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On the Analysis of the Average Height of a Digital Trie: Another Approach

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ON THE ANALYSIS OF THE AVERAGE HEIGHT OF A
DIGITAL TRIE: ANOTHER APPROACH

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Abstract

The average height of a digital trie has been recently investigated in many papers [2]–[8]. In most works on binary digital tries, a Bernoulli model and independent keys are assumed. We relax these assumptions in that V-ary asymmetric tries, Bernoulli and Poisson models, and dependent keys are considered. We show that the average height of the trie is asymptotically equal to 2 \( l_{g_u} n \) (for the Bernoulli model) and 2 \( l_{g_u} \mu \) (the Poisson model) where \( n \) and \( \mu \) are the number of records and the average number of records respectively. The parameter \( u \) is defined as \( u^{-1} = \sum_{i=1}^{V} p_i^2 \), and the \( V \) elements of the alphabet are distributed according to probabilities \( p_i \), \( i = 1, \ldots, V \). Finally, a generalization to the so called \( b \)-tries is discussed. In contrast to the previous analysis, our approach is very simple since we avoid explicit computation of the height distribution.

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1. INTRODUCTION

Let $A$ be a $V$-ary alphabet, i.e., $A = \{\alpha_1, \ldots, \alpha_V\}$ and let $S$ denote the set of $n$ strings (keys) built over the alphabet $A$. A trie (digital search tree) is a $V$-ary digital search tree in which edges are labeled by elements from $A$ and leaves (external nodes) contain the keys (records) [1]. The access path from the root to a leaf is a minimal prefix of the information contained in the leaf. An important variant of tries is obtained using sequential storage algorithm for subtries with the size less than or equal to a fixed bound $b$, i.e. external node is capable of storing at most $b$ keys. Such a trie is called $b$-trie [2], [3].

Digital tries find many applications in computer science. A trie is used as an index to access data in a secondary memory (e.g. extendible hashing) [2], [7], [8], [18], it can be used in the pattern-matching algorithms (position trees and string identifiers) [1] and in sorting algorithms like triesort [4], [16] and radix exchange sort [16],[19]. Some other applications of the digital tries include: conflict resolution algorithms for broadcast communications, polynomial factorization and Huffman’s algorithm [1], [3], [16], [17], [19]. We analyze a random family of tries with $n$ stored records from the height viewpoint. It is assumed that each key consists of (possible infinite) elements from the alphabet $A$, and the element $\alpha_k \in A, k = 1, 2, \ldots, V$, occurs with probability $p_k$ at any position of a key (asymmetric $V$-ary trie). In most analyses (see [2]–[4], [7], [8]) binary symmetric tries were investigated which restricts the applications of the analysis (e.g., see matching-string problem where English characters occur with very different probabilities).

This paper provides a new methodology to study the average height of general asymmetric digital tries. Using a simple inequality for order statistics we prove that the average height $EH_n$, of a trie is $EH_n \approx 2\log n$, where $u^{-1} = \sum_{i=1}^{V} p_i^2$. This result is generalized in three different directions. At first, we drop the assumption that the fixed number of keys are stored in the trie.
Assuming that a trie is built over random number of keys distributed according to Poisson process with parameter \( \mu \) (Poisson model), we prove that \( EH_\mu = 2 lg_2 \mu \). Secondly, for \( b \)-tries we show that the average height is asymptotically equal to \( (1 + \frac{1}{b}) lg_2 n \), where \( u^{-1} = \sum_{i=1}^{V} p_i b^{-i} \).

Finally, we assume that there exists some statistical dependency between keys. Then, it is proved that \( EH_n = O(lg_2 n) \), where \( u \) is a constant which reflects statistical dependency among the keys.

