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A. Hadjidimos

D. Noutsos

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A. Hadjidimos[+,*]
D. Noutsos[*,+]

Computer Sciences Department
Purdue University

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+ Visiting, Department of Computer Sciences, Purdue University, West Lafayette, IN 47907.
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* Department of Mathematics, University of Ioannina, GR-451 10 Ioannina, Greece.

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Abstract

In this paper it is assumed that the point (or block) Jacobi matrix B associated with the matrix A is weakly 2-cyclic consistently ordered with complex, in general, eigenvalue spectrum, $\sigma(B)$, lying in the interior of the infinite unit strip. It is then our objective to apply and extend the Young-Eidson algorithm in order to determine the real optimum relaxation factor in the following two cases: i) In the case of the Successive Overrelaxation (SOR) matrix associated with A when $\sigma(B)$ lies in a "bow-tie" region and ii) In the case of the Symmetric SOR (SSOR) matrix associated with A. It is noted that as a by-product of (ii) above both the relaxation factor for the SSOR matrix corresponding to a "bow-tie" spectrum $\sigma(B)$ and the optimum pairs of the relaxation factors for the Unsymmetric SOR (USSOR) matrix associated with A are also obtained.

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1. Introduction and Preliminaries

In 1970 an algorithm for the determination of the real optimum relaxation factor for the Successive Overrelaxation (SOR) matrix associated with a weakly 2-cyclic consistently ordered Jacobi matrix B (see e.g. [12], [15], [4] or [9]) whose eigenvalue spectrum, $\sigma(B)$, was complex, was developed and proposed by Young and Eidson [16] (see also [15]). To the best of our knowledge, so far, the powerfulness and the simplicity of the Young-Eidson algorithm has been explored by few researchers (see e.g. [2], [3], [1], etc.). So, problems, which could have been solved by the aforementioned algorithm in a much simpler, clearer and more efficient way, have been attacked with more complicated methods while others have simply remained unsolved. Here we mention i) the problem of the optimum SOR parameter when $\sigma(B)$ lies in a "bow-tie" region obtained by Chin and Manteuffel [5] (see also [7]) and ii) the "unsolved" problem of the optimum relaxation factor for the Symmetric SOR (SSOR) method.

It is the purpose of this paper to strictly follow the reasoning behind the Young-Eidson algorithm and "extend" it in order to give the solutions to both aforementioned problems (i) and (ii). These problems are solved under the assumption made in the beginning that is the Jacobi matrix B is weakly 2-cyclic consistently ordered and that $\sigma(B) \subset S$, where

$$S := \{z \in \mathbb{C} : |Re z| < 1\} \quad . \tag{1.1}$$

They are presented in Sections 3 and 4 respectively. As a by-product of our analysis the optimum relaxation factor for the SSOR matrix for a "bow-tie" $\sigma(B)$ and the optimum pairs for the relaxation factors of the Unsymmetric SOR (USSOR) method are also obtained in Section 4. Meanwhile in Section 2 we give a synopsis of the background material on which the Young-Eidson algorithm is based, so that the interested reader can follow its extensions in the later sections with not much difficulty.

2. Presentation of Background Material

Assume that

$$A := D - L - U \tag{2.1}$$

is a 2-cyclic consistently ordered matrix with nonsingular corresponding diagonal (or block diagonal) part D and strictly lower and upper triangular parts L and U. Denote by

$$B := D^{-1}(L + U) \tag{2.2}$$

and

$$\mathcal{L}_\omega := (D - \omega L)^{-1} \ [(1-\omega)D + \omega U] \quad , \tag{2.3}$$

where $\omega \in (0,2)$ is the relaxation factor, the Jacobi and the SOR matrices associated with A. Let H be the smallest convex polygon symmetric about the axes such that $\sigma(B) \subset H$ and let $P_j(\alpha_j, \beta_j)$, $j = 1(1)s$, be its vertices in the first quadrant, in increasing order of magnitude of their abscissas. Obviously our basic assumption $\sigma(B) \subset S$ implies $H \subset S$.

