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ABSTRACT

In this report, we present a new approach to the design of perfect reconstruction filter banks (PRFB's) which have equal length FIR analysis and synthesis filters. To achieve perfect reconstruction, necessary and sufficient conditions are incorporated directly in a numerical design procedure as a set of quadratic equality constraints among the impulse response coefficients of the filters. Any symmetry inherent in a particular application, such as quadrature mirror symmetry, linear phase, or symmetry between analysis and synthesis filters, may be exploited to reduce the number of variables and constraints in the design problem. A novel feature of our new approach is that it allows the design of filter banks that perform functions other than flat passband band-splitting.
Figure 1. \( M \)-channel filter bank.

Figure 2. Polyphase form of filter bank.
\[ F_k(z) = \sum_{i=0}^{M-1} F_{kl}(z^M)z^{-l}, \quad (3) \]

and the analysis and synthesis polyphase matrices by

\[
H_p(z) = \\
\begin{bmatrix}
H_{0,0}(z) & H_{0,M-1}(z) \\
& \ddots & \ddots \\
H_{M-1,0}(z) & H_{M-1,M-1}(z)
\end{bmatrix},
\]

and

\[
F_p(z) = \\
\begin{bmatrix}
F_{0,0}(z) & F_{0,M-1}(z) \\
& \ddots & \ddots \\
F_{0,0}(z) & F_{M-1,0}(z)
\end{bmatrix}.
\quad (4)
\]

The elements in \( F_p(z) \) are indexed as shown in (4) so that (3) and the matrix multiplication implied by Figure 2 are consistent. Perfect reconstruction occurs if and only if [10]

\[ F_p(z)H_p(z) = D(z), \quad (5) \]

where \( D(z) \) has the form

\[
D(z) = z^{-k_{01}} M \begin{bmatrix}
0 & I_{M-k_{00}} \\
0 & 0
\end{bmatrix} \quad (6)
\]

The matrix \( I_{k_{00}} \) is the \( k_{00} \times k_{00} \) identity matrix. The delay \( k_0 \) of the overall analysis/synthesis filter bank is related to the quantities in (6) by

\[ k_0 = k_{00} + k_{01} M + M - 1. \quad (7) \]

We emphasize that (2) and (5) are simply different expressions of the same condition.

The first demonstration that nontrivial two-channel PRFB's exist was given by Smith and Barnwell in [11,12] and by Mintzer in [13]. They begin by specifying a certain relationship among the four filters in the filter bank. This has the effect of canceling aliasing exactly, and in addition, leads to perfect reconstruction as long as the analysis/synthesis filter pairs are spectral factors of a half-band filter which has nonnegative amplitude response. The design procedure consists of two steps. First, a half-band filter with the desired property is obtained, then this filter is spectrally factored to yield the analysis and synthesis filters.

For an arbitrary number of channels, Vaidyanathan, et al. have presented a design framework based on lossless or paraunitary polyphase matrices [10,14,15]. Their approach begins by parameterizing the analysis polyphase matrix in such a way that it is guaranteed to be paraunitary. Under this condition, PR is achieved by taking the synthesis polyphase matrix to be the conjugate transpose of the analysis polyphase matrix. To achieve good analysis bank performance, the parameters of the analysis polyphase matrix are optimized.

In addition to determining conditions under which PR is possible, Vetterli [8,16] has proposed a PRFB design procedure known as the complimentary filter method. In this approach, all but one of the filters are designed in an unconstrained fashion. Then,
a complimentary filter is designed to yield a PRFB. In [17], we presented a PRFB design approach based on analysis polyphase matrices formed by cascading constant matrices with diagonal delay matrices, in a manner similar to Vaidyanathan's approach in [10].

In this paper, we present a new approach to the numerical design of PRFB's, which is applicable for an arbitrary number of channels. The filter banks which result have FIR analysis and synthesis filters of equal length. Our approach to PRFB design differs from those mentioned above in that, rather than build perfect reconstruction into the filter bank structure through parameterization, we optimize directly over the impulse response coefficients of the analysis and synthesis filters, expressing the perfect reconstruction conditions embodied in (5) as a set of equality constraints among the coefficients. Symmetries among various filters in the filter bank or within a single filter serve not only to reduce the number of variables in the design problem, but also manifest themselves in the form of automatically satisfied constraints and redundancies among the constraints. In both cases, the total number of constraints in the design problem is reduced. Within this framework, very general PRFB's may be designed, including systems in which the analysis bank performs functions other than simple flat passband band-splitting. Possible applications of this feature include performing linear prediction in some or all of the channels, or combining equalization with band-splitting, an example of which is given in Section V B.

The paper is organized as follows. In Section II, the design framework is established, including the error function and the perfect reconstruction constraints. Section III is a brief review of the nonlinearly constrained optimization methods employed in the design. In Section IV, we analyze the effect of several types of symmetry in a filter bank on the possibility of perfect reconstruction and on the number of independent constraints. Finally, in Section V, we present two numerical design examples.

II. PRFB DESIGN FRAMEWORK

In this section we outline our approach to the design of maximally decimated perfect reconstruction filter banks. Stated simply, given a measure of the analysis filter error $E_a$, we will minimize $E_a$, subject to the nonlinear equality constraints in (5).

Depending on the filter being designed, we choose $E_a$ in one of two ways. One formulation for $E_a$ is based on the integral-squared error between the analysis filters and their desired response. The second approach we use for specifying $E_a$ is drawn from the statistical filter design procedure of Farden and Scharf [18]. In each of these cases, $E_a$ is a positive definite quadratic form; so there is no real difference between these two approaches from the design algorithm viewpoint. We present two design examples in Section V, one using each error criterion.

A. Error Criteria

Let the set of filter coefficients for analysis filter $H_k(z)$ be represented by the $N$-length vector $h_k$. Let the set of all analysis filter coefficients be represented by $h$. The $NM$-length vector $h$ is formed by concatenating all the $h_k$ vectors. The set of synthesis filter coefficients will be denoted similarly. In the integral-squared error method, we select a desired frequency response $D_k(\epsilon^{j\omega})$ for each channel and define $E_a$ by
where \( P_k \) and \( S_k \) are the passband and stopband of the \( k \)-th channel, and \( \alpha_k \) is a stopband weighting factor.

In the statistical design approach, each filter in the \( M \)-channel analysis filter bank is treated as the linear minimum mean-squared error (MMSE) estimator of a hypothetical input signal in the presence of noise. If the signal and noise are taken to be statistically uncorrelated with flat spectra in the desired passband and stopband(s) respectively, then the best linear estimator approximates a conventional flat-passband filter. Let \( s_k(l) \) and \( n_k(l) \) be the hypothetical signal and noise inputs to the \( k \)-th analysis filter, and let \( y_k(l) \) be the output of that filter. Then \( E_a \) is given by

\[
E_a(h) = \sum_{k=0}^{M-1} \sum_{l=0}^{M-1} \left| y_k(l) - s_k(l - l_k) \right|^2.
\]

The delays \( l_k \) are design parameters. Both (8) and (9) can be rewritten as

\[
E_a(h) = \sum_{k=0}^{M-1} \left( h_k^T R_k h_k - 2 h_k^T r_k + \sigma_k^2 \right).
\]

In the statistical approach, \( R_k \) is the autocorrelation matrix associated with \( x_k = s_k + n_k \), \( r_k \) is the autocorrelation vector associated with \( s_k \), and \( \sigma_k^2 \) is the power in \( s_k \). In either case, the matrices \( R_k \) are positive definite; and in the statistical approach they are Toeplitz. Equation (10) may be rewritten in the form

\[
E_a(h) = h^T R h - 2 h^T r + \sigma^2.
\]

The matrix \( R \) is block diagonal with the \( R_k \) matrices along the diagonal, and \( r \) is formed by concatenating \( r_k \) in the same fashion as \( h \) is formed from \( h_k \).

