1988

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Mo Mu

John R. Rice
Purdue University, jrr@cs.purdue.edu

Report Number:
88-738

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CSD-TR-738
January 1988

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Mo Mu* **and John R. Rice****
Department of Computer Sciences
Purdue University

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ABSTRACT

This paper presents an experimental performance analysis for the rate of convergence of *collocation* on general domains using a bicubic Hermite basis. Twenty domains are selected for the experiment from the population of PDE's on nonrectangular domains in [8], including one rectangle for comparison. The result shows that the convergence of the ELLPACK module COLLOCATION behaves as $O(h^4)$ on all 19 of the nonrectangular domains. This set includes a large variety of nonrectangular domains (only two have reentrant corners). We conclude that, with very high probability, this collocation module has $O(h^4)$ convergence on general domains.

We also enlarge the PDE population in PES (performance evaluation system) by adding to it twenty five nonrectangular domains with two parameters for each. In addition, two new performance analysis tools are used to analyze convergence rates in three kinds of norms.

* Work supported in part by National Science Foundation grant CCR-8619817

** Work supported in part by the Air Force Office of Scientific Research grant 84-0385

CONTENTS

1. INTRODUCTION

The Hermite bicubic collocation method has been well investigated theoretically and experimentally for the rectangular case ([1], [2], [5], [6] [7], [9]). The order of convergence there is $O(h^4)$ when Gauss points are used as collocation points and the true solution is smooth enough. The proof of this result is very difficult and no one has obtained even partial results for nonrectangular domains. However, one of the most attractive potentials of the collocation method is its simplicity when applied to general domains even though the theoretical analysis becomes much more difficult. This paper makes an experimental study of the convergence rate of this collocation method for nonrectangular domains and presents data to support the hypothesis that it is as efficient in nonrectangular cases as it is in the rectangular case.

2. THE NONRECTANGULAR PDE POPULATION

Ribbens and Rice [8] present twenty five domains, each with two parameters, for the nonrectangular PDE population and incorporate them into ELLPACK. We make them available now in the Performance Evaluation System (PES) by adding to it a new file mac.500 (see Appendix A.1) where the equation is Poisson equation:

$$U_{xx} + U_{yy} = f(x,y) \quad in\ \Omega \tag{2.1}$$

with Dirichlet boundary condition:

$$U = g(x,y) \qquad on\ \partial\Omega\ ; \tag{2.2}$$

three parameters a, b, and c specify the domain number, domain parameter1 and parameter2, respectively (for more details see [8]). The right hand sides f and g in (2.1) and (2.2) are defined such that the true solution is a smooth function as in Problem 4 of the PDE population in Appendix A of [10]:

$$U(x,y) = 3e^{x+y}x(1-x)y(1-y) \tag{2.3}$$

In addition, the discretization module COLLOCATION [3], [4] which we use requires that domains should be specified *clockwise* while some of domains given in [8], with the domain numbers 3, 19, 20, 22, 23 and 25, respectively, are specified *counter-clockwise*. They are now re-implemented *clockwise* and their exact definitions are provided within PES by the programs in files mac.503, mac.519, mac.520, mac.522, mac.523 and mac.525, respectively. Appendix A.1 includes mac.503 as an example.

The file prob.500 (Part of it is given in Appendix A.1) contains the information needed to generate the equation, boundary and subprograms segments of the ELLPACK programs for the nonrectangular PDE population. It can easily be extended as needed and includes now 72 records which cover all sets of parameters appearing at the end of [8]

which includes figures of these domains.

All files mentioned above reside now under the directory /usr/mu/pes/pdepop/.

Grids suitable for the domain ranges can be obtained by using the interactive ELLPACK program in the file *nrecdom* under the directory /usr/mu/ellpack/ellpacki/ (see Appendix A.2).

3. THE EXPERIMENT

3.a The PDE Problems Solved

We use PES to make the experimental performance analysis for the rate of convergence of *collocation* on general domains. Twenty problems with general domains are selected from prob.500 and the record numbers are 1, 4, 5, 15, 17, 21, 22, 25, 27, 28, 37, 43, 53, 55, 57, 59, 61, 69, 70 and 71, respectively. One exception among them is Problem 500-1 which is actually defined on a rectangle. It is used for comparison. The true solutions are all the same and very smooth (see (2.3)). Most domains have no re-entrant corners and some of them are rather complicated. They are of reasonable generality and represent a large variety of nonrectangular domains. For their exact definitions readers are referred to [8], their shapes are seen in Appendix C along with the performance plots.

The problems chosen are solved by the double precision version ELLPACK [10] on a VAX 8600. The discretization, indexing and solution modules are COLLOCATION, AS IS and BAND GE, respectively and the corresponding module numbers defined in the file modata of PES are 56,14 and 46.

Nine levels of uniform grids of size 5×5, 7×7, 9×9, 11×11, 13×13, 15×15, 17×17, 19×19 and 21×21 are first used on which all 20 problems are solved. There are 168 successful runs among the 180 ELLPACK jobs. The reasons for the other 12 failed runs are due to one of the following:

- fatal error in the ellpack module DOMAIN PROCESSOR - the grid is not appropriate for the domain. This usually occurs when the grid is too coarse or there is an unfortunate location of one of the grid lines relative to the domain boundary.
- fatal error in the ellpack module COLLOCATION - the boundary enters some element more than once. This usually occurs when the grid is too coarse.
- fatal error in the ellpack module BAND GE - the BAND GE factor routine executed successfully, but the BAND GE back solve routine will divide by zero if called.

In the third case, the resulting matrices are singular. It seems from the plots of collocation points for Problem 500-43 that while the element in the corner which is near (-2.100, 0.174) has no interior collocation points, the singular matrix is caused.

The number of failures was small enough that they have little effect on the conclusions for most problems. However, for some problems this initial set of data was not completely conclusive so we solved more problems. The following are the added combinations of problems and grids.

500-5: $22 \times 22, 23 \times 23, 25 \times 25, 27 \times 27, 29 \times 29$

500-15: $22 \times 22, 23 \times 23$

500-17: $22 \times 22, 23 \times 23, 25 \times 25, 27 \times 27, 29 \times 29$

500-21: $22 \times 22, 23 \times 23, 25 \times 25$

500-22: $22 \times 22, 23 \times 23, 25 \times 25$

500-25: $22 \times 22, 25 \times 25, 27 \times 27$

500-37: $22 \times 22, 23 \times 23, 25 \times 25, 27 \times 27, 29 \times 29$

500-43: $8 \times 8, 10 \times 10, 12 \times 12, 14 \times 14, 16 \times 16, 20 \times 20, 22 \times 22, 24 \times 24$

500-53: $4 \times 4, 6 \times 6, 8 \times 8, 18 \times 18, 22 \times 22, 23 \times 23, 25 \times 25, 27 \times 27, 29 \times 29$

500-55: $22 \times 22, 23 \times 23, 25 \times 25, 27 \times 27, 29 \times 29$

500-61: $4 \times 4, 6 \times 6, 8 \times 8, 10 \times 10, 12 \times 12, 14 \times 14, 16 \times 16, 18 \times 18, 20 \times 20, 22 \times 22, 23 \times 23, 24 \times 24$

500-70: $23 \times 23, 27 \times 27, 29 \times 29$

500-71: $25 \times 25, 29 \times 29$

3.b Analysis of the Rates of Convergence

Three tools are used to analyze the rate of convergence. One of them is CONV0 which is in the original PES with the name CONV and estimates the order of convergence by

$$cerr_{n,n+1} = \frac{\log \dfrac{rerr_n}{rerr_{n+1}}}{\log \dfrac{ngridx_{n+1}}{ngridx_n}} \tag{3.1}$$

and

$$cerrmax_{n,n+1} = \frac{\log \frac{rerrmax_n}{rerrmax_{n+1}}}{\log \frac{ngridx_{n+1}}{ngridx_n}} \tag{3.2}$$

Here *rerr* and *rerrmax* stand for the relative errors of the current approximation in maximum norms on the current and 20×20 grid levels, respectively, and *ngridx* is the number of x-grid points. These measures assume that the grid spacing h_n is $1/ngridx_n$ which is inaccurate for the coarser grids used in this experiment.

The second tool is the newly added CONV1 which estimates the order of convergence by

$$conv1mx_n = \frac{\log \frac{rerr_{n-1}}{rerr_n}}{\log \frac{h_{n-1}}{h_n}} \tag{3.3}$$

where h is defined as

$$h = \frac{1}{ngridx-1}. \tag{3.4}$$

Similarly *conv1mm* and *conv1l2* are defined with *rerr* substituted by *rerrmax* and *errl2*, respectively, where *rerr* and *rerrmax* have the same meanings as above and *errl2* is the approximation error in the discrete L_2 norm on the current grid as defined in [10]. All of these estimates are defined as zero on the coarsest grid. Asymptotically, *conv1mx* and *conv1mm* are equivalent to *cerr* and *cerrmax*, respectively. But when the number of grid points is not too large they are quite different. We recommend CONV1 because theoretically the convergence rate is determined by the spacing size h rather than the number of grid points.

The third measure is another new program CONV2 which uses the data from the first and the current grid levels rather than adjacent grid levels. It estimates the order of convergence by

$$conv2mx_n = \frac{\log \frac{rerr_1}{rerr_n}}{\log \frac{h_1}{h_n}} \tag{3.5}$$

and similarly for *conv2mm* and *conv2l2*. Roughly speaking, CONV0 and CONV1 reflect the convergence rate locally while CONV2 is global and avoids some effects of oscillation.

These three programs reside now under the directory /usr/mu /pes/seedata/ and their usages are described in the file README under the same directory. They shortly will be incorporated into PES permanently.

For our 20 nonrectangular test problems, the outputs from CONV0, CONV1 and CONV2 are listed in Appendix B.1, Appendix B.2 and Appendix B.3, respectively. In addition, we include in Appendix C the plots of slopes of log(*err-nodes*) versus log(*nx*) where *err-nodes* and *nx* denote *rerrmax* and *ngridx-1*, respectively. The shapes of the domains are also given in Appendix C. The data suggest the following

1. The estimates for convergence rate generated by CONV0 are usually larger than those by CONV1 and CONV2 on most grids. For the rectangular problem (Problem 500-1) the latter two agree with the theoretical value 4.0 much better than the former.
2. The data from CONV1 show unreasonable oscillation much more frequently than those from CONV2 do.
3. The estimated convergence rate of COLLOCATION measured by CONV2 has the following distribution. About 42.6% lie in the interval [3.5, 4.5], 36.5% in (4.5, $+\infty$), 19.1% in [1., 3.5) and 1.9% in ($-\infty$, 1.).
4. These tools are useful to quantify the convergence rates observed, but too simple to capture the full complexity of the behavior of the error as the grid is refined. We prefer instead to rely on a visual analysis of plots of *err-nodes* versus *ngridx-1* on a *log-log* scale. Convergence of exactly order 4 results in a straight line with negative slope of -4. Almost exact $O(h^4)$ convergence is seen on some cases, but most error plots show some "random" variation due to unpredictable interaction between the domain shape and the grid location. We have collected enough data on each problem so that, we believe, one sees that it is highly probable that the convergence rate is close to $O(h^4)$. Certainly none of the data sets suggest something much different from $O(h^4)$ convergence.

   We realize that the above analysis is subjective, but we believe that more reliable than using tools such as we used earlier. Further, since all the approaches strongly support a hypothesis of $O(h^4)$ convergence, we have further confidence in our conclusions. Finally, we note that there is no applicable standard statistical test known to us which can be applied to establish confidence levels for our conclusion. Nevertheless, if domains without reentrant corners exist with convergence rate less than $O(h^4)$, they must be relatively rare for our sample to have included none of them.

4. CONCLUSIONS

From the above observation we conclude that

1. The program CONV2 is most appropriate among three tools for estimating convergence rates.
2. The convergence of the COLLOCATION module is $O(h^4)$ in cases of general domains.
3. The COLLOCATION module works well for general domains as long as grids are not too coarse.

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APPENDIX A. The PDE Problems

1. Example macro files

File 1: mac.500

```
*************
* macro 500 *
*************
*         2020020002000112
1         uxx + uyy = f(x,y)
2         u=true(x,y) on x=r9nrxc(p,1),y=r9nryc(p,1) &&
2                     for p=r9nrmn(&a,&b,&c) to r1brng(2,1)
2         u=true(x,y) on x=r9nrxc(p,2),y=r9nryc(p,2) &&
2                     for p=r1brng(1,2) to r1brng(2,2)
2         u=true(x,y) on x=r9nrxc(p,3),y=r9nryc(p,3) &&
2                     for p=r1brng(1,3) to r1brng(2,3)
2         u=true(x,y) on x=r9nrxc(p,4),y=r9nryc(p,4) &&
2                     for p=r1brng(1,4) to r1brng(2,4)
2         u=true(x,y) on x=r9nrxc(p,5),y=r9nryc(p,5) &&
2                     for p=r1brng(1,5) to r1brng(2,5)
2         u=true(x,y) on x=r9nrxc(p,6),y=r9nryc(p,6) &&
2                     for p=r1brng(1,6) to r1brng(2,6)
2         u=true(x,y) on x=r9nrxc(p,7),y=r9nryc(p,7) &&
2                     for p=r1brng(1,7) to r1brng(2,7)
2         u=true(x,y) on x=r9nrxc(p,8),y=r9nryc(p,8) &&
2                     for p=r1brng(1,8) to r1brng(2,8)
2         u=true(x,y) on x=r9nrxc(p,9),y=r9nryc(p,9) &&
2                     for p=r1brng(1,9) to r1brng(2,9)
2         u=true(x,y) on x=r9nrxc(p,10),y=r9nryc(p,10) &&
2                     for p=r1brng(1,10) to r1brng(2,10)
2         u=true(x,y) on x=r9nrxc(p,11),y=r9nryc(p,11) &&
2                     for p=r1brng(1,11) to r1brng(2,11)
2         u=true(x,y) on x=r9nrxc(p,12),y=r9nryc(p,12) &&
2                     for p=r1brng(1,12) to r1brng(2,12)
2         u=true(x,y) on x=r9nrxc(p,13),y=r9nryc(p,13) &&
2                     for p=r1brng(1,13) to r1brng(2,13)
2         u=true(x,y) on x=r9nrxc(p,14),y=r9nryc(p,14) &&
2                     for p=r1brng(1,14) to r1brng(2,14)
2         u=true(x,y) on x=r9nrxc(p,15),y=r9nryc(p,15) &&
2                     for p=r1brng(1,15) to r1brng(2,15)
2         u=true(x,y) on x=r9nrxc(p,16),y=r9nryc(p,16) &&
2                     for p=r1brng(1,16) to r1brng(2,16)
3     function true(x,y)
3     true = 3.*exp(x+y)*x*y*(1.-x)*(1.-y)
3     return
3     end
3     function f(x,y)
3     f = 6.*x*y*exp(x+y)*(x*y+x+y-3.)
3     return
3     end
```

APPENDIX A. The PDE Problems

1. Example macro files

File 2: mac.503

```
*************
* macro 503 *
*************
*         2020020002000112
1         uxx + uyy = f(x,y)
2         u=true(x,y) on x=xc(p,1),y=yc(p,1) for p=0.0 to 2.*pi
3     function true(x,y)
3     true = 3.*exp(x+y)*x*y*(1.-x)*(1.-y)
3     return
3     end
3     function f(x,y)
3     f = 6.*x*y*exp(x+y)*(x*y+x+y-3.)
3     return
3     end
3     function xc(p,ipiece)
3     idom=&a
3     xc=&b*cos(p)
3     return
3     end
3     function yc(p,ipiece)
3     idom=&a
3     yc=-&c*sin(p)
3     return
3     end
```

APPENDIX A. The PDE Problems

1. Example macro files

File 3: part of prob.500

```
***************
* problem 500 *
***************
*parameter set   1(a=1,b=.8,c=.6)
*         000.00    000.00    000.00    000.00
expand 500/1,0.8,0.6/
-
*parameter set   2(a=1,b=.22,c=.88)
*         000.00    000.00    000.00    000.00
expand 500/1,0.22,0.88/
-
*parameter set   3(a=2,b=-2.,c=.5)
*         000.00    000.00    000.00    000.00
expand 500/2,-2.,.5/
-
*parameter set   4(a=2,b=1.,c=1.)
*         000.00    000.00    000.00    000.00
expand 500/2,1.,1./
-
*parameter set   5(a=3,b=2.,c=1.)
*         000.00    000.00    000.00    000.00
expand 503/3,2.,1./
-
*parameter set  71(a=25,b=2.5,c=-1.)
*         000.00    000.00    000.00    000.00
expand 525/25,2.5,-1./
-
*parameter set  72(a=25,b=-3.,c=-.5)
*         000.00    000.00    000.00    000.00
expand 525/25,-3.,-.5/
```

APPENDIX A. The PDE Problems

2. The generic ELLPACK program

File: nrecdom

```
OPTIONS. terminal = tek $ max x points = 65 $ max y points = 65
EQUATION.     uxx + uyy = 1.0
BOUNDARY. u=0.0 on x=r9nrxc(p,1),y=r9nryc(p,1) &
                    for p=r9nrmn(idom,apar,bpar) to r1brng(2,1)
          u=0.0 on x=r9nrxc(p,2),y=r9nryc(p,2) &
                    for p=r1brng(1,2) to r1brng(2,2)
          u=0.0 on x=r9nrxc(p,3),y=r9nryc(p,3) &
                    for p=r1brng(1,3) to r1brng(2,3)
          u=0.0 on x=r9nrxc(p,4),y=r9nryc(p,4) &
                    for p=r1brng(1,4) to r1brng(2,4)
          u=0.0 on x=r9nrxc(p,5),y=r9nryc(p,5) &
                    for p=r1brng(1,5) to r1brng(2,5)
          u=0.0 on x=r9nrxc(p,6),y=r9nryc(p,6) &
                    for p=r1brng(1,6) to r1brng(2,6)
          u=0.0 on x=r9nrxc(p,7),y=r9nryc(p,7) &
                    for p=r1brng(1,7) to r1brng(2,7)
          u=0.0 on x=r9nrxc(p,8),y=r9nryc(p,8) &
                    for p=r1brng(1,8) to r1brng(2,8)
          u=0.0 on x=r9nrxc(p,9),y=r9nryc(p,9) &
                    for p=r1brng(1,9) to r1brng(2,9)
          u=0.0 on x=r9nrxc(p,10),y=r9nryc(p,10) &
                    for p=r1brng(1,10) to r1brng(2,10)
          u=0.0 on x=r9nrxc(p,11),y=r9nryc(p,11) &
                    for p=r1brng(1,11) to r1brng(2,11)
          u=0.0 on x=r9nrxc(p,12),y=r9nryc(p,12) &
                    for p=r1brng(1,12) to r1brng(2,12)
          u=0.0 on x=r9nrxc(p,13),y=r9nryc(p,13) &
                    for p=r1brng(1,13) to r1brng(2,13)
          u=0.0 on x=r9nrxc(p,14),y=r9nryc(p,14) &
                    for p=r1brng(1,14) to r1brng(2,14)
          u=0.0 on x=r9nrxc(p,15),y=r9nryc(p,15) &
                    for p=r1brng(1,15) to r1brng(2,15)
          u=0.0 on x=r9nrxc(p,16),y=r9nryc(p,16) &
                    for p=r1brng(1,16) to r1brng(2,16)
MENU.'Problem Solving Menu'
       'cd:change domain'
        fortran.
         print *,'enter domain number:'
         read *,idom
         l0hvgr=.false.
       'cdp:change domain parameters'
        fortran.
         print *,'enter domain parameter 1:'
         read *,apar
         print *,'enter domain parameter 2:'
         read *,bpar
       'ig:interactive grid'                grid.  interactive
       'sp:solve problem'                   disc.  5 point star
                                            solu.  linpack band
END.
```

