1987

Ultimate Characterizations of the Burst Response of an Interval Searching Algorithm

Philippe Jacquet

Wojciech Szpankowski
Purdue University, spa@cs.purdue.edu

Report Number:

87-711

ULTIMATE CHARACTERIZATIONS OF THE
BURST RESPONSE OF AN INTERVAL
SEARCHING ALGORITHM

Philippe Jacquet
Wojciech Szpankowski

CSD-TR-711
September 1987

ULTIMATE CHARACTERIZATIONS OF THE BURST RESPONSE OF AN INTERVAL SEARCHING ALGORITHM

Philippe Jacquet[+]
INRIA
Rocquencourt
78153 Le Chesnay
FRANCE

Wojciech Szpankowski[*]
Department of Computer Science
Purdue University
West Lafayette, IN 47907
USA

ABSTRACT

The interval searching algorithm for broadcast communications of Gallager, Tsybakov and Mikhailov is analyzed. We present ultimate characterizations of the burst response of the algorithm, that is, when the number of collided packets becomes large. Three quantities are of interest: the *conflict resolution interval* (CRI), the fraction of the *resolved interval* (RI) and the number of *resolved packets* (RP). If n is the multiplicity of a conflict, then it is proved that the m-th moments of CRI, RI and RP are $O(\log^m n)$, $O(n^{-m})$ and $O(1)$ respectively. In addition, for the first two moments of the parameters precise asymptotic approximations are presented. The methodology proposed in this paper is, in particular, applicable to asymptotic analysis of other interval searching algorithms, and in general, to other tree-type data structure algorithms.

Keywords. interval searching algorithms, analysis of algorithms, data structure, asymptotic analysis, functional equations, Mellin transform

1. INTRODUCTION

In a broadcast packet-switching network a number of users share a common communication channel. Since the channel is the only way of communications among the users, packet collisions are inevitable if a central coordination is not provided. The problem is to find an efficient algorithm for retransmitting conflicting packets. In recent years *conflict resolution algorithms* (CRA) [1], [2], [5], [8]–[10], [13]–[16] have become increasingly popular, mainly due to a nice stability property [10]. The basic idea of CRA is to solve each conflict by splitting it into smaller conflicts

---

[+] The research of this author was done primarily while visiting Purdue University.

[*] This research was supported in part by the National Science Foundation under grant NCR-8702115.

(divide-and-conquer algorithm). This is possible if each user observes the channel and learns whether in the past it was idle, success or collision transmissions. The partition of a conflict can be made on the basis of a random variable (flipping a coin) [2], [8], [10], [13], [15] or on the basis of the time a user became active [2], [5], [9], [10], [14], [16]. The former algorithm is known as Capetanakis-Tsybakov-Mikhailov algorithm [2], [15] (stack algorithm) while the latter as Gallager-Tsybakov-Mikhailov algorithm [5] [16] known also as an interval-searching algorithm.

For interval searching algorithms, three parameters are of interest: the *conflict resolution interval* (CRI), the fraction of the *resolved interval* (RI) and the number of *resolved packets* (RP). For the initial multiplicity of a conflict n (i.e., n packets collide) the m-th moments of the above parameters are denoted as T_n^m, W_n^m and C_n^m respectively. We prove that $T_n^m = O(\log^m n)$, $W_n^m = O(n^{-m})$ and $C_n^m = O(1)$ for large n. However, for the first two moments, which are the most important from a practical viewpoint, we offer precise estimations. In particular, we show that asymptotically $W_n^1 \sim a/(n+1) + P(\log n)$, $C_n^1 \sim a + P(\log n)$ and $T_n^1 \sim \log n + c + P(\log n)$, where a, c are constants which will be analytically determined, and $P(\log n)$ is a fluctuating function with a small amplitude. For the variance, we obtain var $W_n \sim a'/(n+1)^2$, var $C_n \sim a''$ and var $T_n \sim c'$, respectively.

To the authors' best knowledge, previous research has been restricted only to the first moments of CRI, RI and RP. Numerical evaluations of T_n^1, W_n^1 are reported in [16] and [9]. In [14], Szpankowski has proposed an analytical method to evaluate modified quantities $\bar{T}_n^1$, $\bar{W}_n^1$ of T_n^1 and W_n^1, and estimated the error functions $\delta_n = T_n^1 - \bar{T}_n^1$ and $\varepsilon_n = W_n^1 - \bar{W}_n^1$. It was proved that the error function $\delta_n = O(1)$ and the modified CRI length is $\bar{T}_n^1 \sim \log_2 n$, hence the asymptotics for T_n^1 is done. Unfortunately, the error function ε_n is $O(n^{-1})$ as the leading factor in W_n^1. This was pointed out by Dr. Andrew Odlyzko when the second author presented his research in AT&T Lab, Murray Hill (also attended by the first author). This criticism is addressed in this paper by obtaining exact asymptotics for W_n^1 and other parameters as stated above.

2. PROBLEM STATEMENT AND MAIN RESULTS

Let us start with a short description of Gallager-Tsybakov-Mikhailov algorithm with ternary feedback [5], [16]. Assume a channel is slotted and a slot duration is equal to a packet transmission time. The algorithm defined below allows the transmission of the packets on the basis of their generation times and we assume that packets are generated according to a Poisson point process with rate λ. Access to the channel is controlled by a window based on the current age of packets. This window will be referred to as the *enabled interval* (EI). Let s_i denote the starting point for the i-th EI, and t_i is the corresponding starting point for the *conflict resolution interval* (CRI), where CRI represents the number of slots needed to resolve a collision. Initially, the enabled interval is set to be $[s_i, \min\{s_i + \tau, t_i\}]$, where τ is a constant which will be further optimized. At each step of the algorithm, we compute the endpoints of the EI based on the outcome of the channel. If at most, one packet falls in the initial EI, then the conflict resolution interval ends immediately, and $s_{i+1} = s_i + \min\{\tau, t_i - s_i\}$. Otherwise, the EI is split into two halves, and three cases must be considered:

(i) all users whose current age of packets fall into the first (left) half are allowed to transmit packets. If it causes next *collision*, all knowledge about the second half is erased, and the first half is immediately split into two halves,

(ii) if enabling the first half causes an *idle* slot, the second half is immediately split into two halves,

(iii) if the first half gives a *success*, the entire second half is enabled.

A CRI that begins with a collision, continues until enabling the second half of some pairs gives a success.

