar DRO, with transfer options between lunar DRO and

$L_{4/5}$ SPO. The outbound trajectory can be obtained by combining a transfer option from LEO to lunar DRO with a solution from lunar DRO to L_4 SPO. Then, to Earth-bound trajectory from the vicinity of L_4 can be obtained by combining the mirror images across the x -axis, as per the *Mirror Theorem* [62], of a transfer solution from lunar DRO to L_5 SPO, and a transfer option from LEO to lunar DRO. Similarly, other such itineraries can be generated with the solutions found in this investigation.

6.2 Recommendations

While this investigation identifies various new transfer options linking the regions near the Earth, Moon and triangular libration points in the Earth-Moon system, there is still much work left ahead. In the future, it may be of interest to explore certain variations of the work presented here. In the construction of initial guesses for transfer trajectories between lunar DRO and $L_{4/5}$ SPO, it may be of interest to allow multiple intersections with the surface of section $\Sigma(\theta)$ to allow transfer options with multiple revolutions about the Earth. It may also be of interest to explore alternate definitions of the surface of section to see if new transfer options can be found. In the computation of transfer options between L_2 NRHO and lunar DROs, it may be of interest to explore longer TOFs that may allow more lunar passes, and enable tangential departure and arrival transfer options. Furthermore, it may be of interest to consider members (other than the ones selected for this investigation) of the lunar DRO, L_2 NRHO, and $L_{4/5}$ SPO families, to see if and/or how transfer options vary from the ones found here.

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APPENDIX

Appendix: Transfer Data

The following tables provide the necessary parameters to reproduce the transfer trajectories discussed in chapter 5. Time of Flight (TOF) is provided in units of days, ΔV 's are given in units of km/s, and all other variables are provided in non-dimensional units. Variables τ provide the time of integration required to reach the departure or insertion location along a specified orbit, where propagation begins at the initial conditions specified in table 4.1.

Table A.1. Data for select transfer options from LEO to lunar DRO with prograde Earth departures

Nro.	x_0	y_0	z_0	$\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$	TOF	$ \Delta V_2 $	C	τ_{DRO}
1	-0.015268	-0.016799	0	10.4798	-1.9447	0	5.9519	0.60954	2.0534	1.1882
2	-0.0050695	0.015549	0	-9.6998	4.4173	0	18.4822	0.60464	2.0632	1.1798
3	-0.019062	-0.015625	0	9.739	-4.3081	0	29.0143	0.52167	2.2539	0.93567
4	-0.019094	-0.015611	0	9.7306	-4.3282	0	31.5691	0.52794	2.2433	1.3771
5	-0.019363	-0.015488	0	9.6529	-4.4953	0	32.5008	0.54588	2.2746	1.6584
6	-0.014254	-0.016956	0	10.6217	-1.3175	0	34.641	1.0467	1.1042	1.549
7	-0.020923	-0.014661	0	9.1322	-5.4642	0	36.3195	0.46111	2.4069	1.5854
8	-0.020484	-0.014915	0	9.2913	-5.1915	0	41.8805	0.48803	2.3814	1.7569
9	0.0031118	0.0076793	0	-4.7863	9.5127	0	42.1711	0.51874	2.2598	1.3746
10	-0.019234	-0.015548	0	9.6905	-4.4146	0	54.5105	0.52834	2.2664	0.8325
11	-0.01924	-0.015545	0	9.6884	-4.4184	0	56.0465	0.51297	2.2734	0.91987
12	-0.019219	-0.015555	0	9.6951	-4.4059	0	56.0972	0.50783	2.2551	1.0992
13	-0.021211	-0.014485	0	9.0218	-5.6429	0	61.5428	0.43625	2.425	1.4724
14	-0.021103	-0.014552	0	9.0642	-5.5758	0	62.1734	0.43308	2.4121	1.3815
15	0.0044708	-0.0039552	0	2.4629	10.3503	0	65.7753	0.3867	2.4667	1.1732