APPENDIX B. The Experimental Data

1. Convergence rates as estimated by CONV0

```
500    1  b4p2unix/56/14/46/
 nx nxl      cerr      cerrmax
 5   7       4.44      4.92
 7   9       4.41      4.10
 9  11       4.27      3.87
11  13       4.15      5.19
13  15       4.36      4.16
15  17       4.08      4.96
17  19       4.27      9.77
19  21       3.87      4.23

500    4  b4p2unix/56/14/46/
 nx nxl      cerr      cerrmax
 5   7       4.78      5.27
 7   9       4.30      4.35
 9  11       4.25      4.31
11  13       4.08      4.67
13  15       4.35      4.01
15  17       4.20      4.80
17  19       4.03      6.23
19  21       4.11      4.35

500    5  b4p2unix/56/14/46/
 nx nxl      cerr      cerrmax
 5   7       5.05      6.42
 7   9       8.27      7.03
 9  11       4.57      3.94
11  13       3.44      6.78
13  15       6.60      0.38
15  17       0.11      5.45
17  19       9.49     16.21
19  21       1.82      5.10
21  22       6.18    -19.21
22  23       1.55      0.
23  25       5.30      0.
25  27       2.54      9.01
27  29       6.35      0.

500   15  b4p2unix/56/14/46/
 nx nxl      cerr      cerrmax
 5   7       7.68      9.00
 7   9       4.49      4.88
 9  11       3.83      3.08
11  13       3.81      5.07
13  15       8.68    -41.70
15  17       4.05     56.01
17  19       4.04      9.40
19  21       4.05      4.25
21  22       4.80     -3.49
22  23       3.00      0.

500   17  b4p2unix/56/14/46/
 nx nxl      cerr      cerrmax
 5   7       5.28      7.75
 7   9       3.26      2.95
 9  11       3.94      2.15
11  13       5.03      6.35
13  15       2.65      4.39
```

```
 15  17       2.51      0.34
 17  19       2.25     15.29
 19  21       0.14      3.70
 21  22      21.38    -10.38
 22  23     -19.76     -4.73
 23  25       5.24      8.73
 25  27       2.37      3.13
 27  29       6.50      9.70

500  21  b4p2unix/56/14/46/
 nx nx1      cerr      cerrmax
  5   7       6.94      5.45
  7   9       4.70      6.28
  9  11       2.05      2.50
 11  13       6.98      6.37
 13  15       1.20      3.47
 15  17       5.54      2.96
 17  19       3.21    -21.86
 19  21       2.87     49.62
 21  22       3.92    -15.36
 22  23       6.69     -7.98
 23  25       2.00      4.00

500  22  b4p2unix/56/14/46/
 nx nx1      cerr      cerrmax
  5   7       6.84      6.41
  7   9       2.58      6.06
  9  11       5.76      3.06
 11  13       4.59      6.09
 13  15       4.84      1.67
 15  17       2.10      6.70
 17  19       7.38     13.16
 19  21       3.18      6.15
 21  22       2.87    -15.43
 22  23      -1.55      1.13
 23  25       4.86      0.96

500  25  b4p2unix/56/14/46/
 nx nx1      cerr      cerrmax
  5   7       7.00      7.11
  7   9       1.78      3.29
  9  11       6.14      6.00
 11  13       2.78      1.44
 13  15       0.58      7.68
 15  17       8.49      0.
 17  19       2.17     19.20
 19  21      -5.97      6.19
 21  22      27.75    -24.62
 22  25       2.30      5.42
 25  27      -5.99      2.90

500  27  b4p2unix/56/14/46/
 nx nx1      cerr      cerrmax
  5   7       1.94     -1.82
  7   9       5.48     14.48
  9  11       2.89      4.04
 11  13       4.97      5.07
 13  15       3.88      4.84
 15  17       3.71      2.85
 17  19       4.73     10.54
 19  21       3.13      7.24

500  28  b4p2unix/56/14/46/
 nx nx1      cerr      cerrmax
```

```
 5   7       4.27      3.70
 7   9       5.57      4.21
 9  11      -1.50     -9.68
11  13      15.59     24.80
13  15       2.74      2.93
15  17       0.33      4.90
17  19       3.65     12.60
19  21       4.70      0.32

500  37  b4p2unix/56/14/46/
 nx nx1      cerr      cerrmax
 5   7       6.58      6.67
 7   9       6.66      5.83
 9  11       3.34      5.11
11  13      -0.50      2.79
13  15       9.86      5.11
15  17      -1.10      2.10
17  19       8.90     10.82
19  21       3.67      6.28
21  22      -3.31    -16.83
22  23      -6.47      0.
23  25       7.49      6.71
25  27       2.90      7.77
27  29       3.72      3.77

500  43  b4p2unix/56/14/46/
 nx nx1      cerr      cerrmax
 5   7       4.17      1.25
 7   8      10.02     19.40
 8   9      -1.13     -3.44
 9  10       8.55     14.28
10  11      -4.52      0.
11  12       1.21      5.56
12  13     -17.55    -26.15
13  14      45.06     42.88
14  15     -29.08    -17.74
15  16      33.91     27.95
16  20      -1.31      2.13
20  22       8.42     13.96
22  24       3.40     -3.05

500  53  b4p2unix/56/14/46/
 nx nx1      cerr      cerrmax
 4   5      23.27     29.40
 5   6       2.22    -11.16
 6   7     -43.91    -31.99
 7   8      62.74     62.45
 8   9     -71.23    -71.26
 9  11      13.06     13.37
11  13      13.69     16.40
13  15      11.25     10.03
15  17       8.66      4.08
17  18      71.95     77.87
18  19    -106.66    -82.32
19  21      23.01     22.21
21  22      93.28     76.51
22  23     -72.41    -57.66
23  25      17.80     17.67
25  27      16.71     18.74
27  29      17.99      7.70

500  55  b4p2unix/56/14/46/
 nx nx1      cerr      cerrmax
 5   7       3.77      6.69
```

```
 7   9     -2.29    -4.14
 9  11     14.93    22.95
11  13      2.24     4.82
13  15      6.09    -5.20
15  19      6.46    11.33
19  21      4.00     4.37
21  22     15.22    -6.03
22  23     -8.89    19.22
23  25      4.38     4.28
25  27      4.00    -0.46
27  29      6.25     5.20

500  57  b4p2unix/56/14/46/
 nx nxl      cerr      cerrmax
 5   7       2.92      3.73
 7   9      10.40     14.75
 9  13      -0.84     -2.33
13  15      18.92     16.76
15  17       3.82      6.68
17  19       3.94     12.89
19  21       3.93      8.68

500  59  b4p2unix/56/14/46/
 nx nxl      cerr      cerrmax
 9  11       6.18     14.34
11  13       5.48      2.95
13  15       4.84     -6.77
15  17       4.84     24.81
17  19       4.98      5.96
19  21       4.50      9.85

500  61  b4p2unix/56/14/46/
 nx nxl      cerr      cerrmax
 4   6      14.50     15.91
 6   7       4.98      9.92
 7   8      -2.44     -8.03
 8   9     -30.96    -20.79
 9  10      58.19     50.29
10  11     -46.82    -33.18
11  12      53.85     60.04
12  13     -52.15    -67.86
13  14      69.27     80.68
14  15     -66.43    -75.68
15  16      78.71     92.63
16  17     -95.40    -97.47
17  18     107.08    115.51
18  19     -89.30    -79.53
19  20     105.79     94.48
20  21     -70.92    -54.87
21  22      79.92     61.86
22  23       9.39      2.93
23  24       3.62     10.28

500  69  b4p2unix/56/14/46/
 nx nxl      cerr      cerrmax
 5   7       5.72      8.16
 7   9       0.77      2.03
 9  11       7.15      0.
11  15       4.74      9.37
15  17       5.20     12.57
17  19       3.31     -1.46
19  21       4.87      8.91

500  70  b4p2unix/56/14/46/
```

```
 nx nxl      cerr      cerrmax
 5   7      14.26     17.58
 7   9       0.29      3.49
 9  11       3.23      5.61
11  15       5.87      5.33
15  17       5.25      1.96
17  19      -2.59     16.41
19  21       7.75      2.32
21  23      11.77     -7.86
23  27      -1.39      4.46
27  29       3.12     10.32

500  71  b4p2unix/56/14/46/
 nx nxl      cerr      cerrmax
 5   7       2.88      1.25
 7  11       3.31      4.31
11  15       5.19      6.30
15  17       8.32     16.34
17  19      -3.03      3.69
19  21       9.04      3.39
21  25       3.04      2.11
25  29       6.01      3.96
```

APPENDIX B. The Experimental Data

2. Errors and convergence rates as estimated by CONV1

1

500 1 b4p2unix/56/14/46/ (a=1,b=.8,c=.6)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	2.0e-01	100	8.9e-05	2.2e-04	1.9e-05	0.0e+00	0.0e+00	0.0e+00	5.7e-01
7	7	1.3e-01	196	2.0e-05	4.2e-05	4.0e-06	3.7e+00	4.1e+00	3.8e+00	5.7e-01
9	9	1.0e-01	324	6.6e-06	1.5e-05	1.3e-06	3.8e+00	3.5e+00	3.9e+00	5.7e-01
11	11	8.0e-02	484	2.8e-06	6.9e-06	5.4e-07	3.8e+00	3.5e+00	3.9e+00	5.7e-01
13	13	6.7e-02	676	1.4e-06	2.9e-06	2.6e-07	3.9e+00	4.7e+00	4.0e+00	5.7e-01
15	15	5.7e-02	900	7.5e-07	1.6e-06	1.4e-07	3.9e+00	4.0e+00	4.0e+00	5.7e-01
17	17	5.0e-02	1156	4.5e-07	8.6e-07	8.3e-08	3.9e+00	4.6e+00	4.0e+00	5.7e-01
19	19	4.4e-02	1444	2.8e-07	2.9e-07	5.2e-08	3.9e+00	9.2e+00	4.0e+00	5.7e-01
21	21	4.0e-02	1764	1.9e-07	1.9e-07	3.4e-08	3.8e+00	4.1e+00	4.0e+00	5.7e-01

500 4 b4p2unix/56/14/46/ (a=2,b=1.,c=1.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	2.5e-01	76	2.8e-04	1.0e-03	4.7e-05	0.0e+00	0.0e+00	0.0e+00	5.1e-01
7	7	1.7e-01	136	5.6e-05	1.7e-04	1.2e-05	4.0e+00	4.4e+00	3.4e+00	5.6e-01
9	9	1.3e-01	212	1.9e-05	5.7e-05	4.2e-06	3.7e+00	3.9e+00	3.6e+00	5.7e-01
11	11	1.0e-01	304	8.1e-06	2.4e-05	1.8e-06	3.9e+00	3.8e+00	3.7e+00	5.7e-01
13	13	8.3e-02	412	4.1e-06	1.1e-05	9.2e-07	3.7e+00	4.4e+00	3.8e+00	5.7e-01
15	15	7.1e-02	536	2.2e-06	6.2e-06	5.1e-07	4.0e+00	3.6e+00	3.8e+00	5.7e-01
17	17	6.3e-02	676	1.3e-06	3.4e-06	3.1e-07	3.9e+00	4.6e+00	3.8e+00	5.7e-01
19	19	5.6e-02	832	8.3e-07	1.7e-06	1.9e-07	4.0e+00	5.8e+00	3.8e+00	5.7e-01
21	21	5.0e-02	1004	5.5e-07	1.1e-06	1.3e-07	3.9e+00	4.3e+00	3.8e+00	5.7e-01

500 5 b4p2unix/56/14/46/ (a=3,b=2.,c=1.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	1.0e+00	100	3.5e-02	6.6e-02	1.7e-02	0.0e+00	0.0e+00	0.0e+00	1.0e+00
7	7	6.7e-01	180	6.4e-03	7.6e-03	5.4e-03	4.2e+00	5.3e+00	2.8e+00	2.9e+00
9	9	5.0e-01	308	8.0e-04	1.3e-03	1.5e-03	7.2e+00	6.1e+00	4.4e+00	4.6e+00
11	11	4.0e-01	436	3.2e-04	5.9e-04	6.8e-04	4.1e+00	3.7e+00	3.6e+00	7.5e+00
13	13	3.3e-01	596	1.8e-04	1.9e-04	3.9e-04	3.3e+00	6.1e+00	3.0e+00	8.0e+00
15	15	2.9e-01	804	7.0e-05	1.8e-04	1.9e-04	6.1e+00	5.7e-01	4.7e+00	8.1e+00
17	17	2.5e-01	996	6.9e-05	9.1e-05	1.3e-04	1.2e-01	5.0e+00	3.1e+00	8.0e+00
19	19	2.2e-01	1252	2.4e-05	1.5e-05	7.4e-05	8.9e+00	1.5e+01	4.4e+00	1.0e+01
21	21	2.0e-01	1492	2.0e-05	9.0e-06	5.2e-05	1.8e+00	5.0e+00	3.3e+00	9.8e+00
22	22	1.9e-01	1664	1.5e-05	2.2e-05	4.0e-05	5.9e+00	-1.9e+01	5.3e+00	8.6e+00
23	23	1.8e-01	1812	1.4e-05	2.2e-05	3.5e-05	1.9e+00	8.9e-02	3.4e+00	9.5e+00
25	25	1.7e-01	2132	9.0e-06	2.2e-05	2.4e-05	4.9e+00	4.1e-01	4.3e+00	9.1e+00
27	27	1.5e-01	2452	7.4e-06	1.1e-05	1.8e-05	2.5e+00	8.2e+00	3.4e+00	1.1e+01
29	29	1.4e-01	2820	4.7e-06	1.1e-05	1.4e-05	6.2e+00	4.1e-01	4.0e+00	1.0e+01

500 15 b4p2unix/56/14/46/ (a=7,b=.4,c=.2)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	2.1e-01	100	4.5e-04	6.4e-04	7.6e-05	0.0e+00	0.0e+00	0.0e+00	3.8e-01
7	7	1.4e-01	188	3.4e-05	3.1e-05	5.9e-06	6.4e+00	7.5e+00	6.3e+00	4.1e-01
9	9	1.0e-01	312	1.1e-05	9.1e-06	1.7e-06	3.8e+00	4.3e+00	4.3e+00	4.3e-01
11	11	8.3e-02	444	5.1e-06	4.9e-06	6.7e-07	3.5e+00	2.8e+00	4.1e+00	4.4e-01
13	13	6.9e-02	624	2.7e-06	2.1e-06	3.1e-07	3.5e+00	4.7e+00	4.1e+00	4.5e-01
15	15	6.0e-02	804	7.8e-07	8.2e-04	1.5e-07	8.1e+00	-3.9e+01	4.9e+00	4.5e-01

17	17	5.2e-02	1032	4.7e-07	7.4e-07	8.8e-08	3.8e+00	5.3e+01	4.0e+00	4.5e-01
19	19	4.6e-02	1264	3.0e-07	2.6e-07	5.5e-08	3.9e+00	9.0e+00	4.0e+00	4.6e-01
21	21	4.2e-02	1544	2.0e-07	1.7e-07	3.6e-08	3.8e+00	4.1e+00	4.1e+00	4.6e-01
22	22	4.0e-02	1680	1.6e-07	2.0e-07	3.0e-08	3.8e+00	-3.6e+00	3.8e+00	4.6e-01
23	23	3.8e-02	1836	1.4e-07	2.0e-07	2.5e-08	3.8e+00	-5.4e-01	4.0e+00	4.6e-01

500 17 b4p2unix/56/14/46/ (a=8,b=1.,c=1.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	5.2e-01	100	1.3e-03	5.7e-03	2.1e-02	0.0e+00	0.0e+00	0.0e+00	3.9e+01
7	7	3.5e-01	180	2.2e-04	4.2e-04	9.0e-03	4.3e+00	6.4e+00	2.1e+00	1.2e+02
9	9	2.6e-01	308	9.7e-05	2.0e-04	2.8e-03	2.8e+00	2.6e+00	4.0e+00	8.9e+01
11	11	2.1e-01	436	4.4e-05	1.3e-04	1.2e-03	3.5e+00	1.9e+00	4.0e+00	7.9e+01
13	13	1.7e-01	580	1.9e-05	4.5e-05	6.4e-04	4.6e+00	5.9e+00	3.2e+00	1.2e+02
15	15	1.5e-01	772	1.3e-05	2.4e-05	3.7e-04	2.6e+00	4.0e+00	3.6e+00	1.0e+02
17	17	1.3e-01	964	9.5e-06	2.3e-05	2.2e-04	2.1e+00	3.9e-01	3.8e+00	9.3e+01
19	19	1.2e-01	1172	7.4e-06	4.2e-06	1.7e-04	2.1e+00	1.5e+01	2.2e+00	1.2e+02
21	21	1.0e-01	1428	7.3e-06	2.9e-06	1.3e-04	1.8e-01	3.4e+00	2.8e+00	1.1e+02
22	22	9.9e-02	1568	2.7e-06	4.7e-06	8.2e-05	2.0e+01	-9.6e+00	9.0e+00	1.2e+02
23	23	9.4e-02	1716	6.5e-06	5.8e-06	7.8e-05	-1.9e+01	-4.8e+00	1.1e+00	1.0e+02
25	25	8.7e-02	2000	4.2e-06	2.8e-06	5.9e-05	5.0e+00	8.3e+00	3.1e+00	1.2e+02
27	27	8.0e-02	2344	3.5e-06	2.2e-06	4.1e-05	2.3e+00	3.4e+00	4.6e+00	1.1e+02
29	29	7.4e-02	2660	2.2e-06	1.1e-06	3.2e-05	6.1e+00	9.4e+00	3.5e+00	1.3e+02

500 21 b4p2unix/56/14/46/ (a=10,b=1.,c=0.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	1.0e+00	76	3.1e-03	1.0e-02	9.6e-02	0.0e+00	0.0e+00	0.0e+00	7.0e+01
7	7	6.9e-01	136	3.0e-04	1.6e-03	1.2e-02	5.8e+00	4.6e+00	5.1e+00	1.2e+02
9	9	5.2e-01	212	9.2e-05	3.3e-04	5.1e-03	4.1e+00	5.5e+00	3.1e+00	1.6e+02
11	11	4.2e-01	300	6.1e-05	2.0e-04	2.6e-03	1.9e+00	2.2e+00	3.1e+00	1.9e+02
13	13	3.5e-01	408	1.9e-05	6.9e-05	1.2e-03	6.5e+00	5.8e+00	4.2e+00	2.1e+02
15	15	3.0e-01	524	1.6e-05	4.2e-05	7.2e-04	1.0e+00	3.3e+00	3.2e+00	2.2e+02
17	17	2.6e-01	664	8.0e-06	2.9e-05	4.0e-04	5.2e+00	2.7e+00	4.4e+00	2.4e+02
19	19	2.3e-01	812	5.6e-06	3.3e-04	2.8e-04	3.0e+00	-2.1e+01	3.0e+00	2.5e+02
21	21	2.1e-01	976	4.2e-06	2.3e-06	1.8e-04	2.6e+00	4.7e+01	4.2e+00	2.6e+02
22	22	2.0e-01	1068	3.5e-06	4.7e-06	1.5e-04	3.8e+00	-1.5e+01	3.4e+00	2.6e+02
23	23	1.9e-01	1160	2.6e-06	6.7e-06	1.3e-04	6.2e+00	-7.5e+00	2.9e+00	2.6e+02
25	25	1.7e-01	1352	2.2e-06	4.8e-06	9.0e-05	2.2e+00	3.8e+00	4.5e+00	2.7e+02

1

500 22 b4p2unix/56/14/46/ (a=10,b=2.,c=0.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	1.0e+00	76	1.7e-03	9.5e-03	6.6e-02	0.0e+00	0.0e+00	0.0e+00	7.0e+01
7	7	6.9e-01	132	1.7e-04	1.1e-03	1.0e-02	5.7e+00	5.3e+00	4.6e+00	1.2e+02
9	9	5.2e-01	200	8.9e-05	2.4e-04	4.9e-03	2.2e+00	5.3e+00	2.6e+00	1.6e+02
11	11	4.2e-01	284	2.8e-05	1.3e-04	2.3e-03	5.2e+00	2.9e+00	3.4e+00	1.9e+02
13	13	3.5e-01	384	1.3e-05	4.7e-05	9.3e-04	4.1e+00	5.4e+00	4.9e+00	2.1e+02
15	15	3.0e-01	492	6.5e-06	3.7e-05	4.9e-04	4.6e+00	1.6e+00	4.2e+00	2.2e+02
17	17	2.6e-01	612	5.0e-06	1.6e-05	3.5e-04	2.0e+00	6.4e+00	2.4e+00	2.4e+02
19	19	2.3e-01	752	2.2e-06	3.7e-06	1.9e-04	6.7e+00	1.2e+01	5.3e+00	2.5e+02
21	21	2.1e-01	900	1.6e-06	2.0e-06	1.2e-04	3.5e+00	5.9e+00	4.3e+00	2.6e+02
22	22	2.0e-01	976	1.4e-06	4.1e-06	1.1e-04	2.0e+00	-1.5e+01	1.2e+00	2.6e+02
23	23	1.9e-01	1060	1.5e-06	3.9e-06	9.6e-05	-1.5e+00	9.5e-01	3.4e+00	2.6e+02
25	25	1.7e-01	1236	1.0e-06	3.6e-06	7.2e-05	4.4e+00	7.8e-01	3.3e+00	2.7e+02

500 25 b4p2unix/56/14/46/ (a=11,b=1.,c=0.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	1.2e+00	96	7.9e-03	3.5e-02	9.0e+02	0.0e+00	0.0e+00	0.0e+00	3.3e+05

7	7	7.8e-01	184	7.5e-04	3.2e-03	3.1e+02	5.8e+00	5.9e+00	2.6e+00	1.2e+06
9	9	5.8e-01	288	4.8e-04	1.4e-03	1.0e+02	1.5e+00	2.9e+00	3.9e+00	9.3e+05
11	11	4.7e-01	424	1.4e-04	4.2e-04	6.1e+01	5.6e+00	5.3e+00	2.4e+00	1.6e+06
13	13	3.9e-01	576	8.8e-05	3.3e-04	2.6e+01	2.5e+00	1.4e+00	4.5e+00	1.3e+06
15	15	3.3e-01	756	8.1e-05	1.1e-04	2.3e+01	5.3e-01	6.9e+00	8.1e-01	1.7e+06
17	17	2.9e-01	964	2.8e-05	1.1e-04	9.5e+00	8.0e+00	2.1e-01	6.8e+00	1.5e+06
19	19	2.6e-01	1192	2.2e-05	1.3e-05	7.5e+00	2.1e+00	1.9e+01	1.9e+00	1.8e+06
21	21	2.3e-01	1444	4.0e-05	7.0e-06	5.9e+00	-5.7e+00	5.5e+00	2.2e+00	1.6e+06
22	22	2.2e-01	1592	1.1e-05	2.2e-05	3.8e+00	2.6e+01	-2.3e+01	9.3e+00	1.8e+06
25	25	1.9e-01	2024	8.2e-06	1.1e-05	2.4e+00	2.4e+00	4.8e+00	3.5e+00	1.7e+06
27	27	1.8e-01	2348	1.3e-05	8.8e-06	2.2e+00	-5.5e+00	3.4e+00	1.2e+00	1.8e+06

500 27 b4p2unix/56/14/46/ (a=11,b=.5,c=0.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	9.3e-01	96	1.9e-03	1.3e-02	2.0e+00	0.0e+00	0.0e+00	0.0e+00	2.2e+03
7	7	6.2e-01	168	9.9e-04	2.4e-02	6.7e-01	1.6e+00	-1.5e+00	2.7e+00	2.7e+03
9	9	4.6e-01	272	2.5e-04	6.3e-04	2.4e-01	4.8e+00	1.3e+01	3.6e+00	4.2e+03
11	11	3.7e-01	396	1.4e-04	2.8e-04	1.2e-01	2.5e+00	3.7e+00	3.2e+00	4.3e+03
13	13	3.1e-01	544	6.1e-05	1.2e-04	5.6e-02	4.6e+00	4.4e+00	4.0e+00	5.1e+03
15	15	2.6e-01	700	3.5e-05	6.0e-05	3.3e-02	3.6e+00	4.7e+00	3.5e+00	5.1e+03
17	17	2.3e-01	888	2.2e-05	4.2e-05	2.0e-02	3.5e+00	2.7e+00	3.8e+00	5.6e+03
19	19	2.1e-01	1092	1.3e-05	1.3e-05	1.2e-02	4.5e+00	1.0e+01	4.0e+00	5.6e+03
21	21	1.9e-01	1308	9.5e-06	6.3e-06	9.3e-03	2.7e+00	6.7e+00	2.7e+00	5.9e+03

500 28 b4p2unix/56/14/46/ (a=11,b=1.5,c=0.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	2.3e+00	92	6.3e-02	6.6e-02	1.6e+07	0.0e+00	0.0e+00	0.0e+00	7.3e+08
7	7	1.5e+00	172	1.5e-02	1.9e-02	3.2e+06	3.5e+00	3.1e+00	3.9e+00	9.0e+08
9	9	1.2e+00	272	3.7e-03	6.6e-03	8.9e+05	4.9e+00	3.6e+00	4.4e+00	1.0e+09
11	11	9.3e-01	400	5.0e-03	4.6e-02	7.7e+05	-1.3e+00	-8.8e+00	6.5e-01	1.1e+09
13	13	7.7e-01	536	3.7e-04	7.3e-04	1.7e+05	1.4e+01	2.3e+01	8.2e+00	2.5e+09
15	15	6.6e-01	692	2.5e-04	4.8e-04	1.1e+05	2.4e+00	2.6e+00	2.9e+00	2.3e+09
17	17	5.8e-01	884	2.4e-04	2.6e-04	7.4e+04	3.9e-01	4.8e+00	3.0e+00	2.1e+09
19	19	5.1e-01	1084	1.6e-04	6.4e-05	4.6e+04	3.6e+00	1.2e+01	4.1e+00	2.0e+09
21	21	4.6e-01	1312	1.0e-04	6.2e-05	2.6e+04	4.0e+00	2.3e-01	5.5e+00	2.0e+09

