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AN M/SM/I VIEW OF A TOKEN RING QUEUE

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Abstract: We study a token ring system with N stations, one of which has an infinite or finite capacity buffer, and the rest have unit-capacity buffers. Assuming that packet arrivals are asymmetric Poisson, and other distributions such as service-times and walking-times are arbitrary, it is shown that the queue-length process at the station with non-unit buffering (reference station) is essentially a Markov renewal process. The states of the Markov renewal process are given by the different types of service cycles made by the token with respect to the reference station. Using the different service cycles initiated by customers at the reference queue to differentiate between customer *types* at this queue, the queueing process at the reference station can be viewed as an $M/SM/1$ queue. We describe how the transition probability submatrices for the block $M/G/1$ structure for the queue length can be computed. Once these submatrices are obtained, we apply known results to obtain the stationary queue-length distribution at the reference queue.

1. Introduction

In building models of communication systems, an increasingly common phenomenon is that of analytic intractability. While one reason for this is the increasing complexity of systems being modelled, it is often possible to handle otherwise difficult models by resorting to algorithmic methods [1]. In such instances, it is convenient to settle performance questions computationally instead of discarding a difficult problem in favour of one obtained via simplifying assumptions (such as independence assumptions). As an example of such an instance, we introduce a simple, but sufficiently detailed model of a token ring queue. Token rings are known to pose formidable problems to analysts, especially when stations offer asymmetric traffic, and messages are transmitted nonexhaustively [2].

A token ring is a local area network [3] that allows N different computer systems to communicate with one another. Messages that are sent from one system to another are, in essence, customers that require service from the network. In this way, each of the N different stations offers customers that require service. A single server (the token) walks around the ring from one station to another, making repeated cycles. At each station, the server serves at most one customer (hence called the *one-at-a-time* service discipline) if the station's buffer is not empty, and then proceeds to the next station. Due to channel propagation delays, the server takes a certain amount of time to walk from one station to the next on this ring.

We are interested in computing stationary distributions of customer-queue lengths and waiting-times, etc. At the very outset, we restrict our model to one in which $(N - 1)$ stations have unit capacity buffers, and the N^{th} station has a buffer of either finite or unlimited capacity. It will be seen the technique is easily generalised, at the expense of increased matrix sizes, to a system in which the unit-capacity buffers are replaced by arbitrary, finite capacity buffers. By making the size of these buffers sufficiently large, one may obtain a theoretically sound approximation to the problem where all buffers are infinite. This generalises the manner in which an $M/GI/1/K$ queue,

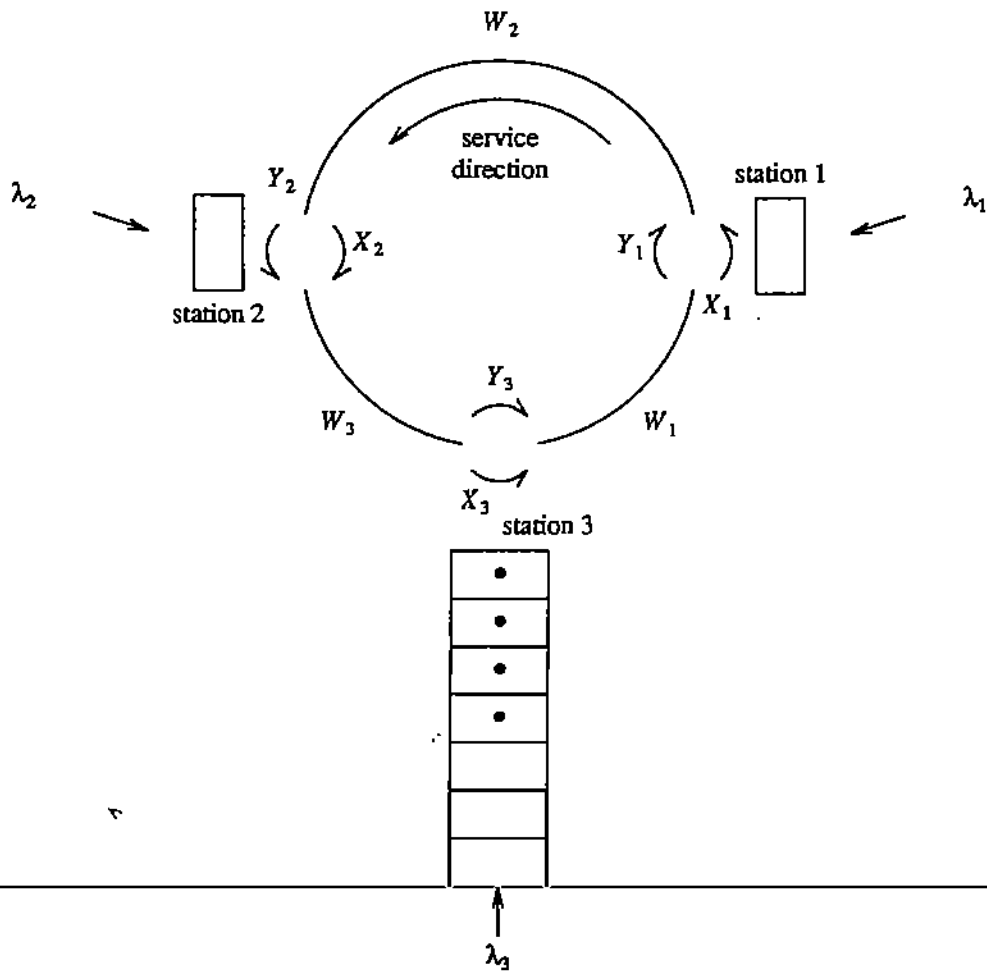


Figure 1. A three station cyclic-server model.

with large K , can be used to approximate an $M/GI/1/\infty$ queue, in essence a truncation.

In section 2, we first present the model and we show that the queueing process at station N is a Markov renewal process. Initially this is done for $N=3$ to demonstrate the ideas involved, and later generalised. Since the queue at station N appears to station- N customers as a single server queue, we label each station- N customer with a *type*, given by the kind of service cycle this customer initiates. Once transitions between customer types are described, known results for $M/SM/1$ theory [6] may be applied to obtain the queueing distributions at station N . In section 3, we describe how these transition probabilities between customer types can be computed, and show that the queueing process at station N is an $M/SM/1$ queue. In section 4, we apply results of Neuts [4], and Lucantoni and Neuts [5] to obtain the stationary queue length distribution of station N customers and present some numerical results.

2. The Cyclic-Server Model

Consider a system of N independent queueing stations arranged in a circle as shown in Figure 1. Each queueing station is given a unique label j , $1 \leq j \leq N$. Each station j , $1 \leq j \leq N-1$, is assumed to have a maximum queue capacity of one, and station N is assumed to have a queue capacity of b , $b \in I^+ = \{1, 2, \dots\}$. Customers arrive at each queue j as a Poisson process with rate λ_j , independently of arrivals to other queues, $1 \leq j \leq N$. Those customers who arrive at a given queue only to find it filled to capacity are lost to the system.