Nro.	x_0	y_0	z_0	$\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$	TOF	$ \Delta V_2 $	C	τ_{DRO}
16	0.0049347	8.5862e-05	0	-0.053474	10.6405	0	67.8851	0.41827	2.437	0.92501
17	0.004929	0.00045	0	-0.28027	10.6372	0	68.8673	0.41735	2.4318	0.96438
18	0.0031737	0.007555	0	-4.709	9.5515	0	70.0123	0.55591	2.2559	1.6538
19	-0.020298	-0.015018	0	9.3554	-5.0757	0	71.6583	0.49101	2.3746	1.814
20	-0.02637	-0.009472	0	5.9025	-8.8608	0	81.8695	0.47849	2.3076	1.164
21	-0.015451	-0.016764	0	10.4516	-2.0577	0	84.6201	0.54385	2.191	1.0525
22	-0.021836	-0.014075	0	8.7652	-6.0321	0	88.3503	0.42037	2.4462	1.4492
23	-0.0049941	-0.015514	0	9.6823	4.4662	0	91.3055	0.68457	1.9678	1.5018

Table A.2. Data for select transfer options from LEO to lunar DRO with retrograde Earth departures

Nro.	x_0	y_0	z_0	$\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$	TOF	$ \Delta V_2 $	C	τ_{DRO}
1	-0.015357	-0.016782	0	-10.5023	2.0068	0	6.0161	0.91379	1.3351	1.1757
2	-0.022597	-0.01352	0	-8.4941	6.5635	0	30.6942	1.2569	0.43174	1.4668
3	-0.02321	-0.013023	0	-8.1784	6.9457	0	31.3839	1.2163	0.53215	0.89125
4	-0.021912	-0.014022	0	-8.792	6.1203	0	31.2546	1.0899	0.90313	1.3911
5	-0.022759	-0.013393	0	-8.3883	6.6447	0	31.2986	0.98756	1.1454	1.0546
6	-0.017226	-0.016314	0	-10.2125	3.1771	0	32.7769	0.94189	1.2724	1.2627
7	-0.01999	-0.015181	0	-9.4885	4.8996	0	36.385	0.85404	1.6226	1.6068
8	-0.022496	-0.013597	0	-8.5417	6.4994	0	59.7054	1.2323	0.4586	0.93321
9	-0.022591	-0.013524	0	-8.4963	6.5591	0	58.3863	1.2175	0.45273	1.2411
10	-0.024233	-0.01208	0	-7.5713	7.5729	0	57.926	1.0552	0.98772	1.3559
11	-0.022184	-0.013829	0	-8.663	6.285	0	58.6911	0.99553	1.1118	1.1648
12	-0.01343	-0.017037	0	-10.6715	0.80169	0	61.1976	0.99287	1.1376	1.0273
13	-0.019981	-0.015186	0	-9.4916	4.8941	0	65.6476	0.86173	1.6184	1.6582
14	-0.023108	-0.013109	0	-8.2333	6.8815	0	84.6145	1.2147	0.51884	1.3833
15	-0.021483	-0.014311	0	-8.9655	5.8467	0	86.2127	1.0119	1.0971	1.3209