The average height of digital tries has been recently investigated in [2]–[8]. In [2] Flajolet studied binary symmetric \( b \)-tries. Based on some classical counting results in occupancy problems, Flajolet derived asymptotic distribution of the height. Using complex analysis (Cauchy integral formula) he also found the average height of a trie. Jacquet and Regnier [3] extended Flajolet’s result to binary asymmetric tries. They have made extensive use of the Mellin transform technique. Devroye [4] analyzed binary symmetric tries, and based on the occupancy problem he derived some inequalities on the asymptotic distribution of the height. The most general results were obtained by Pittel [5] (see also [6]), where \( V \)-ary asymmetric tries with \( b = 1 \) were investigated. Unfortunately, the proofs in [5] and [6] are not constructive, and the results are well hidden. For some more results, see also [7] and [8]. Our approach to the problem is essentially different. In contrast to the previous analysis we use elementary calculus, and we avoid explicit computation of the height distribution. In this paper we only concentrate on the asymptotic results for the average height of digital tries, however, the methodology can be extended to the analysis of digital trees and Patricia tries.

2. MAIN RESULTS

Let us consider a set of all digital tries with \( n \) records, \( X_1, X_2, \ldots, X_n \), over an alphabet \( A = \{a_1, a_2, \ldots, a_v\} \). Each record consists of (possible infinite) string of elements (digits) from \( A \), e.g., \( X_k = (x_{k1}, x_{k2}, \ldots, x_{kj}, \ldots) \) where \( x_{kj} \in A \), \( j = 1, 2, \ldots \). For a given keys \( X_1, X_2, \ldots, X_n \) the digital trie is built in a usual manner (see [1]). For example, in Figure
we show a 3-ary trie built over $A = \{1, 2, 3\}$ with 6 records $A, B, \ldots, F$. Note that a trie consists of two types of nodes, namely internal nodes and external nodes. The internal nodes are used to determine branching strategy, while keys (records) are stored in the external nodes.

The common assumptions under which the random family of tries is analyzed, are specified below:

Figure 1. Example of 3-ary digital trie with n=6.

(i) A key $X_k = (x_{k1}, x_{k2} \ldots)$ is a sequence of elements (digits) from $A$ which form an independent sequence of Bernoulli trials with $Pr\{x_{kj} = \alpha_i\} = p_i$, $k = 1, 2, \ldots, n$, $i = 1, 2, \ldots, V$.

(ii) The keys $X_1, X_2, \ldots, X_n$ are statistically independent.

(iii) The number of records stored in a trie is fixed and equal to $n$.

These assumptions create the so called Bernoulli model. In addition, we also assume that

(iv) the external node is capable to store only one record, i.e., regular tries ($b = 1$) are analyzed in this section.
In a trie three quantities are of particular interests: the depth of a leaf (the paths from the root to a randomly chosen leaf), the height, \( H_n \), of a trie (the maximum over all depths), and the smallest path from the root to a leaf. The depth of a leaf was previously analyzed in [3], [5], [6] and [9]. Here we concentrate on the average height, \( E H_n \).

Let us define a common path of two keys, \( \text{com}(X_i, X_j) \), \( i, j = 1, 2, \ldots, n \), as the common prefix of \( X_i \) and \( X_j \), that is, \( \text{com}(X_i, X_j) = k \) if \( X_i \) and \( X_j \) agree exactly on their first \( k \) digits, but differ in their \((k + 1)\)-st. Let \( Y_{ij} = \text{com}(X_i, X_j) \), \( i \neq j \). Note that \( Y_{ij} = Y_{ji} \), hence we restrict the indices to \( i = 1, 2, \ldots, n \), \( j = i+1, i+2, \ldots, n \). Sometimes, for simplicity, we renumber the random variables \( Y_{ij} \) and we write \( Y_1, Y_2, \ldots, Y_m \) where \( m = n(n - 1)/2 \).

There is, of course, a one-to-one correspondence between \( Y_{ij} \) and \( Y_k \). Under the assumptions (i)-(iv) the random variable \( Y_{ij} \), for any \( i \) and \( j \), is geometrically distributed with parameter \( p_1^2 + p_2^2 + \cdots + p_n^2 \), that is

\[
Pr \{ Y_{ij} = k \} = Pr \{ \text{com} (X_i, X_j) = k \} = \left[ \sum_{l=1}^{V} p_l^2 \right]^k \left[ 1 - \sum_{l=1}^{V} p_l^2 \right] \quad k = 0, 1, \ldots. \quad (1)
\]

Let \( u^{-1} = \sum_{l=1}^{V} p_l^2 \). Note that although \( X_i, i = 1, \ldots, n \) are independent, the random variables \( Y_{ij} \) are dependent.