B. Perfect Reconstruction Constraints

Let the product of the analysis and synthesis polyphase matrices be denoted by \( P(z) \):

\[
P(z) = F_p(z) H_p(z).
\]

Then the elements of \( P(z) \) are given by

\[
P_{kl}(z) = \sum_{i=0}^{M-1} F_{i,M-1-k}(z) H_{i,l}(z).
\]

From (5), the perfect reconstruction condition is

\[
P(z) = D(z).
\]

This polynomial matrix equation may be decomposed into \( M^2 \) polynomial equations of the form

\[
P_{kl}(z) = D_{kl}(z).
\]

By equating the coefficients of corresponding powers of \( z \) from both sides of (15), each of these polynomial equations can be decomposed into \( (2N/M) - 1 \) equations not involving \( z \). Hence, the perfect reconstruction condition amounts to a set of \( M^2[(2N/M) - 1] = 2NM - M^2 \) quadratic equality constraints. Several general
statements can be made about these constraint equations. First, each product term in each equation involves exactly one analysis and one synthesis filter coefficient. Second, there is at least one coefficient from each of the \(2M\) filters in each constraint equation. Finally, by (6), all but \(M\) of these equations has 0 on the right-hand side and the remaining \(M\) equations have \(1/M\).

In Section IV we will work with the constraints in the form of (14) and (15); but in Section III it will be convenient to adopt a different notation. Forming a length \(2NM - M^2\) vector of constraint functions \(c(h, f)\), the constraint equations can be written

\[
c(h, f) = b, \quad \text{(16)}
\]

where \(b\) contains mostly zeros and \(M\) nonzero elements. The constraints are bilinear in \(h\) and \(f\). That is, if \(f\) is fixed at \(f'\), then (16) may be written in the form

\[
\mathcal{F} h = b, \quad \text{(17)}
\]

and if \(h\) is fixed at \(h'\), it may be written

\[
\mathcal{H} f = b, \quad \text{(18)}
\]

where \(\mathcal{F}\) is a sparse matrix with a banded structure which contains the elements of \(f'\), and \(\mathcal{H}\) has a form similar to \(\mathcal{F}\).

To make these ideas concrete, we will illustrate them with a simple example: a two-channel filter bank with filters of length two. The filters are written

\[
H_0(z) = h_0(0) + h_0(1)z^{-1},
\]

\[
H_1(z) = h_1(0) + h_1(1)z^{-1},
\]

\[
F_0(z) = f_0(0) + f_0(1)z^{-1},
\]

and

\[
F_1(z) = f_1(0) + f_1(1)z^{-1}. \quad \text{(19)}
\]

The polyphase matrices are

\[
H_p(z) = \begin{bmatrix} h_0(0) & h_0(1) \\ h_1(0) & h_1(1) \end{bmatrix},
\]

and

\[
F_p(z) = \begin{bmatrix} f_0(1) & f_1(1) \\ f_0(0) & f_1(0) \end{bmatrix}. \quad \text{(20)}
\]

Let us suppose that \(D(z)\) is given as in (6) with \(k_{00} = 0\) and \(k_{01} = 0\). Using this along with (20), the PR constraints in (14) are given by

\[
\begin{bmatrix}
  f_0(1)h_0(0) + f_1(1)h_1(0) & f_0(1)h_0(1) + f_1(1)h_1(1) \\
  f_0(0)h_0(0) + f_1(0)h_1(0) & f_0(0)h_0(1) + f_1(0)h_1(1)
\end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}. \quad \text{(21)}
\]

We may obtain (16) from (21) using any ordering we desire. One example is
\[ c(h, f) = \begin{bmatrix} f_0(1)h_0(0) + f_1(1)h_1(0) \\ f_0(1)h_0(1) + f_1(1)h_1(1) \\ f_0(0)h_0(0) + f_1(0)h_1(0) \\ f_0(0)h_0(1) + f_1(0)h_1(1) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0.5 \end{bmatrix} = b. \] (22)

The version of \( c(h, f) \) corresponding to (17) which displays its bilinear nature is

\[ c(h, f) = \mathcal{F} h = \begin{bmatrix} f_0(1) & 0 & f_1(1) \\ 0 & f_0(1) & 0 & f_1(1) \\ f_0(0) & 0 & f_1(0) & 0 \\ 0 & f_0(0) & 0 & f_1(0) \end{bmatrix} \begin{bmatrix} h_0(0) \\ h_0(1) \\ h_1(0) \\ h_1(1) \end{bmatrix}. \] (23)

In Section II, we have formulated the design of PRFB’s as the minimization of a positive definite quadratic function of the \( MN \) analysis filter coefficients, subject to a set of \( 2NM - M^2 \) quadratic equality constraints which involve both the analysis and synthesis filter coefficients (a total of \( 2NM \) coefficients). The next section concerns the methods we employ to solve this problem.

### III. NONLINEARLY CONSTRAINED OPTIMIZATION

It is generally accepted that optimization under nonlinear equality constraints is significantly more difficult than unconstrained optimization or optimization under linear equality constraints. The main source of difficulty is that motion along any straight line starting from a feasible point, i.e. a point which satisfies the constraints, leads immediately to an infeasible point. Since most optimization methods have at their heart an algorithm which performs minimization along a line, this poses a significant problem.

#### A. Penalty Function Method

The simplest method that we have used for dealing with nonlinear constraints is known as the penalty function method. The idea behind this method is to convert the nonlinearly constrained problem into a sequence of unconstrained subproblems, whose solutions converge to a solution of the original problem. This is done by adding a penalty term to the original error function. We use a quadratic penalty term

\[ E_{pf}(h, f, \rho) = E_a(h) + \rho [c(h, f) - b]^T [c(h, f) - b]. \] (24)

The variable \( \rho \) is the penalty parameter. The process begins by minimizing \( E_{pf}(h, f, \rho) \) for a small value of \( \rho \). The solution to that subproblem is used as the starting point for another subproblem with a larger value of \( \rho \). This continues until an acceptable solution is obtained. Under mild conditions on the error and constraint functions, the sequence of solutions converges to the minimum of \( E_a(h) \). Any method for unconstrained optimization may be applied to minimize \( E_{pf}(h, f, \rho) \).

One major drawback of the penalty function method is that the solutions to the subproblems do not in general satisfy the constraints exactly. Since this is a requirement for perfect reconstruction, we have also employed two more sophisticated algorithms capable of generating feasible solutions. These methods are based on the idea of creating a sequence of linearly constrained subproblems. In each subproblem, the constraints are linearized using a series expansion about the solution to the previous subproblem. The difference between these methods is the function being minimized.
B. Projected Error Function Method

In this approach we minimize $E_a(h)$ subject to the constraints linearized about the solution to the previous subproblem. We shall refer to this as the projected error (PE) function method. This method is most useful when the solution to the previous subproblem nearly satisfies the nonlinear constraints. There are two drawbacks to this approach. The sequence of solutions essentially tracks the constraint curve and can therefore converge rather slowly. In addition, if the starting point is too far away from the true solution, the linearized constraints are not a good approximation to the actual constraints and the subproblem solutions diverge. Nonetheless, we have applied this method with some success to the design of PRFB's.