Assume that an initial collision of a CRI is of multiplicity n, that is, n packets collide in the first slot of CRI. Then all packets whose generation times fall into an interval (s_i, s_{i+1}) are successfully sent in the i-th CRI. The interval (s_i, s_{i+1}) is called the i-th *resolved interval* (RI). The parameters of interest are: the length of a CRI, T, the fraction of the resolved interval (i.e., the ratio of the resolved interval and τ), W, and the number of resolved packets (successfully transmitted) in an enabled interval, C. Let also N denote the multiplicity of a conflict. Then the m-th conditional moments of the above quantities are defined as $T_n^m = E\{T^m \mid N = n\}$, $W_n^m = E\{W^m \mid N = n\}$ and $C^m = E\{C^m \mid N = n\}$ respectively. For $m = 0$, we define $T_n^0 = W_n^0 = C_n^0 = 1$. Then

Theorem 1. (i) The m-th conditional moment of T satisfies the following recurrence

$$
\begin{aligned}
T_0^m &= T_1^m = 1 \\
(2^n - 2)T_n^m &= 2^n + n(2^m - 2) + \sum_{i=1}^{m-1} \binom{m}{i} [T_n^i + n2^{m-i}T_{n-1}^i] \\
&+ \sum_{j=2}^{n} \sum_{i=1}^{m-1} \binom{n}{j} \binom{m}{i} T_j^i + nT_{n-1}^m + \sum_{j=1}^{n-1} \binom{n}{j} T_j^m \quad n \geq 2.
\end{aligned}
\tag{2.1}
$$

(ii) The recurrence for W_n^m is

$$
\begin{aligned}
W_0^m &= W_1^m = 1 \\
(2^{n+m} - 2)W_n^m &= 1 + \sum_{i=1}^{m-1} \binom{n}{j} [W_n^i + nW_{n-1}^i] + nW_{n-1}^m + \sum_{j=1}^{n-1} \binom{n}{j} W_j^m \quad n \geq 2.
\end{aligned}
\tag{2.2}
$$

(iii) The recurrence for the m-th conditional moment of C becomes

$$
\begin{aligned}
C_0^m &= 0 \quad C_0^m = 1 \\
(2^n - 2)C_n^m &= n\sum_{i=1}^{m-1} \binom{m}{j} C_{n-1}^i + nC_{n-1}^m + \sum_{j=1}^{n-1} \binom{n}{j} C_j^m \quad n \geq 2.
\end{aligned}
\tag{2.3}
$$

Proof: Note that an n-conflict is resolved by splitting n into two groups, and according to the algorithm description three basic cases must be considered: (1) $n \to (0, n)$, (2) $n \to (1, n-1)$, (3) $n \to (j, n-j)$, $j \geq 2$. The probabilities of these cases are equal to 2^{-n}, $n2^{-n}$ and $\binom{n}{j} 2^{-n}$ respectively. Therefore, we obtain the following recurrences for

$m = 1, 2, \ldots,$

$$T_n^m = 2^{-n} E\{(1+T)^m \mid N = n\} + n2^{-n} E\{(2+T)^m \mid N = n-1\} + 2^{-n} \sum_{j=2}^{n} \binom{n}{j} E\{(1+T)^m \mid N = j\},$$

$$W_n^m = 2^{-n} E\{(½ + ½\, W)^m \mid N = n\} + n2^{-n} E\{(½ + ½\, W)^m \mid N = n-1\} + 2^{-n} \sum_{j=2}^{n} \binom{n}{j} E\{(½\, W)^m \mid N = j\},$$

$$C_n^m = 2^{-n} E\{C^m \mid N = n\} + n2^{n-1} E\{(1+C)^m \mid N = n-1\} + 2^{-n} \sum_{j=2}^{n} \binom{n}{j} E\{C^m \mid N = j\}.$$

After some algebraic manipulations, we prove the recurrences (2.1) (2.2) and (2.3).

□

We now give a summary of the main results, delaying more complicated proofs to the next two sections. Let $W_m(x)$, $C_m(x)$ and $T_m(x)$ represent the generating functions of W_n^m, C_n^m and T_n^m respectively, e.g., $T_m(x) = \sum_{n=0}^{\infty} T_n^m x^n/n!$. It turns out that to analyze recurrences (2.1)–(2.3), it is more convenient to deal with modified generating functions defined as: $w_m(x) = W_m(x)e^{-x}$, $c_m(x) = C_m(x)e^{-x}$ and $t_m(x) = T_m(x)e^{-x}$. Then recurrences (2.1)–(2.3) can be transformed into functional equations for $w_m(x)$, $c_m(x)$ and $t_m(x)$ (see next two sections) for which a generic form is

$$f(x) = 2^s f(x/2)a(x) + b(x) \tag{2.4}$$

where s is an integer, $a(x) = ½[1 + (1 + x/2)e^{-x/2}]$, and the nonhomogeneous term $b(x)$ depends on the particular recurrence (see Sections 3 and 4). The general solution for (2.4) is given in the next lemma.

Lemma 1. (i) Functional equation (2.4) possesses the following solution

$$f(x) = f^*(x) + \sum_{n=0}^{\infty} 2^{sn}\, b(x2^{-n}) \prod_{k=0}^{n-1} a(x2^{-k}) \tag{2.5}$$

where

$$f^*(x) = \lim_{n \to \infty} 2^{sn}\, f(x2^{-n}) \prod_{k=0}^{n-1} a(x2^{-k}) \tag{2.6}$$

assuming $f^*(x)$ exists and the series in (2.5) is convergent.

(ii) Let $w(x) = \prod_{k=0}^{\infty} a(x2^{-k})$ exist and $f(0) = f'(0) = \cdots = f^{(s-1)} = 0, f^{(s)}(0) \neq 0$. Then

$$f^*(x) = \begin{cases} 0 & s < 0 \\ x^s w(x) f^{(s)}(0)/s! & s \geq 0 \end{cases} \tag{2.7}$$

where $f^{(s)}(0)$ denotes the s-th derivative of $f(x)$ at $x = 0$.

(iii) If $a(0) < 2$ and $b(x) = O(x^{s+1})$ for $x \to 0$, then the series in (2.5) is convergent.

Proof: Part (i) follows immediately from formal iteration of (2.4). For (ii) assume first $s \geq 0$. Let $u = x2^{-n}$, then

$$f^*(x) = x^s w(x) \lim_{u \to 0} \frac{f(u)}{u^s} \tag{2.8}$$

Applying l'Hospital rule s-times one proves (2.7). The case $s < 0$ is easy and left to the reader.

For part (iii), we use D'Alembert's criterion, that is, to prove that $\sum_{n \geq 0} \alpha_n$ is convergent, it is sufficient to show that $\lim_{n \to \infty} \alpha_{n+1}/\alpha_n < 1$. In our case, assuming $u = x2^{-n}$

$$\lim \frac{\alpha_{n+1}}{\alpha_n} = a(0)2^s \lim_{u \to 0} \frac{b(u/2)}{b(u)} = \frac{a(0)}{2} < 1$$

where the last equality follows from the assumption $b(x) = O(x^{s+1})$ for $x \to 0$.

□

Using Lemma 1, we prove in the next sections our first main result.

