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ABSTRACT

In this report, we show that an arbitrarily given quadrilateral can always be interpreted as an image of a parallelogram and that the interpretation is unique aside from a multiplicative constant. Several applications of this theorem are discussed. It could be used to prove an old, geometrical theorem; it could facilitate the matching process when a sequence of images are available; it could be used as a simple technique for passive ranging in industrial environment or autonomous landing of an aircraft on a moving platform.

## 1. INTRODUCTION

This study is somewhat related to line drawing interpretation or "shape from contour" which is considered to be one computational module in intermediate vision. Detailed discussions and further references can be found in [1] [2] [3].

In this report, we show that an arbitrarily given quadrilateral can always be interpreted as an image of a parallelogram and that the interpretation is unique aside from a multiplicative constant. Several applications of this theorem are discussed. It could be used to prove an old, geometrical theorem; it could facilitate the matching process when a sequence of images are available; it could be used as a simple technique for passive ranging in industrial environment or autonomous landing of an aircraft on a moving platform.

## 2. INTERPRETATION OF QUADRILATERAL

All vectors used throughout this study will be in terms of a camera coordinate system as shown in Figure 1. A point $(x, y, z)$ appears in the image plane $z = f$ at $(fx/z, fy/z, f)$ under central projection. It is well known that if the projections of two parallel lines in 3D spaces are not parallel in the image plane and have intersection at $(\alpha, \beta, f)$ then the direction of the parallel line is $(\alpha, \beta, f)$. Using this fact, we first show the following Lemma and then the main observation.

**Lemma 1:** Let segments $A_1 A_2$, $A_4 A_3$ (see Figure 2) be the projections of two parallel segments $a_1 a_2$ and $a_4 a_3$ where $a_i$'s are the object coordinates of $A_i$'s in the 3D-space. Then the depth of $a_1$ and $a_2$ can be derived in terms of the length, denoted by $l_1$, of the segment between $a_1$ and $a_2$. Also, the depth of $a_3$ and $a_4$ can be derived in terms of the length, denoted by $l_2$, of the segments between $a_3$ and $a_4$.

**Proof:** Assuming the intersection of $A_1 A_2$ and $A_4 A_3$ is $P = (\alpha, \beta, f)$. It is evident that the unit direction of $a_1 a_2$ and $a_4 a_3$ is $(\alpha, \beta, f)/\sqrt{\alpha^2 + \beta^2 + f}$. Assuming the ratio between $|A_1 P|$ and $|A_1 A_2|$ is $s$. Then

$$(\alpha, \beta, f) = A_1 + s(A_2 - A_1). \tag{1}$$

Further, the line passing through $a_1$ and $a_2$ can be written as

$$\Gamma(t) = \frac{z_1}{f} A_1 + t(\alpha, \beta, f).$$

It is clear that there exists $t_0$ such that

$$\Gamma(t_0) = \frac{z_2}{f} A_2.$$

Thus,

$$\frac{z_1}{f} A_1 + t_0(\alpha, \beta, f) = \frac{z_2}{f} A_2.$$

Using (1), one obtains

$$(\frac{z_1}{f} + t_0 - t_0 s)A_1 = (\frac{z_2}{f} - t_0 s)A_2.$$

Thus,

$$\frac{z_1}{f} + t_0 - t_0 s = 0; \ \ t_0 = \frac{1}{s-1} \frac{z_1}{f}$$

and

$$z_2 = \frac{s}{s-1} z_1.$$

Since

$$z_2 - z_1 = l_1 f / \sqrt{f^2 + \alpha^2 + \beta^2}.$$

One obtains that

$$z_1 = (s-1)\frac{l_1 f}{\sqrt{f + \alpha^2 + \beta^2}}; \ \ z_2 = s \ \frac{l_1 f}{\sqrt{f + \alpha^2 + \beta^2}}.$$

Using the same reasoning, one can derive, where $t$ is the ratio between $|A_4 P|$ and $|A_4 A_3|$,

$$z_3 = t \ \frac{l_2 f}{\sqrt{t + \alpha^2 + \beta^2}}; \ \ z_4 = (t-1) \ \frac{l_2 f}{\sqrt{f + \alpha^2 + \beta^2}}. \qquad \textbf{Q.E.D.}$$

**Theorem:** Given a quadrilateral and a focal length $f$, the quadrilateral can always be interpreted as an image of a parallelogram in 3D space. This interpretation is unique aside from a multiplicative constant.