To find a relationship between the height \( H_n \) and \( \text{com}(X_i, X_j) \) note that the common prefix of a particular key \( X_k \) and all other keys \( X_j, j = 1, 2, \ldots, n, j \neq k \), determine the position of \( X_k \) in the trie. Hence

\[
H_n = 1 + \max_{1 \leq i \leq n} \max_{j \neq i} \{ Y_{ij} \} = 1 + \min_{1 \leq k \leq m} \{ Y_k \}. \quad (2)
\]

To illustrate (2), let us consider the trie in Figure 1. We find that \( H_6 = 3 \), and \( \text{com}(A, B) = \text{com} (A, C) = 1, \text{com}(A, D) = \text{com} (A, E) = \text{com} (A, F) = 0; \text{com}(B, C) = 2, \text{com}(B, D) = 0, \) etc. Hence \( 3 = H_6 = 1 + \max \{ \text{com} (X_i, X_j) \} = 1 + \text{com} (D, C) = 3. \)
Eq. (2) suggests that to compute $H_n$ we need to know some statistics of the maximum of $m$ dependent random variables, $Y_1, Y_2, \ldots, Y_m$. Such a statistic is known in the literature as the order statistic. In the next subsection, we derive some simple properties of the average of max $\{Y_k\}$, that is, $E \max \{Y_k\}$.

**The average value of max $\{Y_i\}$**

Let $Y_1, Y_2, \ldots, Y_m$ be identically distributed random variables with the distribution function $F(y)$. Define

$$M_m = \max_{1 \leq i \leq m} \{Y_i\}.$$  

It is easy to see that

$$M_m \leq a_m + \sum_{i=1}^{m} (Y_i - a_m)^+$$  

where $a_m$ is a parameter dependent on $m$, and $x^+ = \max \{0, x\}$. For a nonnegative random variable $Y$ with distribution function $F(y)$ the average $EY$ may be computed as

$$EY = \int_{0}^{\infty} [1 - F(y)]dy.$$  

Hence, by (3) the average $EM_m$ is

- for continuous random variables

$$EM_m \leq a_m + m \int_{a_m}^{\infty} [1 - F(x)]dx$$  

- for discrete random variables

$$EM_n \leq a_m + m \sum_{k=a_n}^{\infty} [1 - F(k)]$$

The RHS of (4) is minimized if $a_m$ is chosen such that

$$a_m = \min \{k : Pr\{Y > k\} \leq \frac{1}{m} \}.$$  

**EXAMPLE 1:** Exponential distribution
Let \( F(y) = 1 - e^{-\lambda y} \), \( \lambda \) is a parameter. Then by (5) \( a_m = \frac{1}{\lambda} \ln m \), and (4a) implies

\[
EM_m \leq \frac{1}{\lambda} \ln m + \frac{1}{\lambda}.
\]

If, in addition, \( Y_1, \ldots, Y_m \) are independent, then [10]

\[
EM_m = \frac{1}{\lambda} \ln m + \frac{\gamma}{\lambda},
\]

where \( \gamma = 0.577 \) is the Euler constant. Note that the difference between (6) and (7) is of order \( O(1) \).

EXAMPLE 2: Geometric distribution

Let \( Y \) be geometrically distributed, i.e. \( Pr\{Y = k\} = p^k(1 - p) \). Then \( Pr\{Y > k\} = p^{k+1}, \) and by (5) \( a_m = \lfloor \frac{\ln m}{\ln p^{-1}} \rfloor \), where \( \lfloor \cdot \rfloor \) is the floor operator. Also by (4b)

\[
EM_m \leq \frac{\ln m}{\ln p^{-1}} + \frac{p}{1 - p}.
\]

Note that the geometric distribution may be approximated by an exponential distribution with parameter \( \lambda = \ln p^{-1} \). Since \( \ln p^{-1} \leq \frac{1 - p}{p} \) one finds that (8) is equivalent to (6) with \( \lambda = \ln p^{-1} \).