Proposition. For large x, i.e., $x \to \infty$, the following holds for:

(i) the first two modified generating functions for the resolved interval

$$w_1(x) \sim \frac{a}{x} + \frac{a}{x} P(\log x) \tag{2.9}$$

$$w_2(x) \sim \frac{a(2+b)}{x^2} + P(\log x) \cdot \frac{a}{x^2} \tag{2.10}$$

where $P(\log x)$ is a fluctuating function with a very small amplitude, and

$$a = \exp[\alpha/\log 2 + \log 2/2] \approx 2.505 \;\; ; \;\; b = \beta/\log 2 \approx 1.692 \tag{2.11}$$

where

$$\alpha = \int_0^\infty \frac{xe^{-x} \log x}{1 + (1+x)e^{-x}}\, dx \approx 1.496 \;\; ; \;\; \beta = \int_0^\infty \frac{e^{-x/2}}{1 + (1+x/2)e^{-x/2}}\, dx = 1.17 \tag{2.12}$$

(ii) the modified generating functions for the number of resolved packets

$$c_1(x) = xw_1(x) \sim a + a\, P(\log x) \tag{2.13}$$

$$c_2(x) \sim a(1+b) + a\, P(\log x) \tag{2.14}$$

(iii) the first two modified generating functions for the conflict resolution interval

$$t_1(x) \sim \log_2 x + c + a\, P(\log x) \tag{2.15}$$

$$t_2(x) \sim \log_2^2 x + 2\,c\, \log_2 x + d + a\, P(\log x) \tag{2.16}$$

where

$$c = 2 - \frac{\mu a}{(\log 2)^2} \approx 4.149 \qquad d = \frac{5}{6} + 3(c-1) - c^2 - \frac{a\nu}{(\log 2)^2} = 17.169 \tag{2.17}$$

and

$$\mu = \int_0^\infty f_1(x) \frac{\log x}{x}\, dx = -0.41 \qquad \nu = -\tfrac{1}{2} \int_0^\infty f_2(x) \log^2 x\, dx = -1.484 \tag{2.18}$$

where the functions $f_1(x)$ and $f_2(x)$ depend on $a(x)$, $b(x)$ and $w_1(x)$, and are given in the next sections.

□

The next stage consists in translating the expansions of $T_m(x)$, $W_m(x)$ and $C_m(x)$ for $x \to \infty$ into information about the asymptotics of their coefficients. That is, if $F(x) = f(x)e^x$ is the exponential generating function and $f(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} e^{-x}$, then by the Cauchy formula [7] the coefficient f_n is given by

$$f_n = \frac{n!}{2\pi i} \oint f(x)e^x \frac{dx}{x^{n+1}} \tag{2.19}$$

where the integration is done on the circle of center 0 and radius n. Note, however, that $f(x) \sim g(x)$ does not necessarily imply $f_n \sim g_n$. We use the following result (see also [11]).

Lemma 2. Let $f(x) = F(x)e^{-x}$ and S_θ be a cone $S_\theta = \{x: |\arg x| < \theta, 0 < \theta < \pi/2\}$. If for x in the cone, that is, for $x \in C_\theta$, $x \to \infty$

$$f(x) \sim x^p (\log x)^q \tag{2.20}$$

for some p and q, and for x outside the cone C_θ, i.e., $x \notin C_\theta$

$$|F(x)| < e^{\alpha|z|} \tag{2.21}$$

for $0 < \alpha < 1$, then for large n the coefficient f_n of $f(x)$ asymptotically satisfies

$$f_n = n^p (\log x)^q + O(n^{p - ½} \log^{q-1} n) \tag{2.22}$$

Proof. The coefficient f_n is computed by Cauchy formula (2.19) where the integration is done along the circle $|x| = n$. Two cases are considered: $x \in S_\theta$, and $x \notin S_\theta$. In the latter case, using (2.21) and Stirling's formula [7] we obtain an estimate

$$\left| \frac{n!}{2\pi i} \oint F(x) \frac{dx}{x^{n+1}} \right| < n!\, e^{\alpha|n|} \frac{1}{n^n} = e^{-(1-\alpha)n}$$

hence for large n the contribution of the integral outside the cone S_θ is negligible.

Assume now $x \in C_\theta$. Using substitution $x = ne^{iv}$ in (2.19) and Stirling's formula one finds

$$f_n = I_n \, [1 + O\,(n^{-1})] \tag{2.23}$$

where

$$I_n = \sqrt{\frac{n}{2\pi}}\; e^{-n} \int_{-\pi}^{\pi} f\,(ne^{iv})\; e^{ne^{iv}}\; e^{-inv}\; dv \quad .$$

The last integral can be further simplified by developing e^{iv} and the change of variable $y = \sqrt{n}\;\; v$. Then

$$I_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} f\,(n + iy\sqrt{n} + O(y^3\sqrt{n}))\; e^{-y^2/2}\; e^{O(y^3/\sqrt{n})}\; dy \tag{2.24}$$

But by (2.20), $f\,(n + iy\sqrt{n}\,) = n^p(\log n)^q + O(n^{p\,-\,1/2}\log^{q-1} n)$, hence the Lebesques dominated convergence theorem, (2.23) and (2.24) readily give (2.22).

□

Now we are prepared to state our second main result. Applying Lemma 2 to Proposition (condition (2.21) will be verified in the next sections) one immediately proves

Theorem 2. For large n, the following holds for

(i) the fraction of the resolved interval

$$W_n^1 = \frac{a}{n+1} + \frac{1}{n+1}\,P\,(\log n) + O\,(n^{-3/2}) \approx \frac{2.505}{n+1} \tag{2.25}$$

$$\begin{aligned} \mathrm{var}\; W_n &= \frac{a\,(2+b)}{(n+1)(n+2)} - \frac{a^2}{(n+1)^2} + \frac{1}{(n+1)^2}\,P_1(\log n) + O\,(n^{-3/2}) \\ &\approx \frac{a\,(2+b) - a^2}{(n+1)^2} + \frac{1}{(n+1)^2}\,P_1(\log n) + O\,(n^{-3/2}) \approx \frac{2.97}{(n+1)^2} \end{aligned} \tag{2.26}$$

$$W_n^m = O\,(n^{-m}) \tag{2.27}$$

(ii) the number of resolved packets

$$C_n^1 = a + P\,(\log n) + O\,(n^{-1/2}) \approx 2.505 \quad . \tag{2.28}$$

$$\mathrm{var}\; C_n = a + ab - a^2 + P_2(\log n) + O\;\,(n^{-1/2}\,) \approx 0.47 \tag{2.29}$$

$$C_n^m = O\,(1) \tag{2.30}$$

(iii) the conflict resolution interval

$$T_n^1 = \log_2 n + c + P(\log n) + O\ (n^{-1/2}) \approx \log_2 n + 4.144 \qquad (2.31)$$

$$\mathrm{var}\ T_n = \frac{5}{6} + 3(c-1) - c^2 - \frac{a\nu}{(\log 2)^2} + P_3(\log n) + O\ (n^{-1/2}) \approx 4.7 \qquad (2.32)$$

$$T_n^m = O(\log^m m) \qquad (2.33)$$

□

The consequences of the main results will be discussed in the last section.

3. ANALYSIS OF THE FIRST MOMENTS

In this section we concentrate on derivations of the asymptotics for W_n^1, C_n^1 and T_n^1. From the methodological view point, we mention here the crucial role in the analysis of the generating function $W_1(x)$ of W_n^1. This is a consequence of the fact, that $w_1(x) = W_1(x)e^{-x}$ satisfies homogeneous functional equation of type (2.4), that is, $b(x) \equiv 0$ in (2.4). We shall see that all other generating functions, which satisfy nonhomogeneous functional equation (2.4), can be expressed in terms of $w_1(x)$. In the derivations we use extensively Mellin transform. A good reference for properties of the Mellin transform is [4] and [3].