Table A.3. Data for select transfer options from lunar DRO to L_4 SPO

Nro.	x_0	y_0	z_0	$\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$	TOF	$\Sigma \Delta V $	C	τ_{DRO}	$\tau_{L_4\text{SPO}}$
1	0.74223	1.0007	0	0.35912	-0.44431	0	21.5816	0.21465	2.826	1.195	3.0364
2	0.047111	1.2517	0	0.61867	0.12151	0	21.0254	0.34062	2.7637	0.73358	1.5726
3	0.77404	0.95753	0	0.33207	-0.49613	0	26.1318	0.32086	2.779	0.72191	3.148
4	0.50632	0.56641	0	-0.6205	0.17804	0	43.8937	0.31772	2.7661	0.22167	5.0684
5	-0.27273	1.0055	0	0.097752	0.5906	0	44.6686	0.61599	2.6442	0.62124	0.5033
6	0.51408	0.56421	0	-0.62151	0.17393	0	50.1324	0.34916	2.7598	-0.44612	5.0527
7	0.77729	0.95261	0	0.31705	-0.48444	0	52.3716	0.29257	2.7982	0.4783	3.1605
8	-0.27413	0.96471	0	-0.044996	0.44605	0	55.4544	0.31862	2.7965	-0.14491	6.8924
9	0.49183	0.57066	0	-0.62063	0.18548	0	70.2859	0.32043	2.7751	0.45359	5.0972
10	0.76724	0.96746	0	0.29189	-0.41707	0	94.4194	0.11362	2.8803	0.024275	9.6511
11	0.31009	0.63209	0	-0.64274	0.22781	0	99.4857	0.16986	2.8416	-0.00029324	5.425
12	-0.27395	0.96306	0	-0.043494	0.38732	0	107.6334	0.21557	2.8456	-0.13277	6.8867
13	0.77669	0.567	0	-0.34115	-0.21643	0	110.937	0.23724	2.8354	1.0775	4.3481
14	0.52653	0.56081	0	-0.53063	0.14192	0	166.992	0.15775	2.8642	0.36297	5.0273
15	0.51294	0.56453	0	-0.53828	0.15122	0	167.5003	0.15637	2.8647	0.37252	5.0551

Nro.	x_0	y_0	z_0	$\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$	TOF	$\Sigma \Delta V $	C	τ_{DRO}	$\tau_{L_4\text{SPO}}$
16	0.79164	0.92947	0	0.24332	-0.41485	0	186.1925	0.097341	2.8927	0.25497	9.7477
17	0.79873	0.917	0	0.22957	-0.41663	0	186.3102	0.096746	2.8925	0.2299	9.7792
18	0.16563	0.68286	0	-0.57151	0.1986	0	247.439	0.041692	2.9503	1.0938	5.6657
19	0.13243	0.69439	0	-0.57398	0.1992	0	248.6702	0.0446	2.9381	1.1744	5.7213
20	0.54196	0.55681	0	-0.49143	0.12294	0	250.0156	0.087463	2.8963	0.22789	4.9952
21	0.56224	0.552	0	-0.47733	0.10683	0	249.8964	0.086237	2.8965	0.19374	4.9519
22	0.803	0.9091	0	0.21514	-0.40682	0	270.3923	0.070251	2.9037	0.14744	9.7991
23	0.79854	0.91735	0	0.22425	-0.40625	0	270.6244	0.068994	2.9036	0.094286	9.7783
24	0.55975	0.55256	0	-0.47916	0.1089	0	276.9059	0.087873	2.8964	0.22963	-1.5713
25	0.5549	0.55368	0	-0.4826	0.11286	0	277.0204	0.086332	2.8963	0.19268	-1.5609
26	0.39715	1.2301	0	0.49828	-0.15542	0	287.2749	0.047618	2.9402	1.135	2.241
27	0.39126	1.2319	0	0.49931	-0.15131	0	290.3846	0.052678	2.9404	0.60579	2.2298
28	0.80204	0.91091	0	0.21719	-0.4068	0	297.4147	0.07064	2.9036	0.15332	3.266
29	0.79847	0.91748	0	0.22443	-0.4063	0	297.6986	0.069058	2.9036	0.09031	3.2494
30	0.39707	1.2301	0	0.49788	-0.15524	0	317.2654	0.0525	2.9407	0.61869	2.2409