In the ring-based queueing system just described, a single server provides service by walking from one station to the next, unidirectionally. The sequence of station visits is specified as $1, 2, \dots, N, 1, 2, \dots$, or otherwise known as *cyclic service*. The arrival instant of the server at queueing station j is called the *scan-instant* of station j , $1 \leq j \leq N$. If station j 's queue is found nonempty on any station- j scan, then *at most one* customer is served at station j , $1 \leq j \leq N$ on that visit. Station j customers require service of random length X_j , with $Pr(X_j \leq t) = B_j(t)$, $1 \leq j \leq N$. Customers leave the system if they have completed service. If the server finds

station j 's buffer empty on any station- j scan, it takes the server a small random time Y_j (called the *switching-time* at station- j) to bypass this station, with $Pr(Y_j \leq t) = S_j(t)$, $1 \leq j \leq N$. On completion of either service or switching at station j , the server must walk over to station $k = (j \text{ mod } N) + 1$, this *walk-time* being random, of length W_k , with $Pr[W_j \leq t] = V_j(t)$, $1 \leq j \leq N$.

Semi-Markov Service Cycles

Assuming that the system is operating at steady-state and all queueing distributions are stationary, let time $t = 0$ correspond to an arbitrary scan-instant at station N . Let the set $\{t_j^k; 1 \leq j \leq N, k > 0\}$ define a strictly increasing sequence of time instants, with t_j^k corresponding to the k^{th} scan-instant of the server at station j , $0 = t = t_N^1 < t_1^1 < \dots < t_{N-1}^1 < t_N^2 < \dots$, etc. The *intervisit* time of the server for any station j is the time between two consecutive scans of the server at station j , $1 \leq j \leq N$. The k^{th} inter-visit time of the server at any station j is defined as $C_j^k = t_j^k - t_j^{k-1}$, $1 \leq j \leq N, k > 1$.

Let Z_j^k denote the number of customers that the server finds queued at station j at scan-instant t_j^k , $1 \leq j \leq N, k \geq 1$. For convenience, we take $Z_j^0 = 0$, $1 \leq j \leq N$. It should be clear that $Z_j^k \in \{0, 1\}$ for $1 \leq j \leq N-1$ and $Z_N^k \in \{0, 1, 2, \dots, b\}$, for each $k, k > 1$. Let $\{Z(t); t > 0\}$ be a continuous-time stochastic process defined by $Z(t) = \langle Z_N(t), Z_1(t), \dots, Z_{N-1}(t) \rangle$, where $Z_j(t) = Z_j^k$ for $t_N^k \leq t < t_N^{k+1}$, $k > 1$. The discrete parameter process $\{Z_k; k > 1\}$ given by $Z_k = \langle Z_N^k, Z_1^k, \dots, Z_{N-1}^k \rangle$ and the continuous parameter process $\{Z(t)\}$ are both defined with respect to embedded instants $t_N^k, k > 1$. At each scan-instant t_N^k of station N , the server has available a new *record* $Z_{k-1} = \langle z_N^{k-1}, z_1^{k-1}, \dots, z_{N-1}^{k-1} \rangle$, of events (i.e., an N -vector description) that occurred in the interval $[t_N^{k-1}, t_N^k)$, $k > 1$. Station N is the first to contribute to this record, with $Z_N^{k-1} = z_N^{k-1} \in \{0, 1, \dots, b\}$. The other $N-1$ stations contribute in increasing order of station indices, with station j contributing

$Z_j^{k-1} = z_j^{k-1} \in \{0, 1\}$, $1 \leq j \leq N-1$. Thus, at time t_N^k , the server completes a record $\langle z_N^{k-1}, z_1^{k-1}, \dots, z_{N-1}^{k-1} \rangle$ and simultaneously begins to define a new record with $Z_N^k = z_N^k \in \{0, 1, 2, \dots, b\}$.

THEOREM

If all queueing distributions are stationary, and the system is in steady state operation, then

- (1) $\{Z(t); t > 0\}$ is a semi-Markov process, and
- (2) $\{Z_k; k > 0\}$ is a time-homogeneous Markov chain

defined on $\{0, 1, 2, \dots, b\} \times \Theta_{N-1}$, where Θ_{N-1} denotes the set of all $(N-1)$ bit binary vectors.

PROOF:

For each k , $k > 0$, the events $\{Z_j^k; 1 \leq j \leq N\}$ are *dependent* events. This is due to the nonempty intersection of each pair of intervals $[t_j^k, t_j^{k+1})$ and $[t_i^k, t_i^{k+1})$ corresponding to server intervisit intervals for stations i and j , $i \neq j$, and every $k > 0$.

Let $A(C_j^{k+1})$ denote the number of customer arrivals at each station j during the intervisit time $C_j^{k+1} = t_j^{k+1} - t_j^k$, $1 \leq j \leq N$. Using a stochastic equation very similar to that of the standard *M/G/1* departure-instant based queueing chain,

$$Z_j^{k+1} = [Z_j^k - 1]^+ + D_j^{k+1} \quad (2.1)$$

for $1 \leq j \leq N$, with

$$D_j^{k+1} = \begin{cases} 1 & A(C_j^{k+1}) > 0 \\ 0 & A(C_j^{k+1}) = 0 \end{cases} \quad (2.2)$$

for $1 \leq j \leq N-1$, and

$$D_N^{k+1} = \min \{ b - [Z_N^k - a]^+, A(C_N^{k+1}) \} \quad (2.3)$$

From (2.1) we see that $\{Z_k; k > 0\}$ is an N -dimensional Markov chain embedded at station- N

scan-instants. The process $\{Z(t)\}$ also makes transitions at the embedded instants t_N^k , $k > 1$. However, $\{Z(t)\}$ spends an arbitrary time in each state, with each complete record defining a state, hence it is semi-Markov process with $\{Z_k\}$ as its embedded Markov chain. ■

The server in our system defines states by recording each station's queue size each time he visits the station. He begins with station N , then visits station 1, station 2, ..., station $(N-1)$. Even though he has completed a record when he obtains station $(N-1)$'s queue status, a state transition of the system occurs only when the server arrives at station N . The new system state is then defined by the record just completed by the server. Once again at station N , the server begins to define a new record, and thus, a new system next state.

In the next section, we focus our attention on the queueing process at station N . When a customer from station N goes into service, this customer initiates a server-cycle of some random length. The actual length of this cycle must depend on events at the various stations visited before the server completes the cycle. Since there are only $(N - 1)$ other stations to be visited, the number of such events is 2^{N-1} . Consequently, a station- N customer may initiate any one of 2^{N-1} server cycles. The above theorem tells us that if we know what events (i.e., service or switching) occurred at the N stations during any particular cycle, say $\langle z_N, z_1, \dots, z_{N-1} \rangle$, the probability of a given sequence of events $\langle z_N', z_1', \dots, z_{N-1}' \rangle$ on the next cycle is obtainable in Markov fashion.