C. Projected Augmented Lagrangian Method

A more robust algorithm is known as the projected augmented Lagrangian (PAL) method. This algorithm is also based on a sequence of subproblems subject to linearized constraints; but there are two major differences between this and the previous algorithm. To begin with, the error function is modified by adding a linear combination of the constraints, creating the Lagrangian function. From optimization theory, the Lagrangian function has a stationary point at any point which is a solution to the original nonlinearly constrained problem. The linear weights used are simply the Lagrange multipliers. Robinson proposed minimizing the Lagrangian subject to the linearized constraints in 1972 [19]. Unfortunately, in many cases this algorithm does not converge to the true solution unless the starting point (which must include starting estimates of the Lagrange multipliers as well as the independent variables), is quite close to the solution. Murtagh and Saunders [20] created a more robust algorithm by augmenting the Lagrangian with a penalty term like the one described above. This promotes convergence from starting points which are outside the radius of convergence of Robinson's method.

The term “projected” in the names of the above algorithms refers to the fact that the minimization in each subproblem is done over a parameterized linear subspace. This property is well-suited to the general PRFB problem, since the difference between the number of variables and the number of constraints is $M^2$, which is independent of the length of the filters. Hence in rough terms, the size of the space over which optimization takes place is related to the number of channels $M$, but not to the filter length $N$.

For more information on the optimization methods used in this work, see [21]. In our work, all three algorithms described here were implemented using the MINOS optimization program [22].

IV. INCORPORATING SYMMETRY

In many filter bank applications, there exists symmetry among the desired frequency responses of the filters. A well-known symmetry is quadrature mirror (QM) symmetry, in which two desired responses mirror each other about the frequency $\omega = \pi/2$. If $D_k(e^{j\omega})$ and $D_{M-1-k}(e^{j\omega})$ are the desired responses of the $k$-th and $M-1-k$-th analysis filters, QM symmetry between the desired responses exists if

$$
|D_k(e^{j\omega})| = |D_{M-1-k}(e^{j(\pi-\omega)})|, \quad 0 \leq \omega \leq \pi.
$$

(25)

Given an actual filter $H_k(z)$ whose Fourier transform magnitude approximates the desired response $|D_k(e^{j\omega})|$, it is possible to obtain a filter $H_{M-1-k}(z)$ whose transform
magnitude approximates \[ D_{M-1-k}(e^{j\omega}) \] by simple operations on the impulse response coefficients of \( H_k(z) \) (for example, multiplying them by \((-1)^n\), which we shall refer to as modulation). Such a relationship is often explicitly imposed on the filters in a filter bank to reduce the number of independent variables in the design problem. We will say that an analysis filter bank has QM symmetry if \( H_k(z) \) and \( H_{M-1-k}(z) \) are related in this way, for all \( k \). If the filters in the synthesis bank also bear such a relationship, then we say the entire filter bank has QM symmetry.

A second kind of symmetry often present in filter banks is equality of desired frequency response magnitudes between a filter in the analysis bank and one in the synthesis bank. We will say that a filter bank has analysis/synthesis (AS) symmetry if

\[
H_k(e^{j\omega}) = P_k(e^{j\omega}), \quad 0 \leq \omega \leq \pi,
\]

for all \( k \).

Both AS and QM symmetries are incorporated either explicitly or implicitly in most filter bank design methods [10, 12, 13, 15, 23]. It is natural to impose AS and QM symmetry on the filters in a filter bank when the application is flat passband band-splitting. We should also point out that other types of symmetry exist, the most notable being that associated with linear phase.

In the design framework that we have proposed, symmetry may be capitalized upon to reduce not only the number of variables in the design problem, but also the number of independent PR constraints. The exact nature of the constraint reduction for each type of symmetry is the subject of the remainder of this section. In each case, the treatment is organized similarly. First, we analyze the effect of imposing that symmetry on the possibility of perfect reconstruction. This will lead to restrictions on the form of \( D(z) \) beyond those already implied by (6). We will also find that PR is impossible under certain symmetries. Next, we analyze the effect of the particular symmetry on \( P(z) \). The effect will always be either to make some of the elements \( P_{kl}(z) \) automatically zero, or to cause pairwise relationships between different \( P_{kl}(z) \) terms. In either case, a reduction in the number of independent PR constraints is achieved.

A. Analysis/Synthesis (AS) Symmetry

Recall that AS symmetric filter banks are those in which each analysis filter has a corresponding synthesis filter with the same frequency response magnitude. Two length-\( N \) FIR filters have the same frequency response magnitude, within a multiplicative constant, if the filters have the same zeros, within a reflection through the unit circle. Of all the possible ways in which this can happen, the cases of complete equality of zeros, and of complete reflection of zeros are the most useful, since these types of symmetry result in symmetry in the discrete-time domain. Reflection of all the zeros corresponds to time-reversal of the impulse response coefficients. Let \( H(z) \) be an arbitrary length-\( L_H \) FIR filter. We shall use \( \hat{H}(z) \) to denote the time-reversed version of \( H(z) \):

\[
\hat{H}(z) = z^{-(L_H-1)}H(z^{-1}).
\]

To insure causality of the time-reversed filter without introducing excessive delay, the delay in time-reversal will always equal the order of the filter being time-reversed, unless it is specifically mentioned otherwise. A matrix may be time-reversed by time-
reversing each of the elements. The delay will be the same for each element.

**AS TR Symmetry.** The analysis and synthesis filters are related by AS time-reversal (TR) symmetry if they satisfy

\[ F_k(z) = z^{-(N-1)} H_k(z^{-1}), \]  

where the length of the filters is \( N \). Let \( N \) be a multiple of the number of channels \( M \), i.e. \( N = LM \). Then, the length of the polyphase components is \( L \), and by inserting the polyphase decomposition of \( H_k(z) \) into (28), we find that

\[ F_k(z) = z^{-(N-1)} \sum_{l=0}^{M-1} H_{kl}(z^{-M}) z^l. \]  

After some manipulation, this becomes

\[ F_k(z) = M^{-1} \sum_{l=0}^{M-1} \left[ z^{-M(L-1)} H_{k,M-1-l}(z^{-M}) \right] z^{-l}. \]  

This shows that the \( l \)-th polyphase component of \( F_k(z) \) is a time-reversed version of the \((M-1-l)\)-th polyphase component of \( H_k(z) \):

\[ F_{kl}(z) = z^{-(L-1)} H_{k,M-1-l}(z^{-1}) = \hat{H}_{k,M-1-l}(z). \]  

Therefore, using (4), the synthesis polyphase matrix is related to \( H_p(z) \) by

\[ F_p(z) = \begin{bmatrix} \hat{H}_{0,0}(z) & \hat{H}_{M-1,0}(z) \\ \vdots & \vdots \\ \hat{H}_{0,M-1}(z) & \hat{H}_{M-1,M-1}(z) \end{bmatrix} = \hat{H}_p^T(z). \]  

To understand the effect on PR of imposing this symmetry, we look at \( P(z) \).

\[ P(z) = F_p(z) H_p(z) = \hat{H}_p^T(z) H_p(z) = z^{-(L-1)} H_p^T(z^{-1}) H_p(z). \]  