500 37 b4p2unix/56/14/46/ (a=15,b=.5,c=1.5)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	3.9e-01	92	2.2e-03	4.9e-03	6.6e-04	0.0e+00	0.0e+00	0.0e+00	6.0e-01
7	7	2.6e-01	164	2.4e-04	5.2e-04	1.7e-04	5.4e+00	5.6e+00	3.3e+00	1.5e+00
9	9	1.9e-01	268	4.5e-05	1.2e-04	5.4e-05	5.8e+00	5.2e+00	4.1e+00	2.2e+00
11	11	1.6e-01	384	2.3e-05	4.3e-05	2.4e-05	3.0e+00	4.5e+00	3.6e+00	2.7e+00
13	13	1.3e-01	524	2.5e-05	2.7e-05	1.5e-05	-4.3e-01	2.5e+00	2.5e+00	2.8e+00
15	15	1.1e-01	704	6.1e-06	1.3e-05	6.4e-06	9.2e+00	4.7e+00	5.6e+00	2.7e+00
17	17	9.7e-02	876	7.0e-06	1.0e-05	4.0e-06	-1.0e+00	2.1e+00	3.5e+00	2.6e+00
19	19	8.7e-02	1080	2.6e-06	3.0e-06	2.4e-06	8.2e+00	1.0e+01	4.2e+00	2.9e+00
21	21	7.8e-02	1320	1.8e-06	1.6e-06	1.7e-06	3.5e+00	6.1e+00	3.6e+00	2.8e+00
22	22	7.4e-02	1428	2.1e-06	3.5e-06	1.4e-06	-2.2e+00	-1.6e+01	2.9e+00	2.7e+00
23	23	7.1e-02	1560	2.8e-06	3.5e-06	1.3e-06	-7.0e+00	9.4e-03	2.9e+00	2.8e+00
25	25	6.5e-02	1840	1.5e-06	2.0e-06	8.1e-07	7.2e+00	6.6e+00	5.1e+00	2.8e+00
27	27	6.0e-02	2120	1.2e-06	1.1e-06	6.4e-07	3.1e+00	6.9e+00	2.9e+00	3.0e+00
29	29	5.6e-02	2432	9.2e-07	8.4e-07	4.6e-07	3.5e+00	3.9e+00	4.5e+00	3.0e+00

500 43 b4p2unix/56/14/46/ (a=18,b=0.,c=0.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	1.3e+00	100	1.3e-02	7.3e-02	9.7e-02	0.0e+00	0.0e+00	0.0e+00	1.7e+01

7	7	8.7e-01	172	3.2e-03	4.8e-02	4.4e-02	3.5e+00	1.0e+00	1.9e+00	3.6e+01
8	8	7.4e-01	216	8.4e-04	3.6e-03	2.1e-02	8.7e+00	1.7e+01	5.0e+00	5.8e+01
9	9	6.5e-01	268	9.6e-04	5.4e-03	2.1e-02	-9.6e-01	-3.0e+00	0.0e+00	7.9e+01
10	10	5.8e-01	340	3.9e-04	1.2e-03	1.0e-02	7.6e+00	1.3e+01	6.2e+00	6.8e+01
11	11	5.2e-01	392	6.0e-04	1.2e-03	1.0e-02	-4.0e+00	4.8e-01	-4.0e-01	8.8e+01
12	12	4.7e-01	452	5.4e-04	7.4e-04	6.8e-03	1.2e+00	4.7e+00	4.5e+00	7.1e+01
13	13	4.3e-01	536	2.2e-03	6.0e-03	2.3e-02	-1.6e+01	-2.4e+01	-1.4e+01	9.0e+01
14	14	4.0e-01	616	7.8e-05	2.5e-04	2.5e-03	4.1e+01	4.0e+01	2.8e+01	1.1e+02
15	15	3.7e-01	696	5.8e-04	8.5e-04	5.9e-03	-2.7e+01	-1.7e+01	-1.2e+01	9.0e+01
16	16	3.5e-01	796	6.5e-05	1.4e-04	1.9e-03	3.2e+01	2.7e+01	1.6e+01	1.1e+02
20	20	2.7e-01	1188	8.7e-05	8.7e-05	1.1e-03	-1.2e+00	1.9e+00	2.3e+00	1.0e+02
22	22	2.5e-01	1428	3.9e-05	2.3e-05	5.6e-04	8.1e+00	1.3e+01	6.9e+00	9.7e+01
24	24	2.3e-01	1668	2.9e-05	3.0e-05	4.0e-04	3.0e+00	-3.0e+00	3.7e+00	9.9e+01

1

500 53 b4p2unix/56/14/46/ (a=21,b=.2,c=.5)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
4	4	6.9e-01	60	2.7e-01	1.2e+00	9.7e-02	0.0e+00	0.0e+00	0.0e+00	6.2e-01
5	5	5.2e-01	84	1.5e-03	1.7e-03	2.2e-03	1.8e+01	2.3e+01	1.3e+01	2.6e+00
6	6	4.2e-01	116	1.0e-03	1.3e-02	6.8e-04	1.8e+00	-8.9e+00	5.3e+00	1.4e+00
7	7	3.5e-01	148	8.7e-01	1.8e+00	7.0e-01	-3.7e+01	-2.7e+01	-3.8e+01	2.6e+00
8	8	3.0e-01	184	2.0e-04	4.3e-04	1.9e-04	5.4e+01	5.4e+01	5.3e+01	1.9e+00
9	9	2.6e-01	228	8.8e-01	1.9e+00	6.8e-01	-6.3e+01	-6.3e+01	-6.1e+01	3.2e+00
11	11	2.1e-01	320	6.4e-02	1.3e-01	2.4e-02	1.2e+01	1.2e+01	1.5e+01	1.8e+00
13	13	1.7e-01	436	6.5e-03	8.4e-03	3.3e-03	1.3e+01	1.5e+01	1.1e+01	2.7e+00
15	15	1.5e-01	564	1.3e-03	2.0e-03	7.3e-04	1.0e+01	9.4e+00	9.8e+00	2.7e+00
17	17	1.3e-01	692	4.4e-04	1.2e-03	2.0e-04	8.4e+00	3.8e+00	9.9e+00	2.6e+00
18	18	1.2e-01	780	7.2e-06	1.4e-05	6.5e-06	6.8e+01	7.3e+01	5.6e+01	2.4e+00
19	19	1.2e-01	864	2.3e-03	1.2e-03	4.6e-04	-1.0e+02	-7.7e+01	-7.5e+01	2.4e+00
21	21	1.0e-01	1028	2.3e-04	1.3e-04	4.6e-05	2.2e+01	2.1e+01	2.2e+01	2.6e+00
22	22	9.9e-02	1124	3.0e-06	3.7e-06	2.9e-06	8.9e+01	7.3e+01	5.7e+01	2.6e+00
23	23	9.5e-02	1204	7.5e-05	4.8e-05	1.3e-05	-6.9e+01	-5.5e+01	-3.2e+01	2.5e+00
25	25	8.7e-02	1408	1.7e-05	1.1e-05	3.3e-06	1.7e+01	1.7e+01	1.5e+01	2.7e+00
27	27	8.0e-02	1632	4.7e-06	2.6e-06	1.5e-06	1.6e+01	1.8e+01	1.0e+01	2.7e+00
29	29	7.4e-02	1860	1.3e-06	1.5e-06	9.8e-07	1.7e+01	7.9e+00	5.5e+00	2.7e+00

500 55 b4p2unix/56/14/46/ (a=21,b=20.,c=.6)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	5.4e-01	84	6.4e-03	5.7e-02	5.7e-03	0.0e+00	0.0e+00	0.0e+00	2.1e+00
7	7	3.6e-01	152	1.8e-03	6.0e-03	7.9e-04	3.1e+00	5.6e+00	4.8e+00	1.5e+00
9	9	2.7e-01	244	3.2e-03	1.7e-02	1.4e-03	-2.0e+00	-3.6e+00	-2.0e+00	2.1e+00
11	11	2.2e-01	344	1.6e-04	1.7e-04	9.2e-05	1.3e+01	2.0e+01	1.2e+01	2.9e+00
13	13	1.8e-01	460	1.1e-04	7.6e-05	4.1e-05	2.3e+00	4.5e+00	4.4e+00	2.1e+00
15	15	1.6e-01	576	4.6e-05	1.6e-04	2.1e-05	5.5e+00	-4.6e+00	4.4e+00	2.6e+00
19	19	1.2e-01	880	1.0e-05	1.1e-05	7.2e-06	6.0e+00	1.0e+01	4.3e+00	3.1e+00
21	21	1.1e-01	1060	6.7e-06	7.1e-06	4.7e-06	3.9e+00	4.3e+00	4.0e+00	3.1e+00
22	22	1.0e-01	1164	3.3e-06	9.4e-06	3.5e-06	1.5e+01	-5.7e+00	5.9e+00	2.6e+00
23	23	9.9e-02	1260	4.9e-06	4.0e-06	3.2e-06	-8.5e+00	1.8e+01	1.9e+00	2.9e+00
25	25	9.0e-02	1472	3.4e-06	2.8e-06	2.3e-06	4.0e+00	4.2e+00	4.0e+00	2.9e+00
27	27	8.3e-02	1712	2.5e-06	2.9e-06	1.6e-06	4.0e+00	-7.6e-01	4.4e+00	2.8e+00
29	29	7.8e-02	1940	1.6e-06	2.0e-06	1.2e-06	5.6e+00	5.4e+00	3.9e+00	3.0e+00

500 57 b4p2unix/56/14/46/ (a=22,b=1.,c=.1)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	6.8e-01	80	4.0e-03	2.0e-02	1.3e-02	0.0e+00	0.0e+00	0.0e+00	6.6e+00
7	7	4.5e-01	136	1.5e-03	5.7e-03	1.4e-02	2.4e+00	3.1e+00	-2.4e-01	3.2e+01
9	9	3.4e-01	204	1.1e-04	1.4e-04	1.7e-03	9.1e+00	1.3e+01	7.4e+00	6.1e+01
13	13	2.2e-01	372	1.5e-04	3.3e-04	2.5e-03	-7.0e-01	-2.1e+00	-9.6e-01	1.1e+02
15	15	1.9e-01	476	1.0e-05	3.0e-05	2.6e-04	1.7e+01	1.5e+01	1.5e+01	1.3e+02

17	17	1.7e-01	600	6.2e-06	1.3e-05	1.6e-04	3.7e+00	6.2e+00	3.8e+00	1.4e+02
19	19	1.5e-01	724	4.0e-06	3.1e-06	9.7e-05	3.7e+00	1.2e+01	4.0e+00	1.6e+02
21	21	1.4e-01	852	2.7e-06	1.3e-06	6.5e-05	3.8e+00	8.2e+00	3.8e+00	1.7e+02

500 59 b4p2unix/56/14/46/ (a=22,b=.8,c=.8)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
9	9	3.4e-01	168	3.8e-04	3.2e-03	2.4e-02	0.0e+00	0.0e+00	0.0e+00	2.3e+02
11	11	2.7e-01	236	1.1e-04	1.8e-04	1.0e-02	5.6e+00	1.3e+01	3.9e+00	3.8e+02
13	13	2.3e-01	304	4.4e-05	1.1e-04	5.2e-03	4.9e+00	2.6e+00	3.6e+00	5.4e+02
15	15	2.0e-01	368	2.2e-05	2.9e-04	3.6e-03	4.6e+00	-6.0e+00	2.5e+00	6.8e+02
17	17	1.7e-01	440	1.2e-05	1.3e-05	2.0e-03	4.6e+00	2.3e+01	4.4e+00	8.0e+02
19	19	1.5e-01	540	6.9e-06	6.7e-06	1.2e-03	4.5e+00	5.6e+00	4.1e+00	9.1e+02
21	21	1.4e-01	640	4.4e-06	2.5e-06	8.0e-04	4.4e+00	9.3e+00	4.0e+00	1.0e+03

500 61 b4p2unix/56/14/46/ (a=23,b=0.,c=.5)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
4	4	1.0e+00	56	1.0e+00	1.9e+01	7.2e+00	0.0e+00	0.0e+00	0.0e+00	1.1e+01
6	6	6.2e-01	120	2.8e-03	3.0e-02	1.6e-01	1.2e+01	1.3e+01	7.4e+00	1.3e+02
7	7	5.2e-01	164	1.3e-03	6.5e-03	1.4e-01	4.1e+00	8.5e+00	8.3e-01	2.3e+02
8	8	4.5e-01	200	1.8e-03	1.9e-02	8.8e-02	-1.9e+00	-6.9e+00	3.0e+00	1.3e+02
9	9	3.9e-01	240	6.9e-02	2.2e-01	3.3e+00	-2.7e+01	-1.8e+01	-2.7e+01	2.1e+02
10	10	3.5e-01	300	1.5e-04	1.1e-03	1.4e-02	5.2e+01	4.5e+01	4.6e+01	2.8e+02
11	11	3.1e-01	344	1.3e-02	2.6e-02	1.1e+00	-4.2e+01	-3.0e+01	-4.1e+01	3.6e+02
12	12	2.8e-01	388	1.2e-04	1.4e-04	8.6e-03	4.9e+01	5.4e+01	5.1e+01	2.4e+02
13	13	2.6e-01	456	7.8e-03	3.2e-02	5.1e-01	-4.8e+01	-6.2e+01	-4.7e+01	3.0e+02
14	14	2.4e-01	520	4.6e-05	8.1e-05	4.8e-03	6.4e+01	7.5e+01	5.8e+01	3.6e+02
15	15	2.2e-01	588	4.5e-03	1.5e-02	3.1e-01	-6.2e+01	-7.0e+01	-5.6e+01	4.2e+02
16	16	2.1e-01	644	2.8e-05	3.8e-05	2.9e-03	7.4e+01	8.7e+01	6.8e+01	4.8e+02
17	17	1.9e-01	728	9.1e-03	1.4e-02	4.4e-01	-9.0e+01	-9.2e+01	-7.8e+01	3.6e+02
18	18	1.8e-01	808	2.0e-05	1.9e-05	1.8e-03	1.0e+02	1.1e+02	9.1e+01	4.1e+02
19	19	1.7e-01	884	2.5e-03	1.4e-03	1.3e-01	-8.5e+01	-7.5e+01	-7.5e+01	4.6e+02
20	20	1.6e-01	960	1.1e-05	1.1e-05	1.2e-03	1.0e+02	9.0e+01	8.7e+01	5.1e+02
21	21	1.6e-01	1048	3.5e-04	1.6e-04	2.0e-02	-6.7e+01	-5.3e+01	-5.5e+01	5.6e+02
22	22	1.5e-01	1128	8.5e-06	9.0e-06	8.2e-04	7.6e+01	6.0e+01	6.5e+01	4.4e+02
23	23	1.4e-01	1232	5.6e-06	7.9e-06	6.8e-04	8.9e+00	2.6e+00	4.0e+00	4.8e+02
24	24	1.4e-01	1316	4.8e-06	5.1e-06	5.7e-04	3.6e+00	1.0e+01	4.0e+00	5.2e+02

500 69 b4p2unix/56/14/46/ (a=25,b=.2,c=0.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	5.5e-01	96	3.5e-03	3.9e-02	1.2e-02	0.0e+00	0.0e+00	0.0e+00	8.5e+00
7	7	3.6e-01	164	5.1e-04	2.5e-03	1.8e-03	4.7e+00	6.8e+00	4.7e+00	1.1e+01
9	9	2.7e-01	260	4.2e-04	1.5e-03	1.6e-03	6.8e-01	1.8e+00	3.7e-01	1.9e+01
11	11	2.2e-01	376	1.0e-04	1.5e-03	3.1e-04	6.3e+00	-2.1e-01	7.5e+00	1.1e+01
15	15	1.6e-01	644	2.3e-05	8.2e-05	9.4e-05	4.4e+00	8.7e+00	3.5e+00	1.8e+01
17	17	1.4e-01	808	1.2e-05	1.7e-05	6.2e-05	5.0e+00	1.2e+01	3.1e+00	2.8e+01
19	19	1.2e-01	1000	8.3e-06	2.0e-05	3.4e-05	3.2e+00	-1.7e+00	5.0e+00	2.2e+01
21	21	1.1e-01	1208	5.1e-06	8.2e-06	2.4e-05	4.6e+00	8.6e+00	3.4e+00	2.9e+01

1

500 70 b4p2unix/56/14/46/ (a=25,b=.2,c=1.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	8.1e-01	88	1.7e-01	3.3e+00	1.1e-01	0.0e+00	0.0e+00	0.0e+00	1.5e+00
7	7	5.4e-01	160	1.4e-03	8.9e-03	1.0e-02	1.2e+01	1.5e+01	5.7e+00	1.8e+01
9	9	4.0e-01	248	1.3e-03	3.7e-03	2.5e-03	2.0e-01	3.1e+00	5.0e+00	6.6e+00
11	11	3.2e-01	356	6.8e-04	1.2e-03	1.6e-03	3.0e+00	5.0e+00	1.9e+00	1.2e+01
15	15	2.3e-01	608	1.1e-04	2.3e-04	3.9e-04	5.4e+00	5.0e+00	4.3e+00	2.0e+01
17	17	2.0e-01	768	5.7e-05	1.8e-04	2.3e-04	4.8e+00	1.8e+00	4.0e+00	2.1e+01

19	19	1.8e-01	948	7.6e-05	2.9e-05	2.2e-04	-2.4e+00	1.5e+01	3.8e-01	2.5e+01
21	21	1.6e-01	1140	3.5e-05	2.3e-05	1.0e-04	7.4e+00	2.4e+00	7.1e+00	2.0e+01
23	23	1.5e-01	1348	1.2e-05	4.7e-05	6.0e-05	1.1e+01	-7.7e+00	5.7e+00	2.6e+01
27	27	1.2e-01	1808	1.5e-05	2.3e-05	4.8e-05	-1.2e+00	4.2e+00	1.3e+00	2.5e+01
29	29	1.2e-01	2076	1.2e-05	1.1e-05	3.5e-05	2.9e+00	9.9e+00	4.2e+00	3.0e+01

500 71 b4p2unix/56/14/46/ (a=25,b=2.5,c=-1.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	5.2e-01	88	1.0e-03	6.4e-03	7.8e-03	0.0e+00	0.0e+00	0.0e+00	1.5e+01
7	7	3.5e-01	148	3.8e-04	4.2e-03	2.4e-03	2.5e+00	1.1e+00	2.9e+00	1.4e+01
11	11	2.1e-01	332	8.5e-05	6.0e-04	5.1e-04	2.9e+00	3.8e+00	3.1e+00	2.3e+01
15	15	1.5e-01	548	1.7e-05	8.5e-05	1.1e-04	4.8e+00	5.8e+00	4.5e+00	3.2e+01
17	17	1.3e-01	688	6.0e-06	1.1e-05	5.7e-05	7.7e+00	1.5e+01	5.0e+00	3.4e+01
19	19	1.2e-01	836	8.4e-06	7.3e-06	4.2e-05	-2.8e+00	3.5e+00	2.5e+00	2.8e+01
21	21	1.0e-01	1020	3.4e-06	5.2e-06	2.4e-05	8.7e+00	3.3e+00	5.6e+00	3.1e+01
25	25	8.7e-02	1372	2.0e-06	3.6e-06	1.4e-05	2.7e+00	2.0e+00	3.0e+00	3.6e+01
29	29	7.4e-02	1816	8.2e-07	2.0e-06	6.2e-06	5.9e+00	3.7e+00	5.1e+00	3.2e+01

*** number of ellpack runs = 237

*** total CPU hours = 1.41

APPENDIX B. The Experimental Data

3. Errors and convergence rates as estimated by CONV2

1

500 1 b4p2unix/56/14/46/ (a=1,b=.8,c=.6)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	2.0e-01	100	8.9e-05	2.2e-04	1.9e-05	0.0e+00	0.0e+00	0.0e+00	5.7e-01
7	7	1.3e-01	196	2.0e-05	4.2e-05	4.0e-06	3.7e+00	4.1e+00	3.8e+00	5.7e-01
9	9	1.0e-01	324	6.6e-06	1.5e-05	1.3e-06	3.8e+00	3.9e+00	3.9e+00	5.7e-01
11	11	8.0e-02	484	2.8e-06	6.9e-06	5.4e-07	3.8e+00	3.8e+00	3.9e+00	5.7e-01
13	13	6.7e-02	676	1.4e-06	2.9e-06	2.6e-07	3.8e+00	3.9e+00	3.9e+00	5.7e-01
15	15	5.7e-02	900	7.5e-07	1.6e-06	1.4e-07	3.8e+00	3.9e+00	3.9e+00	5.7e-01
17	17	5.0e-02	1156	4.5e-07	8.6e-07	8.3e-08	3.8e+00	4.0e+00	3.9e+00	5.7e-01
19	19	4.4e-02	1444	2.8e-07	2.9e-07	5.2e-08	3.8e+00	4.4e+00	3.9e+00	5.7e-01
21	21	4.0e-02	1764	1.9e-07	1.9e-07	3.4e-08	3.8e+00	4.4e+00	3.9e+00	5.7e-01

500 4 b4p2unix/56/14/46/ (a=2,b=1.,c=1.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	2.5e-01	76	2.8e-04	1.0e-03	4.7e-05	0.0e+00	0.0e+00	0.0e+00	5.1e-01
7	7	1.7e-01	136	5.6e-05	1.7e-04	1.2e-05	4.0e+00	4.4e+00	3.4e+00	5.6e-01
9	9	1.3e-01	212	1.9e-05	5.7e-05	4.2e-06	3.9e+00	4.2e+00	3.5e+00	5.7e-01
11	11	1.0e-01	304	8.1e-06	2.4e-05	1.8e-06	3.9e+00	4.1e+00	3.5e+00	5.7e-01
13	13	8.3e-02	412	4.1e-06	1.1e-05	9.2e-07	3.9e+00	4.1e+00	3.6e+00	5.7e-01
15	15	7.1e-02	536	2.2e-06	6.2e-06	5.1e-07	3.9e+00	4.1e+00	3.6e+00	5.7e-01
17	17	6.3e-02	676	1.3e-06	3.4e-06	3.1e-07	3.9e+00	4.1e+00	3.6e+00	5.7e-01
19	19	5.6e-02	832	8.3e-07	1.7e-06	1.9e-07	3.9e+00	4.3e+00	3.6e+00	5.7e-01
21	21	5.0e-02	1004	5.5e-07	1.1e-06	1.3e-07	3.9e+00	4.3e+00	3.7e+00	5.7e-01