Consider a server cycle corresponding to the record of events $z = \langle z_N, z_1, \dots, z_{N-1} \rangle$, where $z_N = i$, and $i > 0$. This means that a station- N customer went into service at the start of the cycle. The probability that the next record of events $z' = \langle z_N', z_1', \dots, z_{N-1}' \rangle$ takes on a particular form depends on the record z . In particular, if $z_N' = m > 0$ (i.e., the server finds station N nonempty on the next visit) then $\langle z_1, \dots, z_{N-1} \rangle$ is said to define the *type* of the next customer to go into service at station N . With this interpretation, the 2^{N-1} different records

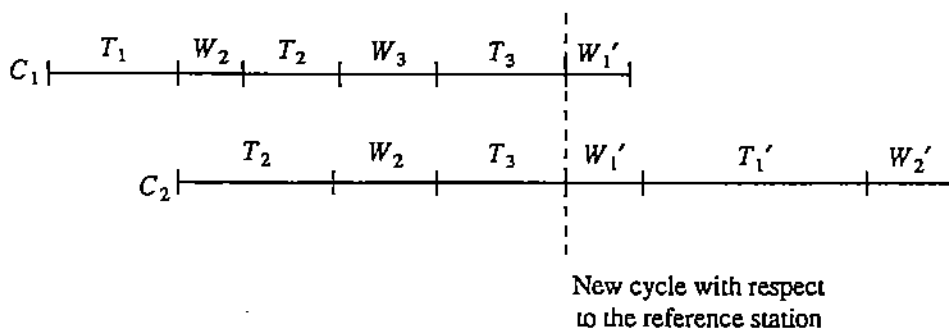
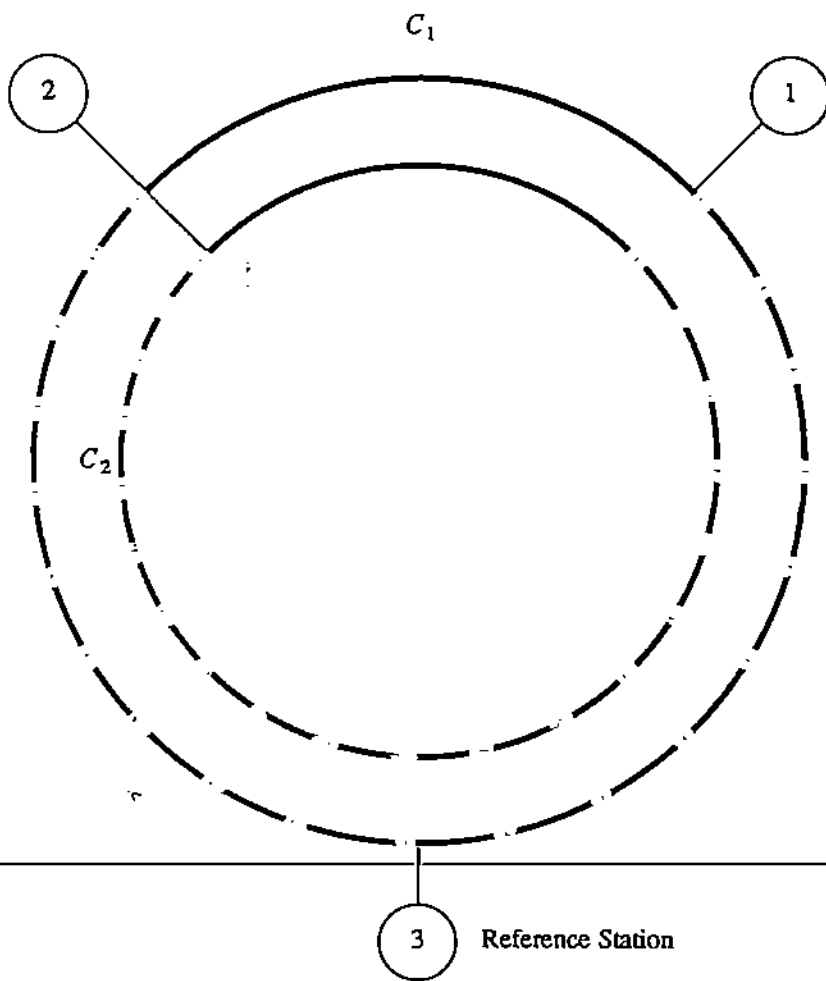


Figure 2. Illustration of cycle-times C_1 and C_2 .

obtainable from stations 1 through $N-1$ define 2^{N-1} possible customer types offered by station N . More importantly, the problem can now be viewed as one in which customers of 2^{N-1} different types arrive at station N as points from a Poisson process, with a customer of type j following a customer of type i with a certain computable probability $p_{i,j}$, $1 \leq i, j \leq 2^{N-1}$. A computational method to obtain $P = [p_{i,j}]$ is outlined in the next two sections, along with a more detailed look at station N 's queue.

3. The $M|SM|1|b$ Queue at Station N

At the k th scan of station N at time t_N^k , the server obtains a new queue-length (i.e., number of customers in the queue), $L_k \equiv Z_N^k$. It is clear that

$$L_{k+1} = [L_k - 1]^+ + D_N^{k+1} \quad (3.1)$$

for each k , $k > 1$. Equation (3.1) bears a strong resemblance to the departure-instant based $M/GI/1$ queueing chain. However, in this case, the sequence of random variables $\{D_N^k; k > 1\}$ is *not* an i.i.d sequence. In fact, by Theorem 1, this sequence is semi-Markovian, thus causing the queue at station N to be a generalisation of the $M/GI/1$ queue, known as the $M|SM|1$ queue [6].

The Transition Probability Matrix

Given that $\mathbf{Z}_k = \langle i, z_1^k, \dots, z_{N-1}^k \rangle$, Theorem 1 says that we can compute the probability of the next state being $\mathbf{Z}_{k+1} = \langle m, z_1^{k+1}, \dots, z_{N-1}^{k+1} \rangle$, for $i \geq 0$, and $\max(i-1, 0) \leq m \leq b$, $k > 1$. Using $(N-1)$ bit binary vectors \mathbf{z} and \mathbf{z}' , we are now interested in computing the probability $Pr[\mathbf{Z}_{k+1} = \langle m, \mathbf{z}' \rangle | \mathbf{Z}_k = \langle i, \mathbf{z} \rangle]$ for $i \geq 0$ and $\max(i-1, 0) \leq m \leq b$. Since transitions are stationary, we can neglect the index k and write this as $Pr[\mathbf{Z}' = \langle m, \mathbf{z}' \rangle | \mathbf{Z} = \langle i, \mathbf{z} \rangle]$.

For each k , $k > 1$, and $1 \leq j \leq N$, the $(k+1)^{th}$ intervisit time of the server at station j (see Figure 2) is given by

$$C_j^{k+1} = \sum_{i=j}^N [T_i^k + W_{(i \bmod N)+1}^k] + W_1^{k+1} + \sum_{i=1}^{j-1} [T_i^{k+1} + W_{(i \bmod N)+1}^{k+1}] \quad (3.2)$$

where

$$T_i^k = \begin{cases} X_i^k & Z_i^k = 1 \\ Y_i^k & Z_i^k = 0 \end{cases} \quad (3.3)$$

where $\{X_i^k; k \geq 1\}$ is an i.i.d sequence of random variables with common law $B_i(\cdot)$, and $\{Y_i^k; k \geq 1\}$ is an i.i.d sequence of random variables with common law $S_i(\cdot)$, $1 \leq j \leq N$. The independence between random variables in the above two sequences is a consequence of the fact that different customers (possibly from the same queue) have independent service-times, and the walk-times of the server between any two consecutive stations, in any two different cycles, are independent. Since transitions are stationary, we can neglect k , and replace Z_i^k by Z_i , and Z_i^{k+1} by Z_i' , for $1 \leq i \leq N$. Note that for each current state and next state pair, we can write

$$\begin{aligned} C_1 &= T_1 + W_2 + T_2 + W_3 + \dots + T_N + W_1' \\ C_2 &= T_2 + W_3 + T_3 + W_4 + \dots + T_1' + W_2' \\ &\vdots \\ C_N &= T_N + W_1' + T_1' + W_2' + \dots + T_{N-1}' + W_N' \end{aligned} \quad (3.4)$$

where all unprimed terms correspond to random times that occur in defining the current state, and primed terms correspond to random times that occur in defining the next state. Note that C_i and C_j are dependent random variables due to the overlap of part of C_i and C_j . Given that the current state is $Z = \langle i, z \rangle$, for any $i \geq 0$ and $z \in \Theta_{N-1}$, we only need to compute $Pr[Z' = \langle m, 0, 0, \dots, 0 \rangle \mid Z = \langle i, z \rangle]$ in order to obtain the desired transition probability matrix. This is best explained with the aid of an example for $N = 3$.