The elements of \( P(z) \) have length \( 2L-1 \), since they are formed by summing products of the elements of \( H_p(z) \) and \( F_p(z) \), and the elements of \( H_p(z) \) and \( F_p(z) \) have length \( L \). Therefore, the proper delay for time-reversing \( P(z) \) is \( 2(L-1) \). Indeed, if we time-reverse and transpose the right-hand side of (33), we find that it is left unchanged. Thus,

\[ P(z) = \hat{P}^T(z). \]  

Since PR requires that \( P(z) = D(z) \), PR is impossible unless \( D(z) \) is also unchanged by time-reversal and transposition. The matrices \( D(z) \) and \( \hat{D}_p^T(z) \) are given by

\[ D(z) = \frac{z^{-k_{01}}}{M} \begin{bmatrix} 0 & I_{M-k_{00}} \\ z^{-1} I_{k_{00}} & 0 \end{bmatrix}, \]  

and

\[ \hat{D}_p^T(z) = \frac{z^{-2(L-1)+k_{01}+1}}{M} \begin{bmatrix} 0 & I_{k_{00}} \\ z^{-1} I_{M-k_{00}} & 0 \end{bmatrix}. \]  

The delay in time-reversing \( D(z) \) is \( z^{-2(L-1)} \) to match that in time-reversing \( P(z) \). The matrix \( \hat{D}_p^T(z) \) can only equal \( D(z) \) if the sizes of the identity matrices in (36) match
those in (35). This occurs only if $k_{00} = 0$ or $k_{00} = M/2$. If $k_{00} = 0$, PR is possible only if the delay terms in $D(z)$ and $D^*(z)$ match, that is, if

$$2(L-1) - k_{01} = k_{01},$$

which means that $k_{01}$ must equal $L-1$. The $k_{00} = M/2$ case is impossible unless $M$ is even, but then in order for the delay terms to match, $k_{01}$ must be a non-integer. Hence, PR is impossible in this case.

Combining these facts, PR is possible under AS TR symmetry only if

$$D(z) = \frac{z^{-(L-1)}}{M} I_M,$$  \hspace{1cm} (38)

in which case, using (7), the overall system delay $k_0$ is

$$k_0 = k_{00} + k_{01} M + M - 1 = (L-1)M + M - 1 = N-1.$$  \hspace{1cm} (39)

Thus the filter bank delay must equal the order of the filters. It is interesting to note that because of (33) and (38), if a filter bank with AS TR symmetry achieves perfect reconstruction, then $H_p(z)$ must satisfy

$$H_p^T(z^{-1})H_p(z) = \frac{1}{M} I_M.$$  \hspace{1cm} (40)

But (40) is the definition of a paraunitary matrix. Therefore, the analysis polyphase matrix is necessarily paraunitary. This result was established by Vaidyanathan through alternate means in [10].

We have determined the necessary form of $D(z)$ and the properties of $P(z)$ imposed by AS TR symmetry (see (38) and (34)). Let us now address the issue of constraint reduction. Since $P(z) = D(z)$ is necessary for PR, and $D(z)$ must be diagonal under AS TR symmetry, all elements not on the diagonal of $P(z)$ must be zero, or perfect reconstruction cannot be achieved in the AS TR case. Also, since $P(z)$ satisfies (34), the elements not on the diagonal satisfy

$$P_{kl}(z) = \hat{P}_{lk}(z).$$  \hspace{1cm} (41)

Thus, if we constrain $P_{kl}(z)$ to be zero, we are guaranteed that $P_{lk}(z)$ will also be zero. Hence, the total number of independent constraints from the non-diagonal elements is reduced by a factor of 2 to $(M^2 - M)(2L - 1)/2 = [(2NM - M^2)/2] + (M/2) - N$.

Turning our attention to the diagonal of $P(z)$, we note from (38) that perfect reconstruction requires that

$$P_{kk}(z) = \frac{z^{-(L-1)}}{M}.$$  \hspace{1cm} (42)

However, under AS TR symmetry, we have from (33) that

$$P_{kk}(z) = z^{-(L-1)} \sum_{l=0}^{M-1} H_{lk}(z^{-1}) H_{lk}(z).$$  \hspace{1cm} (43)

Therefore, $P_{kk}(z)$ is a symmetric polynomial of order $2L-2$. For each diagonal term $P_{kk}(z)$, we find by matching powers of $z$ in (42), that $L-1$ of the $2L-1$ constraints are redundant. Therefore, a total of $ML = N$ independent constraints results from all the diagonal terms of the perfect reconstruction condition. Combining everything, we conclude that AS TR symmetry reduces the total number of independent constraints by
almost a factor of two, from 2NM—M^2 to [(2NM—M^2)/2]+M/2.

**AS EN Symmetry.** A second type of AS symmetry occurs when each analysis filter has a corresponding synthesis filter with exactly the same zeros. In such a case, the filters may be equal to or negatives of each other. We will refer to this as AS EN symmetry. Let s be an M-length vector of binary digits, with s_k = 1 if F_k(z) is the negative of H_k(z) and s_k = 0 otherwise. Then

\[ F_k(z) = (-1)^{s_k} H_k(z), \quad k=1, \ldots, M-1, \tag{44} \]

and the polyphase components are related by

\[ F_{kl}(z) = (-1)^{s_k} H_{kl}(z). \tag{45} \]

Using this in (4), we find that

\[ F_p(z) = J H_p^T(z) I_s, \tag{46} \]

where \( I_s = \text{diag}\{(-1)^{s_0}, \ldots, (-1)^{s_{M-1}}\} \) and J is the permutation matrix with 1's along the anti-diagonal:

\[ J = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \tag{47} \]

Using (46) and (12), \( P(z) \) may be written

\[ P(z) = J H_p^T(z) I_s H_p(z), \tag{48} \]

and satisfies the property

\[ P(z) = J P^T(z) J. \tag{49} \]

In terms of the entries of \( P(z) \), (49) can be written

\[ P_{kl}(z) = P_{M-1-l,M-1-k}(z). \tag{50} \]

Hence, under AS EN symmetry, the matrix \( P(z) \) is symmetric with respect to the anti-diagonal, regardless of the value of \( s \). It is easy to show that any \( D(z) \) given by (6) will satisfy (50); so this alone does not inhibit PR, or place any restriction on \( D(z) \). However, by examining more closely the form of the diagonal entries of \( P(z) \) in (50), it can be shown that the specific choices of \( I_s = \pm I \) render PR impossible. Hence, any filter bank with AS EN symmetry must include some equality and some negation symmetry.

As for constraint reduction, using the same reasoning as we did with AS TR symmetry, we see that under AS EN symmetry, the elements not on the anti-diagonal of \( P(z) \) are pairwise redundant. However, the anti-diagonal elements are not symmetric polynomials, so there is no redundancy within each of them, as there is for diagonal elements in the AS TR case.

All the results for AS TR and AS EN symmetry are summarized in Table I.

**B. Quadrature Mirror (QM) Symmetry**

While AS symmetry relates a filter in the analysis section of an analysis/synthesis filter bank to a filter in the synthesis section, QM symmetry relates filters within a
Table I
Summary of AS symmetry.