Let now E_p denote an ellipse passing through the point P, in the first quadrant of S, symmetric about the axes and contained in S. Let also $\mathrm{int}E_p$ and $\overline{\mathrm{int}}E_p$ denote the interior and the closure of the interior of E_p. Then an analysis based on the Young's famous relationship [14]

$$(\lambda + \omega - 1)^2 = \omega^2 \mu^2 \lambda \tag{2.4}$$

which connects the two sets of eigenvalues $\mu \in \sigma(B)$ and $\lambda \in \sigma(\mathcal{L}_\omega)$, shows that (see [15, pp. 191-200]): If a and b are the "real" and the "imaginary" semiaxes of an E_p passing through the vertex P of H and is such that $\sigma(B)$ (and H) $\subset \overline{\mathrm{int}}E_p$ then the parameter ω and the spectral radius of $\mathcal{L}_\omega, \rho(\mathcal{L}_\omega)$, are given by the expressions

$$\begin{aligned} &\omega = 2/(1 + (1 - a^2 + b^2)^{1/2}), \ \rho(\mathcal{L}_\omega) = \rho^2 \quad , \\ &\rho = (a+b)/(1 + (1 - a^2 + b^2)^{1/2}) \quad . \end{aligned} \tag{2.5}$$

Out of the infinitely many ellipses E_p which satisfy $H \subset \overline{\mathrm{int}}E_{p_j}$, $j = 1(1)s$, there exists a unique "optimum" one $\hat{E}$ for which ρ is a minimum. For $s > 2$ the optimum ellipse is determined by means of the Young-Eidson algorithm. The latter is, in turn, based on the optimum results for $s = 1$ and 2. For $s = 1$, let $P_1(\alpha_1, \beta_1) \equiv P(\alpha, \beta)$. Then one can find out that ρ, in (2.5), as a function of $a \in [\alpha_1, 1]$ strictly decreases in $[\alpha_1, \hat{a}]$ from 1 to $\hat{\rho}$ and strictly decreases in $[\hat{a}, 1]$ from $\hat{\rho}$ to 1. The optimum value for ρ, $\hat{\rho}$, is the unique root of

$$\left[(1+\rho^2)/(2\rho) \right]^{2/3} \alpha^{2/3} + \left[(1-\rho^2)/(2\rho) \right]^{2/3} \beta^{2/3} - 1 = 0 \tag{2.6}$$

in $(0,1)$, where it is noted that (2.6) is equivalent to a cubic equation (see e.g. [3]), while the optimum values for a and b, $\hat{a}$ and $\hat{b}$, are given by

$$\hat{a} = \left[2\hat{\rho}\alpha^2 / (1+\hat{\rho}^2) \right]^{1/3} \ , \ \ \hat{b} = \left[2\hat{\rho}\beta^2 / (1-\hat{\rho}^2) \right]^{1/3} \quad . \tag{2.7}$$

Finally the optimum values $\hat{\omega}$ and $\rho(\mathcal{L}_{\hat{\omega}})$ are obtained through (2.5) by using (2.7). In the very special cases $\beta_1 = 0$ and $\alpha_1 = 0$ the well-known results

$$\hat{\omega} = 2 / \left[1 + (1 - \rho^2(B))^{1/2} \right] \ , \ \ \rho(\mathcal{L}_{\hat{\omega}}) = \hat{\omega} - 1 \tag{2.8a}$$

due to Young [14] and

$$\hat{\omega} = 2 \Big/ \left[1 + (1 + \rho^2(B))^{1/2} \right] \ , \quad \rho(\mathcal{L}_{\hat{\omega}}) = 1 - \hat{\omega} \ , \qquad (2.8b)$$

a special case of Kredell's result [10], are easily recovered.

For $s = 2$ let $E_{p_1p_2}$ be the ellipse symmetric about the axes which passes through both vertices P_1 and P_2. Its semiaxes $a_{1,2}$ and $b_{1,2}$ are given by

$$a_{1,2} = \left[(\alpha_2^2\beta_1^2 - \alpha_1^2\beta_2^2) / (\beta_1^2 - \beta_2^2) \right]^{1/2} \ , \quad b_{1,2} = \left[(\alpha_2^2\beta_1^2 - \alpha_1^2\beta_2^2) / (\alpha_2^2 - \alpha_1^2) \right]^{1/2} \ . \qquad (2.9)$$

The optimum ellipse $\hat{E}$ for H (and $\sigma(B)$) is obtained after an analysis based on the previous arguments takes place (see [15]). If $\hat{E}_{p_j}$ is the optimum ellipse corresponding to P_j and $\hat{a}_j$, $\hat{b}_j$ its semiaxes ($j = 1,2$) then $\hat{E}$ can be determined by the following simple algorithm given in pseudo-code:

Alg. 1: Determine $E_{p_1p_2}(a_{1,2})$;
Determine $\hat{E}_{p_2}(\hat{a}_2)$;
if $\hat{a}_2 \le a_{1,2}$ **then** $\hat{E} \equiv \hat{E}_{p_2}$; **stop;**
else Determine $\hat{E}_{p_1}(\hat{a}_1)$;
if $a_{1,2} \le \hat{a}_1$ **then** $\hat{E} \equiv \hat{E}_{p_1}$; **stop;**
else $\hat{E} \equiv E_{p_1p_2}$; **stop;**
endif;
endif;
end of Alg. 1;

The Young-Eidson algorithm is an ingenious systematic extension of Alg. 1 to $s \ge 3$ (see [16] and [15]). It is taken into consideration that two distinct ellipses symmetric about the axes can not have more than one common point in the first quadrant. Thus by virtue of the analysis presented so far the optimum ellipse $\hat{E}$ is the one out of the $\hat{E}_{p_j}$'s, $j = s(-1)1$, for which $H \subset \overline{\mathrm{int}\hat{E}_{p_j}}$, provided such an ellipse exists, or, in case it does not exist, it is the ellipse $E_{p_jp_k}$ out of $E_{p_jp_k}$'s, $j = s(-1)2$, $k = j - 1(-1)1$, satisfying $H \subset \overline{\mathrm{int}} E_{p_jp_k}$, which corresponds to the smallest ρ. The existence and uniqueness of $\hat{E}$ readily follow.

3. Optimum Relaxation Factor for a Bow-Tie Region

In a recent paper Chin and Manteuffel [5] determined, after a rather complicated analysis, the optimum SOR factor in case $\sigma(B)$ lies in a bow-tie region $R \subset S$ (see Figures 1, 2 and 3). A solution to the same problem was provided by Eiermann et al [7] by applying asymptotically optimal hybrid Semi-Iterative methods. Here we present a much simpler solution based on the strict reasoning of the Young-Eidson algorithm.

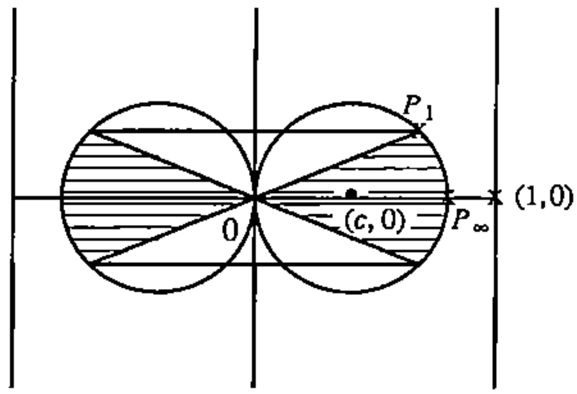

Figure 1

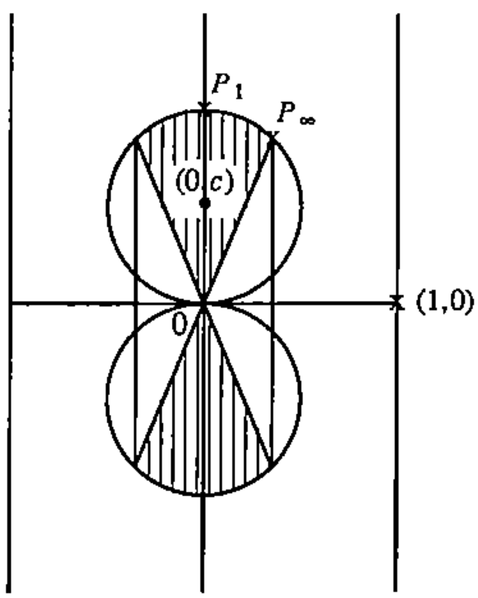

Figure 2

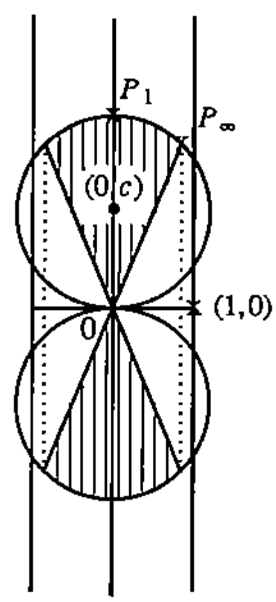

Figure 3

For the algorithm to be applied it is assumed that the convex polygon H, defined in the previous section, has innumerably many vertices V in the first quadrant, that is all the points of the arc $P_\infty P_1$, with abscissas α strictly decreasing from α_∞ to α_1. The coordinates (α, β) of each vertex V satisfy the equation

