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SOME EXPLICIT FORMULAS FOR MIXED EXPONENTIAL SERVICE SYSTEMS

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Abstract

We present some explicit formulas for queue length and waiting time distributions of customers in the $M/HE_m/1$ queue. The formulas are obtained with the aid of roots of quadratic, cubic, and quartic polynomials constructed from a recurrence equation. With an example, we demonstrate that the formulas for queueing distributions are extremely accurate, while the corresponding infinite history $M/GI/1$ recurrence equation is not. Applications include computation of queueing distributions, accurate tail probabilities, in qualitative studies, and in systems where exponentiality can be replaced by hyperexponentiality. The explicit solutions are easier to use than the problem-specific partial fraction expansions of the Pollachek-Khinchin transform.

I. INTRODUCTION

For $M/GI/1$ systems in which service-time distributions are mixtures of exponentials, we present a method that enables us to obtain explicit formulas for steady-state queueing distributions. This is done by associating a real-valued, monic polynomial of degree m, with each $M/GI/1$ system having an m-component mixture as a service-time distribution. When $m \leq 4$, the roots of these polynomials are known explicitly. Denoting these roots as $r_1, \cdots, r_m$, we show that explicit formulas are easily had for such systems, in terms of these roots. When $m \geq 5$, explicit formulas for roots are not known. In this case, the method can still be applied, but one will first need to apply a root-finding algorithm to determine the roots.

Though algorithmic methods are available for $M/GI/1$ systems (see [1],[2]), the method we develop has some interesting features. First, the explicit formulas are more *accurate* than the standard $M/GI/1$ recurrence equation [1] simply due to catastrophic cancellation effects [11] in the use of the latter. Example 1 in section IV demonstrates this effect numerically. Second, the behaviour of the polynomial associated with each queueing system says something about the behaviour of the steady-state queueing distribution of the system. With the aid of graphical examples (see Example 2, section IV), we illustrate the fact that given two polynomials corresponding to two different queueing systems, the one whose largest root is greater yields a system whose queue length distribution possesses a longer tail. Additionally we give an example which shows that a larger traffic-intensity *does not* necessarily imply a longer tail in queue length distribution. Finally, the explicit formulas we derive can be useful in qualitative analysis (where a formula *may* tell more about system behaviour than a computational procedure), and in systems where exponentiality can meaningfully be replaced by hyperexponentiality.

The earliest work we are aware of that proceeded along these lines is that of Greenberg [3]. In an elegant result, Greenberg demonstrated that if the service-time distribution of a Poisson arrival, single-server queue was HE_2 (i.e., two-component hyperexponential), then the steady-state customer queue-length and waiting time distributions must also each be two-component generalised mixtures of geometric and exponential distributions, respectively. By generalised mixture, we mean that the coefficients in the linear combination may be arbitrary constants that sum to one. The techniques used in [3] involved Laplace-Stieltjes transforms. We show that Greenberg's results, and generalisations, can be obtained quite independently via polynomial equations in a single *real variable*.

II. PRELIMINARIES

In this section we introduce some notation through definitions and outline the general approach to our results, which are presented in detail in the next section. First we review the fact that the Poisson process maps distinct continuous probability distributions uniquely onto distinct discrete probability distributions on $(0,\infty)$. This result is applied to determine that the Poisson process maps distinct hyperexponentials onto distinct mixtures of geometrics. Next, the standard $M/GI/1$ recurrence equation is used, along with the mixture of geometrics just indicated, to prove that steady-state probabilities of customer queue-lengths must be distributed as generalised mixtures of geometrics. In doing this, we restrict our attention to the case $2 \le m \le 4$, and obtain the result via two theorems. We remark on the effects of generalisation to $m \ge 5$. The case $m = 1$ is one of purely exponential service, otherwise known the $M/M/1$ queue whose explicit steady-state forms are well known.

The two theorems mentioned above serve the following purpose. The first demonstrates that each $M/HE_m/1$ queue is *uniquely* associated with a monic, real-valued polynomial of degree m. It is shown that for each m, the unique polynomial (that we call the *characteristic polynomial* of the corresponding queue) has m real and positive roots. The second theorem obtains the steady-state queue length distribution as a function of the roots of the characteristic polynomial, and in so doing shows that under certain conditions, the roots must all be distinct and lie in the interval

(0, 1). The queue-length distribution is found to be a generalised mixture of geometrics. A third theorem uses a result of Haji and Newell [10] to show that steady-state waiting-time distributions must also be generalised m-component hyperexponentials.

In the sequel, we use the sets $\mathbf{B}=\{\alpha_1, \ldots, \alpha_m\}$, $\mathbf{x}=\{a_1, \ldots, a_m\}$, and $\mathbf{y}=\{\mu_1, \ldots, \mu_m\}$, where $0<\alpha_j<1$, $\sum_{i=1}^{m} \alpha_i=1$, $0<a_j<1$, and $\mu_j>0$, $1 \leq j \leq m$. Here $\mathbf{B}$ contains a set of probabilities that is to be used as the finite, discrete, mixing distribution, $\mathbf{x}$ contains the set of first terms of different geometric distributions, and $\mathbf{y}$ contains parameters of different exponential distributions.

Definitions

A discrete distribution $\{r_i\}_{i=0}^{\infty}$ on the nonnegative integers satisfying $r_i = a(1-a)^i$ for $0 < a < 1$, and $1 \leq i \leq \infty$, is a *geometric distribution* with first term a, denoted by $G(a)$.

A continuous distribution $dS(\cdot)$ given by $dS(t)=\mu e^{-\mu t}, 0 \leq t < \infty$, is an *exponential distribution* with parameter μ, $\mu > 0$, denoted by $E(\mu)$.

A finite mixture of m geometric distributions is defined to be a distribution of the form $\sum_{j=1}^{m} \alpha_j G(a_j)$. We denote this mixture of geometrics by $G_m(\mathbf{B}, \mathbf{x})$.

A continuous distribution $dF(t)$ on the nonnegative reals, $t \geq 0$, satisfying $dF(t) = \sum_{j=1}^{m} \alpha_j E(\mu_j)$, is a mixture of exponential distributions. We denote this mixture as $HE_m(\mathbf{B}, \mathbf{y})$, i.e., an m-component hyperexponential distribution.

If $\mathbf{C}=\{\sigma_1, \cdots, \sigma_m\}$ is a set of arbitrary constants that sum to one, then $\sum_{j=1}^{m} \sigma_j G(a_j)$ is a generalised mixture of geometrics, denoted by $G_m^*(\mathbf{C}, \mathbf{x})$.

Similarly, $dF^*(t)=\sum_{j=1}^{m} \sigma_j E(\mu_j)$ is a generalised mixture of exponentials, denoted by $HE_m^*(\mathbf{C}, \mathbf{y})$.