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>AS TR</th>
<th>AS EN</th>
</tr>
</thead>
<tbody>
<tr>
<td>Definition</td>
<td>$F_k(z) = z^{-(N-1)} H_k(z^{-1})$</td>
<td>$F_k(z) = (-1)^k H_k(z)$</td>
</tr>
<tr>
<td>Conditions</td>
<td>$N = LM; \quad k_0 = N-1$</td>
<td>$\alpha_k = 0$ for some $k$.</td>
</tr>
<tr>
<td></td>
<td>$D(z) = z^{-(L-1)} I$</td>
<td>$\alpha_k = 1$ for some $k$.</td>
</tr>
<tr>
<td>Consequences</td>
<td>$F_p(z) = \hat{H}_p^T(z)$</td>
<td>$F_p(z) = J \hat{H}_p^T(z) I_s$</td>
</tr>
<tr>
<td></td>
<td>$P(z) = P^T(z)$</td>
<td>$P(z) = J P^T(z) J$</td>
</tr>
</tbody>
</table>
section. For this reason, we cannot expect to impart a property to $P(z)$ simply by imposing QM symmetry on one of the sections; we should simultaneously impose it on both. Fortunately, as we shall see, this does not conflict with the possibility of perfect reconstruction.

**QM MTR Symmetry.** Suppose that the analysis filters are pairwise related by modulation time-reversal (MTR) symmetry, defined by

$$H_k(z) = z^{-(N-1)} H_{M-1-k}(-z^{-1}).$$

(51)

We prefer to define MTR symmetry by (51) rather than using the time-reversal notation of (27), in order to avoid confusion which might result because the actual operations of modulation and time-reversal do not commute if $M$ is even. As with AS TR symmetry, we stipulate that $N = LM$. Using (51), we find that

$$H_k(z) = z^{-(N-1)} \sum_{l=0}^{M-1} H_{M-1-k,l}((-z)^{-M}) (-z)^l$$

(52)

$$= (-1)^{M-1} \sum_{l=0}^{M-1} \left[ (-1)^{-l} z^{-M(l-1)} H_{M-1-k,M-1-l}((-1)^{-M}z^{-M}) \right] z^{-l}.$$ 

Therefore, the analysis polyphase components are related by

$$H_{kl}(z) = (-1)^{M-1+l} z^{-(L-1)} H_{M-1-k,M-1-l}((-1)^Mz^{-1}).$$

(53)

This expression holds for all $M$. We note parenthetically that if $M$ is odd, the filter length must also be odd. Otherwise, $H_{(M-1)/2}(z) \equiv 0$. The proof of this is in the Appendix.

Let us now suppose that the synthesis filters are related by MTR symmetry, but with possible negation, which we refer to as NMTR symmetry:

$$F_k(z) = (-1)^s z^{-(N-1)} F_{M-1-k}(-z^{-1}).$$

(54)

Here $s$ is a bit which controls whether negation is present and applies for all $k$. The synthesis polyphase components are related by

$$F_{kl}(z) = (-1)^{M-1+s+l} z^{-(L-1)} F_{M-1-k,M-1-l}((-1)^Mz^{-1}).$$

(55)

Using (53) and (55) in (13) and manipulating yields

$$P_{kl}(z) = (-1)^{M-1+s+l-k} z^{-2(L-1)} P_{M-1-k,M-1-l}((-1)^Mz^{-1}).$$

(56)

In matrix form, this becomes

$$P(z) = (-1)^{M-1+s} I_{mod} J \hat{P}((-1)^Mz) J I_{mod},$$

(57)

where $I_{mod} = \text{diag} \{1, -1, 1, -1, \ldots\}$. Equation (57) holds for any $M$. To explore the restrictions this places on PR systems, we look for conditions under which $D(z)$ satisfies the same matrix identity.

$$D(z) = (-1)^{M-1+s} I_{mod} J \hat{D}((-1)^Mz) J I_{mod}.$$ 

After some manipulation, this becomes
\[
D(z) = \frac{(-1)^{M-1+s+k_{01}M}}{M} z^{-2(L-1) + k_{01} + 1} I_{mod} \begin{bmatrix} 0 & I_{k_{00}} \\ z^{-1} I_{M-k_{00}} & 0 \end{bmatrix} I_{mod}.
\]  

(58)

Ignoring for the moment the effect of the pre- and post-multiplication by \(I_{mod}\) and using the same argument as that which follows (36), it can be shown that the dimensions of \(D(z)\) are the same as those of the quantity on the right-hand side of (58) only if \(D(z)\) is given by

\[
D(z) = \frac{z^{-(L-1)}}{M} I_{M}.
\]

(59)

This implies that \(k_{01} = L-1\) and \(k_{00} = 0\). Substituting \(k_{01} = L-1\) and simplifying, we find that (58) reduces to

\[
I_{M} = (-1)^{N-1+s} I_{M}.
\]

(60)

Because \(N = LM\), \(M\) even implies that \(N\) must also be even. Thus, if \(M\) is even, (60) is satisfied with \(s = 1\); and perfect reconstruction is possible. On the other hand, PR with \(M\) even and \(s = 0\) is impossible. From the Appendix, the filter length \(N\) must be odd if \(M\) is odd. Therefore, \(s = 0\) is necessary for PR. With these restrictions, (56) becomes

\[
P_{kl}(z) = (-1)^{l-k} z^{-2(L-1)} P_{M-1-k,M-1-l}((-1)^M z^{-1}).
\]

(61)

QM symmetry also leads to a reduction by about a factor of two in the number of independent PR constraints. From (61), we see that every above-diagonal element of \(P(z)\) is NMTR-symmetric with respect to a below-diagonal element. Therefore, if we force an above-diagonal element to be zero, the corresponding below-diagonal element is automatically zero as well. Furthermore, each diagonal element is NMTR-symmetric with respect to another diagonal element, and for PR both must equal the same monomial \(z^{-(L-1)}/M\) (which is self-NMTR-symmetric). Hence, the diagonal element constraints are also redundant. Finally, if \(M\) is odd, NMTR symmetry causes \(P_{(M-1)/2,(M-1)/2}(z)\) to be a symmetric polynomial; and \(L\) of the \(2L-1\) constraints embodied in it may be ignored.

**QM M Symmetry.** The above discussion pertains to QM symmetry achieved through modulation and time-reversal; it is also possible to obtain QM symmetry through modulation alone (QM M symmetry). We shall treat this case in somewhat less detail than we did the MTR case, leaving details to the reader. It should be noted that, as with AS EN symmetry, we place no restriction on the filter length in this case. Such a restriction is only important if time-reversal is involved.

Suppose that the analysis bank filters obey modulation symmetry

\[
H_k(z) = H_{M-1-k}(-z),
\]

(62)

and that the synthesis bank filters do also, but with possible negation

\[
F_k(z) = (-1)^s F_{M-1-k}(-z).
\]

(63)

Then, the polyphase matrices obey the properties

\[
H_p(z) = J H_p((-1)^M z) I_{mod},
\]

(64)

and
\[ F_p(z) = (-1)^{M-1+s} I_{\text{mod}} F_p((-1)^M z) J, \]  

and their product satisfies

\[ P(z) = (-1)^{s+M-1} I_{\text{mod}} P((-1)^M z) I_{\text{mod}}, \]  

regardless of \( M \). We can determine which choices of system parameters (number of channels, overall delay, and \( s \)) allow the possibility of perfect reconstruction, if we impose the property given in (66) on \( D(z) \). We find, after some work, that \( D(z) \) must satisfy

\[ \frac{(-1)^{s+k_0}}{M} z^{-k_0} \begin{bmatrix} 0 & I_{M-k_0} \\ (-1)^M z^{-1} I_{k_0} & 0 \end{bmatrix}, \]  

where \( k_0 \) is the overall system delay defined in (7). The choices \( s = 0 \) and \( k_0 \) even or \( s = 1 \) and \( k_0 \) odd allow PR if \( M \) is even, since in those cases, (67) is satisfied. The same holds true for \( M \) odd, but we must impose the additional restriction that \( k_{00} \) be zero as well.