**Proof:** Let $A_i$'s denote the vertices of the quadrilateral as in Figure 2 and $z_i$ be the depth of $A_i$.

Let $P$ be the intersection of $A_1 A_2$ and $A_4 A_3$; $s$ be the ratio between $|A_1 P|$ and $|A_1 A_2|$; $t$ be the ratio between $|A_4 p|$ and $|A_4 A_3|$. Now choose

$$z_1 = (s-1) \cdot \frac{l_1 f}{\sqrt{f^2 + \alpha^2 + \beta^2}}; \quad z_2 = s \ \frac{l_1 f}{\sqrt{f^2 + \alpha^2 + \beta^2}},$$

$$z_3 = t \ \frac{l_1 f}{\sqrt{f^2 + \alpha^2 + \beta^2}}; \qquad z_4 = (t-1) \ \frac{l_1 f}{\sqrt{f^2 + \alpha^2 + \beta^2}},$$

where $l_1 \geq \max(|A_1 A_2|, |A_3 A_4|)$.

Since there are infinite values of $l_1$, there exists $l_1$ such that $z_1$, $z_2$, $z_3$, $z_4$ will be greater than the given focal length $f$.

It is evident that $a_1 a_2$ is parallel to $a_3 a_4$ since $a_1 a_2$ and $a_3 a_4$ are parallel to $(\alpha, \beta, f)$. It is also clear that $a_1 a_2 a_3 a_4$ is a parallelogram since $|a_1 a_2| = |a_3 a_4|$. The existence is thus shown.

One observes that cosine of angle $\theta$ formed between $a_2 a_1$ and $a_1 a_4$, is

$$\cos\theta = \frac{(\alpha, \beta, f)}{\sqrt{\alpha^2 + \beta^2 + f^2}} \cdot \frac{(r, \delta f)}{\sqrt{\alpha^2 + \beta^2 + f_2}}.$$

The dimension $D$, which is defined to be the ratio of two adjacent side i.e., $\frac{|a_1 a_2|}{|a_2 a_3|}$ can be found as:

$$D = \frac{\left|\left|\frac{z_1}{f} A_1 - \frac{z_2}{f} A_2\right|\right|}{\left|\left|\frac{z_3}{f} A_3 - \frac{z_2}{f} A_2\right|\right|}$$

$$= \frac{||(s-1)A_1 - s\, A_2||}{||t\, A_3 - s\, A_2||}$$

$$= \frac{||(\alpha, \beta, f)||}{||t\, A_3 - s\, A_2||}.$$

It can now be seen that, given a focal length $f$, the dimension and the angle are determined. Also, the dimension and the angle determine the parallelogram up to a scalar. Therefore, the interpretation is unique, up to a scalar. **Q.E.D.**

## 3. APPLICATIONS

From the above theorem, the parallelogram would change if the focal length varies. Also, a parallelogram in the image plane can only be interpreted as the parallelogram facing the viewer.

In other words,, what one sees in the image plane is what it is in the 3D-space except a unknown scale. This is different from the conclusion, using orthogonal projection, where it is suggestive that people perceive it as a slanted rectangle. Below three potential applications are described.