III. MAIN RESULTS

In an M/GI/1 queue operating at steady-state, let X_n be a random variable representing the number of customers remaining in the queue as the n^{th} customer departs from the system. Then $\{X_n\}$ is a well-known Markov chain [9]. The arrival rate of customers to the system is taken to be λ, $\lambda > 0$, and the service-time distribution is $dS(\cdot)$. The probability that j customers arrive during an arbitrary customer's service is given by

$$k_j = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^j}{j!}\, dS(t) \qquad (1)$$

where for the sake of generality the Stieltjes integral is used. As a convention, we henceforth take $\{k_j\}$ to mean $\{k_j\}_{j=0}^{\infty}$. Clearly, $\{k_j\}$ is a discrete distribution defined on the nonnegative integers. For $dS(\cdot)$ an exponential distribution, it can be shown [5] that a necessary condition for stability is equivalent to the condition $\frac{1}{2} < k_0 < 1$. In the next two lemmas, we establish the result that $\{k_j\}$ takes on special forms when $dS(t)$, $t \geq 0$, is an exponential distribution or of the form $HE_m(\mathrm{B}, \mathrm{y})$. Both results are consequences of the fact [4] that a Poisson process maps each probability distribution on $[0, \infty)$ *uniquely* onto a discrete probability distribution defined on $\{0, 1, 2, \cdots\}$. The first Lemma is reproduced from [4] without proof and is easily proved.

Consider a homogeneous Poisson process with parameter λ (i.e., the customer arrival process), and let $N(Y)$ be the number of points in the time interval $[0, Y)$, with Y a non-negative random variable.

LEMMA 1

The random variable Y is exponentially distributed with parameter μ if and only if $N(Y)$ has a geometric distribution with parameter $\frac{\lambda}{\mu + \lambda}$, for λ, $\mu > 0$.

Interpreting (1) in the context of Lemma 1, we see that if $dS(\cdot)$ is exponential with mean μ, then $\{k_j\}$ must be $G\left(\frac{\mu}{\mu+\lambda}\right)$. Next, consider what must happen if $dS(\cdot)$ is $HE_m(\mathbf{B}, \mathbf{y})$.

LEMMA 2

The random variable Y is distributed as $HE_m(\mathbf{B}, \mathbf{y})$ if and only if $N(Y)$ is distributed as $G_m(\mathbf{B}, \mathbf{x})$, with $a_j = \frac{\mu_j}{\mu_j + \lambda}$, $\mu_j > 0$, for $j = 1, 2, \cdots, m$ and $\lambda > 0$.

Proof:

Assume that Y is distributed as $HE_m(\mathbf{B}, \mathbf{y}) \equiv \sum_{j=1}^{m} \alpha_j \mu_j e^{-\mu_j t}$, for $\mu_j > 0$, $j = 1, 2, \cdots, m$. Using (1) with $HE_m(\mathbf{B}, \mathbf{y})$ in place of $dS(\cdot)$, we obtain

$$\begin{aligned} k_j^{(m)} &= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^j}{j!}\left[\sum_{i=1}^{m} \alpha_i \mu_i e^{-\mu_i t}\right] dt \\ &= \sum_{i=1}^{m}\left[\int_0^\infty \frac{e^{-\lambda t}(\lambda t)^j \alpha_i \mu_i e^{-\mu_i t}\, dt}{j!}\right] \\ &= \sum_{i=1}^{m} \alpha_i \frac{\mu_i \lambda^j}{(\lambda+\mu_i)^{j+1}} = \sum_{i=1}^{m} \alpha_i a_i (1-a_i)^j , \end{aligned} \tag{2}$$

where taking $a_j = \frac{\mu_j}{\mu_j + \lambda}$, we find that $\{k_j^{(m)}\}$ is distributed as $G(\mathbf{B}, \mathbf{x})$. Conversely, assume that $\{k_j^{(m)}\}$ is distributed as $G(\mathbf{B}, \mathbf{x})$. It follows that

$$k_j^{(m)} = \sum_{i=1}^{m} \alpha_i \left(\frac{\mu_i}{\lambda+\mu_i}\right)\left(\frac{\lambda}{\lambda+\mu_i}\right)^j = \sum_{i=1}^{m} \alpha_i a_i (1-a_i)^j , \tag{3}$$

for $j \geq 0$. An application of Lemma 1 to each of the m terms of the summation in (3) yields Y distributed as $HE_m(\mathbf{B}, \mathbf{y})$ with $\mathbf{y} = (\mu_1, \cdots, \mu_m)$.

■

From Lemma 1, it is clear that if the service-time distribution of a single-server queue is

$\mu e^{-\mu t}$, $t \geq 0$, then the probability of j customer arrivals during an arbitrary customer's service-time is given by $k_0(1-k_0)^j$, for $j \geq 0$, with $k_0 = \frac{\mu}{\lambda+\mu}$. More generally, if the service-time distribution is $HE_m(\mathbf{B}, \mathbf{y})$, then by Lemma 2, the probability of j customer arrivals during an arbitrary customer's service-time is given by $k_j^{(m)} = \sum_{i=1}^{m} a_i b_i^j$, where $b_i = 1 - a_i$, and $a_i = \frac{\mu_i}{\lambda+\mu_i}$, for $1 \leq i \leq m$ and $j \geq 0$. The next result is a theorem that converts the infinite history $M/GI/1$ recurrence equation [1] for an $M/HE_m(\mathbf{B}, \mathbf{y})/1$ system into one of history $(m+2)$, for $1 \leq m \leq 4$.

The notation that is used in the rest of the discussion is explained as follows. A symbol of the form $c_i^{(m)}$ denotes a particular quantity c_i that comes from an $M/HE_m(\mathbf{B}, \mathbf{y})/1$. Thus, $\{p_n^{(m)}\}$ is the equilibrium queue length distribution, $\{k_n^{(m)}\}$ is the stationary distribution of customer arrivals in a service-time, and $dW_m(\cdot)$ is the equilibrium distribution of waiting time in an $M/HE_m(\mathbf{B}, \mathbf{y})/1$ system, respectively. Since we will need the explicit forms of $p_j^{(m)}$, for $j \geq 0$ and each m, $1 \leq m \leq 4$, these are obtained from the M/GI/1 recurrence equation [1] as

$$
\begin{aligned}
p_0^{(m)} &= 1 - \rho_m \\
p_1^{(m)} &= \frac{p_0^{(m)}\,(1 - k_0^{(m)})}{[k_0{}^{(m)}]} \\
p_2^{(m)} &= \frac{p_0^{(m)}\,(1 - k_0^{(m)} - k_1^{(m)})}{[k_0^{(m)}]^2} \\
p_3^{(m)} &= \frac{p_0^{(m)}\,[(1 - k_1^{(m)})^3 + k_0^{(m)}\,(k_1^{(m)} - k_2^{(m)} - 1)]}{[k_0^{(m)}]^3} \\
p_4^{(m)} &= \frac{p_0^{(m)}\,[(1 - k_1^{(m)})^3 - k_0^{(m)}\,(1 - k_1^{(m)})^2 + (k_0^{(m)})^2 (k_2^{(m)} - k_3^{(m)}) - 2k_0^{(m)}\,k_2^{(m)}\,(1 - k_1^{(m)})}{[k_0^{(m)}]^4}
\end{aligned}
\tag{4}
$$

where $k_j^{(m)}$ has already been defined, and ρ_m is the traffic intensity of the $M/HE_m(\mathbf{B}, \mathbf{y})/1$ queue.