Example. $N = 3$ stations.

In order to obtain $Pr[Z' = \langle m, z' \rangle \mid Z = \langle i, z \rangle]$ for a given pair $z, z' \in \Theta_2$, $i \geq 0$, $\max(i - 1, 0) \leq m \leq b$, we proceed as follows. We keep $z \in \Theta_2$ fixed, and first compute $Pr[Z' = \langle m, 0, 0 \rangle \mid Z = \langle i, z \rangle]$ separately, for $i = 0$ and $i > 0$ respectively. With $x = \langle 0, 0 \rangle$, define the 4×4 (in general, $2^{(N-1)} \times 2^{(N-1)}$) matrices of transition probabilities

$$B_m(x, t \mid z) = Pr[Z' = \langle m, x \rangle, C_N \leq t \mid Z = \langle 0, z \rangle]$$

$$A_{m-i+1}(x, t \mid z) = Pr[Z' = \langle m, x \rangle, C_N \leq t \mid Z = \langle i, z \rangle] \quad (3.5)$$

Observing that the three different station intervisit times can be written as

$$\begin{aligned} C_1 &= T_1 + W_2 + T_2 + W_3 + T_3 + W_1' \\ C_2 &= T_2 + W_3 + T_3 + W_1' + T_1' + W_2' \\ C_3 &= T_3 + W_1' + T_1' + W_2' + T_2' + W_3' \end{aligned} \quad (3.6)$$

where the partial overlap of these times is clearly displayed, we obtain

$$\begin{aligned} B_m(x, t \mid z) &= \int_0^t e^{-\lambda_1 y} dF_{T_1}(y) \int_0^t e^{-\lambda_2 y} dF_{W_2}(y) \\ &\quad \times \int_0^t e^{-(\lambda_1 + \lambda_2)y} dF_{T_2}(y) \int_0^t e^{-(\lambda_1 + \lambda_2)y} dF_{W_3}(y) \\ &\quad \times \left\{ \sum_{S(m)} \left[\int_0^t e^{-(\lambda_1 + \lambda_2 + \lambda_3)y} \frac{(\lambda_3 y)^{m_1}}{m_1!} dF_{T_3}(y) \int_0^t e^{-(\lambda_1 + \lambda_2 + \lambda_3)y} \frac{(\lambda_3 y)^{m_2}}{m_2!} dF_{W_1}(y) \right. \right. \\ &\quad \times \int_0^t e^{-(\lambda_2 + \lambda_3)y} \frac{(\lambda_3 y)^{m_3}}{m_3!} dF_{T_1}(y) \int_0^t e^{-(\lambda_2 + \lambda_3)y} \frac{(\lambda_3 y)^{m_4}}{m_4!} dF_{W_2}(y) \\ &\quad \left. \left. \times \int_0^t e^{-\lambda_3 y} \frac{(\lambda_3 y)^{m_5}}{m_5!} dF_{T_2}(y) \int_0^t e^{-\lambda_3 y} \frac{(\lambda_3 y)^{m_6}}{m_6!} dF_{W_1}(y) \right] \right\} \end{aligned} \quad (3.7)$$

where $S(m) = \{(m_1, \dots, m_6) \mid m_i \geq 0, 1 \leq i \leq 6, \sum_{i=1}^6 m_i = m\}$. Similarly, we obtain

$$\begin{aligned}
 A_m(\mathbf{x}, t | \mathbf{z}) = & \int_0^t e^{-\lambda_1 y} dF_{T_1}(y) & \int_0^t e^{-\lambda_1 y} dF_{W_2}(y) \\
 & \times \int_0^t e^{-(\lambda_1 + \lambda_2)y} dF_{T_2}(y) & \int_0^t e^{-(\lambda_1 + \lambda_2)y} dF_{W_3}(y) \\
 \times \left\{ \sum_{S(m)} \left[\int_0^t e^{-(\lambda_1 + \lambda_2 + \lambda_3)y} \frac{(\lambda_3 y)^{m_1}}{m_1!} dF_{T_3}(y) \int_0^t e^{-(\lambda_1 + \lambda_2 + \lambda_3)y} \frac{(\lambda_3 y)^{m_2}}{m_2!} dF_{W_1}(y) \right. \right. & (3.8) \\
 & \times \int_0^t e^{-(\lambda_2 + \lambda_3)y} \frac{(\lambda_3 y)^{m_3}}{m_3!} dF_{T_1}(y) & \int_0^t e^{-(\lambda_2 + \lambda_3)y} \frac{(\lambda_3 y)^{m_4}}{m_4!} dF_{W_2}(y) \\
 & \left. \left. \times \int_0^t e^{-\lambda_3 y} \frac{(\lambda_3 y)^{m_5}}{m_5!} dF_{T_2}(y) \int_0^t e^{-\lambda_3 y} \frac{(\lambda_3 y)^{m_6}}{m_6!} dF_{W_3}(y) \right] \right\}
 \end{aligned}$$

We next take a case-by-case approach in showing how an arbitrary transition probability $A_m(\mathbf{z}', t | \mathbf{z})$, or $B_m(\mathbf{z}', t | \mathbf{z})$ can be computed, $\mathbf{z}, \mathbf{z}' \in \Theta_2$. Given any two vectors \mathbf{z}, \mathbf{z}' in Θ_2 , we first obtain the distribution functions F_{T_j} and F_{T_j}' , for $1 \leq j \leq N-1$. Additionally, in computing $A_m(\mathbf{z}', t | \mathbf{z})$ we require the distribution $B_3(\cdot)$, and in computing $B_m(\mathbf{z}', t | \mathbf{z})$ we require the distribution $S_3(\cdot)$. Next, keep \mathbf{z} fixed and compute $A_m(\mathbf{x}, t | \mathbf{z})$ and $B_m(\mathbf{x}, t | \mathbf{z})$ for $\mathbf{x} = \langle 0, 0 \rangle$, using the integral products in (3.8) and (3.7), respectively. We now examine how $A_m(\mathbf{z}', t | \mathbf{z})$ is computed for \mathbf{z} fixed, and any $\mathbf{z}' \in \Theta_2$.

CASE 1 : \mathbf{z}' contains a single nonzero bit.

In this case,

$$A_m(\langle 0, 1 \rangle, t | \mathbf{z}) = A_m(\langle 0, \cdot \rangle, t | \mathbf{z}) - A_m(\langle 0, 0 \rangle, t | \mathbf{z}) \quad (3.9)$$

$$A_m(\langle 1, 0 \rangle, t | \mathbf{z}) = A_m(\langle \cdot, 0 \rangle, t | \mathbf{z}) - A_m(\langle 0, 0 \rangle, t | \mathbf{z}) \quad (3.10)$$

Note that the above computations require the conditional joint probability that any one of two stations (excluding station $N = 3$) is found empty by the server. This is easily computed using (3.8). For example, $A_m(\langle 0, \cdot \rangle, t | \mathbf{z})$ is obtained from (3.8) by setting $\lambda_2 = 0$.

CASE 2 : z' contains two nonzero bits

In this case we are merely computing the complement of the probability computed in case 1.