(I). Consider a quadrilateral $A_1 A_2 A_3 A_4$ as in Figure 3. Assuming $P$ is the intersection of $A_1 A_2$ and $A_4 A_3$; $Q$ is the intersection of $A_1 A_4$ and $A_2 A_3$. Since one can always interpret $A_1 A_2 A_3 A_4$ as an image of a parallelogram, we will choose this interpretation. From Lemma 1, one has

$$z_2 = \frac{|A_1 p|}{|A_2 p|} z_1 \quad (2); \qquad z_3 = \frac{|A_4 p|}{|A_3 p|} z_4 \quad (3)$$

Also

$$z_3 = \frac{|A_2 Q|}{|A_3 Q|} z_2 \quad (4); \qquad z_4 = \frac{|A_1 Q|}{|A_4 Q|} z_1 \quad (5)$$

Therefore, using (4) and (2), one derives

$$z_3 = \frac{|A_2 Q|}{|A_3 Q|} \cdot \frac{|A_1 P|}{|A_2 P|} z_1$$

and, using (5) and (3), one derives

$$z_3 = \frac{|A_4 P|}{|A_3 P|} \cdot \frac{|A_1 Q|}{|A_4 Q|} z_1$$

Hence

$$\frac{|A_2 Q|}{|A_3 Q|} \cdot \frac{|A_1 p|}{|A_2 p|} = \frac{|A_4 p|}{|A_3 p|} \cdot \frac{|A_1 Q|}{|A_4 Q|}$$

Thus

$$\frac{|A_2 Q|}{|A_3 Q|} \cdot \frac{|A_1 p|}{|A_2 p|} \cdot \frac{|A_3 p|}{|A_4 p|} \cdot \frac{|A_4 Q|}{|A_1 Q|} = 1$$

The above is a special form of Menelaus' Theorem [6] which is a classical theorem in plane geometry discoverd by Menelaus, a Greek astronomer, in the first centrury A.D. This also unveils the fact that seemingly unrelated branches of science are often interwoven in terms of mathematics.

(II). If four vertices of a parallelogram are marked and the dimension is known (see Figure 4), then one can use the formula

$$D = \frac{||(\alpha, \beta, f)||}{||tA_3 - sA_2||}$$

to derive the focal length, where the notation is the same as those in Lemma 1. Furthermore, one can use $(\alpha, \beta, f) \times (r, \delta, f)$, where $x$ is a cross product, to derive the orientation of the parallelogram with respect to camera coordinate systems. This is similar to the idea in [5] where they discuss passive ranging to known planar point sets. With the restriction of planar point sets to a parallelogram, a priori knowledge can be reduced to the knowledge of dimension of a parallelogram, as opposed to [5] where the exact distance between any two points and the focal length are required.

(III). In industrial environment, scene usually consists of many line segments, corners, circles, parallelograms and etc. The formula $D = \frac{||(\alpha, \beta, f)||}{||tA_3 - sA_2||}$ can be used to facilitate the matching process among those quadrilaterals as initial matches.

## 4. DISCUSSION AND CONCLUSION

Many techniques are proposed to interpret a line drawing. For instance, [4] proposes the heuristic assumption that a skew symmetry is interpreted as an oriented real symmetry; [2] proposes to minimize $\int_C [(d\kappa/ds)^2 + \kappa^2\tau^2]ds$, where $\kappa$ and $\tau$ are curvature and torsion, over all curves $C$ consistent with the data in the image plane; [3] proposes to minimize the measure $A/P^2$ where $P$ is the parameter (total arc length) of $C$, ranging over all planar curves consistent with the projection in the image plane, and $A$ is the plane area enclosed. These techniques all try to formalize (more or less) the belief that an ellipse is interpreted as an image of some circle in 3D spaces; a triangle is interpreted as an image of an equitriangle in 3D spaces; a parallelogram is interpreted as a rectangle.

In this report, we show that one can interpret a quadrilateral as a parallelogram and that the interpretation is unique. However, we do not claim that human perception will always interpret it as such; this remains to be investigated from a psychological aspect. A parallelogram in the image plane can only be interpreted as facing the camera as opposed to the many interpretations exist in the case of parallel projection.

