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Abstract

The bottleneck (capacity) assignment problem seeks for a set of entries in a matrix A, one for each column and row, that minimizes (maximizes) the largest (smallest) element over all such sets. Our interest lies in finding the asymptotically exact solution to these problems in a probabilistic framework, that is, under the assumption that elements in the matrix are independent random variables with common distribution function $F(\cdot)$. It is proved that the optimal values for the bottleneck and capacity assignment problems asymptotically become $F^{-1}(\log n/n)$ and $F^{-1}(1-\log n/n)$ *in probability* respectively, where n is the size of the matrix. Finally, we shall show that a greedy version of the problem produces asymptotically the optimal solution, however, the cost of the greedy algorithms is much cheaper.

Keywords: assignment problems, heuristics, greedy algorithms, order statistics.

1. Introduction

Most algorithms are designed to optimize the worst-case performance. Many elegant constructions have been set up in this endeavor. We note, however, that such a construction has to cope efficiently with unrealistic, even pathological inputs and the possibility is neglected that a simpler algorithm might perform just as well, or even better in practice. One possible solution is to look at an algorithm from the probabilistic view point, so rather typical inputs instead of pathological ones are investigated. This probabilistic approach to design algorithms was practically fulfilled a decade ago when it became clear that the prospects for showing the existence of polynomial algorithms for NP-hard problems were very dim. This fact and apparently a high success rate of heuristic approaches applied not only to NP-hard problems, led computer

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scientists to undertake a more serious investigation of probabilistic approximation algorithms [KA].

In this paper we investigate the capacity and bottleneck assignment problems (we further refer to them as CAP and BAP respectively) in a probabilistic framework [GG,AV]. These problems can be formulated as follows. Let $A = \{a_{ij}\}_{i,j=1}^{n}$ be a $n \times n$ matrix of real numbers which we further call *weights*. In the bottleneck (capacity) assignment problem, we ask to minimize (maximize) the largest (smallest) element over all possible sets of n entries in A, one from each row and column. In our probabilistic framework we assume that the elements of the matrix A are selected randomly and independently with distribution function $F(\cdot)$. Under this assumption, we shall show that the optimal values of the BAP and CAP converge *in probability* and *in mean* to $F^{-1}(\log n/n)$ and $F^{-1}(1 - \log n/n)$ respectively. This settles the problems left open in [GG] and [WE]. As a consequence of our solution, we obtain a greedy algorithm which works in probability as good as the optimal one, that is, the relative error between the greedy and the optimal solutions tends to zero as n becomes large. This leads to $O(n^2)$ heuristic algorithm, and it suggests a more general problem: under what conditions a greedy algorithm can match in a probabilistic sense the quality of the optimal one. We are working along these lines, and some progress is reported in [SZ].

The bottleneck assignment problem (BAP) in a probabilistic framework was discussed in [GG,AV], and also shortly in [LU,WE]. In particular, Garfinkel and Gilbert [GG] have proved a lower bound for uniformly distributed weights that matches our lower bound. Weide [WE] has shown also an upper bound of the form $F^{-1}(C \log n/n)$, however, the constant C was not determined. Actually, for Weide to prove his result he had to use a powerful result of Pósa from a random graph theory [BO]. Our approach is completely different and we use only elementary property of the problem, and some results from order statistics [GA] (see also Section 3). Nevertheless, the methodological approach adopted in this paper is *not* restricted to the assignment problems. In particular, it can be applied to the analysis of traveling salesman problems, spanning trees, geometric location problems, minimum weighted clique problems and so on [SZ].

The paper is organized as follows. In the next section, we present our main results and discuss some consequences of them. All proofs are delayed to Section 3, where we establish a general framework to deal with the maximum of a set of random variables.

2. Main Results

In this section we give a precise formulation of the problem, present our main results and discuss some consequences of our findings.