$$(x-c)^2 + y^2 = c^2 \qquad \text{(Fig. 1)} \tag{3.1}$$

or

$$x^2 + (y-c)^2 = c^2 \qquad \text{(Figs. 2, 3)} \tag{3.1'}$$

First it is noticed that the optimum ellipse $\hat{E}$ for the aforementioned infinite set of vertices V can not be an ellipse passing through two vertices V_2 and V_1 since then the arc $V_2V_1 \not\subset \mathrm{int}\hat{E}$. Consequently, $\hat{E}$ must be an optimum ellipse passing through one vertex only. Apparently this leaves us with three possibilities: $\hat{E}$ is either of $\hat{E}_{p_\infty}$, $\hat{E}_{p_1}$, passing through the end point-vertices P_∞ or P_1, or the osculating ellipse which is tangent to the arc $P_\infty P_1$, at a point V of it and is, at the same time, the $\hat{E}_v$.

In the case of Figure 1, $\hat{E}_{p_\infty}$ is excluded since it is the line segment $[-2c, 2c]$, $c < 1/2$, and $H \not\subset \hat{E}_{p_\infty}$. To examine the possibility of the optimum osculating ellipse, if such an ellipse exists, let $\hat{a}, \hat{b}$ be its semiaxes and $(\hat{\alpha}, \hat{\beta})$ the coordinates of the point of contact V. Substituting $(\hat{\alpha}, \hat{\beta})$ for (α, β) in (2.6) – (2.7) and recalling that $\hat{E}_v$ and (3.1) have a common tangent at $(\hat{\alpha}, \hat{\beta})$ defined by

$$\hat{\alpha}x / \hat{a}^2 + \hat{\beta}y / \hat{b}^2 = 1 \quad \text{and} \quad (\hat{\alpha} - c)(x - c) + \hat{\beta}y = c^2 \tag{3.2}$$

it is obtained that

$$\hat{a}^2 = \hat{\alpha}^2 c / (\hat{\alpha} - c), \quad \hat{b}^2 = \hat{\alpha} c \ . \tag{3.3}$$

Hence by using the fact that $(\hat{\alpha}, \hat{\beta})$ lies on (3.1), the expressions (3.3) and then equating the two roots of the two equations (2.7) in the interval $(0,1)$, namely

$$\hat{\rho}_1 = \left[\frac{(\hat{\alpha} - c)^3}{\hat{\alpha}^2 c^3} \right]^{1/2} - \left[\frac{(\hat{\alpha} - c)^3}{\hat{\alpha}^2 c^3} - 1 \right]^{1/2} \tag{3.4a}$$

and

$$\hat{\rho}_2 = - \left[\frac{\hat{\beta}^4}{\hat{\alpha}^3 c^3} \right]^{1/2} + \left[\frac{\hat{\beta}^4}{\hat{\alpha}^3 c^3} + 1 \right]^{1/2} \ , \tag{3.4b}$$

one obtains

$$\hat{\alpha} = \frac{c(1 + (5 - 4c^2)^{1/2})}{2(1 - c^2)} \quad \left[= \frac{2c}{(5 - 4c^2)^{1/2} - 1} \right] \tag{3.5}$$

Therefore if $\alpha_1 \le \hat{\alpha}$, the optimum ellipse for H is $\hat{E}_v$. Using (3.5) in (3.3) and also in one of (3.4) and then the resulting expressions in (2.5) the values of $\hat{\rho}$, $\hat{\omega}$ and $\rho(\mathcal{L}_{\hat{\omega}})$, in terms of c, are obtained. It is checked that either of (3.4) coincides with (3.34) in [5] and that (2.6) yields (3.35b) in [5], where in the latter the sign in the constant term should be minus instead of plus. If, on the other hand, $\hat{\alpha} < \alpha_1$ the optimum ellipse for H is $\hat{E}_{p_1}$ and all the optimum parameters associated with it are obtained from (2.6), (2.7) and (2.5), with $(\alpha, \beta) = (\alpha_1, \beta_1)$