Now we will address constraint reduction in QM M symmetry. It can be shown that one half of the coefficients of the polynomial elements of \( P(z) \) are guaranteed to be zero by selecting \( k_0 \) and \( s \) as explained above. This leads to half the PR constraints being automatically satisfied. To see this, we rewrite (66) in terms of the individual elements of \( P(z) \).

\[ P_{kl}(z) = (-1)^{M-1+s+k+l} P_{kl}((-1)^M z). \]  

For even \( M \), this implies that every other element in each row and column of \( P(z) \) is identically zero. If we have chosen \( k_0 \) and \( s \) properly, then the corresponding elements of \( D(z) \) are also zero. The PR constraints corresponding to those elements are therefore automatically satisfied. For odd \( M \), if we look inside each \( P_{kl}(z) \) term, we find that every other coefficient of every \( P_{kl}(z) \) is identically zero. Again, the proper choice of system parameters prevents inconsistencies from arising between the zeros in \( P(z) \) and nonzero terms in \( D(z) \), so that the PR constraints are automatically satisfied.

There is, however, at least one case in which further analysis shows that PR is inconsistent with QM modulation symmetry; and that is the two-channel case. Looking at the \( s = 1, k_0 \) odd case, the matrix \( P(z) \) is given by

\[ P(z) = \begin{bmatrix} F_{01}(z) & F_{01}(z) \\ F_{00}(z) & -F_{00}(z) \end{bmatrix} \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ H_{00}(z) & -H_{01}(z) \end{bmatrix} \]

\[ = \begin{bmatrix} 2H_{00}(z)F_{01}(z) & 0 \\ 0 & 2H_{01}(z)F_{00}(z) \end{bmatrix}. \]  

We can see from this that PR is impossible except with trivial transfer functions, because the polynomials on the right side cannot be monomials, although they can approximate them arbitrarily closely. This is exactly what prevents the original QMF scheme proposed in [2] from achieving PR.

Tables II and III encapsulate the QM symmetry results.
Table II
Summary of QM symmetry.

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>QM (N)MTR</th>
<th>QM (N)M</th>
</tr>
</thead>
<tbody>
<tr>
<td>Definition</td>
<td>$H_k(z) = H_k(-z)$</td>
<td>$H_k(z) = H_{M-1-k}(-z)$</td>
</tr>
<tr>
<td></td>
<td>$F_k(z) = (-1)^k F_{M-1-k}(-z)$</td>
<td>$F_k(z) = (-1)^k F_{M-1-k}(-z)$</td>
</tr>
<tr>
<td>Conditions</td>
<td>$N = LM; \quad k_0 = N-1; \quad s = 1.$</td>
<td>$M &gt; 2.$</td>
</tr>
<tr>
<td></td>
<td>$M$ even $\rightarrow s = 1; \quad M$ odd $\rightarrow s = 0.$</td>
<td>Also see Table 3.</td>
</tr>
<tr>
<td></td>
<td>$D(z) = \frac{z^{(2-1)}}{M}$</td>
<td></td>
</tr>
<tr>
<td>Consequences</td>
<td>$H_M(z) = (-1)^{M-1} F_{M-1-M-1}(-z)$</td>
<td>$H_p(z) = J H_p((-1)Mz)I_{mod}$</td>
</tr>
<tr>
<td></td>
<td>$F_M(z) = (-1)^{M-1} F_{M-1-M-1}((-1)^Mz)$</td>
<td>$F_p(z) = (-1)^{M-1} I_{mod} F_p((-1)^Mz)J.$</td>
</tr>
<tr>
<td></td>
<td>$P_M(z) = (-1)^{M-1} P_{M-1-M-1}((-1)^Mz).$</td>
<td>$P_p(z) = (-1)^{M-1} P_{p((-1)^Mz)}.$</td>
</tr>
</tbody>
</table>

Table III
Values of system delay $k_0$, number of channels $M$, and negation parameter $s$ for which PR is possible under QM M symmetry.

<table>
<thead>
<tr>
<th>M</th>
<th>s</th>
<th>$k_0$</th>
<th>Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even</td>
<td>0</td>
<td>Even</td>
<td>$k_{0c}$ must be 0.</td>
</tr>
<tr>
<td>Even</td>
<td>1</td>
<td>Odd</td>
<td></td>
</tr>
<tr>
<td>Odd</td>
<td>0</td>
<td>Even</td>
<td>$k_{0c}$ must be 0.</td>
</tr>
<tr>
<td>Odd</td>
<td>1</td>
<td>Odd</td>
<td>$k_{0c}$ must be 0.</td>
</tr>
</tbody>
</table>
C. Simultaneous AS and QM Symmetry

Since it is natural to impose both AS and QM symmetry on a filter bank in the band-splitting case, we next examine the effect of simultaneously imposing them. We have defined two variations of each type of symmetry, so there are four possible symmetry combinations. We will consider two of the combinations. The first can be applied when $M$ is even, and the second when $M$ is odd. We will first determine whether each of these combinations is compatible; then we will examine the nature of the constraint reduction.

AS TR and QM (N)MTR Compatibility. We will apply the symmetries in a cyclic fashion so that a relationship is obtained between a filter and itself. This relationship coupled with restrictions due to the individual symmetries gives us more information about the possibility of PR. Starting with $H_k(z)$, we apply AS followed by QM symmetry.

$$H_k(z) = z^{-(N-1)} F_k(z^{-1})$$  \hspace{1cm} [using (28)]

$$= -F_{M-1-k}(-z).$$  \hspace{1cm} [using (54), with $s=1$]

Now, we repeat the step:

$$= (-1)^N z^{-(N-1)} H_{M-1-k}(-z^{-1})$$  \hspace{1cm} [using (28)]

$$= (-1)^N H_k(z).$$  \hspace{1cm} [using (51)]

From this, we can see that these two symmetries are incompatible with PR if $N$ is odd. Therefore, since $N$ must be odd when $M$ is odd (see Appendix), AS TR and QM (N)MTR symmetries should not be simultaneously applied to a filter bank with an odd number of channels. This may be the phenomenon that was described by Nguyen and Vaidyanathan in the introduction of [23] with reference to switching symmetry conditions (1a) and (1b) of that paper. However, it is obviously possible to apply these two symmetries when $M$ is even. In fact, they are the ones that hold for the two-channel Smith-Barnwell solution [12].

AS TR and QM (N)M Compatibility. Repeating the steps above for this case yields

$$H_k(z) = z^{-(N-1)} F_k(z^{-1})$$  \hspace{1cm} [using (28)]

$$= (-1)^s z^{-(N-1)} F_{M-1-k}(-z^{-1})$$  \hspace{1cm} [using (63)]

$$= (-1)^{N-1+s} H_{M-1-k}(-z)$$  \hspace{1cm} [using (28)]

$$= (-1)^{N-1+s} H_k(z).$$  \hspace{1cm} [using (62)]

These are compatible provided $N-1+s$ is even. When $M$ is even, this holds if $s = 1$, i.e. negation is present. When $M$ is odd, negation is necessary if the filter length is even, but not if the filter length is odd.

