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Chandrajit L. Bajaj

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**APPROXIMATION METHODS FOR
ALGEBRAIC CURVES AND SURFACES**

Chandrajit L. Bajaj

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Local Parameterization, Implicitization and Inversion of Real Algebraic Curves (*Extended Abstract*)

Chanderjit L. Bajaj*
Computer Science Department,
Purdue University

1 Introduction

Preliminaries

An algebraic plane curve of degree n is implicitly defined by a single polynomial equation $f(x, y) = 0$ of degree n . A rational algebraic curve of degree n can additionally be defined by rational parametric equations which are given as $(x = G_1(u), y = G_2(u))$, where G_1 and G_2 are rational functions in u of degree n , i.e., each is a quotient of polynomials in u of maximum degree n . An algebraic space curve, defined by the intersection of two algebraic surfaces¹ can be given either as a pair of polynomial equations ($f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$) or as two sets of parametric equations ($x = G_{1,1}(u_1, v_1), y = G_{2,1}(u_1, v_1), z = G_{3,1}(u_1, v_1)$) and ($x = G_{1,2}(u_2, v_2), y = G_{2,2}(u_2, v_2), z = G_{3,2}(u_2, v_2)$), where the $G_{i,j}$, $i = 1, 2, 3$, $j = 1, 2$, are rational functions. Rational algebraic space curves are additionally representable as $(x = G_1(u), y = G_2(u), z = G_3(u))$, where G_1, G_2 and G_3 are rational functions in u .

Rational curves are only a subset of implicit algebraic curves of the same degree. While all degree two curves (conics) are rational, only a subset of degree three (cubics) and higher degree curves are rational. In general, a necessary and sufficient condition for the global rationality of an algebraic curve of arbitrary degree is given by the Cayley-Riemann criterion: a curve is rational if and only if $g = 0$, where g , the genus of the curve is a measure of the deficiency of the curve's singularities from its maximum allowable limit [18].

The Problem

Here we wish to consider all algebraic curves, and specifically of genus higher than zero. For all these curves we wish to compute rational parameterizations in the local neighborhood of a

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¹In modeling the boundary of physical objects it suffices to consider only space curves defined by the intersection of two surfaces. Space curves in general can be defined by several surfaces, however this representation is difficult to handle equationally. General space curves is a topic with various unresolved issues of mathematical and computational interest and an area of future research.

point on the curve. This is always possible. At simple points of the curve a straightforward Taylor series expansion, followed by a truncation or a rational Padé approximation, proves sufficient. At a curve's singular point the problem is slightly more complex, as the Taylor series is not defined. Nevertheless, the curve can be factored into a finite number of power series at the singular point, and rational approximations can be constructed from those.

In particular then our problems are:

- For an implicitly defined algebraic plane curve, compute a rational parametric representation $(x = H_1(t), y = H_2(t))$, for each real branch incident at a point $p = (a_0, b_0)$ on the curve, where $H_1(t), H_2(t)$ are rational functions over the Reals.
- For a parameterically defined algebraic plane curve, compute an implicit representation $f(x, y) = 0$, and an inverse relation $t = F(x, y)$ valid about a point $p = (a_0, b_0)$ on the curve, where $f(x, y)$ is a polynomial and $F(x, y)$ is a rational function over the Reals.
- For an implicitly defined algebraic space curve, compute a rational parametric representation $(x = H_1(t), y = H_2(t), z = H_3(t))$, for each real branch incident at a point $p = (a_0, b_0, c_0)$ on the curve, where $H_1(t), H_2(t), H_3(t)$ are rational functions over the Reals.
- For a parameterically defined algebraic plane curve, compute an implicit representation $(f_1(x, y, z) = 0, f_2(x, y, z) = 0)$, and an inverse relation $t = F(x, y, z)$ valid about a point $p = (a_0, b_0, c_0)$ on the curve, where $(f_1(x, y, z) = 0, f_2(x, y, z) = 0)$ is a polynomial and $F(x, y, z)$ is a rational function over the Reals.

Applications:

Rationality of the algebraic curve or surface is a restriction where advantages are obtained from having both the implicit and rational parametric representations [3, 15]. While the rational parametric form of representing a curve or surface allows greater ease for transformation and shape control, the implicit form is preferred for testing whether a point is on the given curve or surface and is further conducive to the direct application of algebraic techniques. Simpler algorithms are possible when both representations are available. For example, a straightforward method exists for computing curve - curve and surface - surface intersections when one of the curves, respectively surfaces, is in its implicit form and the other in its parametric form. Global parameterization algorithms for plane curves of genus zero, are presented in [1].