That is,

$$A_m(\langle 1,1 \rangle, t | z) = 1 - A_m(\langle 0,0 \rangle, t | z) - A_m(\langle 0,1 \rangle, t | z) - A_m(\langle 1,0 \rangle, t | z) \quad (3.11)$$

■

The computation of $B_m(z', t | z)$ for z fixed and arbitrary $z' \in \Theta_2$ proceeds in exactly the same manner as described for $A_m(z', t | z)$, except that (3.7) is used in place of (3.8). Repeating the above procedure for each $z \in \Theta_2$ yields the complete probability transition matrix. This matrix of transition functions is given as

$$Q(t) = \begin{bmatrix} B_0(t) & B_1(t) & B_2(t) & B_3(t) & \dots \\ A_0(t) & A_1(t) & A_2(t) & A_3(t) & \dots \\ 0 & A_0(t) & A_1(t) & A_2(t) & \dots \\ 0 & 0 & A_0(t) & A_1(t) & \dots \\ 0 & 0 & 0 & A_0(t) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (3.12)$$

where $A_m(t)$ is the 4×4 matrix,

$$A_m(t) = \begin{bmatrix} A_m(\langle 0,0 \rangle, t | \langle 0,0 \rangle) & A_m(\langle 0,1 \rangle, t | \langle 0,0 \rangle) & A_m(\langle 1,0 \rangle, t | \langle 0,0 \rangle) & A_m(\langle 1,1 \rangle, t | \langle 0,0 \rangle) \\ A_m(\langle 0,0 \rangle, t | \langle 0,1 \rangle) & A_m(\langle 0,1 \rangle, t | \langle 0,1 \rangle) & A_m(\langle 1,0 \rangle, t | \langle 0,1 \rangle) & A_m(\langle 1,1 \rangle, t | \langle 0,1 \rangle) \\ A_m(\langle 0,0 \rangle, t | \langle 1,0 \rangle) & A_m(\langle 0,1 \rangle, t | \langle 1,0 \rangle) & A_m(\langle 1,0 \rangle, t | \langle 1,0 \rangle) & A_m(\langle 1,1 \rangle, t | \langle 1,0 \rangle) \\ A_m(\langle 0,0 \rangle, t | \langle 1,1 \rangle) & A_m(\langle 0,1 \rangle, t | \langle 1,1 \rangle) & A_m(\langle 1,0 \rangle, t | \langle 1,1 \rangle) & A_m(\langle 1,1 \rangle, t | \langle 1,1 \rangle) \end{bmatrix} \quad (3.13)$$

and $B_m(t)$ is a 4×4 matrix with structure identical to $A_m(t)$, for each $m, m \geq 0$. The matrix $Q(t)$ is the transition probability matrix of an embedded Markov renewal sequence, given by the queue length (at station 3), types of service-cycles (which in effect define customer types) and times between consecutive server scans at station N.

Define $A(t) = \sum_{k=0}^{\infty} A_k(t)$ to be a matrix of transition functions. The matrix $A(\cdot)$ is the transition probability matrix of a 4 state Markov renewal sequence describing the successive (station N) customer types and service-times. Observe that $A(\infty) = P$ is a stochastic matrix that defines consecutive customer types (or equivalently, server cycle types), given that station 3 is not empty. With $z = \langle z_1, z_2 \rangle$ and $z' = \langle z_1, z_2 \rangle$ in Θ_2 representing the current and next type of service cycle, there are at most four kinds of service cycles (i.e., $\langle 0,0 \rangle$, $\langle 0,1 \rangle$, $\langle 1,0 \rangle$ and $\langle 1,1 \rangle$) involving events at stations 1 and 2. Thus, station 3 can be viewed as a station that offers four different types of customers for service. When a customer does go into service at station 3, the following events $\langle z_1, z_2 \rangle$ at stations 1 and 2, respectively, define the length of service of this customer, and also the type of the next customer. Correspondingly, $A_m(z', \infty | z)$ gives the probability that m station-3 customers arrive while a station-3 customer, with service length described by z' , is being served. Since it is possible for station-3 customers to arrive during a cycle for which station 3 was found empty at its scan-instant, $B_m(z', t | z)$ gives the probability of m station-3 arrivals during such a cycle. The traffic intensity ρ of the queue at station 3 is given by

$$\rho = \lambda \pi E(C) \tag{3.14}$$

where $\pi = \langle \pi_{00}, \pi_{01}, \pi_{10}, \pi_{11} \rangle$ is the invariant probability vector of P , and $E(C) = \langle E(C_{00}), E(C_{01}), E(C_{10}), E(C_{11}) \rangle$ is a vector of the expected lengths of the four-different service cycles, *conditioned* on station 3 being nonempty. That is, $E(C_{00}) = E(W) + E(Y_1) + E(Y_2)$, $E(C_{01}) = E(W) + E(Y_1) + E(X_2)$, $E(C_{10}) = E(W) + E(X_1) + E(Y_2)$, $E(C_{11}) = E(W) + E(X_1) + E(X_2)$, and $E(W) = E(W_1) + E(W_2)$. It is necessary for ρ to be less than one for well-defined steady-state distributions to exist.

Extension of Computational Scheme to General N

For the case of general N , we can develop a similar scheme to compute transition probabilities for transitions from any fixed $z \in \Theta_{N-1}$ to all possible $z' \in \Theta_{N-1}$, with m station- N customer arrivals during the server cycle defined by z . In actuality, the extension of (3.7) and (3.8) to the case of general N is fairly straightforward. However, the number of individual cases to investigate is now $(N-1)$ i.e., case k would be the case in which the $(N-1)$ -bit vector z' contained exactly k nonzero bits, $1 \leq k \leq N-1$. Within case k , the number of distinct transition probabilities to be computed would be equal to the number of ways of choosing k bits from $(N-1)$ -bits without repetition, i.e., $\binom{N-1}{k}$.

Let $z \in \Theta_{N-1}$ be fixed. In order to compute $A_m(z', t | z)$ or $B_m(z', t | z)$ for any $z' \in \Theta_{N-1}$, we must first develop general expressions for (3.8) and (3.7), respectively, that is, for $z' = \langle 0, 0, \dots, 0 \rangle$. We first introduce some compact notation. Define vectors $\Lambda^{(j)} = \langle \lambda_1, \dots, \lambda_j \rangle$ and $\Lambda_{(j)} = \langle \lambda_j, \dots, \lambda_N \rangle$ for $1 \leq j \leq N$. Next, for generic, nonnegative random variables X and Y , with distributions $F_X(\cdot)$ and $F_Y(\cdot)$, respectively, we define the joint integral products

$$G_k(\Lambda^{(j)}, t, X, Y) = \int_0^t e^{-\left(\sum_{r=1}^j \lambda_r\right)x} dF_X(x) \cdot \int_0^t e^{-\left(\sum_{r=1}^j \lambda_r\right)y} dF_Y(y) \quad (3.15)$$

for $1 \leq j \leq N-1$, and

$$G_k(\Lambda^{(N)}, t, X, Y) = \int_0^t e^{-\left(\sum_{r=1}^{N-1} \lambda_r\right)x} \frac{(\lambda_N x)^i}{i!} \cdot dF_X(x) \cdot \int_0^t e^{-\left(\sum_{r=1}^{N-1} \lambda_r\right)y} \frac{(\lambda_N y)^{k-i}}{(k-i)!} dF_Y(y) \quad (3.16)$$