In the case of Figures 2 and 3, $\hat{E}_{p_1}$ is excluded since it is the line segment $[-2ic, 2ic]$ and $H \not\subset \hat{E}_{p_1}$. The possibility of osculating ellipse has to be examined only in the case of Figure 2 and this is what is done very briefly in the sequel. The analysis almost duplicated the one made previously, where, instead of (3.1), (3.1') is used. Thus one can obtain

$$\hat{\alpha}x / \hat{a}^2 + \hat{\beta}y / \hat{b}^2 = 1 \quad \text{and} \quad \hat{\alpha}x + (\hat{\beta} - c)(y - c) = c^2 \quad , \tag{3.2'}$$

$$\hat{a}^2 = \hat{\beta}c \quad , \quad \hat{b}^2 = \beta^2 c / (\beta - c) \quad , \tag{3.3'}$$

$$\hat{\rho}_1 = \left[\frac{\alpha^4}{\beta^3 c^3} \right]^{1/2} - \left[\frac{\alpha^4}{\beta^3 c^3} - 1 \right]^{1/2} \quad , \tag{3.4a'}$$

$$\hat{\rho}_2 = - \left[\frac{(\beta - c)^3}{\beta^2 c^3} \right]^{1/2} + \left[\frac{(\beta - c)^3}{\beta^2 c^3} + 1 \right]^{1/2} \tag{3.4b'}$$

and finally

$$\hat{\beta} = \frac{c(1 + (5 + 4c^2)^{1/2})}{2(1 + c^2)} \quad \left[= \frac{2c}{(5 + 4c^2)^{1/2} - 1} \right] \quad . \tag{3.5'}$$

Consequently if $\beta_\infty \le \hat{\beta}$, the optimum ellipse for H is $\hat{E}_v$. So the values for $\hat{\rho}$, $\hat{\omega}$ and $\rho(\mathcal{L}_{\hat{\omega}})$ are derived in exactly the same way as before, where, however, the corresponding primed expressions are used. It is again checked that either of (3.4') coincides with (3.49) in [5] and that (2.6) yields (3.50b) in [5]. It should be mentioned that the numerator in the last fraction under the last square root in the denominator should read 4 instead of 1. If $\hat{\beta} < \beta_\infty$ or if we are in the case of Figure 3, $\hat{E}_{p_-}$ is the optimum ellipse for H. The associated optimum parameters are obtained from (2.6), (2.7) and (2.5) with $(\alpha, \beta) = (\alpha_\infty, \beta_\infty)$.

4. Optimum Relaxation Factor for the SSOR Matrix.

4.1. Development of the Basic Theory

As is known [15],

$$S_\omega := (D - \omega U)^{-1} \left[(1 - \omega)D + \omega L \right] (D - \omega L)^{-1} \left[(1 - \omega)D + \omega U \right]$$

is the SSOR matrix associated with A in (2.1), where $\omega \in (0,2)$ is the relaxation factor. For A 2-cyclic consistently ordered the sets of eigenvalues $\mu \in \sigma(B)$ and $\lambda \in \sigma(S_\omega)$ are connected through the relationship

$$\left[\lambda - (1 - \omega)^2 \right]^2 = \omega^2 (2 - \omega)^2 \mu^2 \lambda \tag{4.1}$$

due to D'Sylva and Miles [6] and Lynn [11] (see also [13]). However, as was proved in [8], when one makes the substitution $\omega' = \omega(2 - \omega) \in (0,1]$ then there exist values of ω', at least in the neighborhood of 0, for which $\rho(S_\omega) < 1$ iff $\sigma(B^2)$ lies in the interior of the parabola $P := y^2 = -4x + 4$, the latter requirement being equivalent to $\sigma(B) \subset S$ (the infinite unit strip). On the other hand, since the aforementioned substitution transforms (4.1) into

$$(\lambda + \omega - 1)^2 = \omega^2 \mu^2 \lambda \quad , \tag{4.2}$$

where primes have been dropped to simplify the notation, which is nothing but (2.4), it is concluded that the problem of the determination of the optimum ω is exactly the same as the one solved in Section 2 with the only exception being that the new ω is now restricted to values in $(0,1]$. This, in turn, implies that for the convex polygon H, defined there, with one vertex $P(\alpha,\beta)$ in the first quadrant out of all the ellipses E_p, such that $H \subset \mathrm{int} E_p$, only those with $a \le b$, equivalent to $\omega \in (0,1]$, have to be considered. Consequently, having in mind the analysis in Section 2 and especially how ρ varies with the semiaxis a varying in $[\alpha,1]$ in order to determine the optimum SSOR factor we work as follows: "Determine the optimum ellipse $\hat{E}_p$ as in Section 2. i) If $\hat{a} \le \hat{b}$ then find $\hat{\omega}$ from (2.5) ii) If $\hat{a} > \hat{b}$ then $\hat{\omega} = 1$ and the circle $\hat{C}_p$ centered at the origin and passing through P gives the optimum "ellipse" for the SSOR problem. In case $\hat{\omega} < 1$ two values $\hat{\omega}_1$ and $\hat{\omega}_2$, the zeros of $\omega(2-\omega) = \hat{\omega}$, are the optimum values for the original ω in the SSOR matrix (4.1). In case $\hat{\omega} = 1$, $\hat{\omega}_1 = \hat{\omega}_2 = 1$."