There are also numerous applications where explicit local parameterizations and implicitizations, which we present here, prove useful in an essential way:

1. Adaptive stepping for curve tracing with initial approximation and Newton correction.
2. Local Intersection Representation
3. Piecewise rational approximation for non-rational algebraic curves, i.e., curves of positive genus.

Prior Work

In [4, 13], power series are constructed to locally approximate plane algebraic curves and surface intersections. The method of [13] technically relies on the Implicit Function Theorem, seeking to represent a curve branch explicitly in one coordinate as function of the other coordinate(s), while [4] uses a Taylor series expansion. Both these methods however do not seem to have a natural extension that handles singular points. Further, [16, 17] also present techniques for curve approximation which work only for special cases.

Methods for computing local branch parameterizations at singular points have been presented in [8, 9], both based on the Newton polygon [see for e.g., 18]. We instead use the iterative lifting technique of Hensel together with the fast univariate Padé algorithm of [5]. Local implicitization is considered in [7] extending the technique of [13] of reducing it to solving a linear system of equations. Our techniques are much more direct, requiring only the efficient power series composition and reversion of [6, 11] and straightforward rational function simplification.

Results:

In this paper we present a combination of both algebraic and numerical techniques to achieve local parameterizations about singular points of algebraic curves. We show how to obtain real Weierstrass and Newton power series factorizations using the technique of Hensel lifting. These, together with rational Padé approximations, are used to efficiently construct locally approximate, rational parametric representations for all real branches of an algebraic plane curve about its singularities. Next we use power series composition and reversion techniques together with rational Padé approximations to efficiently construct locally approximate implicit and inverse representations for parametric algebraic plane curves. Extensions are then given to construct locally approximate, rational parameterizations, implicitizations and inversions for branches of surface intersection space curves. Implementations of these methods and our experiences with them are also discussed.

2 Power Series Computations

2.1 Hensel Lifting

Consider $f(x, y)$ of degree n . Assume it is monic in y . Otherwise, factor out the largest common power of x amongst the terms of f .

$$f(x, y) = f_0(y) + f_1(y)x + \cdots + f_k(y)x^k + \cdots$$

We wish to compute real power series factors $g(x, y)$ and $h(x, y)$ where $f(x, y) = g(x, y)h(x, y)$. The technique of Hensel lifting allows one to reconstruct the power series factors

$$\begin{aligned} g(x, y) &= g_0(y) + g_1(y)x + \cdots + g_i(y)x^i + \cdots \\ h(x, y) &= h_0(y) + h_1(y)x + \cdots + h_j(y)x^j + \cdots \end{aligned} \tag{1}$$

from initial factors $f(0, y) = f_0(y) = g_0(y)h_0(y)$.

Consider the factorization of $f(0, y) = f_0(y)$ as the base case of $k = 0$. Assume $f_0(y)$ is of degree n . Choose real coprime factors $g_0(y)$ of degree p and $h_0(y)$ of degree q satisfying: $p + q = n$. Real coprimeness is achieved by ensuring that g_0 and h_0 contain distinct real roots of f_0 and that complex conjugate pairs are not split up. For the case $n = 2$ however, it may arise that the only coprime factors of f_0 are complex, i.e., the distinct roots are complex conjugates. In that case there only exist complex power series solutions. Since $GCD(g_0(y), h_0(y)) = 1$ using the fast GCD algorithm we can also compute $\alpha(y)$ and $\beta(y)$ such that $\alpha(y)g_0(y) + \beta(y)h_0(y) = 1$

In the iterative Case of $k \geq 1$, we compute $g_k(y)$ and $h_k(y)$ of the desired factorization (1), with degree of $g_k(y) < p$ and degree of $h_k(y) < q$, as follows. We note from (1) that

$$f_k(y) = \sum_{i+j=k} g_i(y)h_j(y)$$

and additionally

$$f_k(y) - \sum_{i < k \wedge j < k} g_i(y)h_j(y) = g_0(y)h_k^*(y) + h_0(y)g_k^*(y) \quad (2)$$