Both inequalities (6) and (8) imply that the leading term in \( EM_m \) is \( a_m \). The question is whether \( EM_m \sim a_m \), i.e. \( \lim_{m \to \infty} EM_m/a_m = 1 \). Lai and Robbin proved [10], [11] that \( EM_m \sim a_m \) if the distribution \( F(y) \) satisfies the following conditions

\[
\lim_{y \to \infty} \frac{1 - F(cy)}{1 - F(y)} = 0 \quad \text{for every} \quad c > 1
\]

(9a)
\[ \int_{-\infty}^{0} |x| \, dF(x) < \infty \text{ for some } r > 0. \]  

(9b)

Note that (9) holds for the exponential and geometric distributions, i.e., \( EM_m \sim \frac{1}{\lambda} \ln m \).

The average height of a trie

By (2) \( EH_n = 1 + E \max_{1 \leq k \leq m} \{ Y_k \} \), where \( m = n(n - 1)/2 \) and \( Y_k \) is geometrically distributed with parameter \( u^{-1} = \sum_{i=1}^{V} p_i^2 \). Define \( h = \ln u \). Then (8) (see also (6)) implies

\[ EH_n \leq 1 + \frac{1}{h} \ln \frac{n(n - 1)}{2} + \frac{1}{u - 1} \]

and after simple algebra one finds

\[ EH_n \leq \frac{2}{h} \ln n + 1 + \frac{1 - \ln 2}{h} + O(n^{-1}). \]  

(10)

Hence by (9) \( EH_n \sim \frac{2}{h} \ln n = 2 \log_2 n \), that is,

\[ \lim_{n \to \infty} \frac{EH_n}{\ln n} = \frac{2}{h}. \]  

(11a)

How tight is the upper bound (10)? For binary symmetric tries \( h = \ln 2 \) Devroy proved that [4]

\[ EH_n \leq 2 \log_2 n + 1 + \frac{\gamma - \ln 2}{\ln 2} \]  

(11b)

hence the upper bound (10) is greater than (11) by 0.61. On the other hand, Flajolet [2] shows that for binary symmetric tries

\[ EH_n = 2 \log_2 n + \frac{\gamma - \ln 2}{\ln 2} + P(\ln n) + o(1) \]  

(12)

where \( P(\ln n) \) is a periodic function with very small amplitude. The derivation of (11) and (12) require, however, much more advanced techniques. In both cases the average \( EH_n \) was obtained.
through the analysis of the asymptotic approximation of the distribution function of $H_n$.

**Some remarks on the asymptotic distribution of $H_n$**

In the subsection we offer some remarks on the asymptotic distribution of $H_n$. We do not pretend to present rigorous proofs. Rather, we give some reasons justifying the form of the asymptotic distribution.

Assume first that $X_1, X_2, \ldots, X_n$ are identically independently distributed random variables with distribution function $F(x)$. Let also $X_{(n)} = \max\{X_1, \ldots, X_n\}$. It is shown [12], [13] that there exist constants $a_n$ and $b_n$ such that $(X_{(n)} - a_n)/b_n$ has a proper distribution $\Lambda(x)$, as $n$ tends to infinity. In fact, it is proved that the extreme distribution $\Lambda(x)$ may have three different forms. If $X_i$ is exponentially distributed with parameters $\lambda$, then $\Lambda(x) = \exp(-e^{-x})$, that is [12], [13]

$$\lim_{n \to \infty} Pr\{(X_{(n)} - \frac{\ln n}{\lambda}) \lambda < x\} = \Lambda(x) = \exp(-e^{-x}).$$

(13)