The above "algorithm" can be directly applied to the cases of i) $\sigma(B)$ real with $\rho(B) < 1$ and ii) $\sigma(B)$ pure imaginary to yield well known results. In case (i) it is $\hat{a} = \rho(B) > 0 = \hat{b}$ implying that

$$\hat{\omega} = 1, \quad \rho(S_{\hat{\omega}}) = \rho^2(B) \quad , \tag{4.3}$$

while in case (ii) it is $\hat{a} = 0 < \rho(B) = \hat{b}$ giving that

$$\hat{\omega}_{1,2} = 1 \pm \frac{\rho(B)}{1 + (1+\rho^2(B))^{1/2}} \quad , \quad \rho(S_{\hat{\omega}_1}) = \rho(S_{\hat{\omega}_2}) = \frac{1 - (1+\rho^2(B))^{1/2}}{1 + (1+\rho^2(B))^{1/2}} \quad , \tag{4.4}$$

(see e.g. [8]).

In the way the Young-Eidson algorithm was developed to determine the optimum ellipse $\hat{E}$, based on the analysis of the special cases when the convex polygon H had one or two vertices in the first quadrant, in a quite analogous way an extension of the algorithm in question can be developed to cope with the SSOR case. In the sequel we give first the algorithm in case H has two vertices in the first quadrant and then the algorithm in the general case. The basic assumptions and the various notations are the ones used so far except that the pair (ν,ξ) is used to denote the semiaxes of an ellipse passing through two points in the first quadrant.

4.2. The Two-Point Algorithm

Alg. 2: Determine $E_{p_1 p_2}(\nu,\xi)$;
  Determine $\hat{E}_{p_2}(\hat{a}_2,\hat{b}_2)$;
  if $\nu > \xi$ **then**
    if $\hat{a}_2 > \hat{b}_2$ **then** $\hat{E} \equiv \hat{C}_{p_2}$; **stop;**
    else $\hat{E} \equiv \hat{E}_{p_2}$; **stop;**
    endif;
  else
    if $\hat{a}_2 \le \nu$ **then** $\hat{E} \equiv \hat{E}_{p_2}$; **stop;**
    else Determine $\hat{E}_{p_1}(\hat{a}_1,\hat{b}_1)$;

if $\hat{a}_1 \geq \nu$ **then**
if $\hat{a}_1 > \hat{b}_1$ **then** $\hat{E} \equiv \hat{C}_{p_1}$; **stop;**
else $\hat{E} \equiv \hat{E}_{p_1}$; **stop;**
endif;
else $\hat{E} \equiv E_{p_1 p_2}$; **stop;**
endif;
endif;
endif;
end of Alg. 2;

4.3. The Many-Point Algorithm

Alg. 3: $\nu_{\text{old}} := \alpha_s$; $\rho_{\text{old}} := 1$;
again: $\nu_{\text{new}} := 1$; $\xi_{\text{new}} := 0$;
for $j := s-1(-1)1$ **do**
Determine $E_{p_s p_j}(\nu_j, \xi_j)$;
if $\nu_{\text{new}} > \nu_j$ **then**
$k := j$; $\nu_{\text{new}} := \nu_k$; $\xi_{\text{new}} := \xi_k$;
endif;
end do;
Determine $\hat{E}_{p_s}(\hat{a}_s, \hat{b}_s)$;
if $\nu_{\text{new}} > \xi_{\text{new}}$ **then**
if $\hat{a}_s > \hat{b}_s$ **then**
Determine $\hat{\rho}_c$($\equiv$ radius of the circle $\hat{C}_{p_s}$)
if $\hat{\rho}_c < \rho_{\text{old}}$ **then**
$\hat{E} \equiv \hat{C}_{p_s}$; $\hat{\rho} := \hat{\rho}_c$; **stop;**
else
$\hat{E} \equiv E_{p_r p_q}$; $\hat{\rho} := \rho_{\text{old}}$; **stop;**
endif;
else
if $\hat{a}_s \geq \nu_{\text{old}}$ **then**
$\hat{E} \equiv \hat{E}_{p_s}$; $\hat{\rho} := \hat{\rho}_s$; **stop;**
else
$\hat{E} \equiv E_{p_r p_q}$; $\hat{\rho} := \rho_{\text{old}}$; **stop;**
endif;
endif;
else
if $\hat{a}_s < \nu_{\text{old}}$ **or** $\hat{a}_s > \nu_{\text{new}}$ **then**
Determine $\rho_{s,k}$ ($\equiv \rho$ corresponding to $E_{p_s p_k}$);
if $\rho_{s,k} < \rho_{\text{old}}$ **then**
$\rho_{\text{old}} := \rho_{s,k}$; $r := s$; $q := k$;
endif;
if $k = 1$ **then**
Determine $\hat{E}_1(\hat{a}_1, \hat{b}_1)$;