AS TR and QM (N)MTR Constraint Reduction, $M$ Even. Since we are imposing two kinds of symmetry, we should expect sets of four elements of $P(z)$ to be related. This is indeed the case for elements which are on neither the diagonal nor the anti-diagonal. By applying the cyclic transformation on only the indices of the elements of $P(z)$, we find that the set associated with a particular $P_{kl}(z)$ is $(P_{kl}, P_{lk}, P_{M-1-k,M-1-l}, P_{M-1-l,M-1-k})$. Each of these resides in a different quadrant formed by partitioning
P(z) along both diagonals. Since the four are not on the diagonal, and given the restriction on D(z) that both of these symmetries impose, PR requires them all to be zero. However, because of their relationship, requiring one to be zero is sufficient to force them all to be zero. We need not worry that the relationship introduces inconsistencies between the constraints and PR; adherence to the restrictions of Sections IV A and B prevents that.

As for the diagonal elements, the cyclic transformation yields pairs (rather than sets of four) of related elements, P_{kk} and P_{M-1-k,M-1-k}. Thus, we may disregard the constraints associated with half of the diagonal elements. In addition, (41) implies that P_{kk}(z) is symmetric under AS TR symmetry and hence, only L of the constraints associated with each P_{kk}(z) are independent. The anti-diagonal elements are automatically zero as can be shown by applying both symmetries to P_{k,M-1-k}(z):

\[ P_{k,M-1-k}(z) = \hat{P}_{M-1-k,k}(z) \quad \text{(ASTR)}, \]  
\[ P_{k,M-1-k}(z) = -\hat{P}_{M-1-k,k}(z) \quad \text{(QM (N)MTR)}. \]  
Adding these two equations shows that P_{k,M-1-k} = 0. Hence, we do not need to enforce constraints on the anti-diagonals at all.

We will derive the independent constraints in the M = 2 case as a concrete example. This is exactly the set of symmetries we employ in the first numerical design example of Section V. The simultaneous application of AS TR symmetry and QM (N)MTR symmetry leads to the following relationships among the four filters in the filter bank:

\[ F_0(z) = z^{-(N-1)} H_0(z^{-1}) \]  
\[ F_1(z) = z^{-(N-1)} H_1(z^{-1}) \]  
\[ H_1(z) = -z^{-(N-1)} H_0(-z^{-1}). \]  
Note that there is negation in the analysis bank, while the discussion in Section IV B seems to imply that negation only takes place in the synthesis bank. The difference is simply a matter of exchanging a negative sign between the two banks, and in no way changes the results. We apply (75)-(77) to the polyphase component matrices and compute P(z), using (4) and (12):

\[
P(z) = \begin{bmatrix} \hat{H}_{00}(z) & H_{01}(z) \\ \hat{H}_{01}(z) & -H_{00}(z) \end{bmatrix} \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ \hat{H}_{01}(z) & \hat{H}_{00}(z) \end{bmatrix} 
= \begin{bmatrix} H_{00}(z)\hat{H}_{00}(z) + H_{01}(z)\hat{H}_{01}(z) & H_{00}(z)H_{01}(z) - \hat{H}_{00}(z)H_{01}(z) \\ H_{00}(z)\hat{H}_{01}(z) - H_{00}(z)\hat{H}_{00}(z) & H_{00}(z)\hat{H}_{00}(z) + H_{01}(z)\hat{H}_{01}(z) \end{bmatrix} 
= \begin{bmatrix} H_{00}(z)\hat{H}_{00}(z) + H_{01}(z)\hat{H}_{01}(z) & 0 \\ 0 & H_{00}(z)\hat{H}_{00}(z) + H_{01}(z)\hat{H}_{01}(z) \end{bmatrix}. \]  

According to Tables II and III, perfect reconstruction under QM and AS symmetry requires that
However from (78), the simultaneous application of AS and QM symmetry causes the off-diagonal elements of $P(z)$ to be zero. Thus, these constraints do not need to be explicitly enforced. Also, the diagonal elements represent redundant constraints, since the expression for $P_{00}(z)$ is identical to that for $P_{11}(z)$. Therefore, the matrix polynomial equation (79) reduces to just one of the diagonal polynomial equations:

$$H_{00}(z)\hat{H}_{00}(z) + H_{01}(z)\hat{H}_{01}(z) = z^{-(L-1)}/2.$$  \hspace{1cm} (80)

The left side of (80) has order $2(L-1)$ (recall that $N = LM$, and $M = 2$); but since it is symmetric about the $z^{-(L-1)}$ term, only the $z^0$ through $z^{-(L-1)}$ terms need be enforced explicitly. Hence, the original $2NM - M^2 = 4(N-1)$ constraints reduce to $L = N/2$ constraints. In addition, because of (75)-(77), the number of variables is reduced by a factor of four.

**AS TR and QM (N)M Constraint Reduction, M Odd.** Applying the cyclic transformation in this case reveals no restrictions on PR beyond those incurred by imposing the symmetries individually (refer to Tables II and III). By the discussion following (41), AS TR symmetry implies that all constraints on non-diagonal elements are pairwise redundant. In addition, AS TR symmetry implies that each diagonal element is symmetric. Now by (68), QM (N)M symmetry implies that

$$P_{kk}(z) = (-1)^s P_{kk}(-z),$$  \hspace{1cm} (81)

and hence, every other term of $P_{kk}(z)$ is equal to zero. Combining these two properties of $P_{kk}(z)$, we find that if $L$ is even, $s$ must be one and if $L$ is odd, $s$ must be zero. Otherwise, the coefficient of $z^{-(L-1)}$ is automatically zero because of (81), while PR requires it to be $1/M$. Thus, if the filter length and hence, $L$ is even, we must have negation in the synthesis bank, while if $N$ is odd we cannot have negation. If one examines any of the odd-$M$ design examples presented in [10,24], they are found to agree with this conclusion, except that the negation sometimes takes place in the analysis rather than synthesis bank.

**D. Linear Phase Related Symmetry**

In many situations, one wishes to impose symmetry or anti-symmetry on a filter itself so as to achieve exact linear phase. The preceding analysis can be carried out in this case as well. We have chosen not to do so here, since it is addressed in [25] and [26]. However, the second example that we present in the next section does incorporate linear phase.

**V. DESIGN EXAMPLES**

In this section, we present the results of applying the proposed design method in two cases. The first is a two-channel flat passband band-splitting filter bank, and the second is a three-channel band-shaping filter bank which performs spectral equalization as it decomposes the signal into subbands.

**A. Two-channel Band-splitting PRFB**

In this example, we have chosen to impose QM and AS symmetries on the filter coefficients. Not only does this reduce the number of variables and constraints, but it
also eliminates serious difficulties associated with the numerical methods we have used. These difficulties arise when some of the constraints are redundant.

We have designed two-channel PRFB's with filter lengths 10, 20, 40, and 60 using the statistical design approach. The details of the specific numerical methods will be discussed shortly. Figure 3 shows the frequency response magnitude of the length 60 lowpass filter, $H_0(z)$. The attenuation can be seen to be at least 80 dB for frequencies higher than $f = 0.3$ cycles/sample. Although there is no identical design in [27], these results compare well with those in Table I of that paper. The frequency response magnitude of $H_1(z)$ is symmetric with $H_0(z)$ about $f = 0.25$ cycles/sample. The lowpass filter impulse response coefficients are shown in Table IV, and verification of the perfect reconstruction property using the test signal given in [10] appears in Table V. We have included a few output samples from immediately before and after the response to the input to show that they are indeed zero to at least ten decimal places.