The situation is a little more delicate for discrete random variables. Anderson [14] showed that if $X_i$ is geometrically distributed with parameter $p$, then

$$\Lambda(x - 1) \leq \lim_{n \to \infty} \inf Pr\{(X_{(n)} - \frac{\ln n}{\ln p^{-1}})\ln p^{-1} < x\} \leq$$

$$\lim_{n \to \infty} \sup Pr\{(X_{(n)} - \frac{\ln n}{\ln p^{-1}})\ln p^{-1} < x\} \leq \Lambda(x).$$

(14)

From the practical viewpoint, the difference between (13) and (14) may be ignored if one assumes $\lambda = \ln p^{-1}$ (i.e., one approximates the geometric distribution with parameter $p$ by the exponential distribution with parameter $\lambda = \ln p^{-1}$). It is also proved [13] that under some assumptions (13) holds for dependent random variables $X_1, X_2, \ldots, X_n$.

The height of a trie $H_n$, is given by (2), where $Y_1, Y_2, \ldots, Y_m, m = n(n - 1)/2 - n^2$ are dependent random variables geometrically distributed with parameter $u^{-1} = \sum_{i=1}^{v} p_i^2$. Approxi-
mating the geometric distribution by the appropriate exponential distribution with parameter 

\[ h = \ln n \] 

and using (13), one may show that

\[ \lim_{n \to \infty} P_r \{ H_n < x + 2 \lg u, n \} = \exp[-\exp(-x \ln u)]. \]  

(15)

A rigorous proof of (15) is given in [6], however, quite a different approach is adopted there. The discrete version of (15) for binary symmetric tries can be found in [4].

Note that for large \( n \) (15) implies the following approximation

\[ P_r \{ H_n < x \} \approx \exp[-\exp[-\ln u (x - 2 \lg n)]. \]  

(16)

Let \( Z \) be a random variable with the distribution function \( \Lambda((x - \xi)\lambda) = \exp[-\exp[-(x - \xi)\lambda]] \).

Then, it is shown [15] that

\[ E Z = \xi + \gamma \lambda, \quad \text{var } X = \frac{\pi^2}{6\lambda^2}, \text{ where } \gamma = 0.577 \text{ is the Euler constant.} \]

By (16) we find that for large \( n \)

\[ E H_n \approx \frac{2}{h} \ln n + \frac{\gamma - \ln 2}{h} + 1 \]  

(17a)

\[ \text{var } H_n \approx \frac{\pi^2}{6h^2} \]  

(17b)

Flajolet in [2] proved that for binary symmetric tries the approximation (17a) is different from the exact asymptotic expression by a fluctuating function with a small amplitude. He also found that the variance, \( \text{var } H_n \), is not a constant, but rather a fluctuating function.

### 3. GENERALIZATIONS

In this section, we generalize the results from Section 2, that is, we investigate Poisson model, consider \( b \)-tries \( (b > 1) \), and finally present some results for dependent keys.

#### 3.1 Poisson model

We replace assumption (iii) by
The number of records stored in a trie, \( N \), is a random variable distributed according to Poisson with parameter \( \mu \), i.e.,

\[
Pr \{ N = n \} = \frac{\mu^n}{n!} e^{-\mu},
\]

Under (iii') the Bernoulli model becomes the Poisson model. Let \( H_\mu, H_n \) denote the height in the Poisson and Bernoulli models, respectively. Then

\[
E H_\mu = \sum_{n=0}^{\infty} E H_n \frac{\mu^n}{n!} e^{-\mu}
\]

and using (10) we find

\[
E H_\mu \leq \frac{2}{h} e^{-\mu} \sum_{n=1}^{\infty} \ln n \frac{\mu^n}{n!} + 1 + \frac{1 - \ln 2}{h}.
\]