500 5 b4p2unix/56/14/46/ (a=3,b=2.,c=1.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	1.0e+00	100	3.5e-02	6.6e-02	1.7e-02	0.0e+00	0.0e+00	0.0e+00	1.0e+00
7	7	6.7e-01	180	6.4e-03	7.6e-03	5.4e-03	4.2e+00	5.3e+00	2.8e+00	2.9e+00
9	9	5.0e-01	308	8.0e-04	1.3e-03	1.5e-03	5.4e+00	5.6e+00	3.5e+00	4.6e+00
11	11	4.0e-01	436	3.2e-04	5.9e-04	6.8e-04	5.1e+00	5.2e+00	3.5e+00	7.5e+00
13	13	3.3e-01	596	1.8e-04	1.9e-04	3.9e-04	4.8e+00	5.3e+00	3.4e+00	8.0e+00
15	15	2.9e-01	804	7.0e-05	1.8e-04	1.9e-04	4.9e+00	4.7e+00	3.6e+00	8.1e+00
17	17	2.5e-01	996	6.9e-05	9.1e-05	1.3e-04	4.5e+00	4.8e+00	3.5e+00	8.0e+00
19	19	2.2e-01	1252	2.4e-05	1.5e-05	7.4e-05	4.8e+00	5.6e+00	3.6e+00	1.0e+01
21	21	2.0e-01	1492	2.0e-05	9.0e-06	5.2e-05	4.6e+00	5.5e+00	3.6e+00	9.8e+00
22	22	1.9e-01	1664	1.5e-05	2.2e-05	4.0e-05	4.7e+00	4.8e+00	3.6e+00	8.6e+00
23	23	1.8e-01	1812	1.4e-05	2.2e-05	3.5e-05	4.6e+00	4.7e+00	3.6e+00	9.5e+00
25	25	1.7e-01	2132	9.0e-06	2.2e-05	2.4e-05	4.6e+00	4.5e+00	3.7e+00	9.1e+00
27	27	1.5e-01	2452	7.4e-06	1.1e-05	1.8e-05	4.5e+00	4.6e+00	3.7e+00	1.1e+01
29	29	1.4e-01	2820	4.7e-06	1.1e-05	1.4e-05	4.6e+00	4.5e+00	3.7e+00	1.0e+01

500 15 b4p2unix/56/14/46/ (a=7,b=.4,c=.2)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	2.1e-01	100	4.5e-04	6.4e-04	7.6e-05	0.0e+00	0.0e+00	0.0e+00	3.8e-01
7	7	1.4e-01	188	3.4e-05	3.1e-05	5.9e-06	6.4e+00	7.5e+00	6.3e+00	4.1e-01
9	9	1.0e-01	312	1.1e-05	9.1e-06	1.7e-06	5.3e+00	6.1e+00	5.5e+00	4.3e-01
11	11	8.3e-02	444	5.1e-06	4.9e-06	6.7e-07	4.9e+00	5.3e+00	5.2e+00	4.4e-01
13	13	6.9e-02	624	2.7e-06	2.1e-06	3.1e-07	4.7e+00	5.2e+00	5.0e+00	4.5e-01

15	15	6.0e-02	804	7.8e-07	8.2e-04	1.5e-07	5.1e+00	-2.0e-01	5.0e+00	4.5e-01
17	17	5.2e-02	1032	4.7e-07	7.4e-07	8.8e-08	5.0e+00	4.9e+00	4.9e+00	4.5e-01
19	19	4.6e-02	1264	3.0e-07	2.6e-07	5.5e-08	4.9e+00	5.2e+00	4.8e+00	4.6e-01
21	21	4.2e-02	1544	2.0e-07	1.7e-07	3.6e-08	4.8e+00	5.1e+00	4.8e+00	4.6e-01
22	22	4.0e-02	1680	1.6e-07	2.0e-07	3.0e-08	4.8e+00	4.9e+00	4.7e+00	4.6e-01
23	23	3.8e-02	1836	1.4e-07	2.0e-07	2.5e-08	4.7e+00	4.7e+00	4.7e+00	4.6e-01

500 17 b4p2unix/56/14/46/ (a=8,b=1.,c=1.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	5.2e-01	100	1.3e-03	5.7e-03	2.1e-02	0.0e+00	0.0e+00	0.0e+00	3.9e+01
7	7	3.5e-01	180	2.2e-04	4.2e-04	9.0e-03	4.3e+00	6.4e+00	2.1e+00	1.2e+02
9	9	2.6e-01	308	9.7e-05	2.0e-04	2.8e-03	3.7e+00	4.8e+00	2.9e+00	8.9e+01
11	11	2.1e-01	436	4.4e-05	1.3e-04	1.2e-03	3.7e+00	4.1e+00	3.2e+00	7.9e+01
13	13	1.7e-01	580	1.9e-05	4.5e-05	6.4e-04	3.8e+00	4.4e+00	3.2e+00	1.2e+02
15	15	1.5e-01	772	1.3e-05	2.4e-05	3.7e-04	3.7e+00	4.4e+00	3.2e+00	1.0e+02
17	17	1.3e-01	964	9.5e-06	2.3e-05	2.2e-04	3.5e+00	4.0e+00	3.3e+00	9.3e+01
19	19	1.2e-01	1172	7.4e-06	4.2e-06	1.7e-04	3.4e+00	4.8e+00	3.2e+00	1.2e+02
21	21	1.0e-01	1428	7.3e-06	2.9e-06	1.3e-04	3.2e+00	4.7e+00	3.2e+00	1.1e+02
22	22	9.9e-02	1568	2.7e-06	4.7e-06	8.2e-05	3.7e+00	4.3e+00	3.3e+00	1.2e+02
23	23	9.4e-02	1716	6.5e-06	5.8e-06	7.8e-05	3.1e+00	4.0e+00	3.3e+00	1.0e+02
25	25	8.7e-02	2000	4.2e-06	2.8e-06	5.9e-05	3.2e+00	4.2e+00	3.3e+00	1.2e+02
27	27	8.0e-02	2344	3.5e-06	2.2e-06	4.1e-05	3.1e+00	4.2e+00	3.3e+00	1.1e+02
29	29	7.4e-02	2660	2.2e-06	1.1e-06	3.2e-05	3.3e+00	4.4e+00	3.3e+00	1.3e+02

500 21 b4p2unix/56/14/46/ (a=10,b=1.,c=0.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	1.0e+00	76	3.1e-03	1.0e-02	9.6e-02	0.0e+00	0.0e+00	0.0e+00	7.0e+01
7	7	6.9e-01	136	3.0e-04	1.6e-03	1.2e-02	5.8e+00	4.6e+00	5.1e+00	1.2e+02
9	9	5.2e-01	212	9.2e-05	3.3e-04	5.1e-03	5.1e+00	5.0e+00	4.2e+00	1.6e+02
11	11	4.2e-01	300	6.1e-05	2.0e-04	2.6e-03	4.3e+00	4.3e+00	4.0e+00	1.9e+02
13	13	3.5e-01	408	1.9e-05	6.9e-05	1.2e-03	4.7e+00	4.6e+00	4.0e+00	2.1e+02
15	15	3.0e-01	524	1.6e-05	4.2e-05	7.2e-04	4.2e+00	4.4e+00	3.9e+00	2.2e+02
17	17	2.6e-01	664	8.0e-06	2.9e-05	4.0e-04	4.3e+00	4.2e+00	4.0e+00	2.4e+02
19	19	2.3e-01	812	5.6e-06	3.3e-04	2.8e-04	4.2e+00	2.3e+00	3.9e+00	2.5e+02
21	21	2.1e-01	976	4.2e-06	2.3e-06	1.8e-04	4.1e+00	5.2e+00	3.9e+00	2.6e+02
22	22	2.0e-01	1068	3.5e-06	4.7e-06	1.5e-04	4.1e+00	4.6e+00	3.9e+00	2.6e+02
23	23	1.9e-01	1160	2.6e-06	6.7e-06	1.3e-04	4.2e+00	4.3e+00	3.9e+00	2.6e+02
25	25	1.7e-01	1352	2.2e-06	4.8e-06	9.0e-05	4.1e+00	4.3e+00	3.9e+00	2.7e+02

1

500 22 b4p2unix/56/14/46/ (a=10,b=2.,c=0.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	1.0e+00	76	1.7e-03	9.5e-03	6.6e-02	0.0e+00	0.0e+00	0.0e+00	7.0e+01
7	7	6.9e-01	132	1.7e-04	1.1e-03	1.0e-02	5.7e+00	5.3e+00	4.6e+00	1.2e+02
9	9	5.2e-01	200	8.9e-05	2.4e-04	4.9e-03	4.3e+00	5.3e+00	3.8e+00	1.6e+02
11	11	4.2e-01	284	2.8e-05	1.3e-04	2.3e-03	4.5e+00	4.7e+00	3.7e+00	1.9e+02
13	13	3.5e-01	384	1.3e-05	4.7e-05	9.3e-04	4.4e+00	4.8e+00	3.9e+00	2.1e+02
15	15	3.0e-01	492	6.5e-06	3.7e-05	4.9e-04	4.4e+00	4.4e+00	3.9e+00	2.2e+02
17	17	2.6e-01	612	5.0e-06	1.6e-05	3.5e-04	4.2e+00	4.6e+00	3.8e+00	2.4e+02
19	19	2.3e-01	752	2.2e-06	3.7e-06	1.9e-04	4.4e+00	5.2e+00	3.9e+00	2.5e+02
21	21	2.1e-01	900	1.6e-06	2.0e-06	1.2e-04	4.3e+00	5.3e+00	3.9e+00	2.6e+02
22	22	2.0e-01	976	1.4e-06	4.1e-06	1.1e-04	4.3e+00	4.7e+00	3.8e+00	2.6e+02
23	23	1.9e-01	1060	1.5e-06	3.9e-06	9.6e-05	4.1e+00	4.6e+00	3.8e+00	2.6e+02
25	25	1.7e-01	1236	1.0e-06	3.6e-06	7.2e-05	4.1e+00	4.4e+00	3.8e+00	2.7e+02

500 25 b4p2unix/56/14/46/ (a=11,b=1.,c=0.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax

5	5	1.2e+00	96	7.9e-03	3.5e-02	9.0e+02	0.0e+00	0.0e+00	0.0e+00	3.3e+05
7	7	7.8e-01	184	7.5e-04	3.2e-03	3.1e+02	5.8e+00	5.9e+00	2.6e+00	1.2e+06
9	9	5.8e-01	288	4.8e-04	1.4e-03	1.0e+02	4.0e+00	4.7e+00	3.1e+00	9.3e+05
11	11	4.7e-01	424	1.4e-04	4.2e-04	6.1e+01	4.4e+00	4.8e+00	3.0e+00	1.6e+06
13	13	3.9e-01	576	8.8e-05	3.3e-04	2.6e+01	4.1e+00	4.2e+00	3.2e+00	1.3e+06
15	15	3.3e-01	756	8.1e-05	1.1e-04	2.3e+01	3.6e+00	4.6e+00	2.9e+00	1.7e+06
17	17	2.9e-01	964	2.8e-05	1.1e-04	9.5e+00	4.1e+00	4.1e+00	3.3e+00	1.5e+06
19	19	2.6e-01	1192	2.2e-05	1.3e-05	7.5e+00	3.9e+00	5.3e+00	3.2e+00	1.8e+06
21	21	2.3e-01	1444	4.0e-05	7.0e-06	5.9e+00	3.3e+00	5.3e+00	3.1e+00	1.6e+06
22	22	2.2e-01	1592	1.1e-05	2.2e-05	3.8e+00	4.0e+00	4.4e+00	3.3e+00	1.8e+06
25	25	1.9e-01	2024	8.2e-06	1.1e-05	2.4e+00	3.8e+00	4.5e+00	3.3e+00	1.7e+06
27	27	1.8e-01	2348	1.3e-05	8.8e-06	2.2e+00	3.4e+00	4.4e+00	3.2e+00	1.8e+06

500 27 b4p2unix/56/14/46/ (a=11,b=.5,c=0.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	9.3e-01	96	1.9e-03	1.3e-02	2.0e+00	0.0e+00	0.0e+00	0.0e+00	2.2e+03
7	7	6.2e-01	168	9.9e-04	2.4e-02	6.7e-01	1.6e+00	-1.5e+00	2.7e+00	2.7e+03
9	9	4.6e-01	272	2.5e-04	6.3e-04	2.4e-01	3.0e+00	4.4e+00	3.1e+00	4.2e+03
11	11	3.7e-01	396	1.4e-04	2.8e-04	1.2e-01	2.9e+00	4.2e+00	3.1e+00	4.3e+03
13	13	3.1e-01	544	6.1e-05	1.2e-04	5.6e-02	3.1e+00	4.2e+00	3.2e+00	5.1e+03
15	15	2.6e-01	700	3.5e-05	6.0e-05	3.3e-02	3.2e+00	4.3e+00	3.3e+00	5.1e+03
17	17	2.3e-01	888	2.2e-05	4.2e-05	2.0e-02	3.2e+00	4.1e+00	3.3e+00	5.6e+03
19	19	2.1e-01	1092	1.3e-05	1.3e-05	1.2e-02	3.3e+00	4.6e+00	3.4e+00	5.6e+03
21	21	1.9e-01	1308	9.5e-06	6.3e-06	9.3e-03	3.3e+00	4.8e+00	3.3e+00	5.9e+03

500 28 b4p2unix/56/14/46/ (a=11,b=1.5,c=0.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	2.3e+00	92	6.3e-02	6.6e-02	1.6e+07	0.0e+00	0.0e+00	0.0e+00	7.3e+08
7	7	1.5e+00	172	1.5e-02	1.9e-02	3.2e+06	3.5e+00	3.1e+00	3.9e+00	9.0e+08
9	9	1.2e+00	272	3.7e-03	6.6e-03	8.9e+05	4.1e+00	3.3e+00	4.1e+00	1.0e+09
11	11	9.3e-01	400	5.0e-03	4.6e-02	7.7e+05	2.8e+00	3.9e-01	3.3e+00	1.1e+09
13	13	7.7e-01	536	3.7e-04	7.3e-04	1.7e+05	4.7e+00	4.1e+00	4.1e+00	2.5e+09
15	15	6.6e-01	692	2.5e-04	4.8e-04	1.1e+05	4.4e+00	3.9e+00	4.0e+00	2.3e+09
17	17	5.8e-01	884	2.4e-04	2.6e-04	7.4e+04	4.0e+00	4.0e+00	3.9e+00	2.1e+09
19	19	5.1e-01	1084	1.6e-04	6.4e-05	4.6e+04	4.0e+00	4.6e+00	3.9e+00	2.0e+09
21	21	4.6e-01	1312	1.0e-04	6.2e-05	2.6e+04	4.0e+00	4.3e+00	4.0e+00	2.0e+09

500 37 b4p2unix/56/14/46/ (a=15,b=.5,c=1.5)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	3.9e-01	92	2.2e-03	4.9e-03	6.6e-04	0.0e+00	0.0e+00	0.0e+00	6.0e-01
7	7	2.6e-01	164	2.4e-04	5.2e-04	1.7e-04	5.4e+00	5.6e+00	3.3e+00	1.5e+00
9	9	1.9e-01	268	4.5e-05	1.2e-04	5.4e-05	5.6e+00	5.4e+00	3.6e+00	2.2e+00
11	11	1.6e-01	384	2.3e-05	4.3e-05	2.4e-05	5.0e+00	5.2e+00	3.6e+00	2.7e+00
13	13	1.3e-01	524	2.5e-05	2.7e-05	1.5e-05	4.1e+00	4.7e+00	3.4e+00	2.8e+00
15	15	1.1e-01	704	6.1e-06	1.3e-05	6.4e-06	4.7e+00	4.7e+00	3.7e+00	2.7e+00
17	17	9.7e-02	876	7.0e-06	1.0e-05	4.0e-06	4.1e+00	4.5e+00	3.7e+00	2.6e+00
19	19	8.7e-02	1080	2.6e-06	3.0e-06	2.4e-06	4.5e+00	4.9e+00	3.7e+00	2.9e+00
21	21	7.8e-02	1320	1.8e-06	1.6e-06	1.7e-06	4.4e+00	5.0e+00	3.7e+00	2.8e+00
22	22	7.4e-02	1428	2.1e-06	3.5e-06	1.4e-06	4.2e+00	4.4e+00	3.7e+00	2.7e+00
23	23	7.1e-02	1560	2.8e-06	3.5e-06	1.3e-06	3.9e+00	4.3e+00	3.7e+00	2.8e+00
25	25	6.5e-02	1840	1.5e-06	2.0e-06	8.1e-07	4.1e+00	4.4e+00	3.7e+00	2.8e+00
27	27	6.0e-02	2120	1.2e-06	1.1e-06	6.4e-07	4.0e+00	4.5e+00	3.7e+00	3.0e+00
29	29	5.6e-02	2432	9.2e-07	8.4e-07	4.6e-07	4.0e+00	4.5e+00	3.7e+00	3.0e+00

500 43 b4p2unix/56/14/46/ (a=18,b=0.,c=0.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax

5	5	1.3e+00	100	1.3e-02	7.3e-02	9.7e-02	0.0e+00	0.0e+00	0.0e+00	1.7e+01
7	7	8.7e-01	172	3.2e-03	4.8e-02	4.4e-02	3.5e+00	1.0e+00	1.9e+00	3.6e+01
8	8	7.4e-01	216	8.4e-04	3.6e-03	2.1e-02	4.9e+00	5.4e+00	2.8e+00	5.8e+01
9	9	6.5e-01	268	9.6e-04	5.4e-03	2.1e-02	3.8e+00	3.7e+00	2.2e+00	7.9e+01
10	10	5.8e-01	340	3.9e-04	1.2e-03	1.0e-02	4.3e+00	5.1e+00	2.8e+00	6.8e+01
11	11	5.2e-01	392	6.0e-04	1.2e-03	1.0e-02	3.4e+00	4.5e+00	2.4e+00	8.8e+01
12	12	4.7e-01	452	5.4e-04	7.4e-04	6.8e-03	3.2e+00	4.5e+00	2.6e+00	7.1e+01
13	13	4.3e-01	536	2.2e-03	6.0e-03	2.3e-02	1.6e+00	2.3e+00	1.3e+00	9.0e+01
14	14	4.0e-01	616	7.8e-05	2.5e-04	2.5e-03	4.3e+00	4.8e+00	3.1e+00	1.1e+02
15	15	3.7e-01	696	5.8e-04	8.5e-04	5.9e-03	2.5e+00	3.6e+00	2.2e+00	9.0e+01
16	16	3.5e-01	796	6.5e-05	1.4e-04	1.9e-03	4.0e+00	4.8e+00	3.0e+00	1.1e+02
20	20	2.7e-01	1188	8.7e-05	8.7e-05	1.1e-03	3.2e+00	4.3e+00	2.9e+00	1.0e+02
22	22	2.5e-01	1428	3.9e-05	2.3e-05	5.6e-04	3.5e+00	4.9e+00	3.1e+00	9.7e+01
24	24	2.3e-01	1668	2.9e-05	3.0e-05	4.0e-04	3.5e+00	4.5e+00	3.1e+00	9.9e+01

1

500 53 b4p2unix/56/14/46/ (a=21,b=.2,c=.5)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
4	4	6.9e-01	60	2.7e-01	1.2e+00	9.7e-02	0.0e+00	0.0e+00	0.0e+00	6.2e-01
5	5	5.2e-01	84	1.5e-03	1.7e-03	2.2e-03	1.8e+01	2.3e+01	1.3e+01	2.6e+00
6	6	4.2e-01	116	1.0e-03	1.3e-02	6.8e-04	1.1e+01	8.8e+00	9.7e+00	1.4e+00
7	7	3.5e-01	148	8.7e-01	1.8e+00	7.0e-01	-1.7e+00	-6.1e-01	-2.9e+00	2.6e+00
8	8	3.0e-01	184	2.0e-04	4.3e-04	1.9e-04	8.5e+00	9.3e+00	7.3e+00	1.9e+00
9	9	2.6e-01	228	8.8e-01	1.9e+00	6.8e-01	-1.2e+00	-4.8e-01	-2.0e+00	3.2e+00
11	11	2.1e-01	320	6.4e-02	1.3e-01	2.4e-02	1.2e+00	1.8e+00	1.2e+00	1.8e+00
13	13	1.7e-01	436	6.5e-03	8.4e-03	3.3e-03	2.7e+00	3.6e+00	2.4e+00	2.7e+00
15	15	1.5e-01	564	1.3e-03	2.0e-03	7.3e-04	3.4e+00	4.1e+00	3.2e+00	2.7e+00
17	17	1.3e-01	692	4.4e-04	1.2e-03	2.0e-04	3.8e+00	4.1e+00	3.7e+00	2.6e+00
18	18	1.2e-01	780	7.2e-06	1.4e-05	6.5e-06	6.1e+00	6.5e+00	5.5e+00	2.4e+00
19	19	1.2e-01	864	2.3e-03	1.2e-03	4.6e-04	2.7e+00	3.8e+00	3.0e+00	2.4e+00
21	21	1.0e-01	1028	2.3e-04	1.3e-04	4.6e-05	3.7e+00	4.8e+00	4.0e+00	2.6e+00
22	22	9.9e-02	1124	3.0e-06	3.7e-06	2.9e-06	5.9e+00	6.5e+00	5.4e+00	2.6e+00
23	23	9.5e-02	1204	7.5e-05	4.8e-05	1.3e-05	4.1e+00	5.1e+00	4.5e+00	2.5e+00
25	25	8.7e-02	1408	1.7e-05	1.1e-05	3.3e-06	4.6e+00	5.6e+00	4.9e+00	2.7e+00
27	27	8.0e-02	1632	4.7e-06	2.6e-06	1.5e-06	5.1e+00	6.0e+00	5.1e+00	2.7e+00
29	29	7.4e-02	1860	1.3e-06	1.5e-06	9.8e-07	5.5e+00	6.1e+00	5.2e+00	2.7e+00

500 55 b4p2unix/56/14/46/ (a=21,b=20.,c=.6)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	5.4e-01	84	6.4e-03	5.7e-02	5.7e-03	0.0e+00	0.0e+00	0.0e+00	2.1e+00
7	7	3.6e-01	152	1.8e-03	6.0e-03	7.9e-04	3.1e+00	5.6e+00	4.8e+00	1.5e+00
9	9	2.7e-01	244	3.2e-03	1.7e-02	1.4e-03	9.9e-01	1.7e+00	2.0e+00	2.1e+00
11	11	2.2e-01	344	1.6e-04	1.7e-04	9.2e-05	4.0e+00	6.3e+00	4.5e+00	2.9e+00
13	13	1.8e-01	460	1.1e-04	7.6e-05	4.1e-05	3.7e+00	6.0e+00	4.5e+00	2.1e+00
15	15	1.6e-01	576	4.6e-05	1.6e-04	2.1e-05	3.9e+00	4.7e+00	4.5e+00	2.6e+00
19	19	1.2e-01	880	1.0e-05	1.1e-05	7.2e-06	4.3e+00	5.7e+00	4.4e+00	3.1e+00
21	21	1.1e-01	1060	6.7e-06	7.1e-06	4.7e-06	4.3e+00	5.6e+00	4.4e+00	3.1e+00
22	22	1.0e-01	1164	3.3e-06	9.4e-06	3.5e-06	4.6e+00	5.2e+00	4.5e+00	2.6e+00
23	23	9.9e-02	1260	4.9e-06	4.0e-06	3.2e-06	4.2e+00	5.6e+00	4.4e+00	2.9e+00
25	25	9.0e-02	1472	3.4e-06	2.8e-06	2.3e-06	4.2e+00	5.5e+00	4.4e+00	2.9e+00
27	27	8.3e-02	1712	2.5e-06	2.9e-06	1.6e-06	4.2e+00	5.3e+00	4.4e+00	2.8e+00
29	29	7.8e-02	1940	1.6e-06	2.0e-06	1.2e-06	4.3e+00	5.3e+00	4.3e+00	3.0e+00