Let $A = \{a_{ij}\}_{i,j=1}^{n}$ be an $n \times n$ matrix of real numbers (weights) and by $\sigma(\cdot)$ we denote a permutation of the set of indices $\{1,2, \ldots, n\}$. The set of all permutations of $\{1,2, \ldots, n\}$ is denoted by B_n, and naturally the cardinality of B_n is $n!$, that is, $|B_n| = n!$ The bottleneck assignment problem (BAP) seeks such a permutation σ that minimizes $\max_{1 \le i \le n} a_{i,\sigma(i)}$. That is, the objective function $Z_{\min}$ for (BAP) is

$$Z_{\min} = \min_{\sigma \in B_n} \{ \max_{1 \le i \le n} a_{i,\sigma(i)} \} \tag{2.1}$$

On the other hand, the objective function $Z_{\max}$ for the capacity assignment problem (CAP) is a reverse to (2.1), that is,

$$Z_{\max} = \max_{\sigma \in B_n} \{ \min_{1 \le i \le n} a_{i,\sigma(i)} \} \tag{2.2}$$

The formulation (2.1) and (2.2) are in fact more general than needed for BAP and CAP problems. For example (2.1) can be interpreted as the bottleneck traveling salesman problem, if one defines B_n as all Hamiltonian circuits in a graph span over n vertices and a_{ij} is understood as a weight assigned to the (i,j)-edge. In the same spirit (2.2) is a correct formulation for the capacity traveling salesman problem [SZ]. In fact, interpreting the weights a_{ij} in the last examples as the capacity of the edge (i,j), we can nicely motivate the names given to these two problems.

We shall analyze both problems in a probabilistic framework by assuming that the weights a_{ij} are *independently* and *identically* distributed (i.i.d) random variables with common distribution functions $F(\cdot)$. In addition, we assume that $F(x)$ is an *increasing* function of x. (Note that $F(\cdot)$ is always nondecreasing as a distribution function, but in addition, we require that $F(\cdot)$ is strictly increasing). By the last assumption, we note that

$$F(Z_{\min}) = \min_{\sigma \in B_n} \{ \max_{1 \le i \le n} F(a_{i,\sigma(i)}) \} \tag{2.1a}$$

and the same holds for (2.2). This reflects the fact that the solution of (2.1) and (2.2) does not depend on the individual values of a_{ij}, but only the ranking of the solution is important. This also implies that proving optimality of the problem for one particular distribution is enough to obtain a general solution. In particular, we shall use the fact that for any random variable X with the distribution $F(\cdot)$ the mapping $F(X) = U$ transforms X into a uniformly distributed random variable U [RE]. It will be convenient, as we shall see later, to work at the beginning with the exponential distribution.

Our main results can be formulated as follows.

THEOREM.(i) For the bottleneck assignment problems, the solution $Z_{\min}$ in (2.1) converges *in probability* to the following constant $F^{-1}(\log n/n)$, where log represents the natural logarithm. More precisely, as n tends to infinity

$$\lim_{n \to \infty} \frac{Z_{\min}}{F^{-1}(\log n/n)} = 1 \quad \textit{in probability} \tag{2.2}$$

In addition, if the r-th moment of the weights a_{ij} exists, then for large n

$$EZ^r_{\min} = (1 + o(1))[F^{-1}(\log n/n)]^r \tag{2.3}$$

where $EZ^r_{\min}$ stands for the r-th moment of $Z^r_{\min}$.

(ii) For the capacity assignment problem, the following holds for large n

$$\lim_{n \to \infty} \frac{Z_{\max}}{F^{-1}(1 - \log n/n)} = 1 \qquad \textit{in probability} \tag{2.4}$$

In addition, assuming that the r-th moment of the weights exists, then

$$EZ_{\max}^r = (1 + o(1))[F^{-1}(1 - \log n/n)]^r \tag{2.5}$$

where $EZ_{\max}^r$ is the r-th moment of $Z_{\max}$.

□

We delay the proof of the theorem and all necessary technicalities to the next section. The rest of this section is devoted to provide an interpretation of our theorem, and to discuss some consequences of it in terms of designing sub-optimal algorithms solving BAP and CAP problems.