3.1 The fraction of the resolved interval

The first moment of the fraction of resolved interval W_n^1, satisfies recurrence (2.2) with $m = 1$. Let $W_1(x)$ be the exponential generating function of W_n^1, and define $w_1(x) = W_1(x)e^{-x}$. Multiplying both sides of (2.2) by $x^n/n!$ and summing up we find

$$w_1(x) = ½\,[1 + (1 + x/2)e^{-x/2}]w_1(x/2) \qquad (3.1)$$

This functional equation falls into (2.4) with $b(x) \equiv 0$ (homogeneous equation) and $a(x) = ½\,[1 + (1 + x/2)e^{-x/2}]$. Let us also define $a_1(x) = a(2x)$. Then, by Lemma 1 and $w_1(0) = 1$ the solution of (3.1) is

$$w_1(x) = \prod_{k=1}^{\infty} a_1(x2^{-k}) = \prod_{k=0}^{\infty} a(x2^{-k}) \tag{3.2}$$

Let $l(x) = \log w_1(x)$ and $g_1(x) = \log a_1(x)$. Then (3.2) becomes

$$l(x) = \sum_{k=1}^{\infty} g_1(x2^{-k}) \tag{3.3}$$

Eq. (3.3) is used to derive asymptotics for $l(x)$ for $x \to \infty$. At this stage, we start plunging into complex analysis, more precisely: into the Mellin transform. Let us first give an overview of the methodology. We use the following facts about Mellin transform [4], [3].

Property 1. If $f(x)$ is a piecewise continuous on $[0,\infty]$, and

$$f(x) = O(x^{\alpha}) \quad x \to 0 \ ; \ f(x) = O(x^{\beta}) \quad x \to \infty \tag{3.4a}$$

then the Mellin transform of $f(x)$ defined as [4], [3]

$$f^*(s) = \int_0^{\infty} f(x)\, x^{s-1}\, dx \tag{3.4b}$$

exists in the *fundamental strip*

$$-\alpha < \mathrm{Re}\ s < -\beta$$

•

Another property of the Mellin transform, which we will use extensively, allows us to compute some intricate sums. Namely

Property 2. A *harmonic sum* of the form

$$F(x) = \sum_{k=0}^{\infty} f(\mu_k x) \tag{3.5a}$$

has a simple Mellin transform

$$F^*(s) = f^*(s) \sum_{k=0}^{\infty} \mu_k^{-s} \tag{3.5b}$$

•

Knowing $F^*(s)$, under fairly general conditions, one obtains an asymptotic expansion of $F(x)$ for $x \to \infty$ from the singularities of $F^*(s)$, that is [4],

Property 3. The following holds for $x \to \infty$

$$F(x) \sim - \sum_{\mathrm{Re}\,\alpha \geq 0} \mathrm{Res}\,\{F^*(s)\,;\; s = \alpha\} \tag{3.6a}$$

where $\mathrm{Res}_{s=\alpha}\,\{F^*(s)\}$ denotes residue of $F^*(s)$ at singularity α.

•

On the other hand, the characteristic properties of the Mellin transform are reflected by the singularities of the transform function.

Property 4. If for $x \to \infty$ $f(x) \sim d\, x^{\beta}$, then

$$f^*(s) \sim - \frac{d}{s+\beta} \qquad s \to -\beta \tag{3.6b}$$

•

Noting that (3.3) falls into the harmonic sum (3.5a) in Property 2 we first compute the Mellin transform of $l(x)$, $l^*(s)$, as suggested in (3.5b). The fundamental strip of $l^*(s)$ is defined in Property 1. Then, by Property 4 we find the asymptotic expansion for $l^*(s)$, and by the inversion formula (3.6a) in Property 3 we determine asymptotics for $l(x)$.

This is the plan for dealing with (3.3). Note that (3.3) is of (3.5a) form, hence for Re $s < 0$ the Melling transform of (3.3) is

$$l^*(s) = \frac{2^s}{1-2^s}\, g_1^*(s) \tag{3.7}$$

But $g_1(x) = O(x^2)$ for $x \to 0$ and $g_1(x) = O(1) = -\log 2$ for $x \to \infty$. Hence, $g_1^*(s)$ exists in the strip $-2 < \mathrm{Re}\, s < 0$. Note also that by (3.6b) $g^*{}_1(s)$ has a pole at $s = 0$ and $g^*{}_1(s) \sim \frac{1}{s} \log 2$. But, the first factor in (3.7), that is, $2^s/(1-2^s)$ has poles at $\chi_k = 2\pi i k/\log 2$, $k = 0, \pm 1, \ldots$. The singularity at $k = 0$, i.e., $\chi_0 = 0$ is the most difficult to treat, since it is a double pole, and it determines the leading component of the asymptotics [4]. Let first $k = 0$. To use (3.5a) we need Taylor expansions for $2^s/(1-2^s)$ and $g_1^*(s)$, that is

$$\frac{2^s}{1-2^s} = \frac{1}{\log 2} \cdot \frac{1}{s} - \frac{1}{2} + O(s) \tag{3.8a}$$

$$g_1^*(s) = \frac{\log 2}{s} + \alpha + O(s) \tag{3.8b}$$

We now determine the constant α. By definition of $g_1^*(s)$ and integration by part

$$g_1^*(s) = \int_0^\infty \log ½[1 + (1+x)e^{-x}]x^{s-1}\, dx = \frac{1}{s}\int_0^\infty \frac{xe^{-x}}{1+(1+x)e^{-x}}\, x^s\, dx \tag{3.9}$$

Using $x^s = 1 + s \log x + O(s^2)$ we immediately obtain from (3.9) $sg_1^*(s) = \log 2 + \alpha s + O(s^2)$ where

$$\alpha = \int_0^\infty \frac{xe^{-x} \log x}{1 + (1+x)e^{-x}}\, dx \tag{3.10a}$$

As a side-effect, we prove also that

$$\int_0^\infty \frac{xe^{-x}}{1 + (1+x)e^{-x}} = \log 2 \tag{3.10b}$$

The rest is easy. Multiplying (3.8a) and (3.8b) and taking into account additional simple poles at $s = \chi_k = \frac{2\pi ik}{\log 2}$ $k \neq 0$, we obtain

$$l^*(s) = -\frac{1}{s^2} - \left[\frac{\alpha}{\log 2} + \frac{\log 2}{2}\right] \cdot \frac{1}{s} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{g_1^*(s)}{s - \chi_k} + O(1) \tag{3.11}$$

Applying Property 3 to (3.11) and computing appropriate residues imply [4], [7]

$$l(x) = -\log x + \left[\frac{\alpha}{\log 2} + \frac{\log 2}{2}\right] + P(\log x) + O(x^{-M}) \tag{3.12}$$

where M, is a large positive constant, and

$$P(u) = \frac{1}{\log 2} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} g_1^*(2\pi ik/\log 2)\exp[-2\pi iku/\log 2] \tag{3.13}$$

The fluctuating function $P(\log x)$ can be safely ignored in practice, since it has a very small amplitude. Finally, (3.12) implies

$$w_1(x) = \exp[l(x)] \sim \frac{a}{x} + \frac{a}{x} P(\log x) \tag{3.14}$$

where $a = \exp\left[\dfrac{\alpha}{\log 2} + \dfrac{\log 2}{2} \right]$.

This proves formula (2.9) of Proposition (i). To estimate coefficient W_n^1 of $W_1(x)$, we apply Lemma 2. We only need to prove that (2.21) holds, that is, $|W_1(x)| < e^{\alpha |z|}$ for $\alpha \in (0,1)$ outside the cone S_θ. Since $W_1(x) = ½ \, (e^{x/2} + 1 + \frac{x}{2}) W_1(x/2)$, hence $|W_1(x)| < e^{|x|/2} |W_1(x/2)|$. This implies $W_1(x) = (1 + O(x^2))e^x$, and for $x \in S_\theta$ $|W_1(x)| = (1 + O(x^2))e^{\alpha |x|}$ where $\alpha = \cos \theta$. Thus, $|W_1(x)| < e^{(\alpha + \varepsilon)|z|}$ and condition (2.21) in Lemma 2 holds. Thus, with (3.14) and (2.22) this proves formula (2.25) in Theorem 2(i).