THEOREM 1

Let $\mathbf{B} = (\alpha_1, \cdots, \alpha_m)$, and $\mathbf{y} = (\mu_1, \cdots, \mu_m)$, with $\alpha_i > 0$, $\sum_{j=1}^{m} \alpha_j = 1$, $\mu_i > 0$, for $i = 1, 2, \cdots, m$, and $1 \leq m \leq 4$. If the arrival rate of an $M/GI/1$ queue is λ, $\lambda > 0$, and the service-time distribution is $HE_m(\mathbf{B}, \mathbf{y})$, then the equilibrium queue length distribution, if it exists, is given by a recurrence of history $(m+2)$. Additionally, corresponding to the queueing system, there exists a unique real-valued polynomial $p_m(z)$ of degree m, with real coefficients depending on $\mathbf{B}$ and $\mathbf{y}$, with all roots real and positive.

Proof:

First, it is necessary to establish that we deal with a stable queue. The traffic intensity ρ_m of the $M/HE_m(\mathbf{B}, \mathbf{y})/1$ queue can be expressed as [5]

$$\rho_m = \sum_{j=1}^{\infty} \left[\sum_{i=1}^{m} \alpha_i \, a_i \, b_i^j \right] \tag{5}$$
$$= \sum_{i=1}^{m} \frac{\alpha_i \, (1-a_i)}{a_i},$$

from where the condition $\rho_m < 1$ is obtained in terms of α_i and a_i, $1 \le i \le m$, as the required stability condition. Using a standard M/GI/1 recurrence equation (see [1]), the steady-state queue length probabilities can be computed as

$$k_0^{(m)} \, p_{n+1}^{(m)} = p_n^{(m)} - p_0^{(m)} \, k_n^{(m)} - \sum_{j=1}^{n} p_j^{(m)} \, k_{n-j+1}^{(m)}, \quad n \ge 0 \tag{6}$$

where $p_0^{(m)} = 1 - \rho_m$ is the probability of an empty queue.

In the following, we choose to suppress the dependence of each distribution on m for convenience, taking pains to handle each case separately. The particular system being handled is made clear from context.

The case $m = 1$

Using $HE_1(\mathbf{B}, \mathbf{y})$, i.e, the exponential distribution $\mu_1 \, e^{-\mu_1 t}$, for $t \ge 0$, and Lemma 1, it follows that $\{k_j\}$ is $G(k_0)$, or $k_j = k_0 \, (1-k_0)^j$ for $j \ge 0$. Thus, (6) can be written as

$$k_0 \, p_{n+1} = p_n - p_0 \, k_0 \, (1-k_0)^n - \sum_{j=1}^{n} p_j \, k_0 \, (1-k_0)^{n-j+1}, \quad n \ge 0 \tag{7}$$

with $p_0 = 1 - \rho_1$. On comparing (7) with a version of itself with n replaced by $(n-1)$, multiplying by $(1-k_0)$ and subtracting, one obtains the simplified recurrence

$$p_{n+1} = \left[\frac{1}{k_0}\right] p_n - \left[\frac{1-k_0}{k_0}\right] p_{n-1}, \quad n \geq 1 \tag{8}$$

with $k_0 = \alpha_1 a_1 = a_1$, since α_1 must equal unity. The stability condition translates into the requirement $\frac{1}{2} < k_0 < 1$ (i.e., if $k_0 = 1$ the arrival process is nonexistent, and if $k_0 = \frac{1}{2}$, the probability transition matrix for the chain $\{X_n\}$ becomes doubly stochastic and the queueing process is unstable). Since (7) is a linear recurrence equation with constant coefficients, it is uniquely associated with a characteristic polynomial $P(z)$ (see [5], [6]), given by

$$P(z) = k_0 z^2 - z + (1 - k_0) \tag{9}$$

with the roots $z = 1$ and $z = \frac{1-k_0}{k_0}$. The degree one polynomial $p_1(z)$ which is the characteristic polynomial of the $M/HE_1(\mathbf{B},\mathbf{y})/1$ queue is obtained by dividing (9) by the factor $(z-1)$, yielding

$$p_1(z) = k_0 z - (1 - k_0) \tag{10}$$

Note that single root of $p_1(z)$ lies in $(0,1)$ since $\frac{1}{2} < k_0 < 1$. Clearly, this root is positive. The uniqueness of $p_1(z)$ in (10) comes from the fact that (9) is the characteristic equation [6] of a recurrence relation with constant coefficients. That is, starting with the characteristic equation in (9) and reversing the procedure that was used to obtain (9) will give the original recurrence in (6).

The case $m = 2$

Using the recurrence in (6) with a service-time random variable distributed as $HE_2(\mathbf{B},\mathbf{y})$, one obtains, for $n \geq 1$,

$$p_{n+1}\left[\alpha_1 a_1 + \alpha_2 a_2\right] = p_n - p_0\left[\alpha_1 a_1 b_1^n + \alpha_2 a_2 b_2^n\right] \tag{11}$$

$$- \sum_{j=1}^{n} p_m \left[\alpha_1 a_1 b_1^{n-j+1} + \alpha_2 a_2 b_2^{n-j+1} \right]$$

where, $p_0 = 1 - \rho_2$. On comparing (11) with a version of itself with n replaced by $(n-1)$ we obtain another recurrence, valid for $n \geq 2$. Multiplying this last recurrence by b_1 and subtracting from (11), we obtain

$$p_{n+1} \left[\alpha_1 a_1 + \alpha_2 a_2 \right] = p_n \left[1 + (\alpha_1 a_1 + \alpha_2 a_2) b_1 \right] - p_{n-1} b_1 + p_0 \left[\alpha_2 a_2 b_2^{n-1} b_1 - \alpha_2 a_2 b_2^n \right]$$
$$+ \alpha_2 a_2 b_1 \sum_{j=1}^{n} p_m b_2^{n-j} - \alpha_2 a_2 \sum_{j=1}^{n} p_m b_2^{n-j+1} \tag{12}$$

for $n \geq 1$. On repeating this procedure with (12), we obtain the simplified four-term recurrence

$$p_{n+1} \left[\alpha_1 a_1 + \alpha_2 a_2 \right] = p_n \left[\alpha_1 a_1 b_2 + \alpha_2 a_2 b_1 + 1 \right] - p_{n-1} \left[b_1 + b_2 \right] + b_1 b_2 p_{n-2} \tag{13}$$

for $n \geq 2$. The terms p_0, p_1, and p_2 can be obtained from (4).