Discussion of Numerical Methods. The filters in this example were generated as follows. We began by computing, for a length ten filter, the unconstrained minimum of the positive definite quadratic form associated with each analysis filter. This is simply the MMSE filter. This step was accomplished by solving a Toeplitz system of linear equations (IMSL routine LSLTO). Now, using those analysis filter coefficients, the quadratic PR constraints reduce to an overdetermined set of linear equations in the synthesis filter coefficients given by (18). We found the least-squares solution to this system using IMSL routine LSQR. Note that if an exact solution existed, the system would achieve PR, but unfortunately that was not the case in any of the examples we tried.

This set of analysis and synthesis coefficients served as the starting point for the projected augmented Lagrangian (PAL) method discussed in Section III. The algorithm was found to converge to the optimum solution using various (but not all) values of the penalty parameter in the range 0.0 - 100.0.

We then used the optimal solution for the $N = 10$ case as the starting point for a larger problem. We started this problem using the projected error (PE) function method to drive the solution close to an optimal point. Since this method typically requires many iterations to obtain an optimal solution, we switched to the PAL method once the sub-problem solutions appeared to be converging.

B. Three-Channel Equalizing PRFB

The purpose of this example is to show that PRFB's which have analysis bank functions other than flat passband band-splitting may be designed within the new framework. We have chosen to design a PRFB which provides log-linear equalization from 0 dB at $f = 0.0$ to 10 dB at $f = 0.5$ cycles/sample. Therefore, according to the discussion in Section II A, each analysis bank filter will attempt to follow $D(e^{j\omega}) = 10^{\omega/2\pi}$ in its passband, and attenuate the stopband(s) as much as possible.

An additional feature of this example is that linear phase related symmetry has been incorporated in all six of the filters. As we would expect, this reduces the number of independent constraints by about a factor of two. It can be shown that $P(z)$ satisfies

$$P(z) = J \hat{P}(z) J,$$

and that the elements of $P(z)$ satisfy
Figure 3. Lowpass filter frequency response for two-channel example.
Table IV
Lowpass filter impulse response coefficients for the two-channel example.

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<tr>
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<th>( h_0(n) )</th>
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Table V
Verification of PR for the two-channel example, using input signal from [10].

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\[ P_{kl}(z) = \hat{P}_{M-1-k,M-1-l}(z). \] (83)

According to [25], if a filter bank has an odd number of channels \( M \), PR is only possible if \( (M+1)/2 \) channels have symmetric impulse responses, \( (M-1)/2 \) channels have anti-symmetric impulse responses, and the length of the filters is odd. Since an odd-length antisymmetric filter has zeros at both \( f = 0 \) and \( f = 0.5 \), the bandpass filter \( H_1(z) \) is clearly the correct one to be anti-symmetric. We note here that the statistical error criterion cannot incorporate anti-symmetry. Therefore, we use the integral-squared error criterion in this example.

The length of the filters in this example is 45; there are 136 independent variables and 131 constraint equations. The frequency response magnitudes of the analysis filters are shown as the solid lines in Figures 4-6. For comparison, the filters which minimize the error function without regard for the PR constraints (the unconstrained optimum) are shown as the dotted lines. The PR analysis filter performance is nearly as good as the unconstrained filter performance. Figure 7 depicts the synthesis filters. Note that there are no undesirably large peaks in the stopbands, as has been seen in other PRFB solutions in which the synthesis filters and analysis filters do not have exactly the same magnitude [9]. In fact, we do not expect to see such strange behavior in any equal-complexity FIR filter banks because the synthesis filters are FIR.

The impulse response coefficients of all six filters appear in Tables VI and VII. Since the filters have symmetry about the center coefficient, only the first 23 out of 45 are shown. Finally, Table VIII verifies that the filter bank achieves PR to five decimal places, which is essentially PR in light of the fact that far more significant distortions would be introduced in any application by the processing that takes place between the analysis and synthesis banks. The reason for this slight imperfection is that we used the penalty function method to generate this filter bank. As discussed in Section IV, the penalty function method yields exact constraint satisfaction only in the limit. We will now discuss the specific numerical methods we employed.

**Discussion of Numerical Methods.** The penalty function method proved to be very robust in our application. Since it is also rather easy to implement, it is an attractive option when perfect reconstruction is necessary to five or fewer decimal places. Given the unconstrained optimum analysis filters and synthesis filters set to zero as a starting point, it was usually possible to use \( \rho = 1000 \) on the first iteration and increase it by an order of magnitude or more with each new subproblem. Other starting points also worked well (for example, embedding the solution to a small problem in a larger one, as we did in the first example). This example is the result of four subproblems, with \( \rho \) equal to 5, \( 10^3 \), \( 10^5 \), and \( 10^6 \), respectively. The stopband weighting was \( \alpha_k = 100 \) for each of the three filters. The passband of \( H_k(z) \) was \([\frac{k}{3}, \frac{k+1}{3}]\) in cycles/sample, and the transition bandwidths were all 0.03 cycles/sample.

**VI. CONCLUSIONS**

In this paper, we have presented an analysis of the effect of imposing symmetries among filters in a filter bank on the possibility of perfect reconstruction. In addition, we have presented a PRFB design framework based on the use of general methods for nonlinear optimization under nonlinear constraints. The approach is flexible enough to allow design of a wide variety of filter banks. Using these methods, we have designed
Figure 4. Lowpass analysis filter frequency response for three-channel example. PR filter is solid, unconstrained optimum is dotted.

Figure 5. Bandpass analysis filter frequency response for three-channel example. PR filter is solid, unconstrained optimum is dotted.
Figure 6. Highpass analysis filter frequency response for three-channel example. PR filter is solid, unconstrained optimum is dotted.

Figure 7. Synthesis filter frequency response for three-channel bandshaping example.
Table VI
Analysis filter impulse response coefficients for three-channel band-shaping example (only the first 23 of 45 are shown).

<table>
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<tr>
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Table VII
Synthesis filter impulse response coefficients for three-channel band-shaping example (only the first 23 of 45 are shown).

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Table VIII
Verification of perfect reconstruction for three-channel band-shaping example, using the input signal from [10].

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APPENDIX: PERTAINING TO QM SYMMETRY

Suppose that a maximally-decimated filter bank has an odd number $M$ of channels and filters of even length $N$. Suppose further that the analysis filters are related by MTR symmetry. Then the middle filter $H_{M-1/2}(z)$ must satisfy self-MTR symmetry:

$$H_{M-1/2}(z) = z^{-(N-1)} H_{M-1/2}(-z^{-1}).$$  \hspace{1cm} (A1)

This implies two conditions on each pair of impulse response coefficients, namely

$$h_{M-1/2}(l) = (-1)^{N-1} h_{M-1/2}(N-1-l), \hspace{1cm} (A2)$$

and

$$h_{M-1/2}(N-1-l) = (-1)^l h_{M-1/2}(l). \hspace{1cm} (A3)$$

But since $l$ and $N-1-l$ cannot simultaneously be even (or odd) when $N$ is even, (A2) and (A3) contradict each other, unless

$$h_{M-1/2}(l) = h_{M-1/2}(N-1-l) = 0. \hspace{1cm} (A4)$$

Therefore, MTR symmetry implies that $H_{M-1/2}(z) = 0$. ■

This difficulty does not occur when $M$ is even, because then no filter is required to be self-symmetric.

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REFERENCES


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