500 57 b4p2unix/56/14/46/ (a=22,b=1.,c=.1)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	6.8e-01	80	4.0e-03	2.0e-02	1.3e-02	0.0e+00	0.0e+00	0.0e+00	6.6e+00
7	7	4.5e-01	136	1.5e-03	5.7e-03	1.4e-02	2.4e+00	3.1e+00	-2.4e-01	3.2e+01
9	9	3.4e-01	204	1.1e-04	1.4e-04	1.7e-03	5.2e+00	7.2e+00	2.9e+00	6.1e+01
13	13	2.2e-01	372	1.5e-04	3.3e-04	2.5e-03	3.0e+00	3.7e+00	1.5e+00	1.1e+02

15	15	1.9e-01	476	1.0e-05	3.0e-05	2.6e-04	4.8e+00	5.2e+00	3.1e+00	1.3e+02
17	17	1.7e-01	600	6.2e-06	1.3e-05	1.6e-04	4.7e+00	5.3e+00	3.2e+00	1.4e+02
19	19	1.5e-01	724	4.0e-06	3.1e-06	9.7e-05	4.6e+00	5.8e+00	3.2e+00	1.6e+02
21	21	1.4e-01	852	2.7e-06	1.3e-06	6.5e-05	4.5e+00	6.0e+00	3.3e+00	1.7e+02

500 59 b4p2unix/56/14/46/ (a=22,b=.8,c=.8)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
9	9	3.4e-01	168	3.8e-04	3.2e-03	2.4e-02	0.0e+00	0.0e+00	0.0e+00	2.3e+02
11	11	2.7e-01	236	1.1e-04	1.8e-04	1.0e-02	5.6e+00	1.3e+01	3.9e+00	3.8e+02
13	13	2.3e-01	304	4.4e-05	1.1e-04	5.2e-03	5.3e+00	8.3e+00	3.8e+00	5.4e+02
15	15	2.0e-01	368	2.2e-05	2.9e-04	3.6e-03	5.1e+00	4.3e+00	3.4e+00	6.8e+02
17	17	1.7e-01	440	1.2e-05	1.3e-05	2.0e-03	5.0e+00	8.0e+00	3.6e+00	8.0e+02
19	19	1.5e-01	540	6.9e-06	6.7e-06	1.2e-03	4.9e+00	7.6e+00	3.7e+00	9.1e+02
21	21	1.4e-01	640	4.4e-06	2.5e-06	8.0e-04	4.9e+00	7.8e+00	3.7e+00	1.0e+03

500 61 b4p2unix/56/14/46/ (a=23,b=0.,c=.5)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
4	4	1.0e+00	56	1.0e+00	1.9e+01	7.2e+00	0.0e+00	0.0e+00	0.0e+00	1.1e+01
6	6	6.2e-01	120	2.8e-03	3.0e-02	1.6e-01	1.2e+01	1.3e+01	7.4e+00	1.3e+02
7	7	5.2e-01	164	1.3e-03	6.5e-03	1.4e-01	9.6e+00	1.1e+01	5.7e+00	2.3e+02
8	8	4.5e-01	200	1.8e-03	1.9e-02	8.8e-02	7.5e+00	8.2e+00	5.2e+00	1.3e+02
9	9	3.9e-01	240	6.9e-02	2.2e-01	3.3e+00	2.7e+00	4.6e+00	8.1e-01	2.1e+02
10	10	3.5e-01	300	1.5e-04	1.1e-03	1.4e-02	8.0e+00	8.9e+00	5.7e+00	2.8e+02
11	11	3.1e-01	344	1.3e-02	2.6e-02	1.1e+00	3.6e+00	5.5e+00	1.6e+00	3.6e+02
12	12	2.8e-01	388	1.2e-04	1.4e-04	8.6e-03	7.0e+00	9.1e+00	5.2e+00	2.4e+02
13	13	2.6e-01	456	7.8e-03	3.2e-02	5.1e-01	3.5e+00	4.6e+00	1.9e+00	3.0e+02
14	14	2.4e-01	520	4.6e-05	8.1e-05	4.8e-03	6.8e+00	8.4e+00	5.0e+00	3.6e+02
15	15	2.2e-01	588	4.5e-03	1.5e-02	3.1e-01	3.5e+00	4.6e+00	2.0e+00	4.2e+02
16	16	2.1e-01	644	2.8e-05	3.8e-05	2.9e-03	6.5e+00	8.1e+00	4.9e+00	4.8e+02
17	17	1.9e-01	728	9.1e-03	1.4e-02	4.4e-01	2.8e+00	4.3e+00	1.7e+00	3.6e+02
18	18	1.8e-01	808	2.0e-05	1.9e-05	1.8e-03	6.3e+00	7.9e+00	4.8e+00	4.1e+02
19	19	1.7e-01	884	2.5e-03	1.4e-03	1.3e-01	3.4e+00	5.3e+00	2.2e+00	4.6e+02
20	20	1.6e-01	960	1.1e-05	1.1e-05	1.2e-03	6.2e+00	7.8e+00	4.7e+00	5.1e+02
21	21	1.6e-01	1048	3.5e-04	1.6e-04	2.0e-02	4.2e+00	6.1e+00	3.1e+00	5.6e+02
22	22	1.5e-01	1128	8.5e-06	9.0e-06	8.2e-04	6.0e+00	7.5e+00	4.7e+00	4.4e+02
23	23	1.4e-01	1232	5.6e-06	7.9e-06	6.8e-04	6.1e+00	7.4e+00	4.7e+00	4.8e+02
24	24	1.4e-01	1316	4.8e-06	5.1e-06	5.7e-04	6.0e+00	7.4e+00	4.6e+00	5.2e+02

500 69 b4p2unix/56/14/46/ (a=25,b=.2,c=0.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	5.5e-01	96	3.5e-03	3.9e-02	1.2e-02	0.0e+00	0.0e+00	0.0e+00	8.5e+00
7	7	3.6e-01	164	5.1e-04	2.5e-03	1.8e-03	4.7e+00	6.8e+00	4.7e+00	1.1e+01
9	9	2.7e-01	260	4.2e-04	1.5e-03	1.6e-03	3.0e+00	4.7e+00	2.9e+00	1.9e+01
11	11	2.2e-01	376	1.0e-04	1.5e-03	3.1e-04	3.8e+00	3.5e+00	4.0e+00	1.1e+01
15	15	1.6e-01	644	2.3e-05	8.2e-05	9.4e-05	4.0e+00	4.9e+00	3.9e+00	1.8e+01
17	17	1.4e-01	808	1.2e-05	1.7e-05	6.2e-05	4.1e+00	5.6e+00	3.8e+00	2.8e+01
19	19	1.2e-01	1000	8.3e-06	2.0e-05	3.4e-05	4.0e+00	5.0e+00	3.9e+00	2.2e+01
21	21	1.1e-01	1208	5.1e-06	8.2e-06	2.4e-05	4.0e+00	5.3e+00	3.9e+00	2.9e+01

1

500 70 b4p2unix/56/14/46/ (a=25,b=.2,c=1.)

nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	8.1e-01	88	1.7e-01	3.3e+00	1.1e-01	0.0e+00	0.0e+00	0.0e+00	1.5e+00
7	7	5.4e-01	160	1.4e-03	8.9e-03	1.0e-02	1.2e+01	1.5e+01	5.7e+00	1.8e+01
9	9	4.0e-01	248	1.3e-03	3.7e-03	2.5e-03	7.0e+00	9.8e+00	5.4e+00	6.6e+00
11	11	3.2e-01	356	6.8e-04	1.2e-03	1.6e-03	6.1e+00	8.6e+00	4.6e+00	1.2e+01
15	15	2.3e-01	608	1.1e-04	2.3e-04	3.9e-04	5.9e+00	7.7e+00	4.5e+00	2.0e+01

17	17	2.0e-01	768	5.7e-05	1.8e-04	2.3e-04	5.8e+00	7.1e+00	4.4e+00	2.1e+01
19	19	1.8e-01	948	7.6e-05	2.9e-05	2.2e-04	5.1e+00	7.7e+00	4.1e+00	2.5e+01
21	21	1.6e-01	1140	3.5e-05	2.3e-05	1.0e-04	5.3e+00	7.4e+00	4.3e+00	2.0e+01
23	23	1.5e-01	1348	1.2e-05	4.7e-05	6.0e-05	5.6e+00	6.5e+00	4.4e+00	2.6e+01
27	27	1.2e-01	1808	1.5e-05	2.3e-05	4.8e-05	5.0e+00	6.3e+00	4.1e+00	2.5e+01
29	29	1.2e-01	2076	1.2e-05	1.1e-05	3.5e-05	4.9e+00	6.5e+00	4.1e+00	3.0e+01

500 71 b4p2unix/56/14/46/ (a=25,b=2.5,c=-1.)

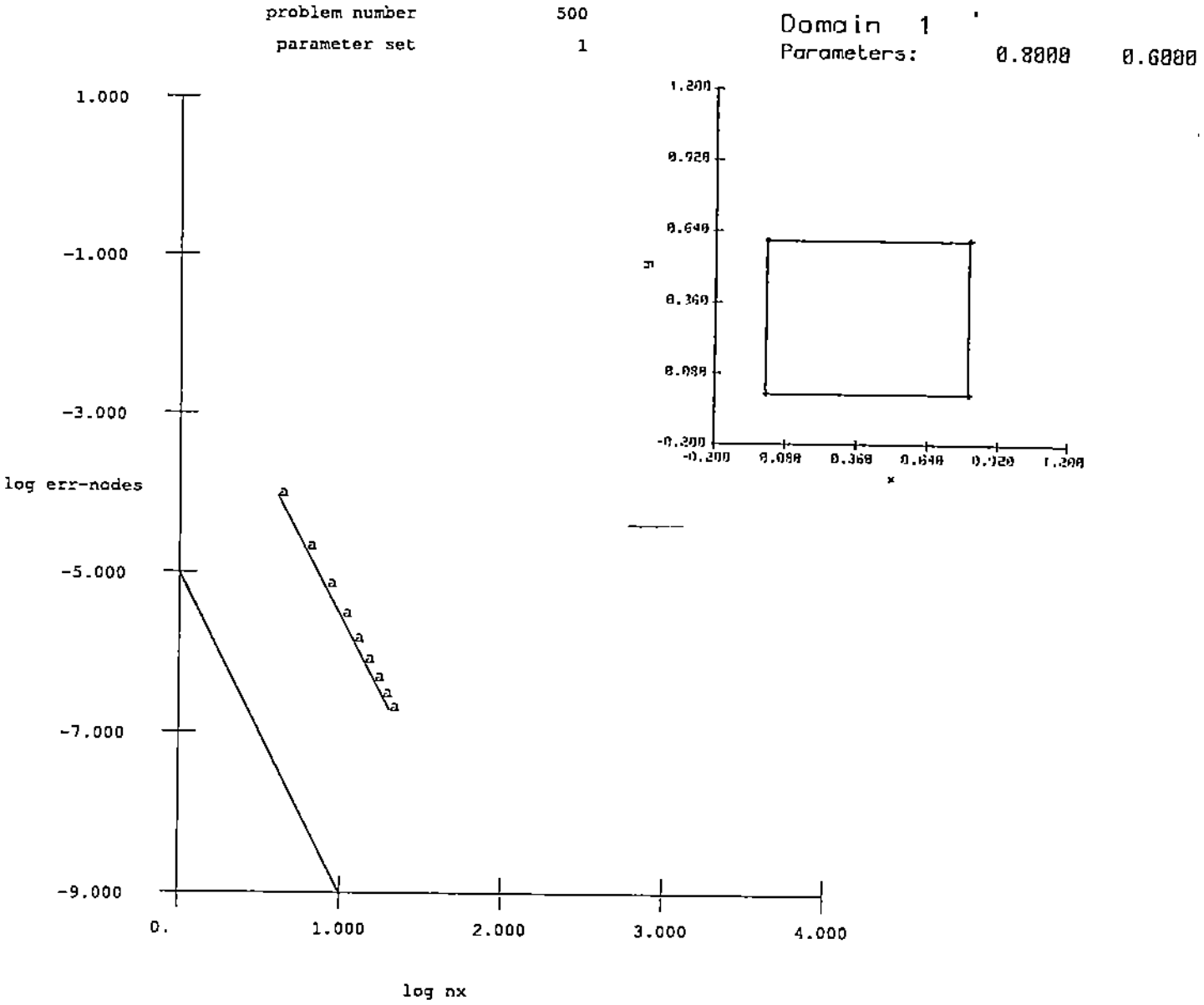
nx	ny	h	nunk	rerr	rerrmax	errl2	convm1	convm2	convl2	solmax
5	5	5.2e-01	88	1.0e-03	6.4e-03	7.8e-03	0.0e+00	0.0e+00	0.0e+00	1.5e+01
7	7	3.5e-01	148	3.8e-04	4.2e-03	2.4e-03	2.5e+00	1.1e+00	2.9e+00	1.4e+01
11	11	2.1e-01	332	8.5e-05	6.0e-04	5.1e-04	2.7e+00	2.6e+00	3.0e+00	2.3e+01
15	15	1.5e-01	548	1.7e-05	8.5e-05	1.1e-04	3.3e+00	3.5e+00	3.4e+00	3.2e+01
17	17	1.3e-01	688	6.0e-06	1.1e-05	5.7e-05	3.7e+00	4.6e+00	3.5e+00	3.4e+01
19	19	1.2e-01	836	8.4e-06	7.3e-06	4.2e-05	3.2e+00	4.5e+00	3.5e+00	2.8e+01
21	21	1.0e-01	1020	3.4e-06	5.2e-06	2.4e-05	3.6e+00	4.4e+00	3.6e+00	3.1e+01
25	25	8.7e-02	1372	2.0e-06	3.6e-06	1.4e-05	3.5e+00	4.2e+00	3.5e+00	3.6e+01
29	29	7.4e-02	1816	8.2e-07	2.0e-06	6.2e-06	3.7e+00	4.1e+00	3.7e+00	3.2e+01

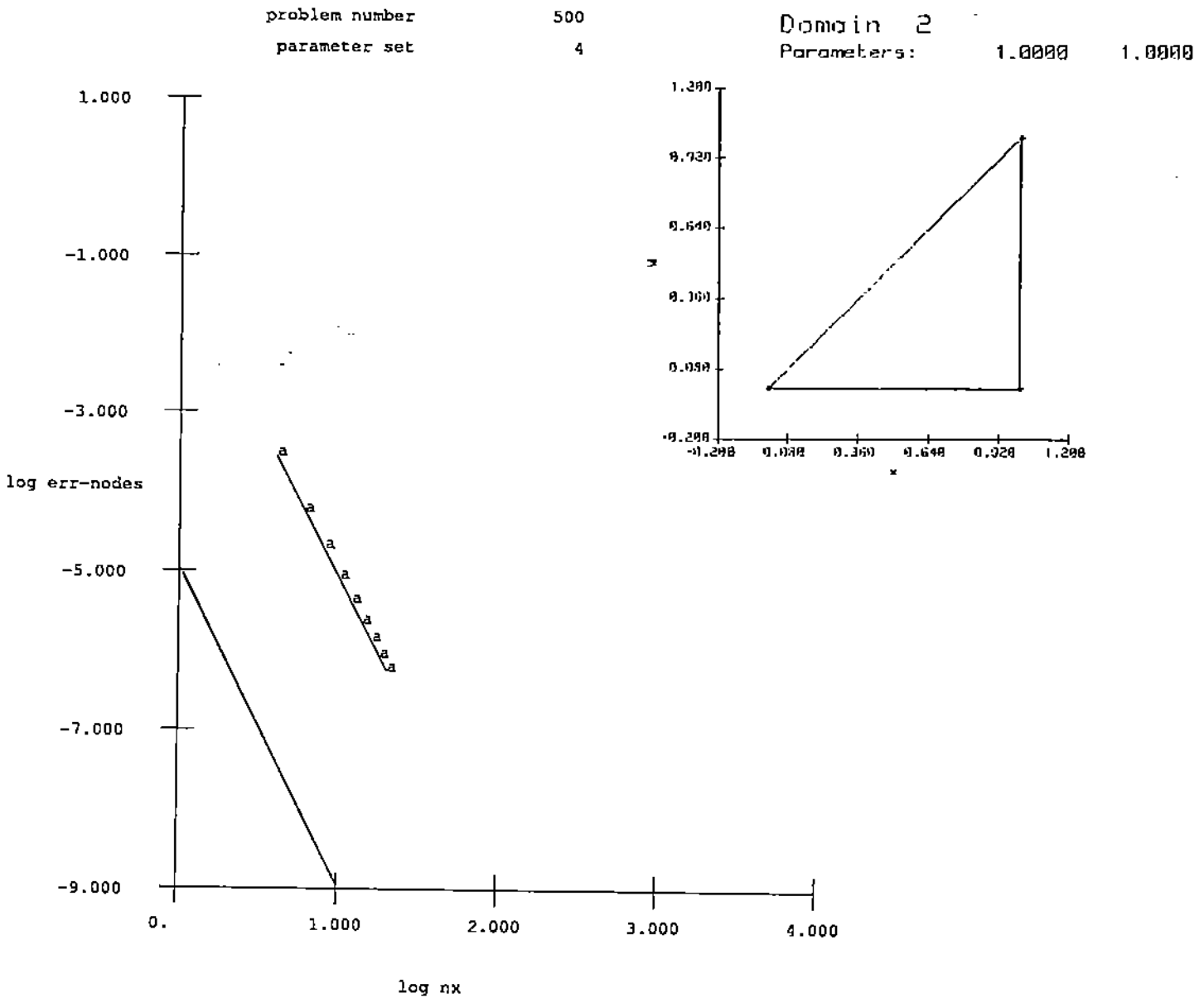
*** number of ellpack runs = 237

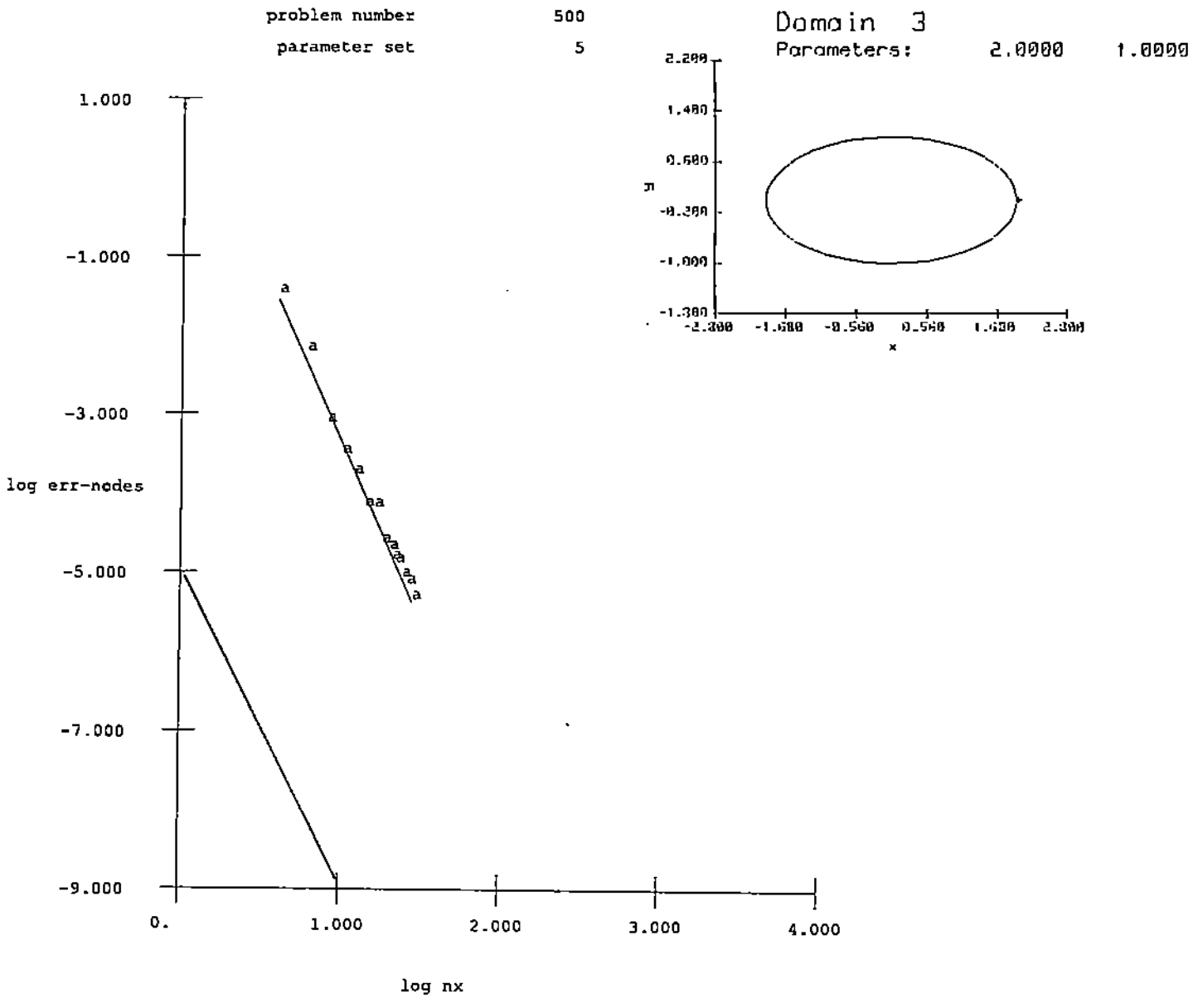
*** total CPU hours = 1.41

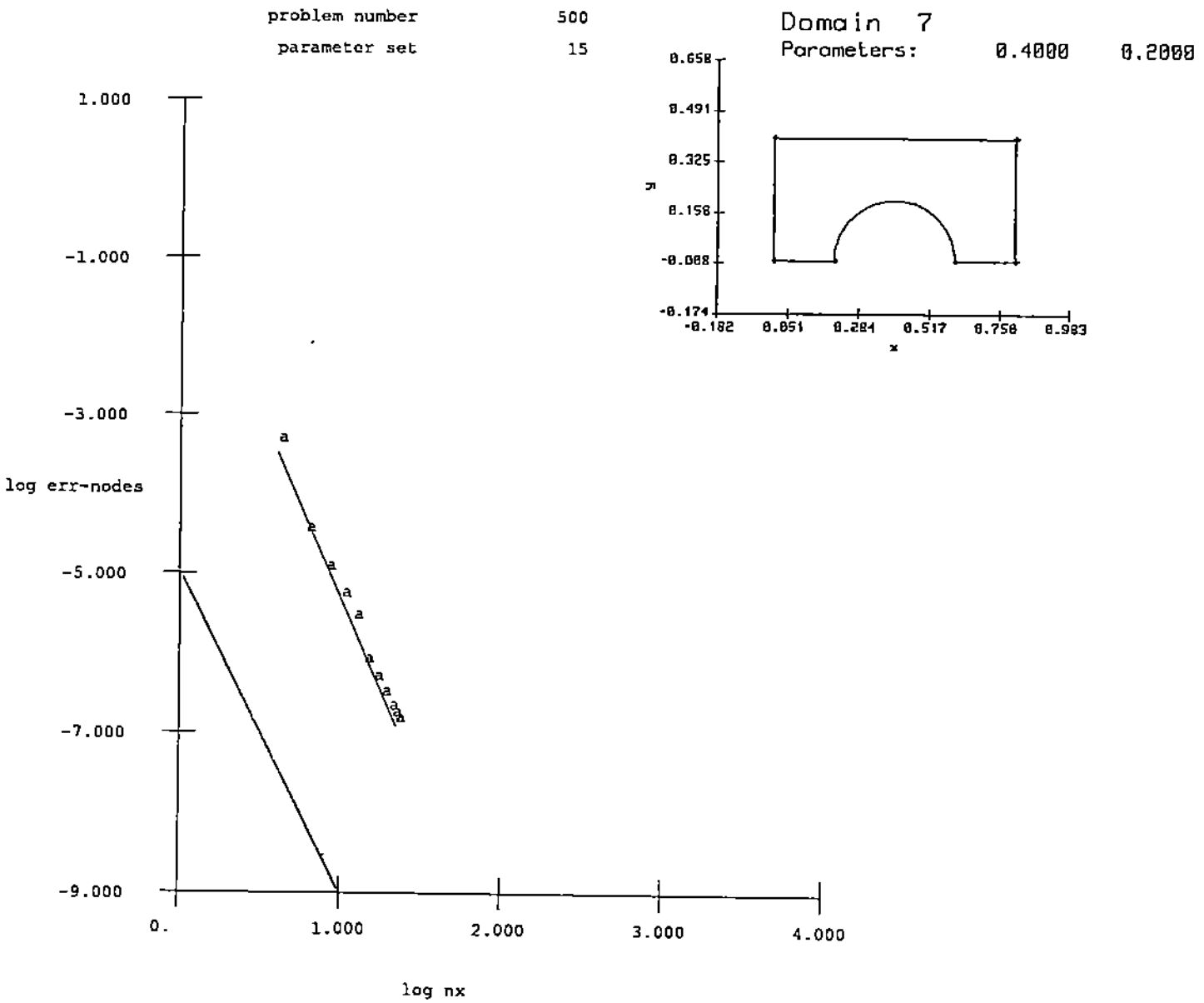
APPENDIX C. Domain Shapes and Convergence Rate Plots