Similarly, define

$$H_k(\Lambda^{(j)}, t, X, Y) = \sum_{i=0}^k \left\{ \int_0^t e^{-\left(\sum_{r=1}^N \lambda_r\right)x} \frac{(\lambda_N x)^i}{i!} \cdot dF_X(x) \cdot \int_0^t e^{-\left(\sum_{r=1}^N \lambda_r\right)y} \frac{(\lambda_N y)^{k-i}}{(k-i)!} dF_Y(y) \right\} \quad (3.17)$$

for $1 \leq j \leq N$. Note that the function $G_k(\Lambda^{(j)}, t, X, Y)$ is independent of k , for each j , $1 \leq j \leq N$. For a fixed $z \in \Theta_{N-1}$, we first must obtain the distributions $F_{T_j}(\cdot)$, $1 \leq j \leq N-1$. In computing $A_m(x^*, t | z)$ and $B_m(x^*, t | z)$, for $x^* = \langle 0, \dots, 0 \rangle \in \Theta_{N-1}$, and $m \geq 0$, we require the distributions $F_{X_N}(\cdot)$ and $F_{Y_N}(\cdot)$, respectively. For a fixed $z \in \Theta_{N-1}$, we obtain

$$B_m(x^*, t | z) = \sum_{S^*(m)} \left\{ \prod_{j=1}^N \left[G_{m_j} \left(\Lambda^{(j)}, t, T_j, W_{(j \bmod N)+1} \right) \right] \times \prod_{k=1}^{N-1} \left[H_{m_k} \left(\Lambda^{(k)}, t, T_k', W'_{(k \bmod N)+1} \right) \right] \right\} \quad (3.18)$$

where $S^*(m) = \{(m_1, \dots, m_N) \mid m_i \geq 0, 1 \leq i \leq N, \sum_{k=1}^N m_k = m\}$

From the integral product in (3.12) it is possible to generate $B_m(z', t | z)$ for all $z' \in \Theta_{N-1}$, with z fixed in Θ_{N-1} . The idea is a generalisation of what was done for the $N = 3$ case. For example,

$$B_m(\langle 0, \dots, 1 \rangle, t | z) = B_m(\langle 0, \dots, 0, \cdot \rangle, t | z) - B_m(\langle 0, \dots, 0, 0 \rangle, t | z) \quad (3.19)$$

where $B_m(\langle 0, \dots, 0, \cdot \rangle, t | z)$ is obtained from (3.17) by setting $\lambda_{N-1} = 0$. The other transition probabilities are obtained similarly. Observe that in (3.17), $T_N = Y_N$ (since for B_m , we take $z_N = 0$). In case $z_N = 1$, then $T_N = X_N$ and (3.17) yields $A_m(x^*, t | z)$ instead of $B_m(x^*, t | z)$. Hence, from (3.17) we can obtain both $A_m(x^*, t | z)$ as well as $B_m(x^*, t | z)$. Once these are obtained, for z fixed, it is a routine matter to obtain $A_m(z', t | z)$ and $B_m(z', t | z)$ for arbitrary z in Θ_{N-1} . This procedure is repeated for each $z \in \Theta_{N-1}$ in order to obtain the transition probability matrix $Q(\cdot)$ for general N .

4. Stationary Queue-Length Distribution

In section 4 we showed that station N may be viewed as an $M/SM/1$ queue in which there are 2^{N-1} different types of customers, and we also showed how the embedded transition probability matrix Q of this queue can be obtained. Let $q = \{q_{ij}, i \geq 0, i \leq j \leq 2^{N-1}\}$ denote the invariant probability vector of the matrix Q . Clearly, q is the stationary joint density of the queue length and the customer type at embedded points. Writing q in the partitioned form q_0, q_1, \dots , where each q_j is a 2^{N-1} dimensional vector, the equilibrium equations $qQ = q$ may be written as

$$q_i = q_0 B_i + \sum_{k=1}^{i+1} q_k A_{i-k+1}, \quad i \geq 0 \quad (4.1)$$

where A_i and B_i have been described in Section 3. If q_0 is known, (4.1) can be applied to obtain the stationary queue length distribution. However, obtaining q_0 is not straightforward. This requires an examination of a sequence of matrices $\{G(k), k \geq 1\}$, with $G(k) = [G_{jj'}(k)]$, and $G_{jj'}(k), 1 \leq j, j' \leq 2^{N-1}, k \geq 1$, representing the conditional probability that, starting in state $(i+1, j)$, the process reaches level i for the first time in the state (i, j') after precisely k transitions. This sequence of matrices defines the first passage time distributions from states in level $(i+1)$ to states in level i . Starting from any state in level $(i+1)$, the process eventually reaches a state in level i if $G = G(1-)$, given by

$$G(z) = \sum_{k=0}^{\infty} G(k) z^k = \sum_{k=0}^{\infty} z A_k G^k(z), \quad (4.2)$$

is a stochastic matrix. The first equality essentially describes a matrix transform for $0 \leq z \leq 1$, and the second is a matrix functional equation obtained by making use of first passage times [7,8]. In [7] it is shown that the equation

$$G = \sum_{k=0}^{\infty} A_k G^k \quad (4.3)$$

has a minimal nonnegative solution which is stochastic if and only if $\rho < 1$, in which case G is the unique nonnegative matrix satisfying (4.3).

Assuming that the Markov chain is recurrent non-null (i.e., $\rho < 1$), the matrix G exists and can be computed using a technique of modified successive substitutions [8]. That is,

$$G(0) = (I - A_1)^{-1} A_0 \quad (4.4)$$

$$G(k+1) = \sum_{\substack{m=0 \\ m \neq 1}}^{\infty} (I - A_1)^{-1} A_m G^m(k),$$

for $k \geq 0$. In [7] it is shown that this sequence of substitutions yields entry-wise strictly increasing matrices that converge to the matrix G .

Let \mathbf{g} be the invariant probability vector of the stochastic matrix G . In [4] it is shown that

$$\mathbf{q}_0 = (1 - \rho) \mathbf{g} \quad (4.5)$$

with the aid of which (4.1) can be used to generate the invariant vector \mathbf{q} . In actuality, the recurrence in (4.1) can be numerically highly unstable when applied directly. An alternate iterative-recursive scheme based on the Gauss-Seidel iterative method for obtaining \mathbf{q} can be found in [5].

First, we obtain the vector \mathbf{q}_1 from [9],

$$\mathbf{q}_1 = \frac{1 - \rho}{\mathbf{h}(I - A_0)^{-1}\mathbf{e} - 1} \mathbf{h} \quad (4.6)$$

where \mathbf{h} is the left-invariant vector of the matrix

$$H = (I - A_0)^{-1} \sum_{k=1}^{\infty} A_k G^{k-1} \quad (4.7)$$

and \mathbf{e} is the vector with all entries equal to one. Once \mathbf{q}_0 and \mathbf{q}_1 are known, the stationary vector can be computed using the Gauss-Seidel iterative scheme,

$$\mathbf{q}_k(0) = (\mathbf{q}_0 B_k + \mathbf{q}_1 A_k)(I - A_1)^{-1} \quad (4.8)$$