The unique polynomial $P(z)$ associated with this recurrence is given by the cubic

$$P(z) = (\alpha_1 a_1 + \alpha_2 a_2) z^3 - (1 + \alpha_1 a_1 b_2 + \alpha_2 a_2 b_1) z^2 + (b_1 + b_2) z - b_1 b_2 \tag{14}$$

for which $z = 1$ is a root. Upon dividing out the factor $(z - 1)$, we obtain the characteristic polynomial $p_2(z)$ of the $M/HE_2(\mathrm{B}, \mathrm{y})/1$ queue as

$$p_2(z) = (\alpha_1 a_1 + \alpha_2 a_2) z^2 - (1 - a_1 a_2) z + (1 - a_1 - a_2 + a_1 a_2). \tag{15}$$

Since $P(z)$ characterises a probability distribution, it will always be the case that $z = 1$ is a zero of $P(z)$. We are left with showing that both roots of $p_2(z)$ are positive. Using the fact that $z = -1$ must be a root of the polynomial $P(-z)$, we apply Descartes rule of signs [8] to (14). The number of sign changes in $P(-z)$ is zero, thus implying that $P(z)$ cannot have any negative real roots. Consequently, $p_2(z)$ cannot have any negative real roots. Finally, the discriminant of (15) is easily see to be always positive, for a stable queue, thus yielding its two real, and by our earlier work, positive roots r_1 and r_2 as the zeros

$$z = \frac{(1 - a_1 a_2) \pm \sqrt{(1 - a_1 a_2)^2 - 4(\alpha_1 a_1 + \alpha_2 a_2)(1 - a_1 - a_2 + a_1 a_2)}}{2(\alpha_1 a_1 + \alpha_2 a_2)}. \tag{16}$$

The cases $m = 3, 4$

The recurrence in (6) now becomes

$$p_{n+1}\left[\sum_{i=1}^{m} \alpha_i \, a_i\right] = p_n - p_0\left[\sum_{i=1}^{m} \alpha_i \, a_i \, b_i^n\right] - \sum_{j=1}^{n} p_m \left[\sum_{i=1}^{m} \alpha_i \, a_i \, b_i^{n-j+1}\right] \qquad (17)$$

which can be simplified in a manner similar to that done previously. The simplified five-term recurrence is

$$A \; p_{n+1} = B \; p_n - C \; p_{n-1} + D \; p_{n-2} - E \qquad (18)$$

for $n \geq 3$. The quantities p_j can be obtained from (4) for $0 \leq j \leq 3$. The polynomial $P(z)$ corresponding to this recurrence is

$$P(z) = A\, z^4 - B\, z^3 + C\, z^2 - D\, z + E \qquad (19)$$

with

$$A = \sum_{i=1}^{3} \alpha_i \, a_i \, , \qquad B = 1 + \sum_{i=1}^{3}\left\{(1-a_i) \sum_{\substack{j=1 \\ j \neq i}}^{3} \alpha_j \, a_j\right\},$$

$$C = \sum_{i=1}^{3}\left\{\alpha_i \, a_i \left[\prod_{\substack{j=1 \\ j \neq i}}^{3} (1-a_j)\right] + (1-a_i)\right\}, \qquad D = \sum_{i=1}^{3}\left[\prod_{j \neq i} (1-a_j)\right],$$

and $E = \prod_{i=1}^{3} (1-a_i)$.

Since $P(z)$ characterises a probability distribution, $z = 1$ is a root of $P(z)$. The number ν of sign changes of $P(z)$ is $\nu=4$, meaning that $P(z)$ has either two or four real roots (since $(\nu - k)$ must be even and nonnegative, where k is the number of roots of $P(z)$). Substituting $-z$ in place of z in (19), we find that the corresponding equation has no sign changes. Correspondingly, $P(z)$ cannot have any negative real roots. Using Descartes rule, we conclude that $P(z)$ has at least two real roots.

We divide $P(z)$ by $(z - 1)$ to obtain the characteristic polynomial

$$p_3(z) = A\,z^3 + (A - B)z^2 + (C - B + A)z + [(A + C) - (B + D)] \tag{20}$$

of the $M/HE_3(\mathrm{B},\mathrm{y})/1$ queue. The term $(B - A)$ reduces to $[1 + \sum_{j=1}^{3} (1 - a_j)\,(\sum_{\substack{i=1\\ i\neq j}}^{3} \alpha_i\, a_i) - \alpha_j\, a_j]$ which is positive, meaning that $(A - B)$ is negative. Similarly, $C - B + A = 2 + a_1\ \ a_2 a_3 - (a_1 + a_2 + a_3)$ which is always positive, and $C - B + A - D = a_1 a_2 a_3 - a_1 a_2 - a_1 a_3 - a_2 a_3 + a_1 + a_2 + a_3$ which is always negative. At this stage, a tedious term by term comparison of terms shows that as long as the stability condition $\rho_3 < 1$ is satisfied, the discriminant of $p_3(z)$ will always be positive. Thus we conclude that (20) has three real roots, and by our previous work, all must be positive. The three roots r_1, r_2, and r_3 can be written explicitly (for example see [8]). We omit these expressions for brevity.

An alternate method to show that all the roots are positive is to compute the leading coefficients of Sturm's remainders directly using (20). These turn out to be $R_1 = 2B^2 + 2AB - 4A^2 - 6AC$ and $R_2 = -\left[\sum_{i=1}^{10} f_i\right] / \left[\sum_{j=1}^{7} h_i\right]$ where the functions f_i and h_i are given in the appendix. An algebraic comparison of terms shows that $R_1 > 0$ and $R_2 > 0$ for a stable queue. Since both leading coefficients of Sturm's remainders are positive, all roots of (20) must be real, and by our previous work, also positive.

When $m = 4$, a little labour yields the finite-history recurrence as

$$A\,p_{n+1} = B\ p_n - C\ p_{n-1} + D\ p_{n-2} - E\ p_{n-3} + F\ p_{n-4} \tag{21}$$

for $n \geq 4$. Again, the terms p_j can be obtained from (4) for $0 \leq j \leq 4$. The polynomial $P(z)$ corresponding to this recurrence is given by

$$P(z) = A\,z^5 - B\,z^4 + C\,z^3 - D\,z^2 + E\,z - F \tag{22}$$

with

$$A = \sum_{i=1}^{4} \alpha_i\, a_i\ , \qquad B = 1 + \sum_{i=1}^{4} \left\{ (1 - a_i) \sum_{\substack{j=1\\ j\neq i}}^{4} \alpha_j\, a_j \right\},$$

$$C = \left[\sum_{\substack{i,j=1 \\ j \neq i}}^{4} (1-a_i)(1-a_j) \sum_{\substack{k=1 \\ k \neq i,j}}^{4} \alpha_k \, a_k \right] + \sum_{i=1}^{4} (1-a_i), \quad D = \left[\sum_{i=1}^{4} (1-a_i) \sum_{\substack{j=1 \\ j \neq i}}^{4} (1-a_j) \right],$$

$E = \sum_{i=1}^{3} \prod_{\substack{j=1 \\ j \neq i}}^{4} (1-a_j)$, and $F = \prod_{i=1}^{4} (1-a_i)$.