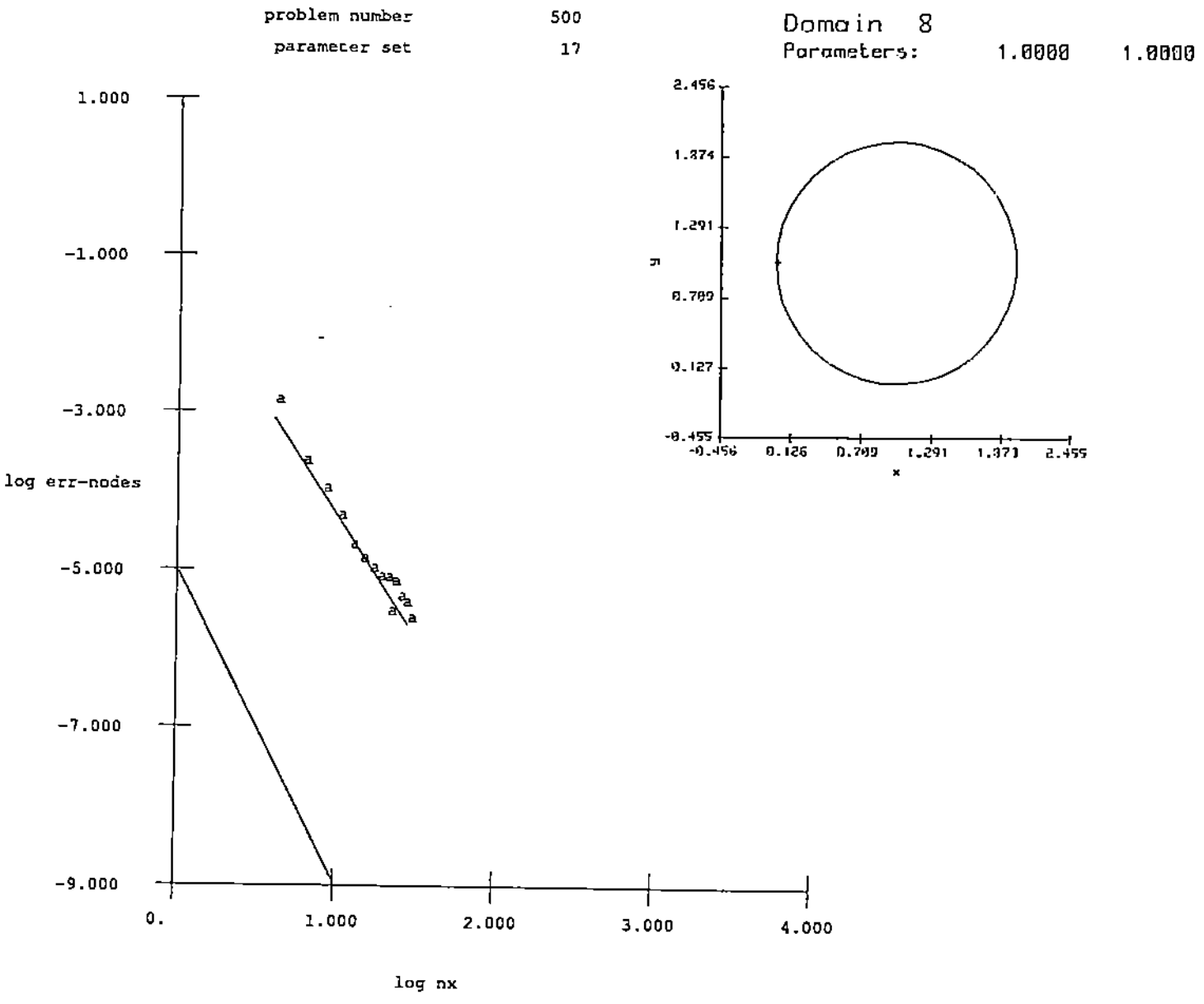
We present 20 plots which show the domain shapes chosen from [8] along with a *log-log* plot of *rerrmax* (the maximum error on the nodes) versus *ngridx-1* (the reciprocal of the spacing h). The ordinates and abscissa are labeled *err-nodes* and *nx* by the plotting software. A straight line is shown along with the data which is determined by a least squares fit. For reference purposes we also show a line in the lower left with slope -4. Our claim is that all the data sets form a "line" parallel to this reference line. For many domains, the sets are considerably perturbed but all generally lie along this line.

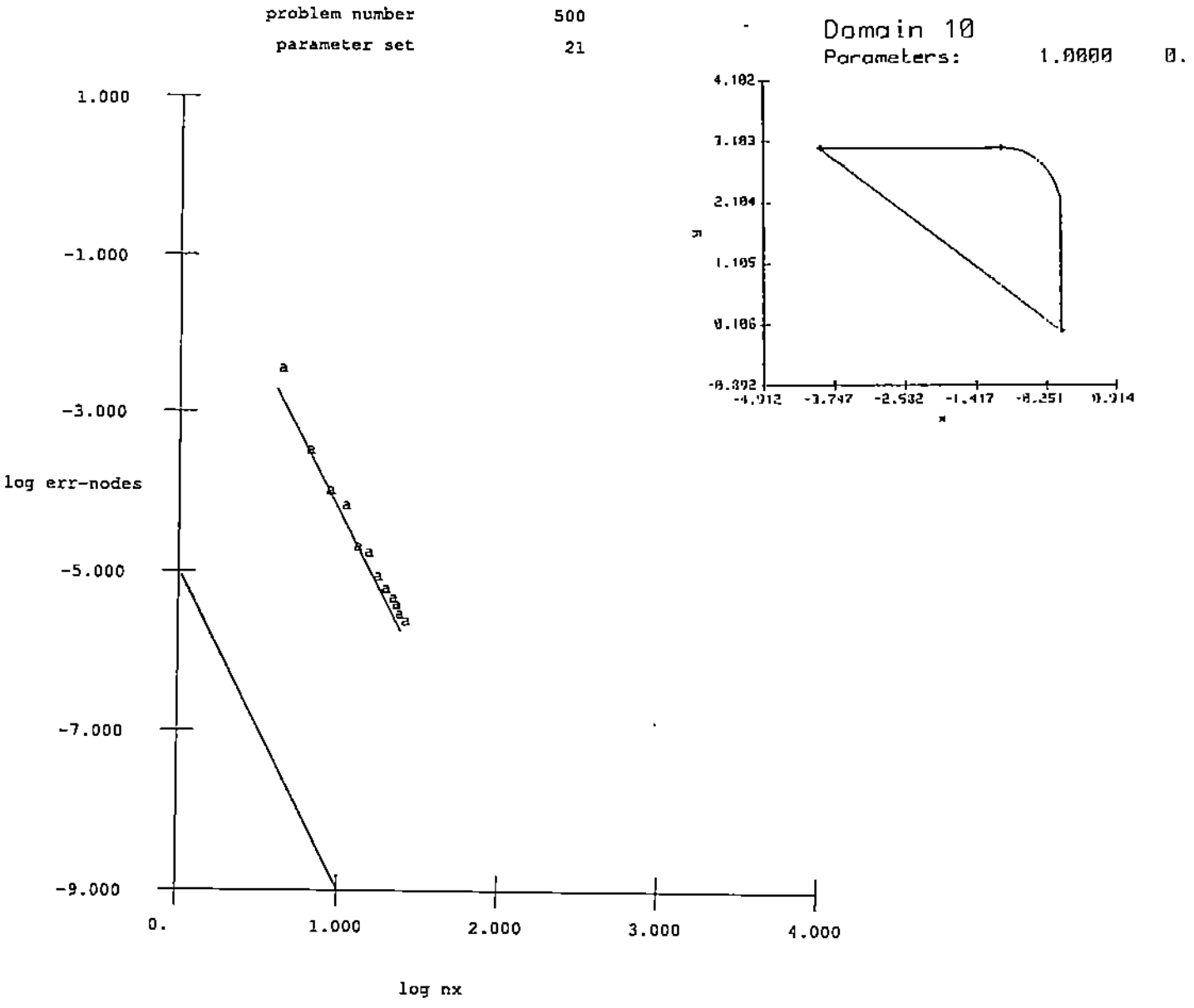

problem number 500
parameter set 1
Domain 1
Parameters: 0.8000 0.6000
1.200
0.928
0.648
0.368
0.088
-0.200
y
-0.200
0.088
0.368
0.648
0.928
1.200
x
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
a
0.
1.000
2.000
3.000
4.000
log nx

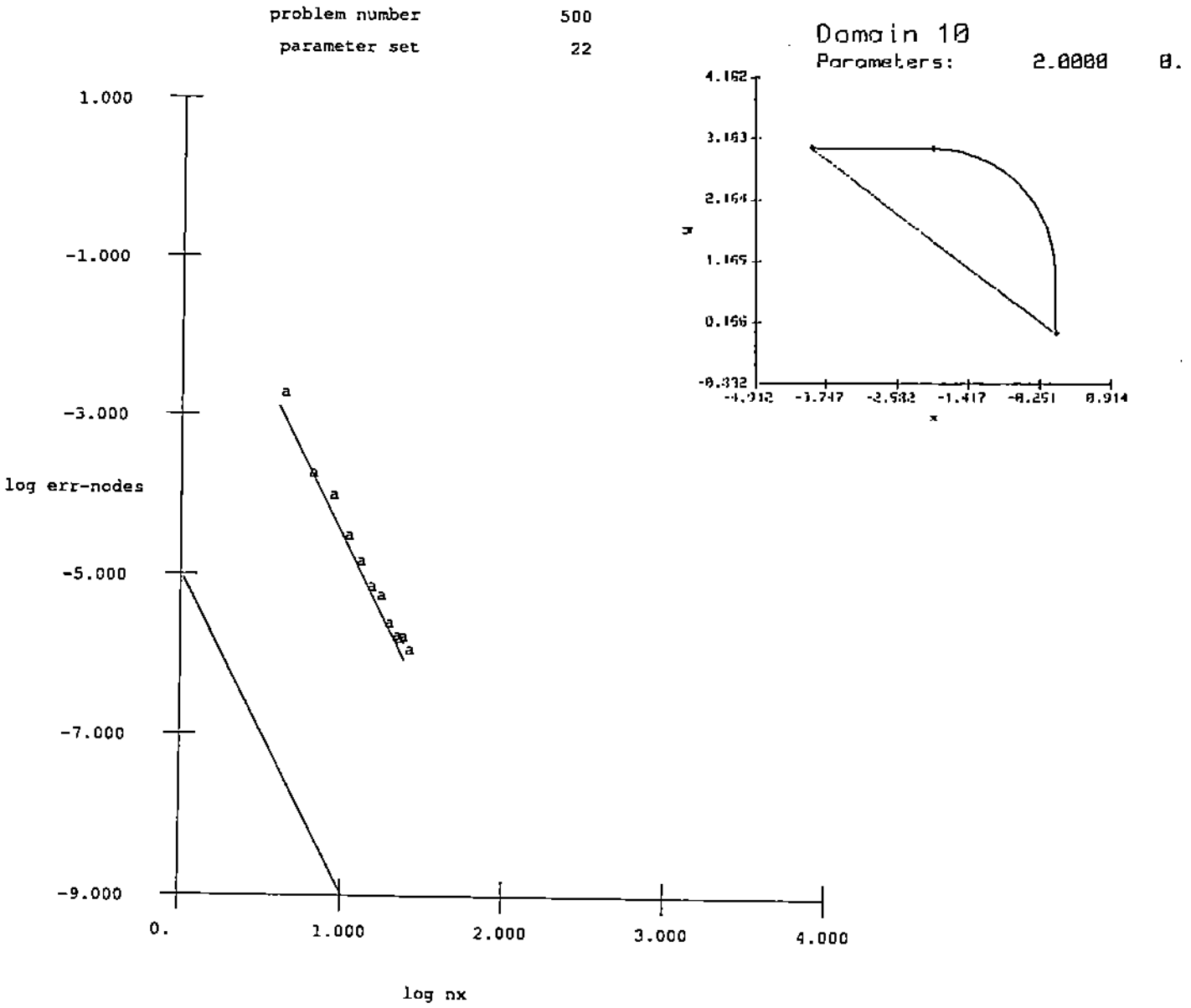

problem number 500
parameter set 4
Domain 2
Parameters: 1.0000 1.0000
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx
1.200
0.920
0.640
0.360
0.080
-0.200
y
-0.200
0.080
0.360
0.640
0.920
1.200
x


problem number 500
parameter set 5
Domain 3
Parameters: 2.0000 1.0000
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx
2.200
1.400
0.600
-0.200
-1.000
-1.300
-2.300
-1.600
-0.900
0.900
1.600
2.300
x
y

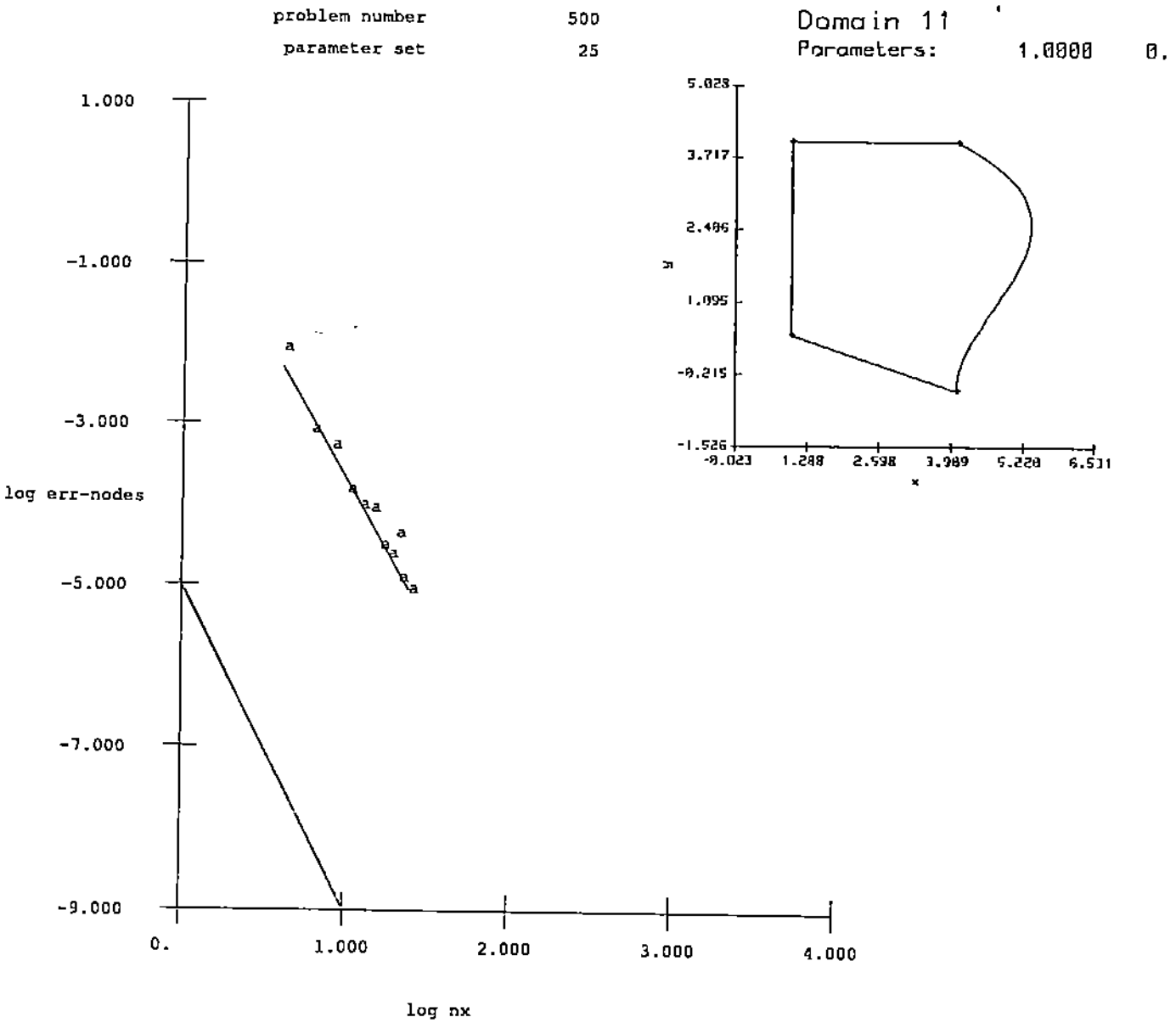

problem number 500
parameter set 15
Domain 7
Parameters: 0.4000 0.2000
0.658
0.491
0.325
0.158
-0.008
-0.174
y
-0.182
0.051
0.284
0.517
0.750
0.983
x
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx

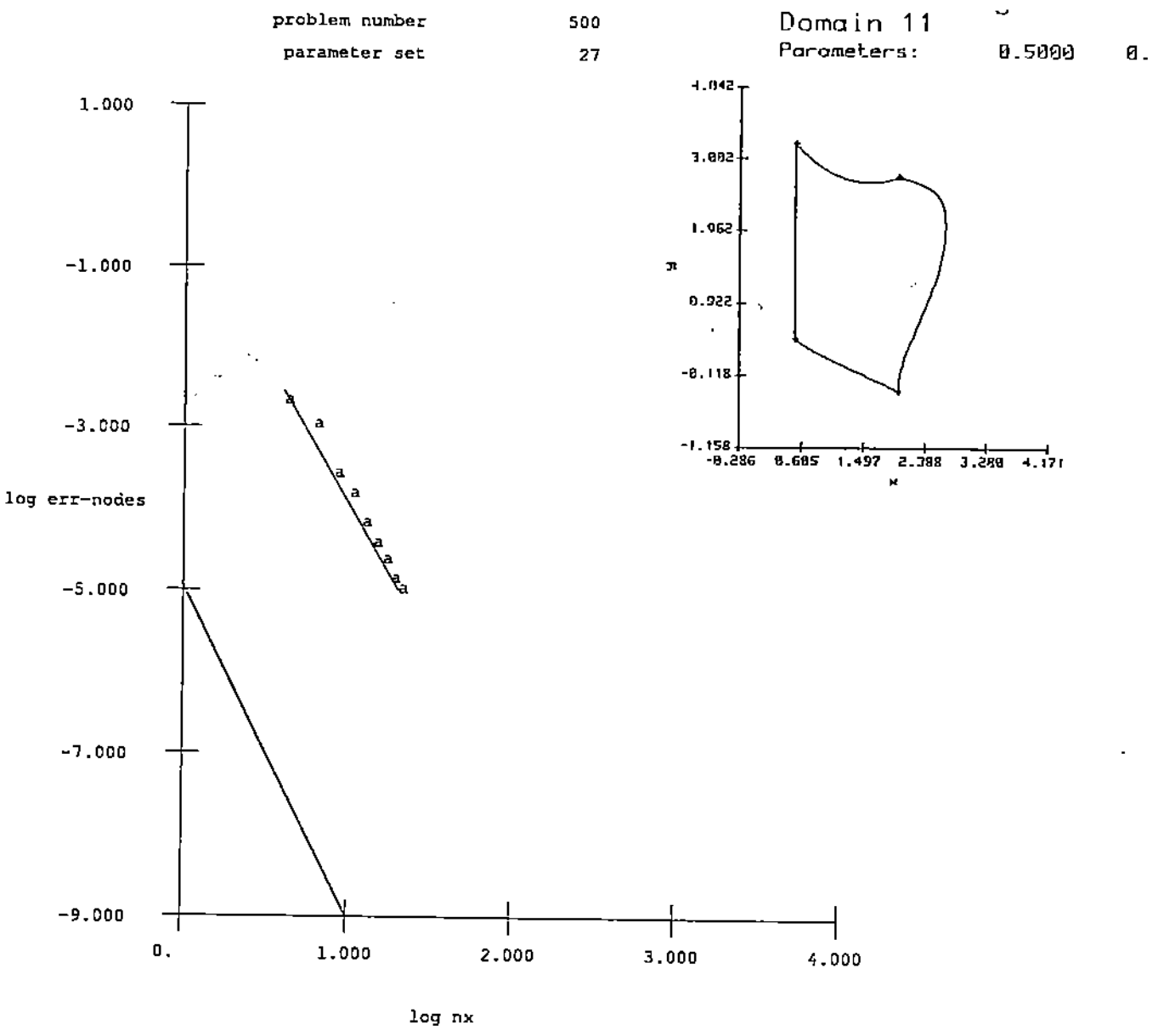

problem number 500
parameter set 17
Domain 8
Parameters: 1.0000 1.0000
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx
2.456
1.874
1.291
0.709
0.127
-0.455
-0.456
0.126
0.709
1.291
1.873
2.455
x
y