Table A.4. Data for select transfer options from lunar DRO to L_5 SPO

Nro.	x_0	y_0	z_0	$\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$	TOF	$\Sigma \Delta V $	C	τ_{DRO}	$\tau_{L_5\text{SPO}}$
1	-0.21438	-0.85507	0	0.30409	0.28049	0	25.8904	0.14859	2.871	-0.17895	6.573
2	0.30949	-0.6323	0	0.62338	0.22095	0	28.5555	0.12416	2.8694	-0.027391	1.1025
3	0.55144	-0.55449	0	0.51134	0.12189	0	33.3994	0.15342	2.8685	-0.41151	1.5534
4	0.40903	-0.59743	0	0.57701	0.19696	0	51.2983	0.10216	2.8846	1.0882	7.8043
5	-0.0077307	-0.74439	0	0.56229	0.21297	0	55.5523	0.14233	2.8663	1.4201	7.0872
6	0.48733	-0.57201	0	0.55027	0.16631	0	58.1748	0.1412	2.8679	1.4185	7.9511
7	0.67191	-1.0747	0	-0.50743	-0.45812	0	56.2968	0.39642	2.7117	1.2596	3.6954
8	0.71518	-1.032	0	-0.464	-0.50298	0	59.0752	0.44129	2.6958	0.81782	3.5757
9	0.44275	-1.2143	0	-0.73495	-0.2816	0	63.1316	0.58656	2.5929	0.70541	4.1993
10	-0.25988	-1.0484	0	-0.20892	0.48201	0	65.9365	0.41312	2.7396	1.052	5.8798
11	0.78282	-0.94399	0	-0.27324	-0.43456	0	120.5524	0.16074	2.8665	0.64023	3.3461
12	-0.27476	-0.97374	0	0.013028	0.32823	0	129.3754	0.11473	2.89	0.91865	6.1336
13	0.77245	-0.95989	0	-0.31693	-0.46848	0	119.4233	0.24615	2.8165	0.28751	3.3866
14	0.76702	-0.96777	0	-0.3208	-0.45774	0	119.5952	0.22995	2.8271	0.33616	3.4068
15	-0.27327	-1.002	0	-0.054016	0.37662	0	133.6262	0.18892	2.857	-0.14825	6.0371
16	-0.27471	-0.98697	0	-0.016844	0.3555	0	134.1757	0.15328	2.8726	-0.13658	6.0882
17	0.74425	-0.99814	0	-0.35992	-0.45009	0	154.2105	0.24005	2.8193	1.4964	3.4855

Nro.	x_0	y_0	z_0	$\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$	TOF	$\Sigma \Delta V $	C	τ_{DRO}	$\tau_{L_5\text{SPO}}$
18	0.76714	-0.9676	0	-0.32236	-0.46033	0	154.019	0.23406	2.8237	1.4965	3.4063
19	0.46832	-0.57788	0	0.53376	0.16814	0	166.6943	0.074196	2.9003	1.1876	7.9144
20	0.47975	-0.57432	0	0.52642	0.16194	0	166.6533	0.073084	2.9011	1.224	7.9364
21	0.79523	-0.92325	0	-0.24321	-0.42764	0	200.9916	0.11864	2.8794	1.1645	3.2937
22	0.51729	-0.56332	0	0.45572	0.12614	0	204.2257	0.050391	2.9501	0.92252	1.4823
23	0.80484	-0.90558	0	-0.19124	-0.36846	0	223.1326	0.054363	2.9417	2.1531	3.2492
24	0.8223	-0.86825	0	-0.17571	-0.42166	0	258.7314	0.1078	2.8895	0.82253	3.1553
25	0.79356	-0.92617	0	-0.2418	-0.41904	0	258.2915	0.10233	2.8886	0.99348	3.301
26	-0.26187	-1.0436	0	-0.13185	0.32667	0	269.2626	0.11181	2.8897	0.94233	5.8959
27	0.6912	-1.0566	0	-0.33822	-0.33024	0	269.5863	0.04984	2.9494	1.0642	3.6438
28	-0.27348	-1.0005	0	-0.030966	0.23154	0	283.2349	0.079871	2.9469	1.2505	6.0423
29	-0.27342	-1.0009	0	-0.031688	0.23156	0	286.5601	0.083326	2.947	0.53574	6.0407
30	0.80268	-0.9097	0	-0.2267	-0.42734	0	285.7853	0.11443	2.8818	1.1342	3.2595
31	-0.25836	-1.0518	0	-0.14768	0.32451	0	296.9454	0.11169	2.8898	0.9244	5.8681
32	-0.019228	-1.2348	0	-0.46668	0.14799	0	295.6054	0.075475	2.9007	1.2288	5.0964
33	0.71242	-1.0349	0	-0.3144	-0.33656	0	314.1935	0.050237	2.953	-0.037237	3.5838
34	0.24764	-1.2606	0	-0.51109	-0.050507	0	316.9438	0.047549	2.9383	2.315	4.5687