if $\hat{a}_1 \geq v_{new}$ **then**
  if $\hat{a}_1 > \hat{b}_1$ **then**
    Determine $\hat{\rho}_c$ ($\equiv$ radius of the circle $\hat{C}_{p_1}$);
    if $\hat{\rho}_c < \rho_{old}$ **then**
      $\hat{E} \equiv \hat{C}_{p_1}$; $\hat{\rho} := \hat{\rho}_c$; **stop**;
    else
      $\hat{E} \equiv E_{p_r p_q}$; $\hat{\rho} := \rho_{old}$; **stop**;
endif;
  else
    $\hat{E} \equiv \hat{E}_{p_1}$; $\hat{\rho} := \hat{\rho}_1$; **stop**;
  endif;
else
  $\hat{E} \equiv E_{p_r p_q}$; $\hat{\rho} := \rho_{old}$; **stop**;
endif;
else
  $s := k$; $v_{old} := v_{new}$; **goto** again;
endif;
else
  $\hat{E} \equiv \hat{E}_{p_s}$; $\hat{\rho} := \hat{\rho}_s$; **stop**;
endif;
endif;
end of Alg. 3;

4.4. Applications

The analysis and the optimum algorithms presented in the previous subsections will be applied i) to the SSOR matrix corresponding to the "bow-tie" spectrum B of Section 2 and ii) to the Unsymmetric (US) SOR matrix associated with a special type block 2-cyclic consistently ordered matrix A in (2.1).

i) The observation made in Section 2, that is the optimum ellipse $\hat{E}$ for the SOR matrix can not be an ellipse passing through two vertices of the "convex polygon" still holds for the SSOR matrix. In the case of Figure 1 we notice that $\hat{E}$ has $\hat{a}_\infty = \rho(B) = 2c > 0 = \hat{b}_\infty$. This simply implies that $\hat{\omega} = 1$ and the problem is solved. In the case of Figures 2 and 3 we have for $\hat{E}$ either $\hat{E}_v$ ($\hat{\alpha} < c < \hat{\beta}$) or E_{p_∞} ($\hat{\alpha}_\infty < c < \hat{\beta}_\infty$). In view of (2.7) it is implied that either $\hat{a} < \hat{b}$ or $\hat{a} = \hat{a}_\infty < \hat{b}_\infty = \hat{b}$ respectively. Therefore $\hat{\omega}$ obtained from (2.5) with $(a,b) = (\hat{a},\hat{b})$ provides us with the two values $\hat{\omega}_1$, $\hat{\omega}_2$ of the optimum SSOR factor through the formulas

$$\hat{\omega}_{1,2} = 1 \pm \frac{(\hat{b}^2 - \hat{a}^2)^{1/2}}{1 + (1 - \hat{a}^2 + \hat{b}^2)^{1/2}} \quad . \tag{4.5}$$

ii) Let A in (2.1) be 2-cyclic consistently ordered matrix of the following block form

$$A = \begin{bmatrix} D_1 & -U \\ -L & D_2 \end{bmatrix} \quad , \tag{4.6}$$

where D_1, D_2 are square nonsingular matrices. If, in (2.1), $D = \text{diag}(D_1, D_2)$ then the USSOR matrix associated with A in (4.6) is defined by

$$C_{\omega_1,\omega_2} := (D - \omega_2 U)^{-1} \left[(1-\omega_2)D + \omega_2 L \right] (D - \omega_1 L)^{-1} \left[(1-\omega_1)D + \omega_1 U \right] \quad (4.7)$$