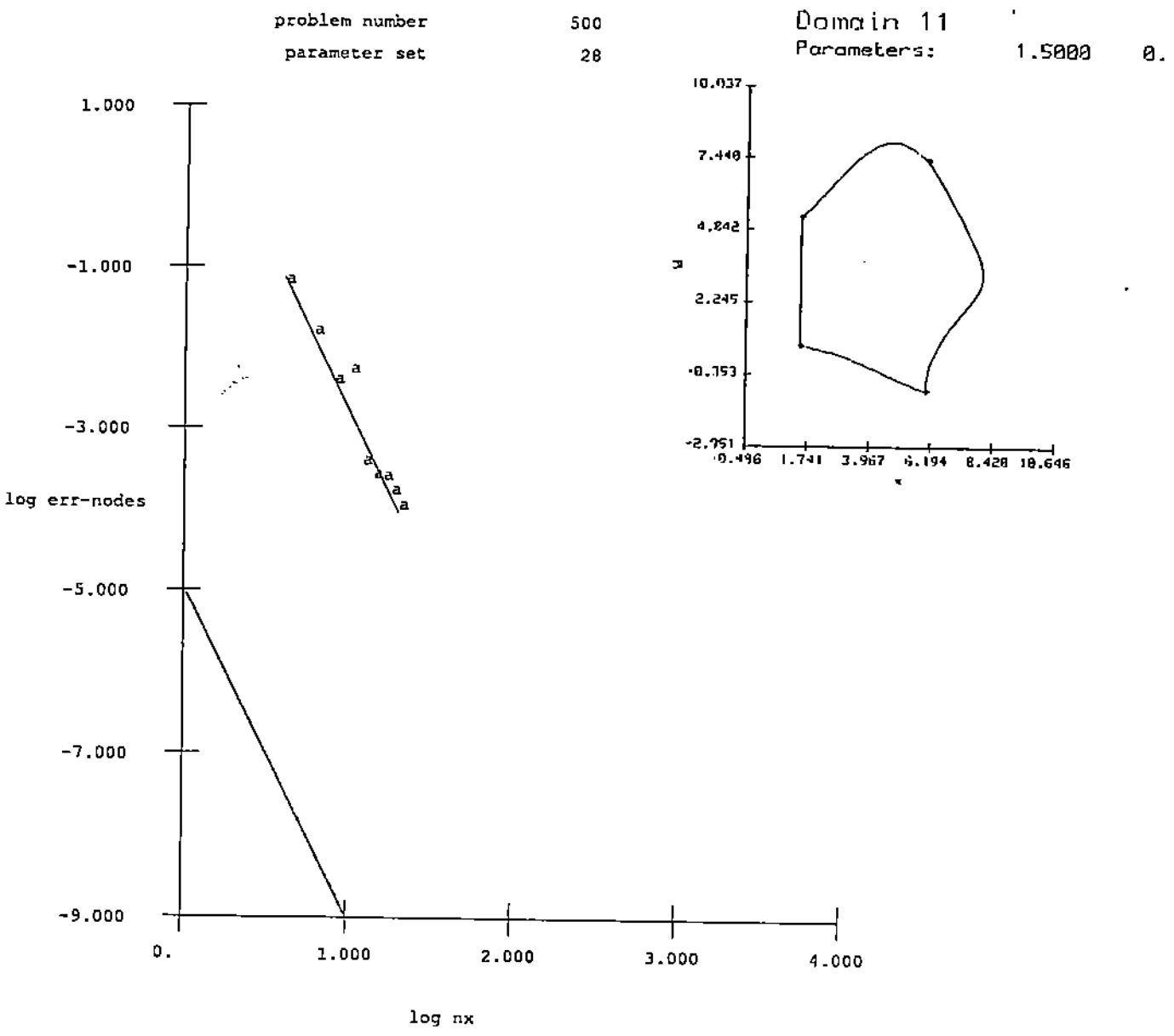

problem number 500
parameter set 21
1.000
-1.000
-3.000
log err-nodes
-5.000
-7.000
-9.000
0.
1.000
2.000
3.000
4.000
log nx
Domain 10
Parameters: 1.0000 0.
4.102
3.103
2.104
1.105
0.106
-0.892
-4.912
-3.747
-2.582
-1.417
-0.251
0.914
y
x

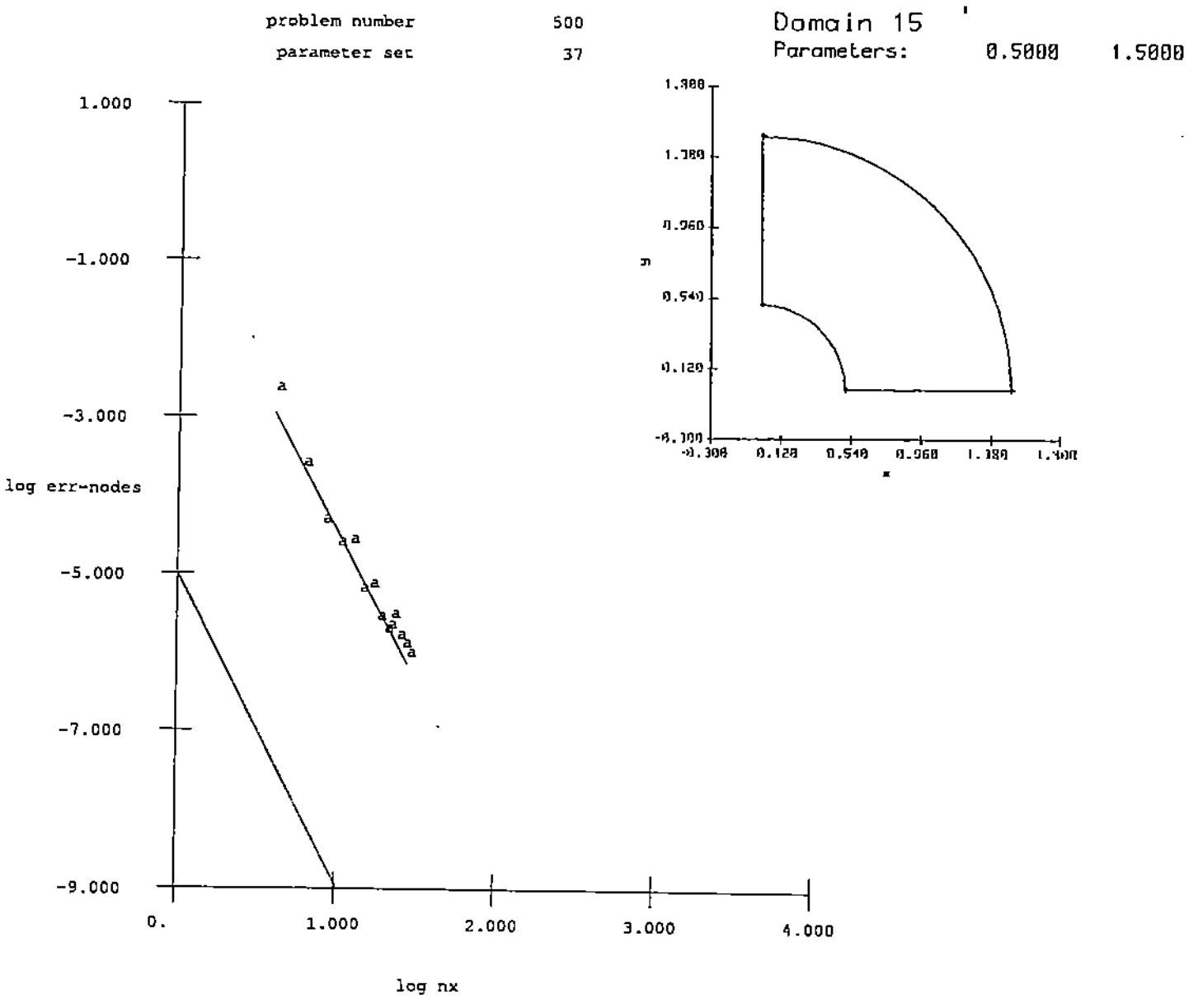

problem number 500
parameter set 22
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx

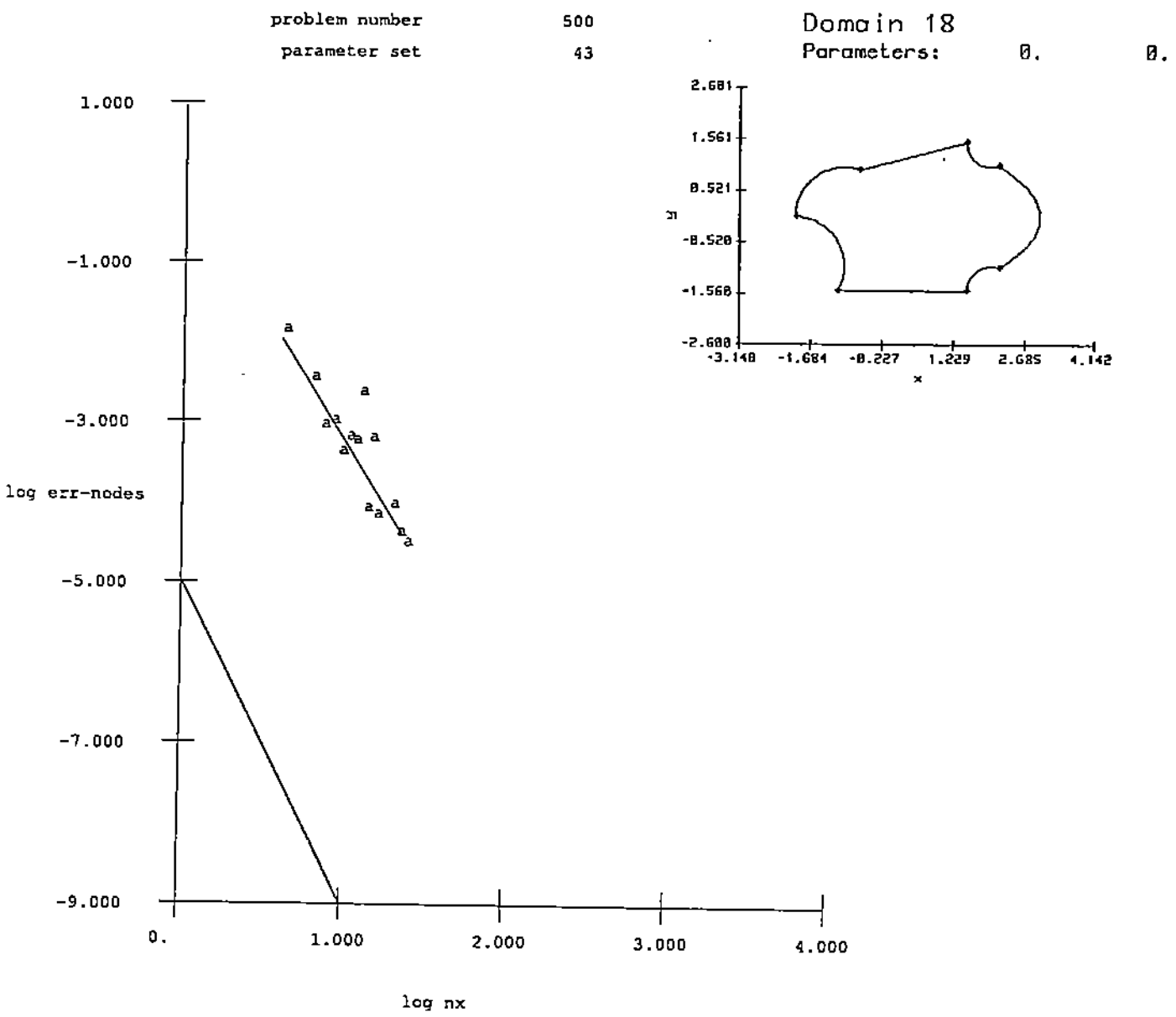
Domain 10
Parameters: 2.0000 0.
4.162
3.163
2.164
1.165
0.166
-0.832
-4.912
-3.747
-2.583
-1.417
-0.251
0.914

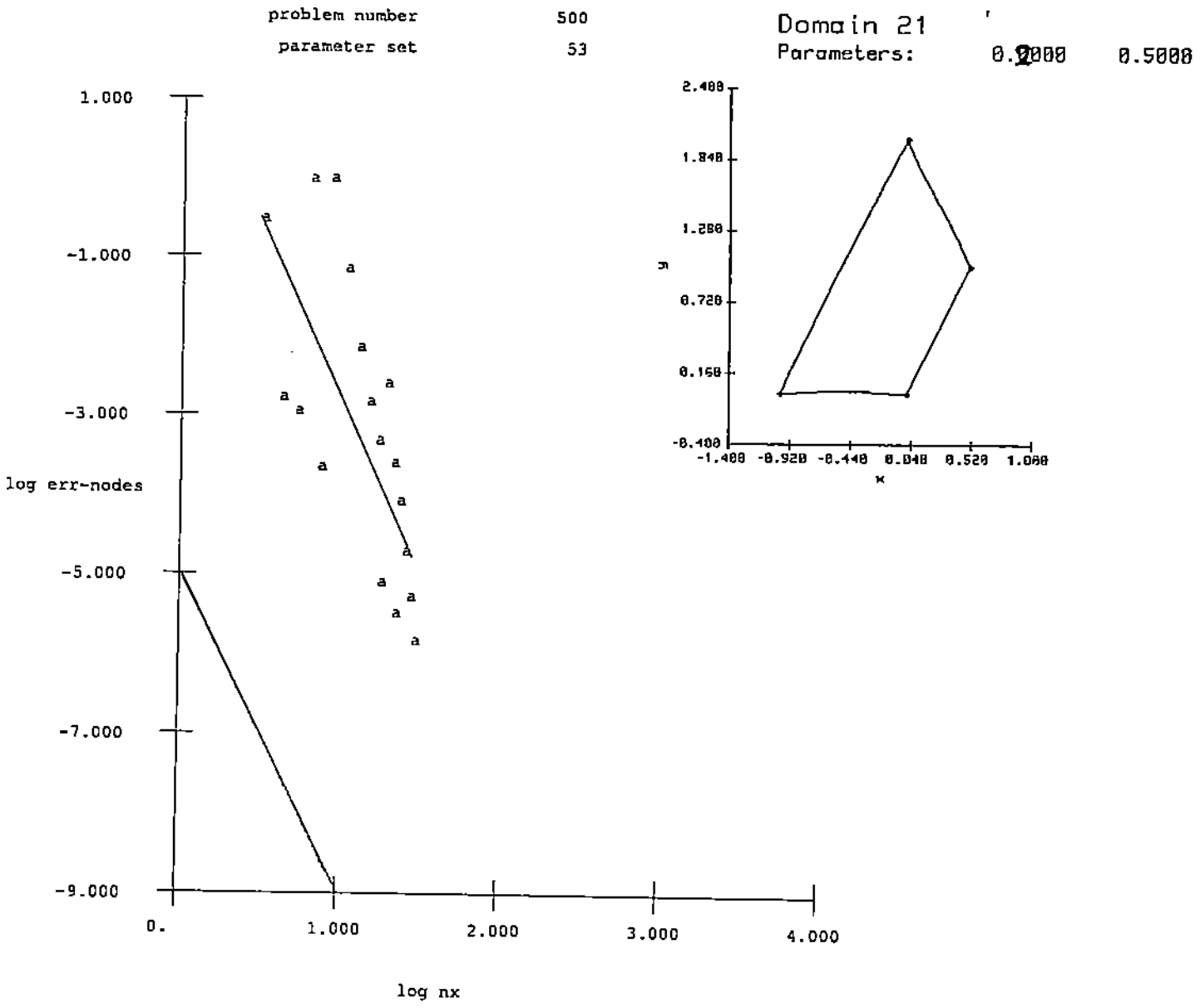

problem number 500
parameter set 25
Domain 11
Parameters: 1.0000 0.
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx
5.028
3.717
2.406
1.095
-0.215
-1.526
y
-0.023
1.288
2.598
3.909
5.220
6.531
x

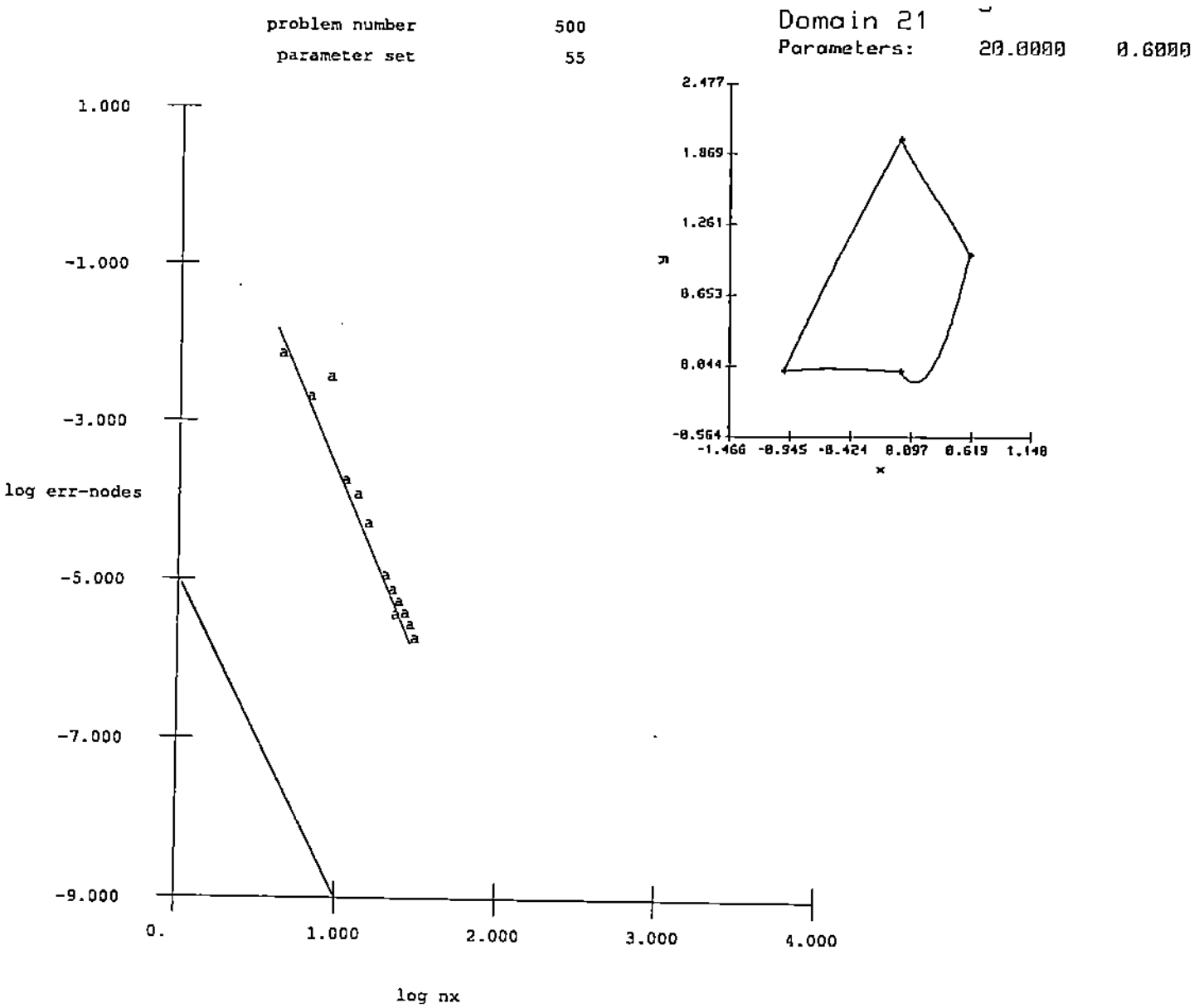

problem number 500
parameter set 27
Domain 11
Parameters: 0.5000 0.
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx
4.042
3.082
1.962
0.922
-0.118
-1.158
-0.286
0.685
1.497
2.388
3.280
4.171

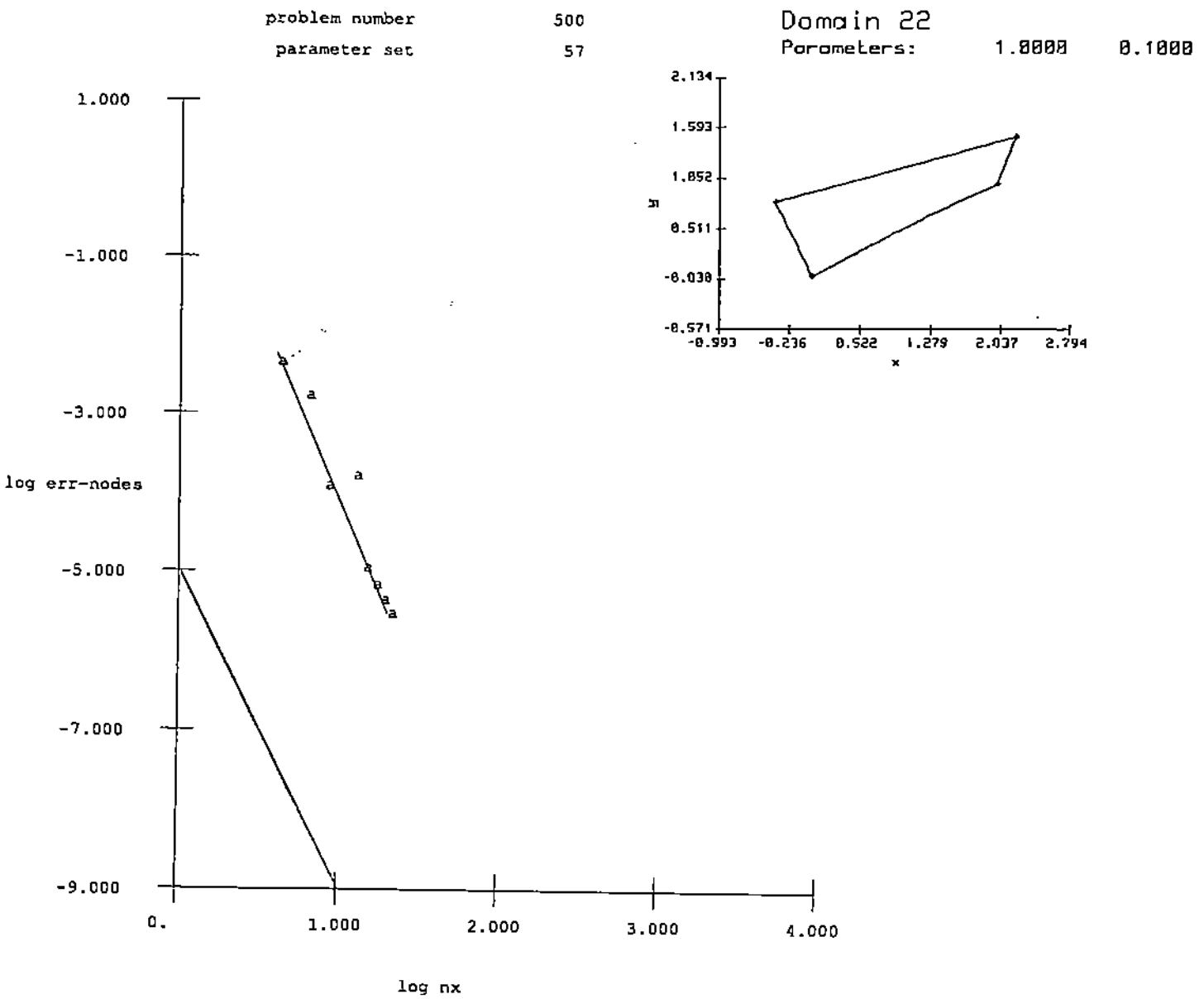

problem number 500
parameter set 28
Domain 11
Parameters: 1.5000 0.
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx
10.037
7.440
4.842
2.245
-0.353
-2.951
-0.496
1.741
3.967
6.194
8.420
10.646