3.2 The number of resolved packets

The first moment of the number of resolved packets, C_n^1, satisfies recurrence (2.3) with $m = 1$. Then, after simple algebra, one checks that the modified generating function $c_1(x) = C_1(x)e^{-x}$ satisfies the functional equation

$$c_1(x) = [1 + (1 + x/2)e^{-x/2}] c_1(x/2) = 2a(x) c_1(x/2). \tag{3.15}$$

Let $c_1(x) = xF_1(x)$ for some function $F_1(x)$. Then (3.15) is transformed into $F_1(x) = a(x)F_1(x/2)$, which is exactly the same as the functional equation for $w_1(x)$ (see (3.17)). Thus $c_1(x) = xW_1(x)$ and $C_n^1 = nW_{n-1}^1$. Proposition (ii) formula (2.13) and Theorem 2(ii), Eq. (2.28) follow immediately.

The constant a (see (2.11)) again appears in our derivation. It depends on α defined by the infinite integral (3.10a). The integral in (3.10a) can be either evaluated numerically using a programming package (e.g., IMSL) or we can transform (3.10a) into infinite series. We illustrate here the second method. Noting that for $|f(x)| < 1$ [6]

$$\frac{1}{1+f(x)} = \sum_{k=0}^{\infty} (-1)^n [f(x)]^n \tag{3.16a}$$

and using the fact (see [6])

$$\int_0^{\infty} x^n \, e^{\mu x} \log x \, dx = \frac{n!}{\mu^{n+1}} \left[H_{n+1} - \gamma - \ln \mu \right] \tag{3.16b}$$

where H_n is the n-th harmonic number, the integral becomes

$$\begin{aligned} \alpha = \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} x e^{-x} (1+x)^n \, e^{-nx} \log x \, dx = \\ \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{n} \binom{n}{k} \frac{(k+1)!}{(n+1)^{k+2}} \left[H_{k+1} - \gamma - \log(n+1) \right] \end{aligned} \tag{3.17}$$

where $\gamma = 0.577$ is the Euler constant. We comment here that the series is rather slowly convergent.

3.3 The conflict resolution interval

The recurrence for T_n^1 is given in (2.1) with $m = 1$. Let $T_1(x)$ be the exponential generating function for T_n^1, and define $t_1(x) = T_1(x)e^{-x} - 1$. Then, $t_1(x)$ satisfies recurrence

$$\begin{aligned} t_1(x) &= [1 + (1 + x/2)e^{-x/2}] t_1(x/2) + 1 - (1 + \frac{3}{2} x) e^{-x} + \frac{x}{2} e^{-x/2} \\ &= 2a(x) \, t_1(x/2) + b(x) \end{aligned} \tag{3.18}$$

where $a(x) = ½ \, [1 + (1 + x/2)e^{-x/2}]$ and

$$b(x) = 1 - (1 + \frac{3}{2} x) e^{-x} + \frac{x}{2} e^{-x/2} \tag{3.19}$$

The recurrence (3.18) falls into (2.4) and by Lemma 1 it has solution (2.5). Note that our definition of $t_1(x)$ implies that $t_1(0) = 0$, thus the solution of (3.18) is

$$t_1(x) = \sum_{n=0}^{\infty} 2^n \, b(x 2^{-n}) \prod_{k=0}^{n-1} a(x 2^{-k}) \tag{3.20}$$

The series is convergent since $b(x) = O(x^2)$ for $x \to 0$ as required in Lemma 1(iii). Define

$$w_1(x) = \prod_{k=0}^{\infty} a(x2^{-k}) \tag{3.21}$$

which falls into our solution of the homogeneous equation (3.2) on the fraction of the resolved interval. Noting that $\prod_{k=0}^{n-1} a(x2^{-k}) = w_1(x)/w_1(x2^{-n})$ one transforms (3.21) into

$$\frac{t_1(x)}{w_1(x)} = \sum_{n=0}^{\infty} 2^n \frac{b(x2^{-n})}{w_1(x2^{-n})} \tag{3.22}$$

Finally, defining

$$Q_1(x) = \frac{t_1(x)}{xw_1(x)} \qquad q_1(x) = \frac{b(x)}{xw_1(x)} \tag{3.23}$$

we obtain from (3.22)

$$Q_1(x) = \sum_{n=0}^{\infty} q_1(x2^{-n}) \tag{3.24}$$

This falls into our general harmonic sum discussed in Property 2, and by (3.5), the Mellin transform of (3.24) is

$$Q_1^*(s) = \frac{q_1^*(s)}{1-2^s} \tag{3.25}$$

for $-1 < \mathrm{Re}\ s < 0$. Unfortunately, straightforward application of the same trick as in Section 3.1 does not work here, since the appropriate integrals as in (3.9) do not exist. This follows from the fact that $w_1(x)$ involved in the definition of $q_1(x)$ has infinity many poles for $\mathrm{Re}\ s = 0$. To avoid this problem, we define a new function

$$f_1(x) = q_1(x) - q_1(x/2) = \frac{b(x) - 2b(x/2)\,a(x)}{xw_1(x)} \tag{3.26}$$

Note now that $f_1(x) = O(e^{-x})$ for $x \to \infty$ and the Mellin transform $f_1^*(s)$ exists in the strip $-1 < \mathrm{Re}\ s < \infty$. For, $f_1^*(s) = q_1(s)/(1-2^s)$, (3.25) becomes

$$Q_1^*(s) = \frac{f_1^*(s)}{(1-2^s)^2} \tag{3.27}$$

But $f_1^*(s)$ is analytical at $s=0$, hence $f_1^*(0)=f_1^*$ and $\left.\frac{d}{ds} f^*(s)\right|_{s=0} = \mu$ exist. Trivial computations show

$$f_1^* = \int_0^\infty f_1(x)\,\frac{dx}{x} \qquad \mu = \int_0^\infty f_1(x)\,\frac{\log x}{x}\,dx \tag{3.28}$$

The second integral must be evaluated numerically, while the first reduces to

$$\begin{aligned} f_1^* = \int_0^\infty \left[\frac{q_1(x)}{x} - \frac{q_1(x/2)}{x} \right] dx = \lim_{k\to\infty} \int_0^{2^{k+1}} \frac{q_1(x) - q_1(x/2)}{x}\,dx = \\ \lim_{k\to\infty} \int_{2^k}^{2^{k+1}} \frac{q_1(x)}{x}\,dx = \frac{\log 2}{a} \end{aligned} \tag{3.29}$$

where the last equality follows from (3.14) and (3.23) noting that $\lim_{x\to\infty} q_1(x) = \frac{1}{a}$.