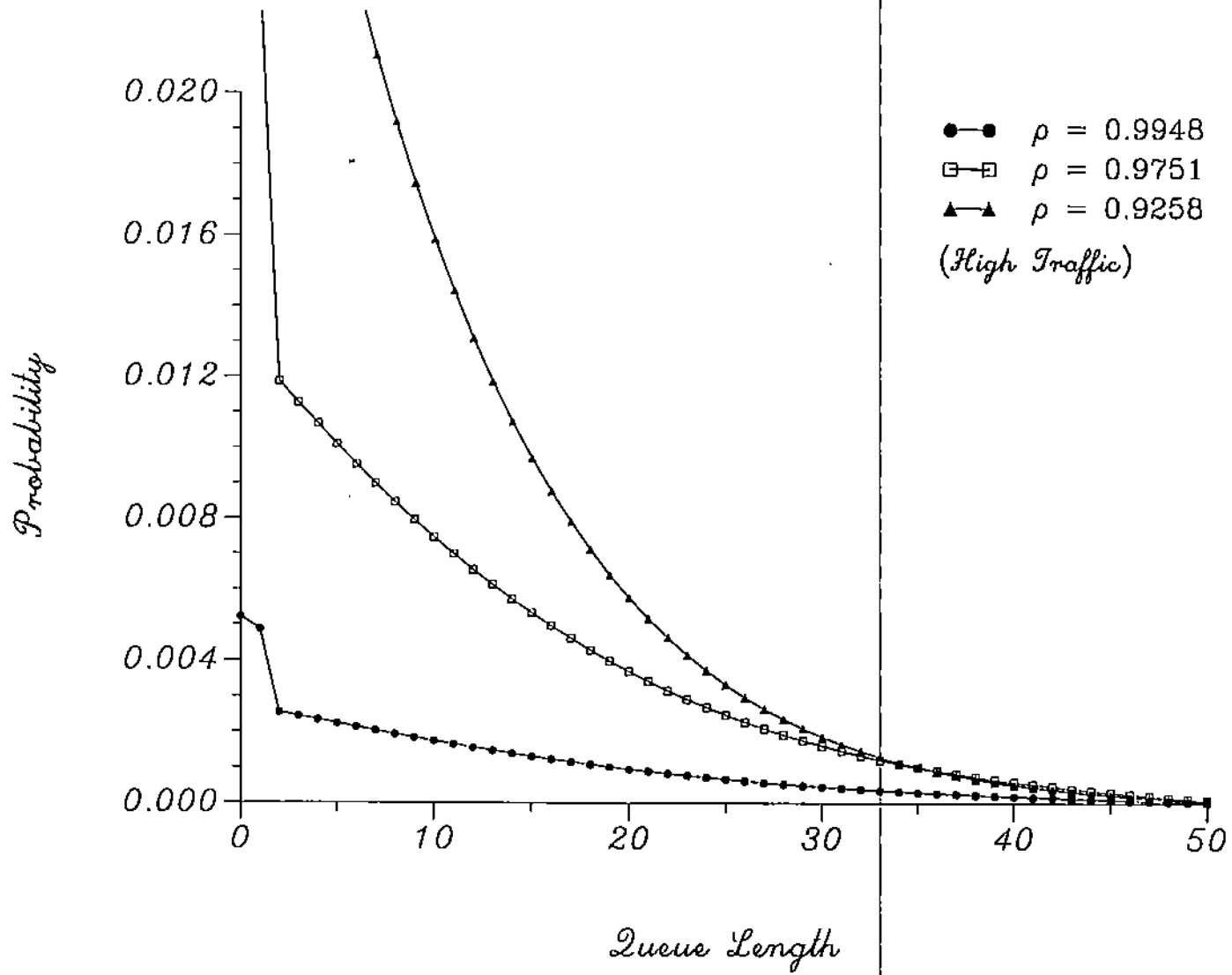
$$\mathbf{q}_k(n+1) = \mathbf{q}_k(0) + \left\{ \sum_{m=2}^{k-1} \mathbf{q}_m(n+1) A_{k+1-m} + \mathbf{q}_{k+1}(n) A_0 \right\} (I - A_1)^{-1}$$

where the iteration index n is large enough to effect a tolerable value of $|q_k(n+1) - q_k(n)|$, $0 \leq k \leq M$, and $\sum_{k=0}^M q_k$ is tolerably close to unity.

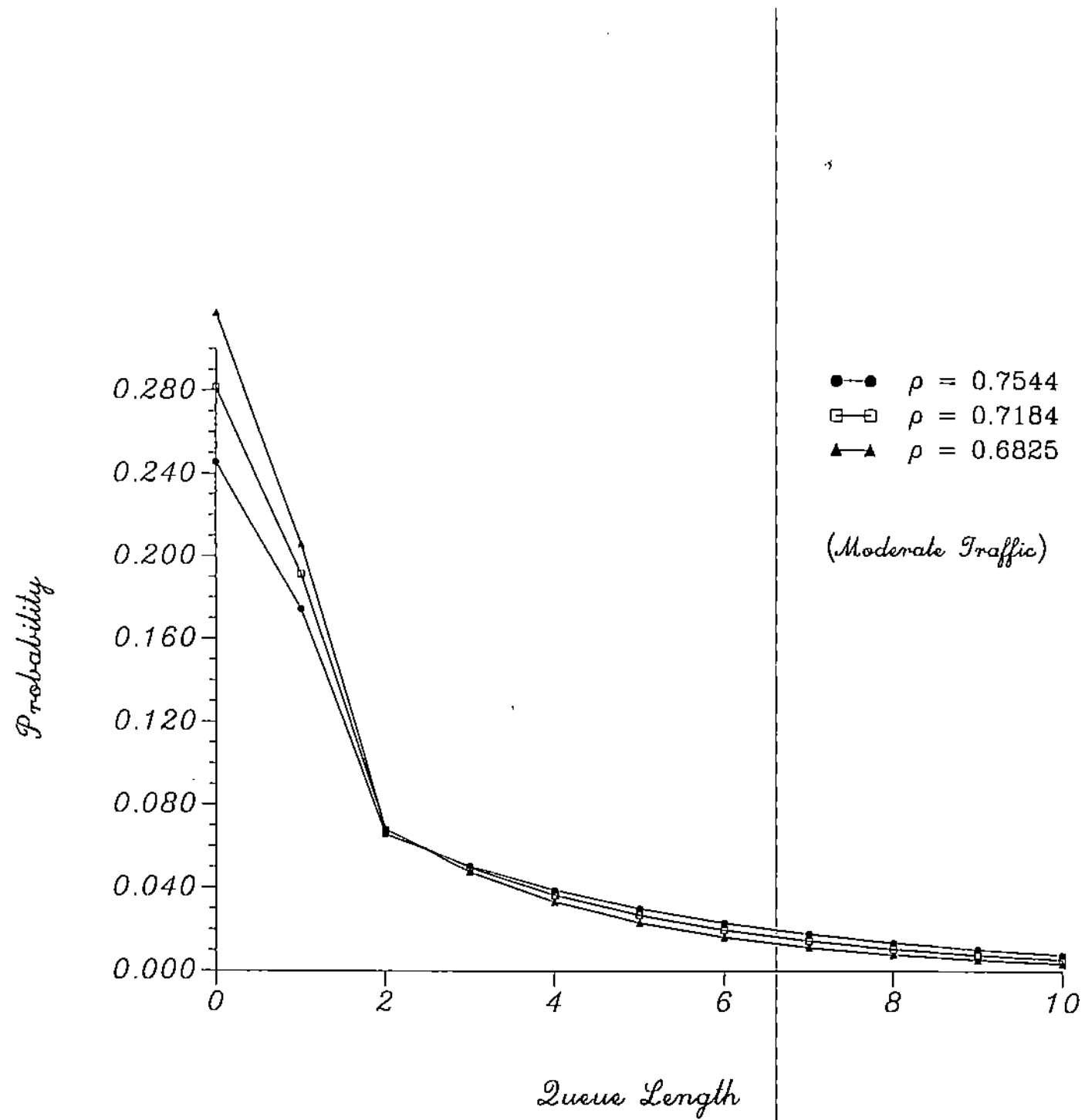
In Figure 3 can be seen a plot of steady-state queue length distributions for three different values of ρ , and $N = 3$. Stations 1 and 2 have unit buffers, and station 3 has unlimited waiting room. Arrivals at stations 1, 2 and 3 are Poisson, with rates $\lambda_1 = 0.004$, $\lambda_2 = 0.0045$, and $\lambda_3 = 0.00505$, 0.005 , and 0.0047 , respectively, for the three different high traffic situations displayed. All other random variables are assumed to be exponential, since the transition functions are most easily computed for this case. The parameters are fixed at $E(W_1) = 1$, $E(W_2) = 1/3$, $E(W_3) = 1/2$, $E(X_1) = 90$, $E(X_2) = 110$, $E(X_3) = 100$, $E(Y_1) = 1/10$, $E(Y_2) = 1/20$, and $E(Y_3) = 1/30$. In Figure 3, we see that the distribution can be fairly sensitive to variation in input traffic. Additionally, these distributions tend to have extremely long tails, which makes computation time-consuming. If Figure 4a and 4b are shown disjoint parts of the same graph for moderate values of ρ . Figure 4a shows the crossing that occurs when lower traffic queue-length distributions fall more steeply than higher traffic queue-length distributions. The fact that the three graphs appear to cross at the same point is merely a coincidence. In Figure 4b can be seen the second half of the same graph, with all distributions taking on an exponential form.