As far as interpretation is concerned, condition (2.2) implies that the probability $Pr\{|Z_{\min}/F^{-1}(\log n/n) - 1| > \varepsilon\}$ becomes smaller and smaller for larger values of n, that is, $\lim_{n \to \infty} Pr\{|Z_{\min}/F^{-1}(\log n/n) - 1| > \varepsilon\} = 0$. Roughly speaking, this means that it is very unlikely that $Z_{\min}$ differs from $F^{-1}(\log n/n)$ by more than ε, whatever $\varepsilon > 0$ is selected. On the other hand, condition (2.3) implies that $\lim_{n \to \infty} EZ_{\min}^r/[F^{-1}(\log n/n)]^r = 1$, so the r-th moment of the objective function $Z_{\min}$ is well approximated by $[F^{-1}(\log n/n)]^r$ for large n. These facts imply that for large n the objective function (a random variable) $Z_{\min}$ becomes a degenerate random variable such that $Z_{\min} - EZ_{\min}$ *in probability*. This suggests also that any algorithm which produces a solution within an order of magnitude of $F^{-1}(\log n/n)$ is acceptable in the sense that the probability of error between this algorithm and the optimal solution becomes zero for large n (for more details see error lemma in [WE]). We next concentrate on this observation, and give some remarks concerning the designing of some heuristic solutions which are suboptimal for our problems.

A brute force solution to either BAP or CAP requires $O(n!)$ time complexity, however, a clever solution of Edmonds [ED] and Gabow [GB] needs only $O(n^3)$. A question arises whether a *greedy* algorithm can produce *faster* a solution which *is close* to the optimal one. In general, by greedy solution, one means a discrete version of the steepest descent algorithm [NW], that is, such that at each step the algorithm gives the greatest (smallest) immediate increase in the value of the objective function. For BAP the greedy solution is as follows. We first select the minimum element from the first column and delete it together with the row containing this element. We repeat the same with the second column, and so on. Finally, we find maximum of the elements just found. This is a feasible algorithm which belongs to a class of greedy ones. Naturally, it upper bounds the optimal solution and in Section 3, we use this upper bound to prove our main result. In particular, if Z_{grd} denotes the value of the objective function for the greedy solution, we prove that $Z_{grd} \sim F^{-1}(\log n/n)$, hence $Z_{\min}/Z_{grd} \sim 1$ *in probability*. The ready conclusion is that the greedy algorithm works as good as the optimal one in the sense that the error between these two tends to zero for large n. Naturally, the greedy algorithm is cheaper, and it is easy to see that the complexity is equal to $O(n^2)$. A more general question, however, arises. For what class of problems a greedy algorithm matches the quality of the optimal, more expensive, one in a probability sense. A solution to this problem will generalize matroid [NW] and greedoid structures [KL] which are proved to produce exact optimal solutions from a class of greedy algorithms. We are working along these lines and some progress is reported in [SZ].

3. Analysis and Some More Results

In this section we prove our theorem. We shall concentrate on BAP problems, and during the analysis we propose a methodology which is easily extended to the CAP problem.

To prove our estimate on $Z_{\min}$ defined in (2.1), we lower bound and upper bound the objective function (a random variable) by two other random variables which are easier to evaluate. For the lower bound $\underline{Z}$ we select the minimum value in each column of A, and then take the maximum over all such elements. That is

$$\underline{Z} \stackrel{def}{=} \max_{1 \le j \le n} \{ \min_{1 \le i \le n} a_{ij} \} \underset{st}{\le} \min_{\sigma \in B_n} \{ \max_{1 \le i \le n} a_{i,\sigma(i)} \} = Z_{\min} \tag{3.1}$$

where $\underset{st}{\le}$ is stochastically smaller [ST]. The upper bound for $Z_{\min}$ is conceptually easy to construct. Indeed, selecting a particular permutation, say σ^*, by the definition of the problem we must have $Z_{\min} \le Z(\sigma^*)$. It seems to be interesting to select the solution σ^* such that the algorithm becomes a greedy one. This can be done as follows in a sequence of n steps. In the first step, we select the minimum element of the first column in A, and delete this column together with the row it contains. The resulting matrix is $A^{(2)}$. We apply the same arguments to $A^{(2)}$ constructing $A^{(3)}$ and so on, and finally $A^{(n)}$ which is just one remaining element in the last column. Let Z_{grd} denote the value of the objective function for such a greedy construction. Since the elements a_{ij} of A are i.i.d. random variables, the equivalent form of Z_{grd} is as follows

$$Z_{grd} = \max_{1 \le j \le n} \{ \min_{j \le i \le n} a_{ij} \} \underset{st}{\ge} Z_{\min} \tag{3.2}$$

so finally

$$\underline{Z} \underset{st}{\le} Z_{\min} \underset{st}{\le} Z_{grd} \tag{3.3}$$

The problem is solved once we establish some asymptotics for order statistics defined in (3.1) and (3.2), namely the maximum of independent, however not necessary identical distributed, random variables.