The rest is really simple and standard. Using the following developments

$$\frac{1}{(1-2^s)^2} = \frac{1}{s^2(\log 2)^2} - \frac{1}{s\,\log 2} + O(1)$$

$$f_1^*(s) = \frac{\log 2}{a} + s\,\mu + O(s^2)$$

one obtains:

$$Q_1^*(s) = \frac{1}{a\,\log 2}\cdot\frac{1}{s^2} - \frac{1}{s}\left[\frac{1}{a} - \frac{1}{(\log 2)^2}\right] + O(1) \tag{3.30}$$

Taking additional poles of $(1-2^s)^2$ at $\chi_k = \frac{2\pi i k}{\log 2}$, $k \neq 0$, and translating (3.30) into $Q_1(x)$ through Property 3, we find for $x\to\infty$

$$Q_1(x) = \frac{\log x}{a\,\log 2} + \left(\frac{1}{a} - \frac{\mu}{\log^2 2}\right) + P(\log x)$$

Finally definition (3.23) of $Q_1(x)$ and estimation (3.14) for $w_1(x)$ imply for $x\to\infty$

$$T_1(x)e^{-x} \sim \log_2 x + 2 - \frac{\mu a}{(\log 2)^2} + a\,P(\log x) \tag{3.31}$$

which proves formula (2.15) of Proposition (iii). Application of Lemma 2 (in the same manner as for $w_1(x)$) to (3.31) establishes formula (2.31) in Theorem 2(iii).

4. ANALYSIS OF VARIANCES AND HIGHER MOMENTS

In the previous section, we have established methodology to study nonhomogeneous functional equation of type (2.4). It consists of two parts: first the solution to a homogeneous equation on $w_1(x)$ is found, then using it and properties of the Mellin transform, we obtain asymptotic for nonhomogeneous one as it was done, for example, in Section 3.3. Finally, we appeal to Lemma 2 to establish asymptotics for the coefficients of the generating function. The same plan is adopted in this section.

4.1 The fraction of the resolved interval

Let us compute first the second moment, W_n^2, of the fraction of the resolved interval, that is, we deal now with recurrence (2.3) for $m = 2$. Let $W_2(x)$ represents the exponential generating function for W_n^2, and, as before, define $w_2(x) = W_2(x)e^{-x}$. Then, simple algebra reveals

$$w_2(x) = 2^{-1}\, a(x) w_2(x/2) + b(x) \tag{4.1a}$$

where $a(x)$ is defined as before (see (3.1)), and

$$b(x) = w_1(x) - ½\, w_1(x/2) \tag{4.1b}$$

with $w_1(x)$ defined in (3.1). The functional equation (4.1) falls into (2.4) with $s = -1$ and $a(0) = ½$. Since $b(x) = O(1)$ for $x \to 0$ the recurrence (4.1) has the following solution

$$w_2(x) = \sum_{n=0}^{\infty} 2^{-n}\, b(x2^{-n}) \prod_{k=0}^{n-1} a(x2^{-k}) = w_1(x) \sum_{n=0}^{\infty} 2^{-n}\, \frac{b(x2^{-n})}{w_1(x2^{-n})} \tag{4.2}$$

where the last equality follows from the same arguments as (3.22) for $t_1(x)$. Define now

$$Q_2(x) = \frac{x\, w_2(x)}{w_1(x)} \qquad q_2(x) = \frac{x\, b(x)}{w_1(x)} \tag{4.3}$$

then (4.2) becomes

$$Q_2(x) = \sum_{n=0}^{\infty} q_2(x2^{-n})$$

which is the desired harmonic sum, and the Mellin transform is

$$Q_2^*(s) = \frac{q_2^*(s)}{1 - 2^s} \tag{4.4}$$

for $-1 < \text{Re } s < 0$. Note also that $q_2^*(s)$ exists for $-1 < \text{Re } s < \infty$. Thus, the only poles are the roots of the denominator, that is, $\chi_k = 2\pi ik/\log 2$. We deal first with $\chi_0 = 0$. The analysis of $q_2^*(s)$ is simple, since by (4.3), (4.1b) and (3.1)

$$q_2(x) = \frac{x\,[w_1(x) - \tfrac{1}{2}\,w_1(x)]}{w_1(x)} = x - \frac{x}{1 + (1 + x/2)e^{-x/2}} \tag{4.5}$$

The Mellin transform $q_2^*(s)$ exists at $s = 0$ and

$$q_2^*(0) = \int_0^{\infty} \frac{q_2(x)}{x}\,dx = \int_0^{\infty} \frac{(1 + x/2)e^{-x/2}}{1 + (1 + x/2)e^{-x/2}}\,dx \tag{4.6}$$

This integral can be computed as follows. By (3.10b)

$$\int_0^{\infty} \frac{e^{-x/2}\,dx}{1 + (1 + x/2)e^{-x/2}} = 2 \log 2$$

and let

$$\beta = \int_0^{\infty} \frac{e^{-x/2}}{1 + (1 + x/2)e^{-x/2}}\,dx \tag{4.7}$$

Then $q_2^*(0) = 2 \log 2 + \beta$. The evaluation of (4.7) can be done either by using a programming package, or through the same derivation as for (3.17), that is,

$$\beta = 2 \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(n+1)^{k+1}} \tag{4.8}$$

Finally, using the Taylor expansions for $1/(1 - 2^s)$, that is,

$$\frac{1}{1-2^s} = -\frac{1}{s\ \log 2} + ½ + O(s) \tag{4.9}$$

and $q_2^*(s) = q_2^*(0) + O(s) = 2\log 2 + \beta + O(s)$ one shows

$$Q_2^*(s) = -\frac{2\log 2 + \beta}{\log 2} \cdot \frac{1}{s} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{q_2^*(s)}{s - \chi_k} + O(1)$$

and this is translated into (see (4.3))

$$w_2(x) \sim a(2+b)\frac{1}{x^2} + \frac{a}{x^2} P(\log x) \tag{4.10a}$$

where $b = \beta/\log 2$, and

$$P(u) = \frac{1}{\log 2} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} q_2^*(2\pi ik/\log 2) \exp[-2\pi iku/\log 2] \tag{4.10b}$$

This proves formula (2.10) in Proposition (i). To show formula (2.26), we apply Lemma 2 and a standard formula var $\mathbf{W}_n = W_n^2 - (W_n^1)^2$, where $P_1(\log n)$ in (2.26) is a combination of (4.10b) and (3.13).

To obtain ultimate asymptotic analysis for $\boldsymbol{W}_n$, we need only to prove formula (2.27) for the m-th moment of the fraction of the resolved interval. Using (2.2) for general m, one shows that the modified generating function, $w_m(x)$, satisfies

$$w_m(x) = 2^{-m+1}\, a(x) w_m(x/2) + b(x)$$

where $b(x) = O(e^{-x})$ for $x \to \infty$. Then, generalizing (4.3), we define

$$Q(x) = \frac{x^{m-1} w_m(x)}{w_1(x)} \qquad q(x) = \frac{x^{m-1} b(x)}{w_1(x)}$$

then $Q^*(s) = q^*(s)/(1-2^s)$. But $q^*(s) = O(1)$ for $s \to 0$, hence $Q^*(s) = O(s^{-1})$, and finally by (4.12) $W_m(x) = O(x^{-m})$. Using Lemma 2, we prove (2.27).