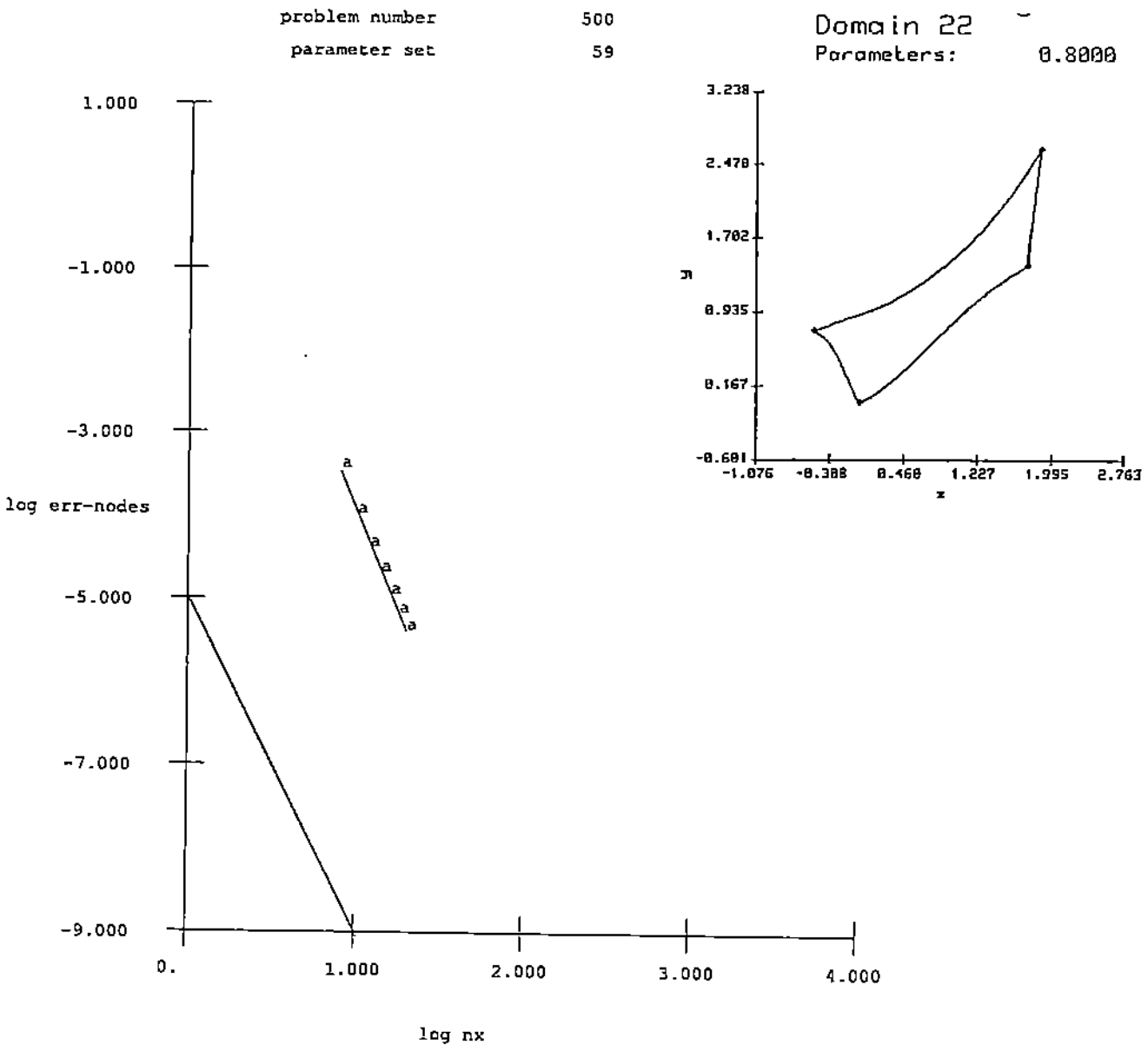

problem number 500
parameter set 37
Domain 15
Parameters: 0.5000 1.5000
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx
1.380
1.160
0.960
0.540
0.120
-0.300
0.120
0.540
0.960
1.380
1.800
y
x

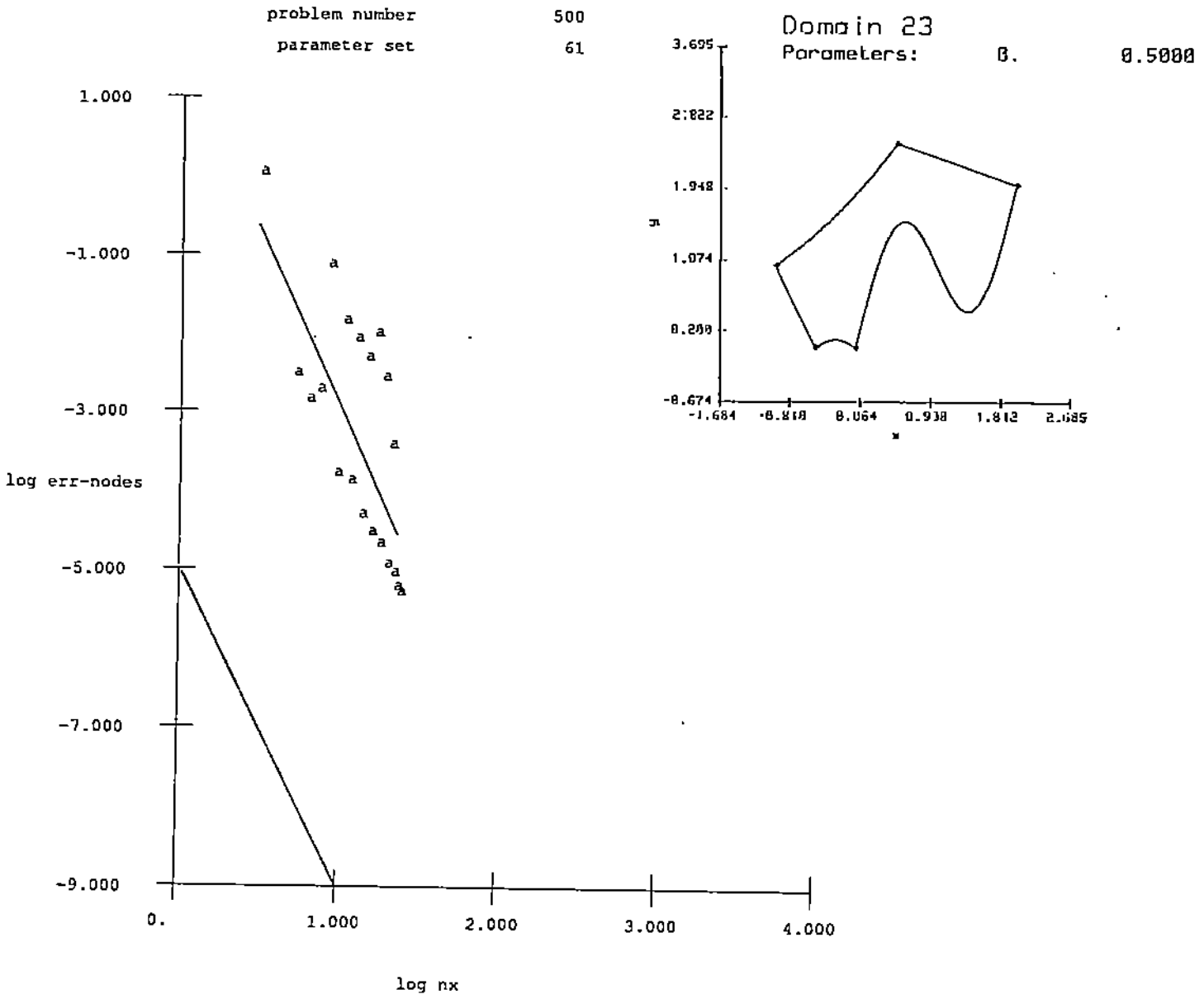

problem number 500
parameter set 43
Domain 18
Parameters: 0. 0.
2.681
1.561
0.521
-0.520
-1.560
-2.600
y
-3.148
-1.684
-0.227
1.229
2.685
4.142
x
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx

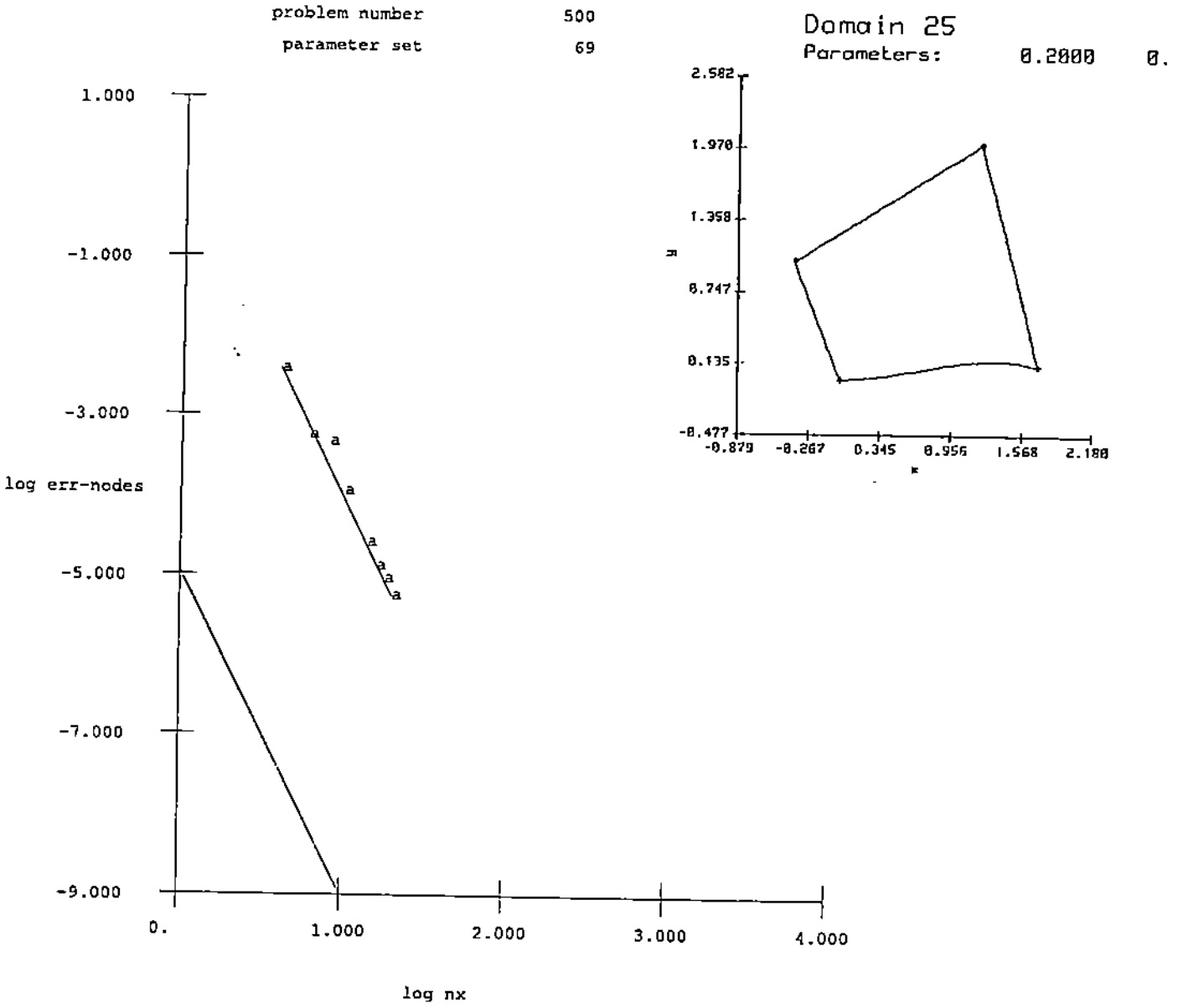

problem number 500
parameter set 53
Domain 21
Parameters: 0.2000 0.5000
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx
2.400
1.840
1.280
0.720
0.160
-0.400
-1.480 -0.920 -0.440 0.040 0.520 1.080
y
x

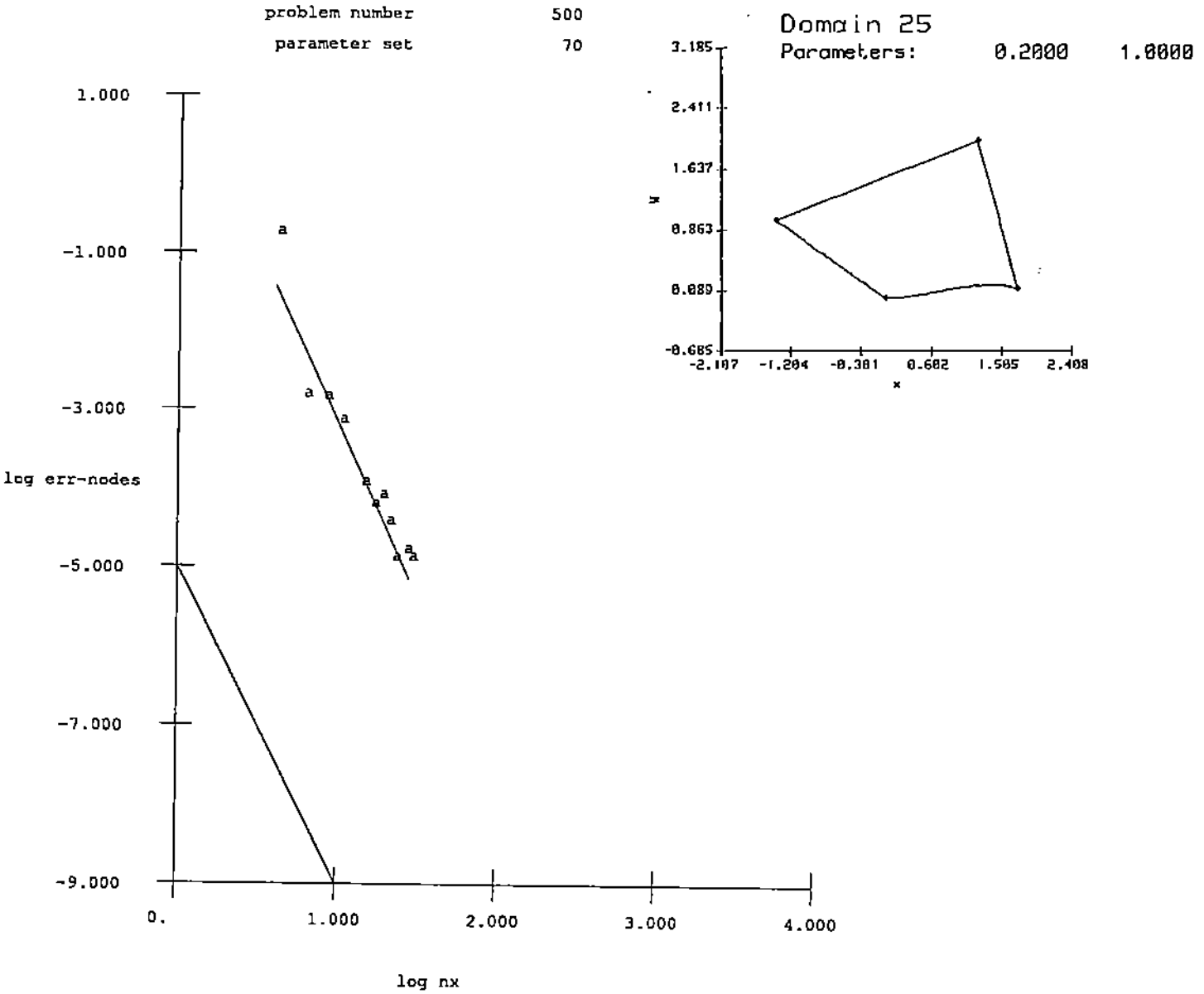

problem number 500
parameter set 55
Domain 21
Parameters: 20.0000 0.6000
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx
2.477
1.869
1.261
0.653
0.044
-0.564
y
-1.466 -0.945 -0.424 0.097 0.619 1.140
x

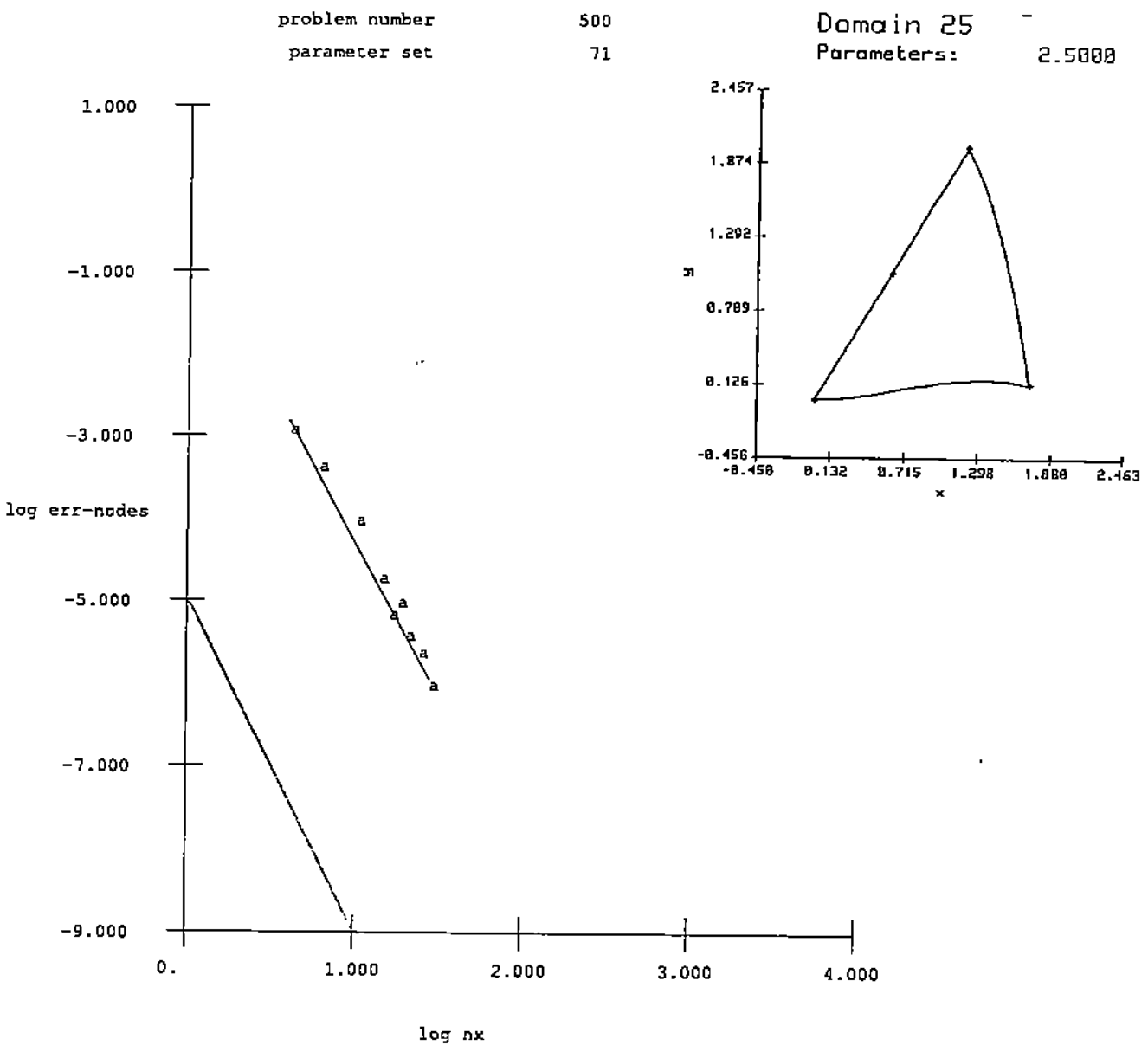

problem number 500
parameter set 57
Domain 22
Porometers: 1.0000 0.1000
2.134
1.593
1.052
0.511
-0.030
-0.571
y
-0.993
-0.236
0.522
1.279
2.037
2.794
x
1.000
-1.000
-3.000
log err-nodes
-5.000
-7.000
-9.000
0.
1.000
2.000
3.000
4.000
log nx


problem number 500
parameter set 59
Domain 22
Parameters: 0.8000 0.8000
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx
3.238
2.470
1.702
0.935
0.167
-0.601
-1.076
-0.388
0.460
1.227
1.995
2.763
y
x


problem number 500
parameter set 61
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx
Domain 23
Parameters: 0. 0.5000
3.695
2.822
1.948
1.074
0.200
-0.674
-1.684
-0.810
0.064
0.938
1.812
2.685


problem number 500
parameter set 69
Domain 25
Parameters: 0.2000 0.
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx
2.582
1.970
1.358
0.747
0.135
-0.477
-0.879
-0.267
0.345
0.956
1.568
2.180


problem number 500
parameter set 70
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx
Domain 25
Parameters: 0.2000 1.0000
3.185
2.411
1.637
0.863
0.089
-0.685
-2.197
-1.204
-0.381
0.682
1.505
2.408
y
x


problem number 500
parameter set 71
1.000
-1.000
-3.000
-5.000
-7.000
-9.000
log err-nodes
0.
1.000
2.000
3.000
4.000
log nx

Domain 25
Parameters: 2.5000 -1.0000
2.457
1.874
1.292
0.709
0.126
-0.456
-0.458
0.132
0.715
1.298
1.880
2.463
x
y