Hence,

$$\begin{aligned} h_k^*(y) &= \alpha(y)[f_k(y) - \sum_{i < k \wedge j < k} g_i(y)h_j(y)] \\ g_k^*(y) &= \beta(y)[f_k(y) - \sum_{i < k \wedge j < k} g_i(y)h_j(y)] \end{aligned}$$

If degree $h_k^*(y) \geq q$ then compute $h_k(y) = h_k^*(y) \bmod h_0(y)$ and set $g_k(y) = \gamma(y)g_0(y) + g_k^*(y)$ where $h_k^*(y) = \gamma(y)h_0(y) + h_k(y)$.

$$f_k(y) - \sum_{i < k \wedge j < k} g_i(y)h_j(y) = g_0(y)h_k(y) + h_0(y)g_k(y) \quad (3)$$

Clearly degree $h_k(y)$ is $< q$. Additionally in (3) the degree of $g_k(y)$ must also be $< p$. This is so because in (3) the degree of the LHS is $< n$ and since degree $g_0(y)h_k(y)$ is $< n$ and degree $h_0(y)$ is $= q$, it must be that degree $g_k(y)$ is $< p$.

Similarly if degree $g_k^*(y) \geq p$ then compute $g_k(y) = g_k^*(y) \bmod g_0(y)$ and set $h_k(y) = \delta(y)h_0(y) + h_k^*(y)$ where $g_k^*(y) = \delta(y)g_0(y) + g_k(y)$. Again, from similar degree arguments as above, is easily seen that the degree bounds of $h_k(y)$ and $g_k(y)$ are met.

2.2 Weierstrass Factorization

Consider $f(x, y)$ with degree n and $ord_y f(0, y) = d < \infty$. An $ord_y f(0, y) = \infty$ corresponds to $f(0, y) = 0$. This can easily be rectified by a simple linear transformation of $f(x, y)$, which yields a nonzero $f(0, y)$ and hence a finite $ord_y f(0, y)$. We wish to compute a power series factorization of the form $f(x, y) = g(x, y) \underbrace{(y^d + a_{d-1}(x)y^{d-1} + \dots + a_0(x))}_{h(x, y)}$ where $g(x, y)$ is a unit power

series, i.e., $g(0,0) \neq 0$ while $h(x,y)$ is a polynomial in y with coefficients $a_i(x)$, $i = 0 \dots n-1$ being non-unit power series, i.e., $a_i(0) = 0$. Such a factorization is known as a Weierstrass preparation and is always possible as we now show.

The Weierstrass preparation can efficiently be achieved via Hensel Lifting. Given

$$f(x,y) = f_0(y) + f_1(y)x + \dots + f_k(y)x^k + \dots$$

with

$$f_0(y) = f_0(y) = \underbrace{(a_0 + a_1y + \dots)}_{g_0(y)} \underbrace{y^d}_{h_0(y)}$$

in general for $k \geq 1$, we wish to compute $h_k(y)$ and $g_k(y)$ using Hensel, yielding factors similar to (1) such that

$$f_k(y) - \sum_{i < k \wedge j < k} g_i(y)h_j(y) = g_0(y)h_k(y) + y^d g_k(y)$$

with degree $h_k(y) < d$.

To achieve this we compute $A(y) = \frac{f_k(y) - \sum_{i < k \wedge j < k} g_i(y)h_j(y)}{g_0(y)}$ and then set $h_k(y) =$ Terms of $A(y)$ with degree $< d$ and $g_k(y) =$ Terms of $A(y)$ with degree $\geq d$.

2.3 Newton Factorization

Consider $f(x,y)$, a monic polynomial in y of degree n , with coefficients polynomial or power series or meromorphic series in x

$$f(x,y) = y^n + a_{n-1}(x)y^{n-1} + \dots + a_0(x)$$

Then it is possible to factor $f(x,y)$ into linear factors

$$f(x,y) = \prod_{i=1}^n (y - \eta_i(t))$$

with $x = t^m$ and m a positive integer and $\eta_i(t)$ power series or meromorphic series. This factorization can also be achieved via Hensel lifting. We precondition the curve so that it admits a non-trivial base factorization, i.e. having at least two coprime factors which can be lifted.