4.2 The number of resolved packets

The corresponding functional equation for the modified generating function, $c_2(x)$, satisfies

$$c_2(x) = 2\, a(x) c_2\, (x/2) + b(x) \tag{4.11}$$

where

$$b(x) = x e^{-x/2} c_1\, (x/2) = \frac{x^2}{2}\, e^{-x/2}\, w_1\, (x/2) \tag{4.12}$$

and the last equality follows from $c_1(x) = x\, w_1(x)$ proved in Section 3.2. Then by Lemma 1, the functional equation (4.11) possesses the solution

$$c_2(x) = c_2'(x) + c_2''(x) \tag{4.13}$$

where by Lemma 1, formula (2.7),

$$c_2'(x) = x\, w_1(x) \tag{4.14a}$$

and, since $b(x) = O(x^2)$ for $x \to 0$,

$$c_2''(x) = \frac{1}{w_1(x)} \sum_{n=0}^{\infty} 2^n\, \frac{b(x 2^{-n})}{w_1(x 2^{-n})} \tag{4.14b}$$

Define

$$Q_3(x) = \frac{c_2''(x)}{x\, w_1(x)} \qquad q_3(x) = \frac{b(x)}{x\, w_1(x)} \tag{4.14c}$$

then

$$Q_3(x) = \sum_{n=0}^{\infty} q_3(x 2^{-n})$$

which implies

$$Q_3^*(s) = \frac{q_3^*(s)}{1 - 2^s} \tag{4.15}$$

and the Mellin transform $Q_3^*(s)$ exists in $-1 < \mathrm{Re}\, s < 0$, while $q_3^*(s)$ exists for $-1 < \mathrm{Re}\ s < \infty$. To evaluate (4.15) we use (4.9) and compute $q_3^*(0)$ as

$$q_3^*(0) = \int_0^\infty \frac{q_3(x)}{x}\, dx = \int_0^\infty \frac{e^{-x/2}}{1 + (1 + x/2)\, e^{-x/2}}\, dx = \beta$$

Thus, appealing to the same arguments as before, $Q_3^*(s) \sim ab/s$; $b = \beta/\log 2$ and finally using (4.14a)

$$c_2(x) \sim a + ab + a\,P(\log x) \tag{4.16}$$

which shows formula (2.14) in Proposition (ii). To prove (2.29) in Theorem 2(ii), we need only to compute var $C_n^2 - (C_n^1)^2$, and to note that $P_2(\log n)$ is a combination of the fluctuating functions $P(\log n)$ defined above.

It remains to prove formula (2.30) in Theorem 2(ii). Note that $C_m(x)$ satisfies (4.11) with $b(x) = O(e^{-x})$ for $x \to \infty$. Hence, an appropriate $q(x)$ function defined as in (4.14c) is $q(x) = O(e^{-x})$ and $q^*(s) = O(1)$. Hence $C_m(x) = O(1)$, and (2.30) is done.

4.3 The conflict resolution interval

The analysis of T_n^2 is much more intricate. Define $t_2(x) = T_2(x)e^{-x} - 1$, then the modified generating function $t_2(x)$ satisfies

$$t_2(x) = 2a(x)\, t_2(x/2) + b(x) \tag{4.17a}$$

where

$$b(x) = 2t_1(x) + xe^{-x/2}\, t_1(x/2) + \frac{x}{2}\, e^{-x/2} - \frac{5}{2}\, xe^{-x/2} - 1 \tag{4.17b}$$

We split $b(x)$ into two functions, $b(x) = b_1(x) + b_1(x)$

$$b_1(x) = 1 - [\,1 + \frac{3}{2}\, x\,]e^{-x} + \frac{x}{2}\, e^{-x/2} \tag{4.18a}$$

$$b_2(x) = 2t_1(x) - 2 + xe^{-x/2}\, t_1(x/2) - xe^{-x} \tag{4.18b}$$

and $t_2(x) = t_2'(x) + t_2''(x)$ where $t_2'(x)$ $t_2''(x)$ satisfies (4.17a) with $b(x)$ replaced by $b_1(x)$ and $b_2(x)$, respectively. Then, by (3.18) $t_2'(x) = t_1(x)$, and

$$t_2(x) = t_1(x) + t_2''(x) \tag{4.19}$$

We concentrate now on $t_2''(x)$. Note that $b_2(x) = O(x^2)$ for $x \to 0$, thus by Lemma 1, after some algebra,

$$\frac{t_2''(x)}{w_1(x)} = \sum_{n=0}^{\infty} 2^n \frac{b_2(x2^{-n})}{w_1(x2^{-n})}$$

Defining $Q_4(x) = \dfrac{t_2''(x)}{x\, w_1(x)}$ and $q_4(x) = \dfrac{b_2(x)}{x\, w_1(x)}$ and computing the Mellin transform one finds

$$Q_4^*(s) = \frac{q_4^*(s)}{(1-2^s)} \tag{4.20}$$

Note that both Mellin transform exists for $-1 < \mathrm{Re}\, s < -\varepsilon$. As in the case of $t_1(x)$, the representation (4.20) is not appropriate for asymptotic extension. Thus, we define a new function

$$f_2(x) = q_4(x) - q_4(x/2) = \frac{b_2(x) - 2b_2(x/2)a(x)}{x\, w_1(x)} \tag{4.21a}$$

and

$$f_2^*(x) = \frac{q_4^*(s)}{(1-2^s)} \tag{4.21b}$$

however, $f_2^*(s)$ exists for $-1 < \mathrm{Re}\, s < 0$. The function $f_2^*(s)$ has pole at $s = 0$. Noting that $\lim_{x \to \infty} f_2(x) = 2/a$, by Property 4, Eq., (3.6b), one shows

$$f_2^*(s) = -\frac{2}{sa} + f_2^* + s\nu + O(s^2) \tag{4.22}$$

where f_2^* and ν will be given below. For,

$$\frac{1}{(1-2^s)^2} = \frac{1}{s^2 \log^2 2} - \frac{1}{s \log 2} + \frac{5}{12} + O(s)$$

then by (4.20) and (4.21b)

$$\begin{aligned} Q_3^*(s) = \frac{-2}{a \log^2 2} \cdot \frac{1}{s^3} + \frac{1}{s^2}\left[\frac{f_2^*}{\log^2 2} + \frac{2}{a \log 2}\right] - \\ \frac{1}{s}\left[\frac{5}{6a} + \frac{f_2^*}{\log 2} - \frac{\nu}{\log^2 2}\right] + O(1) \end{aligned} \tag{4.23}$$

To find explicit formula for $Q_4^*(s)$, we need f_2^* and ν. Integrating by part the formula on the Mellin transform, and using Taylor expansion for x^s, we find

$$sf_2^*(s) = \frac{2}{a} - s\int_0^\infty f_2'(x)\log x\,dx - \frac{s^2}{2}\int_0^\infty f_2'(x)\log^2 x\,dx + O(s^3)$$

thus

$$f_2^* = -\int_0^\infty f_2'(x)\log x\,dx, \quad \nu = -\tfrac{1}{2}\int_0^\infty f_2'(x)\log^2 x\,dx \tag{4.24}$$

The quantity ν must be evaluated numerically. For f_2^* we have

$$f_2^* = -\int_0^\infty f_2'(x)\log x\,dx = -\lim_{k\to\infty}\left\{ f_2(x)\log x\Big|_0^{2^{k+1}} - \int_0^{2^{k+1}} \frac{f_2(x)}{x}\,dx \right\}$$