Since $(z-1)$ is a factor of $P(z)$, we divide out this factor to obtain the characteristic polynomial

$$p_4(z) = A\,z^4 + (A-B)z^3 + (C-B+A)z^2 + (A+C-B-D)z + (A+C+E-D-B) \quad (23)$$

of the $M/HE_4(\mathrm{B},\mathrm{y})/1$ queueing system. We proceed with the same constructive argument as in the case of $m = 3$ to show that all four roots of $p_4(z)$ are real and positive. Explicit expressions for the roots r_1, r_2, r_3, and r_4 of the quartic $p_4(z)$ can be found in [8].

■

Remark

When $m \geq 5$, the expression in (22) generalises to a polynomial $P(z)$ of degree m in the real variable z. The coefficients $A_1, A_2, \ldots, A_{(m+1)}$ of this polynomial can be obtained by generalising the coefficients shown for the case $m=4$. On dividing $P(z)$ by the factor $(z-1)$, we obtain the characteristic polynomial $p_m(z)$ of the $M/HE_m/1$ queue. The roots $r_1, r_2, \cdots, r_m$ of $p_m(z)$ must be determined with the aid of a root-finding procedure. Due to the lack of a general formula for the roots of a degree m polynomial, $m > 4$, the queue-length and waiting-time distributions do not have an explicit representation for $m \geq 5$.

In [9], the queue length distribution for an $M/HE_2/1$ system is obtained explicitly via partial fraction decomposition of the P-K transform equation. However, note that the partial-fraction expansion method must be used on a problem-specific basis. That is, the expansion itself depends on the constants involved. On the contrary, our next theorem shows how one can avoid such a problem-specific expansion via an application of Theorem 1 (i.e., using known formulae for the roots of quadratic, cubic, and quartic polynomials of a single variable).

■

We consider an example (see [9], p. 189) where arrivals are Poisson with rate λ, and service times are distributed as

$$dS(t) = \frac{1}{4}\,\lambda e^{-\lambda t} + \frac{3}{4}(2\lambda)\,e^{-2\lambda t} \tag{24}$$

In order to motivate the following theorem, we demonstrate how this result can be obtained as a special case of a more general result. From (24) we obtain the parameters $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{3}{4}$, and $a_1 = \int_0^\infty \lambda e^{-2\lambda t}\,dt = \frac{1}{2}$, and $a_2 = \int_0^\infty \lambda e^{-3\lambda t}\,dt = \frac{2}{3}$. Let L_2 denote the stationary queue-length random variable for this $M/HE_2(\mathbf{B},\mathbf{y})/1$ queue, where $\mathbf{B} = \{\frac{1}{4}, \frac{3}{4}\}$, and $\mathbf{y} = \{\lambda, 2\lambda\}$. Since this is an $M/GI/1$ queue, the queue-length distribution must be of the form

$$p_k = Pr[L_2 = k] = (1-\rho_2)\,g_k\,, \qquad k = 0, 1, 2, \cdots \tag{25}$$

where g_k , $0 < g_k < 1$, is some discrete function of k, with $g_0 = 1$. Corresponding to the service-time distribution in (24), g_k takes the form

$$g_k = c_1\,r_1^k + c_2\,r_2^k \tag{26}$$

where r_1 and r_2 are the two roots of the characteristic polynomial $p_2(z)$, and c_1 and c_2 are certain constants. Since the two roots are positive, it is clear that (26) will define a probability distribution only if each root lies in $(0,1)$, and the constants c_1 and c_2 sum to one. On computing the roots of the polynomial from (16), we obtain $r_1 = \frac{2}{5}$, $r_2 = \frac{2}{3}$. The constants c_1 and c_2 can be obtained by using boundary conditions (i.e., probabilities that we already know from (4)). Using $p_0 = 1-\rho_2$, and $p_1 = (1-\rho_2)\frac{(1-\alpha_1 a_1 - \alpha_2 a_2)}{\alpha_1 a_1 + \alpha_2 a_2}$ we obtain $c_1 = \frac{1}{4}$, $c_2 = \frac{3}{4}$. Finally, computing $\rho_2 = \frac{5\lambda}{8\lambda} = \frac{5}{8}$, we arrive at

$$p_k = \frac{3}{32}\left[\frac{2}{5}\right]^k + \frac{9}{32}\left[\frac{2}{3}\right]^k, \quad k = 0, 1, 2 \cdots \tag{27}$$

Thus, without resorting to partial-fraction expansion, we have an explicit formula for the $M/HE_2(\mathbf{B}, \mathbf{y})/1$ queue. We now present the general result.

THEOREM 2 (Queue-length distributions)

Let $r_1, \cdots, r_m$ be the zeros of the characteristic polynomial $p_m(z)$ of an $M/HE_m(\mathbf{B}, \mathbf{y})/1$ queue with arrival rate $\lambda > 0$, and $2 \le m \le 4$. If the queueing system is stable, then the equilibrium queue length distribution is given by a generalised mixture of geometrics $G_m^*(\mathbf{C}, \mathbf{r})$, i.e.,

$$p_k^{(m)} = Pr[L_m = k] = \sum_{i=1}^{m} \beta_i^{(m)} r_i^k \tag{28}$$

where $\mathbf{C} = \{\frac{\beta_1^{(m)}}{1-r_1}, \cdots, \frac{\beta_m^{(m)}}{1-r_m}\}$, $\mathbf{r} = \{(1-r_1), \cdots, (1-r_m)\}$, with $r_1, \cdots, r_m$ all distinct,

$0 < r_i < 1$, and the $\beta_i^{(m)}$ given constants, $i = 1, .., m$, satisfying $\sum_{i=1}^{m} \frac{\beta_i^{(m)}}{1-r_i^{(m)}} = 1$.