Table A.5. Data for select transfer options from L_2 NRHO to lunar DRO

Nro.	x_0	y_0	z_0	$\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$	TOF	$\Sigma \Delta V $	C	$\tau_{L_2\text{Halo}}$	τ_{DRO}
1	1.0316	0.053858	0.1585	0.039477	-0.19215	0.19261	7.8724	0.53912	3.001	1.4267	1.1972
2	1.0404	0.034731	0.18176	0.10214	-0.26761	0.11705	5.8983	0.50931	2.9629	1.5872	2.0243
3	1.0431	-0.024069	0.18871	0.064343	-0.10479	-0.11865	6.5577	0.4772	3.0247	1.993	0.67553
4	1.0216	0.064677	0.13009	0.060682	0.00211	0.2882	10.9195	0.46506	3.0166	1.2965	0.70148
5	0.98712	0.0039637	-0.019785	0.011811	1.063	0.11292	12.2209	0.45925	3.0118	0.91815	0.78815
6	1.0098	0.068535	0.092002	-0.15203	0.23731	0.25137	17.0848	0.59631	3.011	-0.66121	-0.38013
7	1.0406	0.034112	0.18226	0.060546	-0.29817	0.12228	17.3403	0.46417	2.9508	-0.23699	0.43267
8	0.98842	0.028418	-0.0087968	0.06918	0.66734	0.55102	20.2046	0.46963	3.0145	0.94621	1.6338
9	1.0355	0.047117	0.16888	0.199	-0.25996	0.1803	16.932	0.44287	2.9285	-0.34061	1.1788
10	1.0179	-0.066873	0.11878	-0.012102	0.097636	-0.39206	17.9841	0.43099	2.9528	0.5747	0.49537
11	1.0244	0.062446	0.13822	-0.088223	-0.012898	0.38054	25.0097	0.4457	2.9422	1.3297	1.5377
12	1.0449	0.01206	0.19318	0.013528	-0.16128	0.037243	32.8411	0.54069	3.0233	1.7476	1.6186
13	1.0388	0.039292	0.17773	0.090314	-0.29876	0.11311	30.2913	0.42608	2.9514	1.5522	-0.091168
14	1.0185	-0.066583	0.12061	0.0029312	0.20754	-0.41055	28.1652	0.41071	2.9024	0.56811	0.2286
15	1.0131	-0.068395	0.10329	-0.10462	0.063459	-0.27729	29.1929	0.40762	3.0445	0.62679	1.3221
16	1.0451	0.0096134	0.19371	0.071811	-0.32811	0.038092	30.6663	0.40262	2.9363	1.7642	-0.72085

Table A.6. Data for select transfer options from LEO to L_4 SPO

Nro.	x_0	y_0	z_0	$\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$	TOF	$ \Delta V_2 $	C	$\tau_{L_4\text{SPO}}$
1	0.0043947	-0.0042622	0	2.6608	10.3287	0	7.0952	0.59861	1.8999	2.224
2	-0.020477	-0.014919	0	9.2939	-5.1866	0	37.8555	0.49409	2.3831	4.1574

VITA

VITA

Lucia Capdevila was born in Buenos Aires, Argentina on December 6, 1981. She began her undergraduate studies at Purdue University during the fall semester of 2000. She has since earned a Bachelor and Master of Science degrees from the School of Aeronautics & Astronautics, and is currently finishing her Ph.D. in the same field under the guidance of Professor Kathleen C. Howell. During her graduate career at Purdue, Lucia has had the opportunity to teach aerospace, mathematics, and first year engineering courses, as well as perform braille transcription for visually impaired students. She has also had the opportunity to work with Professor Hiroshi Yamakawa at Kyoto University in Kyoto, Japan, and with Nathan Strange at NASA's Jet Propulsion Laboratory in Pasadena, California. Lucia is currently finishing her dissertation (in absentia from Purdue) from Redwood City, California, where she lives with her husband.