Note that

$$f_2(x)\log x\Big|_0^{2^{k+1}} = \frac{2\log 2}{a}\cdot(k+1) \tag{4.25a}$$

and (see (3.29))

$$\begin{aligned}\int_0^{2^{k+1}} \frac{f_2(x)}{x}\,dx &= \int_{2^k}^{2^{k+1}} \frac{q_4(x)}{x}\,dx = \frac{2}{a}\int_{2^k}^{2^{k+1}} \frac{t_1(x)-1}{x}\,dx + O(k^{-1}) \\ &= \frac{2}{a}(c-1)\log 2 + \frac{\log 2}{a}(2k+1) + O(k^{-1})\end{aligned} \tag{4.25}$$

thus,

$$f_2^* = \frac{2}{a}(c-1)\log 2 - \frac{\log 2}{a} \tag{4.26}$$

Using (4.19), (2.15), (4.23) and (4.26), we prove finally formula (2.16) in Proposition (iii). By Lemma 2, also the following holds

$$T_n^2 = \log_2^2 n + 2c\log_2 n + \frac{5}{6} + 3(c-1) - \frac{a\nu}{\log^2 2} + P(\log n) + O(n^{-1/2}) \tag{4.27}$$

and formula (2.32) in Theorem 2(iii) follows from var $T_n = T_n^2 - (T_n^1)$.

Now we need only a final touch to prove (2.33) in Theorem 2(iii). Using recurrence (2.1) for T_n^m we prove first that $t_m(x) = T_m(x)e^{-x} - 1$ satisfies functional equation of type (4.17a) with $b(x) = O(\log^{m-1} x)$ for $x \to \infty$. Then, defining $Q(x)$, $q(x)$ and $f(x)$ as in (4.20) and (4.21a) we show that $f^*(s) = O(1/s^{m-1})$ which implies $Q^*(s) = O(1/s^{m+1})$ and $T_m(x) = (\log^m x)$. This by Lemma 2 proves (2.33).

5. FINAL REMARKS

In the last section, we offer some final remarks on the consequences of the obtained results for the performance evaluation of interval searching algorithms.

Maximum throughput for the algorithm

Let $x = \lambda\tau$, where λ is the input rate from the Poisson arrival process, and τ is the algorithm parameter defined in Section 2. The *maximum throughput*, $\lambda_{\max}$, is defined as the maximum rate of successful transmissions which assures that the average packet delay is finite. In other words, for $\lambda < \lambda_{\max}$ the algorithm is stable. But, the throughput is also the ratio of the average number of successful transmissions to the average conflict resolution length, hence (see [16])

$$\lambda_{\max} = \sup_x \frac{C_1(x)}{T_1(x)} = \sup_x \frac{x\, W_1(x)}{T_1(x)} \tag{5.1}$$

It is easy to compute $\lambda_{\max}$ numerically for some values of x, and taking the first eight or nine values of T_n^1 and W_n^1 reveals $\lambda_{\max} = 0.48771$ for $x = 1.26$. This comes from the fact that the maximum in (5.1) occurs for small values of x. Thus, from the numerical point of view, there is nothing to add. From the mathematical viewpoint, however, there is some interest in finding an analytical solution for $\lambda_{\max}$, e.g., to show whether some maximizing x is a global or a local maximum. Using Proposition, we immediately see that for large x

$$\lambda_{\max} \sim \sup_x \frac{a}{\log_2 x + c} \tag{5.2}$$

and λ_{max} is a decreasing function of x. Thus, $x_{op} = 1.26$ is a global one.

Non-poisson arrival processes and limiting distribution

The reason why the asymptotic behavior of T_n^1 and W_n^1 is not essential for the algorithm, follows from the fact that the probability of more than four packets collide, is less than 0.01 if x is near optimal value. This, however, holds only for the Poisson assumption model. If this assumption is dropped and the input process is more biased (e.g., multimodal or the peak of the distribution is for large values), then the asymptotic behavior becomes more important. In such a case, limiting distributions for T_n^1 plays a role. From our analysis (Theorem 2(iii)) it follows that $T_n^{'}$ is *not* normally distributed, since $T_n^1 = O(\log n)$ and var $T_n = O(1)$. In fact, the limiting distribution can be obtained using the idea from [11].

Analysis of other interval searching algorithms

The analysis of this paper has a methodological flavor. Other interval searching algorithms, like Berger's one [2], are analyzed exactly in the same manner (see [14] for more details). Furthermore, some algorithms in data structure can be analyzed by the proposed methodology. For example, the unsuccessful search in a Patricia trie satisfies a similar recurrence to (2.1). In that case, however, an exact solution of the recurrence is possible (see [12]). Nevertheless, modifying slightly the Patricia trie, that is, adding some weights to the unsuccessful search, disturb the recurrence so much that direct application of the method presented in [12] is not possible. Then, however, the idea explained in that paper can be applied.

REFERENCES

[1] Berger, T., Poisson multiple-access contention with binary feedback, *IEEE Trans. on Inf. Theory*, vol. IT-30, 5 (1984), pp. 745–751.

[2] Capentanakis, J., Tree algorithms for packet broadcast channels, *IEEE Trans. Inf. Theory*, IT-25 (1979), pp. 505–515.

[3] Davies, B., Integral transforms and their applications, Springer-Verlag, New York (1985).

[4] Flajolet, P., Regnier, M., Sedgewick, R., Mellin transform techniques and the analysis of algorithms, MIT Press, (1987).

[5] Gallager, R., Conflict resolution in a random access broadcast networks, *Proceedings AFOSR Workshop in Communication Theory and Applications*, (1978), pp. 74–76.

[6] Gradshteyn, I., Ryzhik, I., Table of integrals, series and products, Academic Press, New York (1980).

[7] Henrici, P., Applied and computational complex analysis, vol. 2, John Wiley and Sons, New York (1977).

[8] Hofri, M., Stack algorithms for collision-detecting channels and their analysis: A limited survey, *Proceedings of Intern. Seminar on Modeling and Performance Evaluation Methodology*, Paris (1983), vol. 1, pp. 53–78.

[9] J.–C. Huang, T. Berger, Delay analysis of 0.487 contention resolution algorithms, *IEEE Trans. on Communication*, vol. COM-34 (1986), pp. 916 - 926.

[10] *IEEE Transaction on Information Theory*, vol. IT-31, March (1985).

[11] Jacquet, P., Regnier, M., Limiting distributions for trie parameters, In: *Lecture Notes in Computer Science*, 214 (1986) pp. 196–210.

[12] Szpankowski, W., Solution of a linear recurrence equation arising in the analysis of some algorithms, *SIAM J. Algebraic and Discrete Methods*, 8 (1987), pp. 233–250.

[13] Szpankowski, W., On a recurrence equation arising in the analysis of conflict resolution algorithms, *Stochastic Models*, 3, (1987), pp. 89 - 114.

[14] Szpankowski, W., An analysis of a contention resolution algorithm - another approach, *Acta Informatica*, 24 (1987), pp. 173–190.

[15] Tsybakov, B., Mikhailov, V., Free synchronous packet access in a broadcast channel with feedback, *Prob. Inf. Trans.*, 14,4 (1978), pp. 259–280.

[16] Tsybakov, B., Mikhailov, V., Random multiple packet access: Part-and-try algorithm, *Prob. Inf. Trans.*, 16,4 (1980), pp. 305–317.