Proof:

From M/GI/1 theory the queueing system $M/HE_m(\mathbf{B}, \mathbf{y})/1$, $2 \le m \le 4$, is stable provided that $\sum_{i=1}^{m} \frac{\alpha_i(1-a_i)}{a_i} < 1$. Without loss of generality, we can assume that $\mathbf{B}$ and $\mathbf{y}$ each contain distinct elements, for otherwise, combining like elements will reduce the number of terms, but still yield distinct elements. The coefficients of the characteristic polynomial $p_m(z)$ are symmetric functions [8] of the m distinct elements $a_1, \cdots, a_m$. But the coefficients must also be symmetric functions of the m roots, and since no collapsing occurs among the a_i and α_i, $i = 1, \cdots, m$, the roots $r_1, \cdots, r_m$ must be all distinct. The theory of finite history linear recurrences [6, 7] tells us that the characteristic polynomial $p_m(z)$ possesses a unique solution of the form

$$p_k^{(m)} = \beta_1^{(m)} r_1^k + \cdots + \beta_m^{(m)} r_m^k, \quad k = 0, 1, 2, \cdots \tag{29}$$

where $\beta_1^{(m)}, \cdots, \beta_m^{(m)}$ are coefficients yet to be determined. In order to discover these coefficients, we make use of known boundary conditions. As before, we suppress superscripts, since each value of m is treated separately, and the relation of each queue length distribution to m is made clear from context.

The case $m = 2$

The boundary conditions are given by $p_0 = 1 - \rho_2$, and $p_1 = (1-\rho_2)\dfrac{(1-k_0)}{k_0}$, for $k_0 = \alpha_1 a_1 + \alpha_2 a_2$. Here p_0 and p_1 are probabilities corresponding to zero length and one customer $M/HE_2(\mathrm{B}, \mathrm{y})/1$ systems, respectively, explicitly given in (4). Upon simplifying these equations, we obtain the system

$$\begin{aligned} \beta_1 + \beta_2 &= 1 - \rho_2 \\ \beta_1 r_1 + \beta_2 r_2 &= (1-\rho_2)\frac{(1 - \alpha_1 a_1 - \alpha_2 a_2)}{\alpha_1 a_1 + \alpha_2 a_2} \end{aligned} \tag{30}$$

which can be solved to yield the coefficients

$$\begin{aligned} \beta_1 &= \left[\frac{(\alpha_1 a_1 + \alpha_2 a_2)(1 + r_2) - 1}{(\alpha_1 a_1 + \alpha_2 a_2)(r_2 - r_1)}\right](1-\rho_2) \\ \beta_2 &= \left[\frac{1 - (\alpha_1 a_1 + \alpha_2 a_2)(1 + r_1)}{(\alpha_1 a_1 + \alpha_2 a_2)(r_2 - r_1)}\right](1-\rho_2) \end{aligned} \tag{31}$$

in terms of the roots r_1, r_2, and the parameters of the distribution $G_m(\mathrm{B}, \mathrm{x})$. Thus we obtain

$$p_k = \beta_1 r_1^k + \beta_2 r_2^k, \quad k = 0, 1, 2, \cdots \tag{32}$$

as the equilibrium queue length distribution of the $M/HE_2(\mathrm{B}, \mathrm{y})/1$ queue. Since (32) defines a distribution, it is clear that r_1 and r_2 must both lie in $(0, 1)$. Otherwise the distribution in (32) will give terms that grow rapidly with increasing n. Since (32) defines a probability distribution, we must have $\sum_{i=1}^{m} \dfrac{\beta_i}{1-r_i} = 1$.

The case $m = 3$

From (29), we can obtain the unique solution to the recurrence in (17) provided that we determine the β_i, for $i=1,2,3$. The roots r_1, r_3, and r_3 are already known (i.e., using the known formula for the zeros of a cubic). Using p_0, p_1 and p_2 for boundary conditions, we obtain the system

$$\begin{aligned}
&\beta_1+\beta_2+\beta_3=1-\rho_3\\
&\beta_1 r_1+\beta_2 r_2+\beta_3 r_3=(1-\rho_3)\left[\frac{1-(\alpha_1 a_1+\alpha_2 a_2+\alpha_3 a_3)}{\alpha_1 a_1+\alpha_2 a_2+\alpha_3 a_3}\right]\\
&\beta_1 r_1^2+\beta_2 r_2^2+\beta_3 r_3^2=(1-\rho_3)\frac{\left[1-\alpha_1 a_1(2-a_1)-\alpha_2 a_2(2-a_2)-\alpha_3 a_3(2-a_3)\right]}{(\alpha_1 a_1+\alpha_2 a_2+\alpha_3 a_3)^2};
\end{aligned} \tag{33}$$

Solving this system yields the coefficients

$$\begin{aligned}
\beta_1&=(1-\rho_3)\left[\frac{[k_0]^2(r_2 r_3+r_3+r_2)-k_0(r_3+r_2)-k_1-k_0+1}{[k_0]^2(r_2 r_3-r_1 r_3-r_1 r_2+r_1^2)}\right]\\
\beta_2&=(1-\rho_3)\left[\frac{[k_0]^2(r_1 r_3+r_3+r_1)-k_0(r_3+r_1)-k_1-k_0+1}{[k_0]^2(r_1 r_3-r_2 r_3-r_1 r_2+r_2^2)}\right]
\end{aligned} \tag{34}$$

and

$$\beta_3=(1-\rho_3)\left[\frac{[k_0]^2(r_1 r_2+r_2+r_1)-k_0(r_2+r_1)-k_1-k_0+1}{[k_0]^2(r_1 r_2-r_2 r_3-r_1 r_3+r_3^2)}\right]$$

where $k_0=\alpha_1 a_1+\alpha_2 a_2+\alpha_3 a_3$, and $k_1=\alpha_1 a_1(1-a_1)+\alpha_2 a_2(1-a_2)+\alpha_3 a_3(1-a_3)$. Thus we obtain

$$p_k=\beta_1 r_1^k+\beta_2 r_2^k+\beta_3 r_3^k\,,\qquad k=0,\,1,\,2,\,\cdots \tag{35}$$

as the equilibrium queue length distribution of the $M/HE_3(\mathrm{B},\mathrm{y})/1$ queue. As in the last case, (35) defines a distribution, and hence each r_i is in $(0,1)$, and $\sum_{i=1}^{m}\frac{\beta_i}{1-r_i}=1$.