Step 1: Make $a_{n-1}(x) = 0$ via substitution $\tilde{y} = y + \frac{a_{n-1}(x)}{n}$

Step 2: Ensure some $a_{n-i}(0) \neq 0$ for $i \geq 2$ via substitution $\check{y} = \frac{\tilde{y}}{x^\lambda}$ with $\lambda = \min_{(2 \leq i \leq n)} \frac{\alpha_i}{i}$ and $\alpha_i = \text{ord}_x a_{n-i}(x)$. Then $f(0,\check{y}) = f_0(\check{y})$ has at least two distinct roots.

Step 3: Now use Hensel lifting to lift the factorization $f_0(\check{y}) = g_0(\check{y})h_0(\check{y})$ to $f(x,\check{y}) = g(x,\check{y})h(x,\check{y})$.

Repeat Steps 1-3 until all factors are linear or all real factors are obtained.

3 Local Parameterization

Consider an implicit plane algebraic curve $f(x, y) = 0$, with a singularity at the origin. (A singularity can be translated to the origin by a straightforward linear transformation). To compute a local parametric approximation of each of the curve's branches incident at the origin, we execute the following steps:

1. Compute a Weierstrass power series factorization of $f(x, y)$ into $f = gh$, where $g((x, y))$ is a unit power series and $h((x))(y)$ is a polynomial in y with coefficients non-unit power series in x . The equation $h = 0$ corresponds to the curve's branches at the origin while the power series equation $g = 0$ corresponds to the portion of the plane curve away from the origin.
2. Recursively apply the Newton factorization to $h((x))(y)$ till all factors are linear in y or all real factors are obtained. Each of these power series factors represent a local branch parameterization of the type $x = t^k$ and $y = b_i((t))$ where b_i is a power series. The minimum of k and $ord_t(b_i)$, say d , is known as the order of the branch, with $d > 1$ implying a singular branch or "place" of the curve.
3. For each distinct branch power series parameterization $y = b_i((t))$, compute a Padé rational function approximation.

4 Local Implicitization

Consider a rational parametric curve given by $(x = H_1(t), y = H_2(t))$ where H_1 and H_2 are rational functions over the Reals. To compute a local implicit approximation of the curve around the origin, we execute the following steps:

1. Let $\tau^k = H_1(t)$ where $k = ord_t(t) =$ power of the lowest degree term of the power series expansion of the rational function. (wlg assume $ord_t H_1 \geq ord_t H_2$)
2. Compute $\tau = (H_1)^{1/k} = g_1((t)) =$ power series of order 1.
3. Next invert the power series equation $\tau = g_1((t))$ to yield $t = g_1^{-1}((\tau))$. This yields $(x = \tau^k, y = H_2(g_1^{-1}((\tau))))$.
4. Next consider the power series $y = H_2(g_1^{-1}((\tau)))$ and construct a suitable Padé rational function approximant, $y = H_3(\tau) = H_3(x^{1/k})$. Rationalize and simplify the equation $y = H_3(x^{1/k}) = 0$, by computing the k^{th} power, to yield $f(x, y) = 0$, the local implicit approximation.

5 Local Inversion

To locally invert a parameterization $(x = H_1(t), y = H_2(t))$ about the origin we first execute steps (1), (2) and (3) of the last section. Next we compute the following:

1. Let $(x = \tau^k, y = H_2(g_1^{-1}(\tau)) = H(\tau))$ represent a branch of the curve through the origin. Compute $\ell = \text{ord}_\tau H(\tau)$ and $m = \text{least common multiple of } \ell \text{ and } k$.
2. Then compute $\frac{y^{m/\ell}}{x^{m/k}} = \frac{H^{m/\ell}}{\tau^m} = \kappa = (1 + c_1\tau + \dots)^{m/\ell} = g_2(\tau)$, a power series of order 1. Note $\frac{m}{\ell}$ and $\frac{m}{k}$ are both integers.
3. Compute the inverse power series $\tau = g_2^{-1}(\kappa)$, and set $t = g_1^{-1}(g_2^{-1}(\kappa))$. This followed by the rational Padé approximant computation to yield $t = G(\kappa) = G(y^{m/\ell}/x^{m/k})$ where G is a rational function.

6 Conclusions and Future Research

I am currently working to extend the results of this paper to efficiently compute

1. Power series expansions about singular points and curves on surfaces.
2. Suitable expansion points and curves for a piecewise rational surface approximation.

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