To evaluate the series in (19) we use the inequality \( \ln n \leq \chi_n \), where \( \chi_n \) is the \( n \)-th Harmonic number. It is known that \([16]\), \([17]\)

\[
\sum_{n=1}^{\infty} \chi_n \frac{\chi_n}{n!} = \left[ \frac{e^x - e^{xy}}{1 - y} \right]_{x=0}^\infty dy
\]

hence

\[
e^{-\mu} \sum_{n=1}^{\infty} \ln n \frac{\mu^n}{n!} \leq e^{-\mu} \sum_{n=1}^{\infty} \chi_n \frac{\mu^n}{n!} = \int_0^\infty \frac{1 - e^{-y}}{y} dy = \ln \mu + \gamma + E_1(\mu)
\]

where \( E_1(\mu) \) is the exponential integral defined as \( E_1(x) = \int_0^x e^{-t} t^{-1} dt \) \(| \arg x | < \pi \). Thus, (11) and (20) implies that

\[
E H_\mu \leq \frac{2}{h} \ln \mu + \frac{E_1(\mu) + \gamma + 1 - \ln 2}{h} + 1
\]

Also, by (11) and (20) we find

\[
\lim_{\mu \to \infty} \frac{E H_\mu}{\ln \mu} = \frac{2}{h},
\]
i.e., $EH_{\mu} - 2 \log \mu$.

3.2 The average height of $b$-tries

We now drop assumption (iv), and consider $b$-tries with $b > 1$, that is, each external node may store at most $b$ keys. Let $X_1, X_2, \ldots, X_n$ be the keys, and for $i_1, i_2, \ldots, i_{b+1} \in \{1, 2, \ldots, n\}$ we denote $Y(i_1, i_2, \ldots, i_{b+1}) = \text{com}(X_{i_1}, X_{i_2}, \ldots, X_{i_{b+1}})$, the common prefix for $X_{i_1}, \ldots, X_{i_{b+1}}$, i.e., the number of digits that $X_{i_1}, \ldots, X_{i_{b+1}}$ agree. Note that we have \( \binom{n}{b+1} \) random variables $Y(i_1, \ldots, i_{b+1})$, and for simplicity we sometimes renumber them and denote $Y_1, Y_2, \ldots, Y_m, m = \binom{n}{b+1}$. Figure 2 shows 2-tries for the same keys as in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{ trie_example.png}
\caption{Example of 3-ary digital 2-trie with $n=6$.}
\end{figure}

Note that the height of $b$-tries is given by

$$H_n = 1 + \max_{1 \leq i \leq m} \{Y_i\}$$

as in (2). The distribution of $Y(i_1, \ldots, i_{b+1})$ is geometric with parameter $u^{-1} = \sum_{i=1}^{b+1} p_i$, that is

$$Pr \{ Y(i_1, \ldots, i_{b+1}) = k \} = u^{-k}(1 - u^{-1}) \quad k = 0, 1, \ldots$$

Let also $h = \ln u$. 
To compute \( E H_n \) we need \( E \max_{1 \leq i \leq m} Y_i \). By (4) and (5) we find

\[
E \max_{1 \leq i \leq m} Y_i \leq a_m + m \sum_{k = a_m}^{\infty} u^{-(k+1)}
\]

where

\[
a_m = \frac{\ln m}{h} = \frac{\ln \left( \frac{n}{b+1} \right)}{h}.
\]

Note that

\[
m \sum_{h = a_m}^{\infty} u^{-(k+1)} \leq \frac{1}{h}
\]

and

\[
\left( \frac{n}{b+1} \right) = \frac{n^{b+1}}{(b+1)!} \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{2}{n} \right) \cdots \left( 1 - \frac{b}{n} \right)
\]

hence, these and (25), (26) imply

\[
E H_n \leq \frac{b+1}{h} \ln n + 1 + \frac{1 - \ln (b+1)!}{h} + O(n^{-1}).
\]

Since the geometric distribution satisfies (9), we also obtain

\[
\lim_{n \to \infty} \frac{E H_n}{\ln n} = \frac{b+1}{h}.
\]

For symmetric V-ary trie, i.e., \( p_1 = p_2 = \cdots = p_V = \frac{1}{V} \) one immediately shows that \( h = b \ln V \) and

\[
E H_n = \left( 1 + \frac{1}{b} \right) \lg V n
\]

which generalizes Flajolet's result.