The case $m=4$

The unique solution to (17) for $m=4$ is given by (29), where the r_i are obtained from the explicit formula for the zeros of a quartic, and the coefficients β_i are left to be determined, $i=1,2,3,4$. Using p_j, $j=0,\,1,\,2,\,3$, given in (4), to define boundary conditions, we get

$$\begin{aligned}
&\beta_1+\beta_2+\beta_3+\beta_4=1-\rho_4\\
&\beta_1 r_1+\beta_2 r_2+\beta_3 r_3+\beta_4 r_4=(1-\rho_4)\frac{\left[1-\sum_{i=1}^{4}\alpha_i a_i\right]}{(\sum^{4}\alpha_i a_i)}
\end{aligned}$$

which can be solved to yield the coefficients β_i, $i = 1, 2, 3, 4$. Introducing the functions

$$h_1(x, y, z) = xyz + xy + xz + yz, \qquad h_2(x, y, z) = xy + xz + yz + x + y + z,$$

$$h_3(x,y,z) = k_0[(1-k_1)(x+y+z)+(1-k_2)]+(1-k_1),$$

$$h(w, x, y, z) = xyz + w^2(x + y + z - w) - w(xy + xz + yz), \quad \text{and}$$

$$H(w,x,y,z) = \frac{[k_0]^3 h_1(r_2, r_3, r_4) - [k_0]^2 h_2(r_2, r_3, r_4) + h_3(r_2, r_3, r_4)}{[k_0]^3 h(r_1, r_2, r_3, r_4)}$$

we obtain the coefficients explicitly as

$$\begin{aligned} \beta_1 &= (1-\rho_4)\, H(r_2, r_3, r_4, r_1) \\ \beta_2 &= (1-\rho_4)\, H(r_3, r_4, r_1, r_2) \\ \beta_3 &= (1-\rho_4)\, H(r_4, r_1, r_2, r_3)' \\ \beta_4 &= (1-\rho_4)\, H(r_1, r_2, r_3, r_4) \end{aligned} \tag{37}$$

where $k_0 = \sum_{i=1}^{4} \alpha_1 a_i$, $k_1 = \sum_{i=1}^{4} \alpha_i a_i(1-a_i)$ and $k_2 = \sum_{i=1}^{4} \alpha_i a_i(1-a_i)^2$. The equilibrium queue length distribution in this case is given by

$$p_k = \beta_1 r_1^k + \beta_2 r_2^k + \beta_3 r_3^k + \beta_4 r_4^k \,, \quad k = 0, 1, 2, \cdots \tag{38}$$

where, just as before, each r_i is in $(0,1)$, and $\sum_{i=1}^{m} \frac{\beta_i}{1-r_i} = 1$.

∎

In the following theorem we make use of results from [5] and [10] to obtain the waiting time distribution of customers in an $M/HE_m(\mathbf{B}, \mathbf{y})/1$ system as a generalised mixture of exponentials. In a useful result Haji and Newell showed [10] that under certain conditions, the equilibrium queue length distribution of customers in a single server queue has the same distribution as the number of customers who arrive during a random time interval distributed as the stationary waiting time. In [5] it was shown that geometric equilibrium queue lengths can result if and only if both interarrival times and service times are exponentially distributed random variables. Armed with these results, we present a theorem.

THEOREM 3

In an $M/HE_m(\mathbf{B}, \mathbf{y})/1$ queue with arrival rate $\lambda > 0$, the stationary waiting-time distribution is explicitly given by

$$dW_m(t) \equiv HE_m^*(\mathbf{D}, \mathbf{z}) = \sum_{j=1}^{m} \gamma_j \, \theta_j \, e^{-\theta_j t} \tag{39}$$

with $\mathbf{D} = (\gamma_1, \cdots, \gamma_m)$, $\mathbf{z} = (\theta_1, \cdots, \theta_m)$, $\sum_{j=1}^{m} \gamma_j = 1 - \rho_m, \theta_i > 0, 2 \le i \le m, 2 \le m \le 4.$

Proof:

The proof treats a general m, $2 \le m \le 4$. Using $W_m(t)$ to denote the cumulative distribution function of a customer's waiting time in an $M/HE_m(\mathbf{B}, \mathbf{y})/1$ queue, it is known [17] that

$$\sum_{k=0}^{j} p_k^{(m)} = \sum_{k=0}^{j} \int_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \, dW_m(t), \quad k = 0, 1, 2, \cdots \tag{40}$$

from which can be obtained an infinite system of equations that uniquely defines $W_m(\cdot)$ for each m, $2 \le m \le 4$. In [5] it was shown that a queue length distribution of the form $(1-\rho)\rho^n$, for $0 < \rho < 1$ and a Poisson arrival process of rate $\lambda > 0$, can result if and only if service-times are exponentially distributed with mean $\mu = \frac{\lambda}{\rho}$. It follows that for a given value of m, an equilibrium queue length distribution of the form $[1 - r_i^{(m)}][r_i^{(m)}]^k$ for $0 < r_i^{(m)} < 1$, must come from a service-time which is exponentially distributed with mean $\frac{\lambda}{r_i^{(m)}}$, $i = 1, \cdots, m$.

Equivalently,

$$[1 - r_i^{(m)}][r_i^{(m)}]^k = \int_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^k \, \theta_i \, e^{-\theta_i t} \, dt}{k!}, \quad k = 0, 1, 2, \cdots \tag{41}$$

for $i = 1, \cdots, 4$. Multiplying (41) by $\frac{\beta_i^{(m)}}{[1 - r_i^{(m)}]}$ and summing over i, we get

$$\sum_{i=1}^{m} \beta_i^{(m)} \left[r_i^{(m)} \right]^k = \sum_{i=1}^{m} \frac{\beta_i^{(m)}}{\left[1 - r_i^{(m)}\right]} \int_0^{\infty} \frac{\theta_i (\lambda t)^k \, e^{-(\lambda + \theta_i)t} dt}{k!} \tag{42}$$

for $k \geq 0$. A term by term comparison of each side of (42) reveals that

$$\beta_i^{(m)}\left[r_i^{(m)}\right]^k = \frac{\beta_i^{(m)}}{\left[1 - r_i^{(m)}\right]} \left[\frac{\theta_i}{\lambda + \theta_i}\right] \left[\frac{\lambda}{\lambda + \theta_i}\right]^k \tag{43}$$

from where we obtain $\theta_i = \dfrac{\lambda\,[1 - r_i^{(m)}]}{r_i^{(m)}}$ for $i = 1, \cdots, m$. As a consequence of Haji and Newell's result,

$$dW_m(t) = \sum_{i=1}^{m} \frac{\beta_i^{(m)}}{\left[1 - r_i^{(m)}\right]}\, \theta_i\, e^{-\theta_i t} \tag{44}$$

so that $dW_m(\cdot) \equiv HE_m^*(\mathbf{D}, \theta)$, with $\mathbf{D} = \left\{ \dfrac{\beta_1^{(m)}}{1 - r_1^{(m)}}, \dfrac{\beta_2^{(m)}}{1 - r_2^{(m)}}, \dfrac{\beta_3^{(m)}}{1 - r_3^{(m)}}, \dfrac{\beta_4^{(m)}}{1 - r_4^{(m)}} \right\}$ and $\mathbf{z} = (\theta_1, \theta_2, \theta_3, \theta_4)$. The fact that the elements in $\mathbf{D}$ sum up to one follows by summing up the left hand side of (42) for values of k ranging from 0 to ∞, to get

$$\sum_{i=1}^{m} \frac{\beta_i^{(m)}}{\left[1 - r_i^{(m)}\right]} = 1 \tag{45}$$

which, of course, comes from the fact that the queue-length probabilities sum to one.