Let $X_1, X_2, \ldots, X_n$ be a sequence of random variables each distributed according to $G_1(\cdot), \ldots, G_n(\cdot)$. Define

$$M_n = \max_{1 \le i \le n} \{X_i\} \tag{3.4}$$

We are interested in the behavior of M_n for large values of n. In particular, we shall study convergence of M_n *in probability* sense. We restrict the class of distribution functions $G_1(x)$, $G_2(x), \ldots, G_n(x)$ to ones satisfying the following two conditions

(i) for every $k = 1,2, \ldots, n \quad G_k(x) < 1 \quad \text{for} \quad x < \infty$

(ii) uniformly in $k = 1,2, \ldots, n$ (3.5)

$$\lim_{x \to \infty} \frac{1 - G_k(cx)}{1 - G_k(x)} = 0 \qquad c > 1 \tag{3.6}$$

Let us also define a_n as the smallest solution to the next equation

$$\sum_{k=1}^{n} [1 - G_k(a_n)] = 1 \tag{3.7}$$

We prove the following lemma.

Lemma.(i) If conditions (3.5) and (3.6) hold, then

$$\lim_{n \to \infty} \frac{M_n}{a_n} \le 1 \quad \textit{in probability} \tag{3.8}$$

where a_n is the solution of (3.7).

(ii) If, in addition to (3.5) and (3.6), $X_1, X_2, \ldots, X_n$ are independent, then

$$\lim_{n \to \infty} \frac{M_n}{a_n} = 1 \quad \textit{in probability} \tag{3.9}$$

(iii) Assume hypotheses of (ii) hold and the r-th moments of $X_1, X_2, \ldots, X_n$ exist, then

$$EM_n^r = (1 + o(1))a_n^r \tag{3.10}$$

and a_n, as before, is defined in (3.7).

Proof.(i) Let $R_k(x) = 1 - G_k(x) = Pr\{X_k > x\}$. Note that, taking also into account (3.5), we can upper bound [RE]

$$\begin{aligned} Pr\{M_n > r\} = Pr\{X_1 > r \ \text{ or } \ X_2 > 2 \ \text{ or } \ \cdots \ \text{ or } \ X_n > r\} \le \\ \le \sum_{k=1}^{n} R_k(r). \end{aligned} \tag{3.11}$$

Put $r = a_n(1 + \varepsilon)$, and note that (3.6) implies $R_k(a_n(1 + \varepsilon)) = o(1)R_k(a_n)$ uniformly in k, so from (3.11) we obtain

$$Pr\{M_n > (1 + \varepsilon)a_n\} = o(1) \sum_{k=1}^{n} R_k(a_n) = o(1)$$

by (3.7). This proves Lemma (i).

(ii) To prove (3.9) we assume, in addition that $X_1, X_2, \ldots, X_n$ are independent, that is,

$$Pr\{M_n < r\} = Pr\{X_1 \le r, X_2 \le r, \ldots, X_n \le r\} = G_1(r) \cdots G_n(r) \tag{3.12}$$