and as is proved in [15, pp. 476-478] the eigenvalues of C_{ω_1,ω_2} are the same as those of $\mathcal{L}_\omega$ with

$$\omega := \omega_1 + \omega_2 - \omega_1\omega_2 \quad . \quad (4.8)$$

On the other hand, necessary conditions for $\rho(C_{\omega_1,\omega_2}) < 1$ are

$$o < \omega_1 + \omega_2 - \omega_1\omega_2 < 2 \quad . \quad (4.9)$$

So, if there is no further restriction on ω_1, ω_2 then the algorithm of Young and Eidson will provide us with an optimum $\omega = \hat{\omega} \in (0,2)$ and (4.8) will give us optimum pairs $(\omega_1, \omega_2) = (\hat{\omega}_1, \hat{\omega}_2)$ lying on the hyperbola

$$\hat{\omega}_1 + \hat{\omega}_2 - \hat{\omega}_1\hat{\omega}_2 = \hat{\omega} \quad . \quad (4.10)$$

If, however, we impose a further restriction on ω_1, ω_2, as for example in the case of the SSOR method where $\omega_1 = \omega_2$, this may restrict the interval for ω from (0,2) to, say, $(\underline{\omega}, \overline{\omega}) \subset (0,2)$. In such a case new restrictions will be yielded on the semiaxes (a,b) of the ellipse E_p passing through the point $P(\alpha,\beta)$ which considered together with the behavior of ρ as a function of a will lead us to a modification of the basic Young-Eidson algorithm as this was done in the SSOR case.

References

1. G. Avdelas, J. de Pillis, A. Hadjidimos and M. Neumann, ''A Guide to the Acceleration of Iterative Methods Whose Iteration Matrix is Nonnegative and Convergent'', *SIAM J. Matrix Anal. Appl.* **9** (1988), 329-342.
2. G. Avdelas, S. Galanis and A. Hadjidimos, ''On the Optimization of a Class of Second Order Iterative Schemes'', *BIT* **23** (1983), 50-64.
3. G. Avdelas and A. Hadjidimos, ''Optimum Second Order Stationary Extrapolated Iterative Schemes'', *Math. Comput. Simulation* XXV (1983), 189-198.
4. A. Berman and R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979.
5. R.C.Y. Chin and T.A. Manteuffel, ''An Analysis of Block Successive Overrelaxation for a Class of Matrices with Complex Spectra'', *SIAM J. Numer. Anal.* **25** (1988), 564-585.
6. E. D'Sylva and G.A. Miles, ''The SSOR Iteration Scheme for Equations with σ_1 Ordering'', *Comput. J.* **6** (1963), 271-273.
7. M. Eiermann, X.-Z. Li and R.S. Varga, ''On Hybrid Semi-Iterative Methods'', paper presented at the *Third International Congress on Computational and Applied Mathematics*, Leuven, Belgium, July 1988.
8. S. Galanis, A. Hadjidimos and D. Noutsos, ''On an SSOR Matrix Relationship and its Consequences'', Technical Report CSD-TR-815, Computer Sciences

Department, Purdue University, West Lafayette, Indiana, October 1988.

9. L.A. Hageman and D.M. Young, *Applied Iterative Methods*, Academic Press, New York, 1981.

10. B. Kredell, "On Complex Successive Overrelaxation", *BIT* **2** (1962), 143-152.

11. M.S. Lynn, "On the Equivalence of SOR, SSOR and USSOR as Applied to σ_1-Ordered Systems of Linear Equations", *Comput. J.* **7** (1964), 72-75.

12. R.S. Varga, *Matrix Iterative Analysis*, Prentice Hall, Englewood Cliffs, New Jersey, 1962.

13. R.S. Varga, W. Niethammer and D.-Y. Cai, "p-Cyclic Matrices and the Successive Overrelaxation Method", *Linear Algebra Appl.* **58** (1984), 425-439.

14. D.M. Young, "Iterative Methods for Solving Partial Difference Equations of Elliptic Type", *Trans. Amer. Math. Soc.* **76** (1954), 92-111.

15. D.M. Young, *Iterative Solution of Large Linear Systems*, Academic Press, New York, 1971.

16. D.M. Young and H.E. Eidson, "On the Determination of the Optimum Relaxation Factor for the SOR Method When the Eigenvalues of the Jacobi Matrix are Complex", Report CNA-1, Center for Numerical Analysis, University of Texas, Austin, Texas, 1970.