To evaluate the asymptotic distribution of \( H_n \) we may use the arguments from the preceding section. Then

\[
\lim_{n \to \infty} Pr \{ H_n < x + (b+1) \lg u n \} = \exp\{-\exp[-x \ln u]\}
\]
which agrees with the result obtained in [6].

3.3 Dependent keys

In many applications the keys are statistically dependent, e.g. see position trees and substring identifiers (suffix trie) [1]. In this subsection, we relax assumption (ii) keeping the other assumptions unchanged (for simplicity we set $b = 1$).

Let $X_k^i, X_l^j$ denote the $i$-th and the $j$-th digit in the $k$-th and the $l$-th keys. We assume that there is a dependency between $X_k^i, X_l^j$, which we express in terms of the joint distribution, that is,

$$p_{n,m}(k, l) = Pr\{X_k^i = \alpha_n, \ X_l^j = \alpha_m\} < 1$$

where $k, l = 1, 2, \ldots, n$, $n, m = 1, 2, \ldots, V, \ i, j = 1, \ldots$. The probability (31) does not depend on $i$ and $j$ because of the assumption (i). Define $Y_{kl} = \text{com}(X_k, X_l)$ as the common prefix of $X_k$ and $X_l$. By assumption (i) the distribution of $Y_{kl}$ is geometric with parameter $[u(k,l)]^{-1} = \sum_{i=1}^{V} p_{i}(k,l)$. Note that now $Y_{kl}$ are not identically distributed, and $F_H(j) = Pr\{Y_{kl} < j\} = 1 - [u(k,l)]^{-j(k+1)}$. The height of the trie is given by (2), and to compute $E H_n$ we need $\max_{k,l} Y_{k,l}$

We now generalize formula (3)–(5). Since inequality (3) holds we also have

$$E \max_{k,l} Y_{kl} \leq a_m + \sum_{k,l=1}^{n} \sum_{j=a_m}^{\infty} [1 - F_H(j)]$$

where $m = n(n-1)/2$. The RHS of (32) is minimized for such $a_m$ that

$$\sum_{k=1}^{n} \sum_{l=k+1}^{n} F_H(a_m) = n - 1$$

(33)

For geometric distribution with parameter $u^{-1}(k,l)$ (33) is equivalent to

$$\sum_{k=1}^{n} \sum_{l=k+1}^{n} [u(k,l)]^{-a_m + 1} = 1$$

(34)
and one finds that

\[ a_m \leq \frac{\ln m}{\ln \min_{k,l} n(k,l)} \]  

(35)

Let \( u_{\min} = \min_{k,l} \{u(k,l)\} = \left[ \max_{k,l} \{\sum_{i=1}^{v} p_i^2(k,l)\} \right]^{-1} \), and \( h_{\min} = \ln u_{\min} \). Noting that \( m = n(n - 1)/2 \) and the contribution of the sum in (32) is \( O(1) \) we prove that

\[ EH_n \leq \frac{2}{h_{\min}} \ln n + O(1), \]  

(36)

hence \( EH_n = O(\ln n) \). This result is easy to generalize to \( b \)-tries and Poisson model. Note also that the assumption \( p_{n,m}(k,l) < 1 \) (see (31)) is important. For example, if one builds a prefix tree (the \( k \)-th key is the prefix of the \( (k - 1) \)st key), then the height is equal to \( n \).

4. CONCLUSIONS

This paper studies the average height of digital tries. Using elementary methods, in contrast to the previous analysis, we proved that \( EH_n \sim 2 \ln u \), where \( u \) is a parameter dependent on the distributions of the digits in a key. This result is next extended to Poisson model, \( b \)-tries and dependent keys. Under quite general assumptions, we show that \( EH_n = O(\ln n) \), even for dependent keys. The methodology proposed here can be applied to analyze some other quantities of the digital tries, as well as to obtain some characteristics for Patricia tries and digital trees.

REFERENCES