It is more convenient to deal with logarithm of (3.12). We first note that for $v \to 1$ [GA] $\log v = \log[1-(1-v)] \le v - 1$, hence for (3.12)

$$\log Pr\{M_n < r\} \le -\sum_{k=1}^{\infty} [1 - G_k(r)] \tag{3.13}$$

Let $r = (1-\varepsilon)a_n$, $\varepsilon > 0$. Then to prove (3.9), we need to show that the RHS of (3.13) tends to $-\infty$ when $n \to \infty$ for $r = (1-\varepsilon)a_n$. But substituting in (3.6) $x = z/c$ for $c > 1$ one finds $1 - G_k(x) = o(1)\,(1 - G_k(z/c))$. Let $1/c = 1 - \varepsilon < 1$, then by (3.7) and the above one shows

$$\sum_{k=1}^{n} [1 - G_k(1-\varepsilon)a_n)] = \frac{1}{o(1)} \sum_{k=1}^{n} [1 - G_k(a_n)] = \frac{1}{o(1)} \tag{3.14}$$

so this proves $\log Pr\{M_n < (1-\varepsilon)a_n\} \to -\infty$, and it completes the derivation of (3.9).

(iii) In Lemma (ii), we have proved that $M_n \xrightarrow{P} a_n$. This would imply convergence *in mean* (3.10) if one proves that $X_1, X_2, \ldots, X_n$ are uniformly integratable [RE]. This fact is proved in [LR] assuming (3.6) holds and $X_1, X_2, \ldots, X_n$ possess r-th moments.

□

Now we are ready to prove our main result, that is, Theorem from Section 2. We consider first the lower bound $\underline{Z}$ given by (3.1). Let $X_k = \min_{1 \le i \le n} a_{ij}$ where a_{ij} is $F(\cdot)$ distributed. Note that $G_k(x) = Pr\{X_k > x\} = [1 - F(x)]^n$, since a_{ij} are i.i.d. To apply our lemma, we must select a distribution $F(\cdot)$ satisfying (3.5) and (3.6). Since, as noticed before, the solution to the BAP problem is distribution-independent, we can select any distribution without loss of generality. Therefore, we assume a_{ij} are exponentially distribution, that is $F(x) = 1 - e^{-x}$. The last step before applying our lemma, is to solve equation (3.7). In our case it becomes

$$ne^{-na_n} = 1$$

so, $a_n = \log n/n$. We note that $U = F(X)$ is uniformly distributed, so for $F(x) = 1 - e^{-x}$ we find that a_n for uniformly distributed weights is

$$a_n = 1 - \exp(-\log n/n) = \frac{\log n}{n} + o(\log n/n) \tag{3.15}$$

This directly implies that $Z_{\min} \geq F^{-1}(\log n/n + o(\log n/n))$ for any distribution function $F(\cdot)$.

To prove upper bound on $Z_{\min}$, we consider the greedy algorithm described before and investigate Z_{grd} defined in (3.2). Now, we define $X_k = \min_{k \leq i \leq n} a_{ij}$, so $G_k(x) = Pr\{X_k > x\} = [1 - F(x)]^{n-k}$. As in the case of the lower bound, we choose to work with exponential distributions $F(x) = 1 - e^{-x}$, then (3.5) and (3.6) are easy to verify. The equation (3.7) for a_n becomes

$$1 = e^{-na_n} + e^{-(n-1)a_n} + \cdots + e^{-a_n} \tag{3.16}$$

To solve (3.16) we note that the equivalent equation (e.g., $y = e^{-a_n}$)

$$y^n + y^{n-1} + \cdots + y = 1$$

has asymptotically the following solution $y = 1 - \frac{\log n}{n}$ which is easy to verify. Hence, $e^{-a_n} = 1 - \log n/n$, and finally $a_n = \log n/n + o(\log n/n)$. This proves upper bound $Z_{\min} \leq Z_{grd} \sim F^{-1}(\log n/n)$ and completes the proof of our Theorem (i).

The proof of the second part of our Theorem is derived in a similar way. This time, however, we must concentrate on minimum of a set $X_1, X_2, \ldots, X_n$ of random variables. A lemma similar to the one proved in the paper can be formulated for $m_n = \min\{X_1, \ldots, X_n\}$. We need, however, to replace in (3.5) – (3.7) $1 - G_k(x)$ by the distribution function $G_k(x)$. In particular, (3.7) becomes

$$\sum_{k=1}^{n} G_k(a_n) = 1 \tag{3.17}$$

The other steps of the previous proof are the same. In particular, a solution to (3.17) for negative exponential distribution becomes $a_n \sim 1 - \log n/n$, as needed in Theorem (ii).

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