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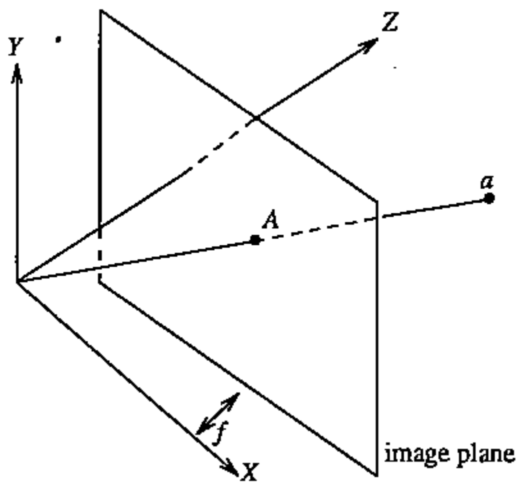


Figure 1: $f$ is the focal length; $A$ is the projection of $a$.

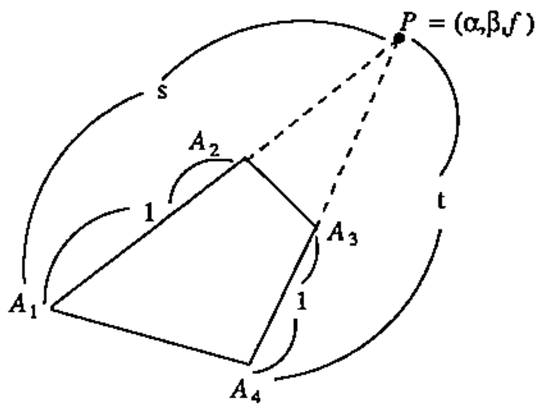


Figure 2: $s$ is the ratio between $|A.P|$ and $|S_1A_2|$ ; $t$ is the ratio between $|A_4P|$ and $|A_3A_4|$ .

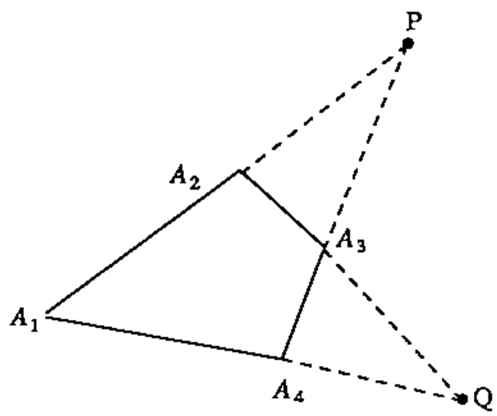


Figure 3: $\frac{|A_2Q|}{|A_3Q|} \cdot \frac{|A_1P|}{|A_2P|} \cdot \frac{|A_3P|}{|A_4P|} \cdot \frac{A_4Q|}{|A_1Q|} = 1$ .

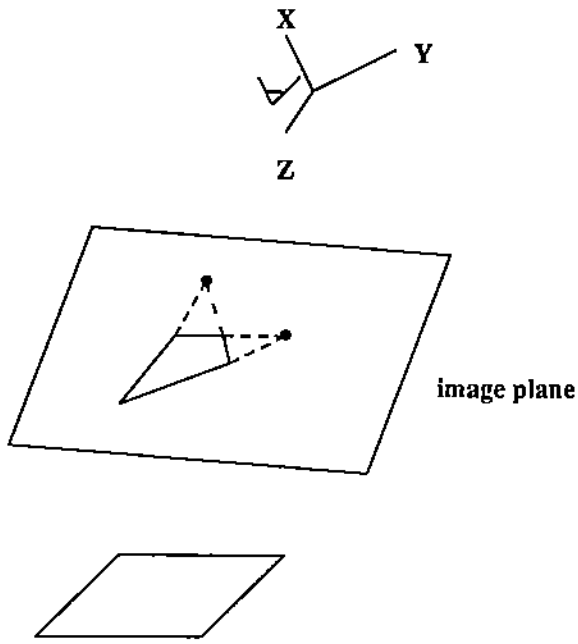


Figure 4: Passing ranging to planar points.