Remarks

The method of successive substitutions in (4.4) converges fairly rapidly, with speed of convergence depending on the sequence of matrices $\{A_m ; m \geq 0\}$. Defining $P_M = \sum_{m=0}^M A_m$, and $P = \lim_{M \rightarrow \infty} P_M$, we see that as $M \rightarrow \infty$, $\|P - P_M\| \rightarrow 0$, and the faster that $\|P - P_m\| \rightarrow 0$, so also the faster will the sequence of modified successive substitutions converge to the stochastic matrix G . It follows that when ρ is close to one, $\|P - P_M\|$ converges to zero very slowly, and



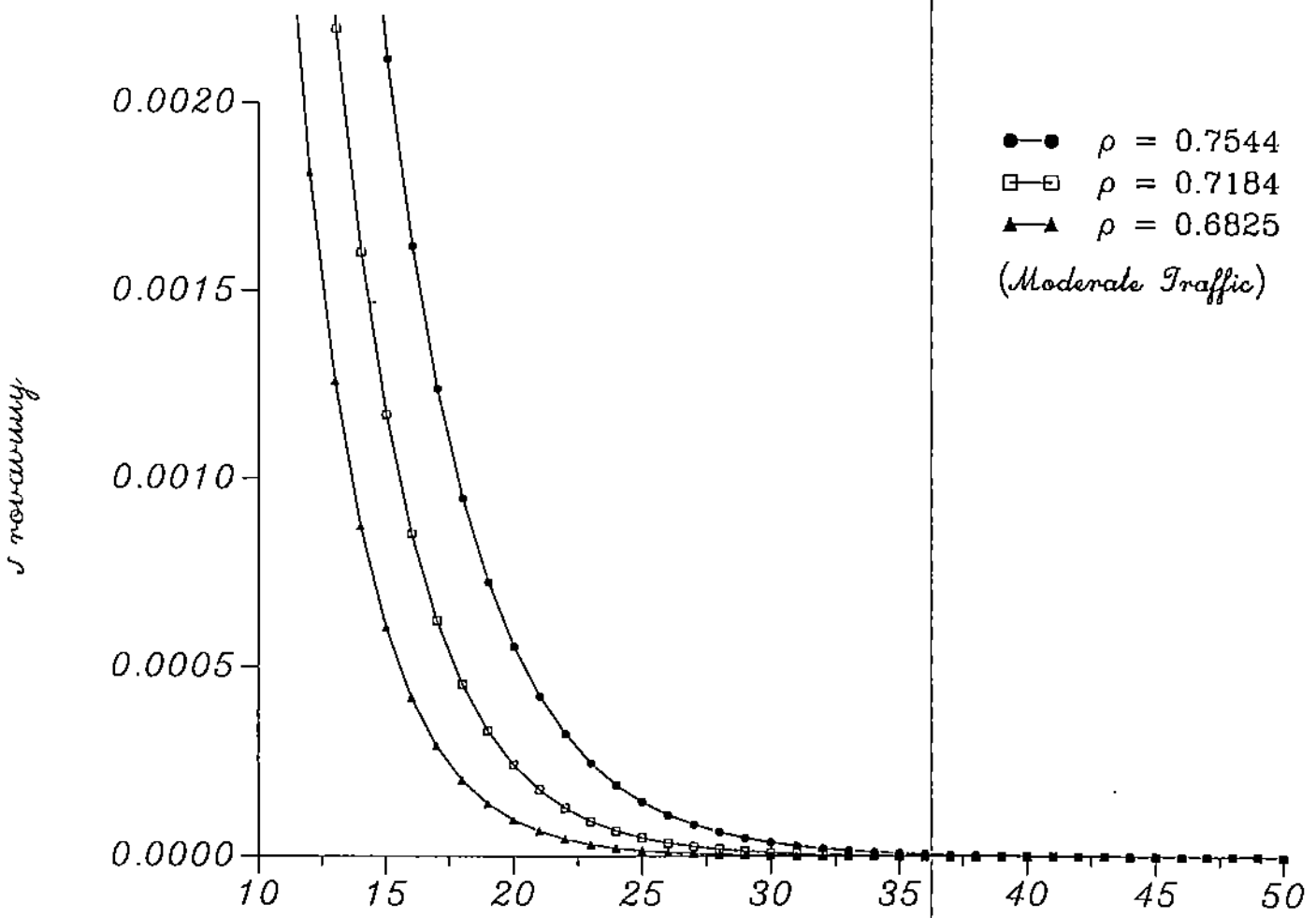
Queue Length
 Figure 3



●—● $\rho = 0.7544$
□—□ $\rho = 0.7184$
▲—▲ $\rho = 0.6825$

(Moderate Traffic)

Queue Length
Figure 4a



Queue Length
Figure 4b.

(4.4) can be time consuming. Neuts suggests the use of a hybrid algorithm that uses a Newton-Kantorovich iteration scheme for the first few G-iterates, after which (4.4) is used. In [8], Neuts demonstrates considerable savings in time with this technique.

Though a Gauss-Seidel iteration can be performed without explicit knowledge of q_1 , it is clear that first computing q_1 , will speed up the convergence of (4.8). From hard experience, the authors advocate the use of (4.6) in computing q_1 instead of resorting to the more tempting matrix relation

$$q_0 = q_0 B_0 + q_1 A_0 \quad (4.9)$$

which yields $q_1 = q_0 (I - B_0) A_0^{-1}$.

The latter computation is extremely sensitive to the condition of A_0 . In our particular token-ring application, the first column of A_0 tends to take on values that are considerably larger than the values in the other columns. This is understandable, since the first column represents the probabilities of no customer arrivals at station N during cycles in which other all stations are found to have empty buffers. Since such cycles tend to be extremely small in relation to cycles that include customer service at these stations, the result is immediate.

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