IV. COMPUTATIONAL RESULTS

EXAMPLE 1: Numerical Accuracy

Consider an $M/HE_3(\mathbf{B}, \mathbf{y})/1$ queueing system with $\mathbf{B} = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\}$, $\mathbf{y} = \{10, 30, 60\}$, and $\lambda = 10$. From (5), the traffic intensity of this system is $\rho_3 = \frac{23}{36}$, ensuring a stable queue. On computing the coefficients A through E and solving for roots of the resulting polynomial given in (20), we obtain

$$r_1 = 0.1457024767005490$$
$$r_2 = 0.2680440436872688$$
$$r_3 = 0.7112534796121822$$

Finally, it is left to determine the three coefficients β_1, β_2, and β_3 from (34), which are easily computed as

$$\beta_1 = 4.0208916737761 \times 10^{-2}$$
$$\beta_2 = 7.5549055768796 \times 10^{-2}$$
$$\beta_3 = 2.4535313860455 \times 10^{-1}$$

The queue length distribution for this system is

$$p_n = \beta_1 r_1^n + \beta_2 r_2^n + \beta_3 r_3^n$$

for $n \geq 0$. In order to verify the correctness of the explicit form given above, the solution to this queueing distribution was also obtained via the classical recurrence equation shown in (6). The results are give in Table 1. In each case the computation was done in double precision on a VAX 8600. The discrepancy between the numbers in each column of Table 1 is due to the phenomenon of catastrophic cancellation that occurs in the use of the recurrence equation. This effect is typical in expressions involving sums of numbers of the same magnitude, but of different signs. The error can be seen to increase as n increases, meaning that the recurrence equation gives a considerably large error for probabilities in the tail of the distribution. In fact, for $n \geq 110$ the recurrence gives a small (constant) *negative* result. The formula yields values that decrease gradually, until finally reaching zero (due to underflow) at $n = 257$ (for p_n roughly 10^{-39}).

n	p_n (*formula* (32))	p_n (*recurrence* (6))
0	0.36111111111111	0.36111111111111
1	0.20061728395062	0.20061728395062
2	0.13040123456790	0.13040123456790
3	8.9859824124772d-02	8.9859825102881d-02
4	6.3197925599803d-02	6.3197927383402d-02
5	4.4766647625568d-02	4.4766649787809d-02
6	3.1792609263146d-02	3.1792611469675d-02
7	2.2599967236732d-02	2.2599969290321d-02
8	1.6070944537531d-02	1.6070946344714d-02
9	1.1429618105014d-02	1.1429619637409d-02
10	8.1291156361587d-03	8.1291169019119d-03
105	7.1179196112653d-17	5.6442588174285d-17
106	5.0626447882063d-17	3.5889798459903d-17
107	3.6008237473062d-17	2.1271555259811d-17
108	2.5610982491771d-17	1.0874278917929d-17
109	1.8215900134402d-17	3.4791799412752d-18
110	1.2956122489793d-17	- 1.7806099379460d-18
111	9.2150865001388d-18	(negative)
112	6.5542623636956d-18	"
113	4.6617417781651d-18	"
114	3.3156800731394d-18	"
251	1.7697612074685d-38	"
252	1.2587488557041d-38	"
253	8.9528944872728d-39	"
254	6.3677769783387d-39	"
255	4.5291034857057d-39	"
256	3.2213406429007d-39	"
257	0.0	"

TABLE 1.

For the waiting-time density, theorem 2 readily yields

$$\begin{aligned}
\gamma_1 &= 4.7066643477471 \times 10^{-2} \\
\gamma_2 &= 1.0321530446486 \times 10^{-1} \\
\gamma_3 &= 8.4971803364994 \times 10^{-1} \\
\theta_1 &= 58.633011001922 \\
\theta_2 &= 27.307300168800 \\
\theta_3 &= 4.0596854750494
\end{aligned}$$

EXAMPLE 2: On the largest root of the characteristic polynomial

In this section we demonstrate a result which shows that two different $M/HE_3(\mathrm{B},\mathrm{y})/1$ systems with the *same traffic intensity* can exhibit stationary characteristics that are *very different.*

With the aid of computational examples, we show that the largest root of the characteristic polynomial plays a role in qualitative behaviour. That is, given the characteristic polynomials of two different $M/HE_m/1$ systems, the queue length distribution possessing a longer tail is given by the polynomial whose largest root is greater. Additionally, we also dmonstrate the result that for an M/GI/1 system, a larger value of traffic intensity *does not* necessarily mean a longer tail.

Consider two different single-server queueing systems with Poisson arrivals of rate $\lambda=10$, and service time distributions $HE_3(\mathbf{B}_j, \mathbf{y}_j)$, $j=1,2$. We take $\mathbf{B}_1=\{0.7, 0.2, 0.1\}$, $\mathbf{B}_2=\{0.5, 0.2341985, 0.2658105\}$, $\mathbf{y}_1=\{10, 15, 20\}$, and $\mathbf{y}_2=\{17, 5, 22\}$. From (5), we compute the traffic intensities to be $\rho_{1,3}=0.8833333$ and $\rho_{2,3}=0.8833335$, for $j=1,2$, respectively. Though the difference in traffic intensities is of the order of 10^{-7}, one would expect the system with $j=2$ to possess a longer tail simply because it does have a larger traffic intensity. For this example, this indeed works out to be the case. In Figure 1a is shown the polynomials (see Eq.(18)) for both $M/HE_3/1$ queueing systems. Observe that the system corresponding to $HE_3(\mathbf{B}_2, \mathbf{y}_2)$ is the one whose largest root is greatest. From Theorem 2 (see Eq.(35)), we see that this particular root effects longer tails in queue length distribution. Correspondingly, Figure 1b displays the queue length distributions for both systems, with the system possessing greater largest root also possessing a longer tail.

In order to demonstrate that a larger traffic intensity does not imply a longer tail, consider the following example. Just as before, we have two queueing systems with $\lambda=10$, but now $\mathbf{B}_1=\{1/3, 1/3, 1/3\}$, $\mathbf{B}_2=\{0.6, 0.047311, 0.352689\}$, $\mathbf{y}_1=\{8, 10, 13.85\}$, and $\mathbf{y}_2=\{15, 1, 30\}$. From (5), we compute the traffic intensities to be $\rho_{1,3}=0.990673880$ and $\rho_{2,3}=0.990673013$ for $j=1,2$, respectively. In Figure 2a we see the polynomials (see Eq.(18)) corresponding to these queueing systems. Since both polynomials have roots clustered near unity (including unity), we display the behaviour of the polynomials in this region in Figure 2b. In Figure 2b, we see that the system corresponding to $HE_3(\mathbf{B}_2, \mathbf{y}_2)$ is the one with the *greater largest root* but with *smaller*

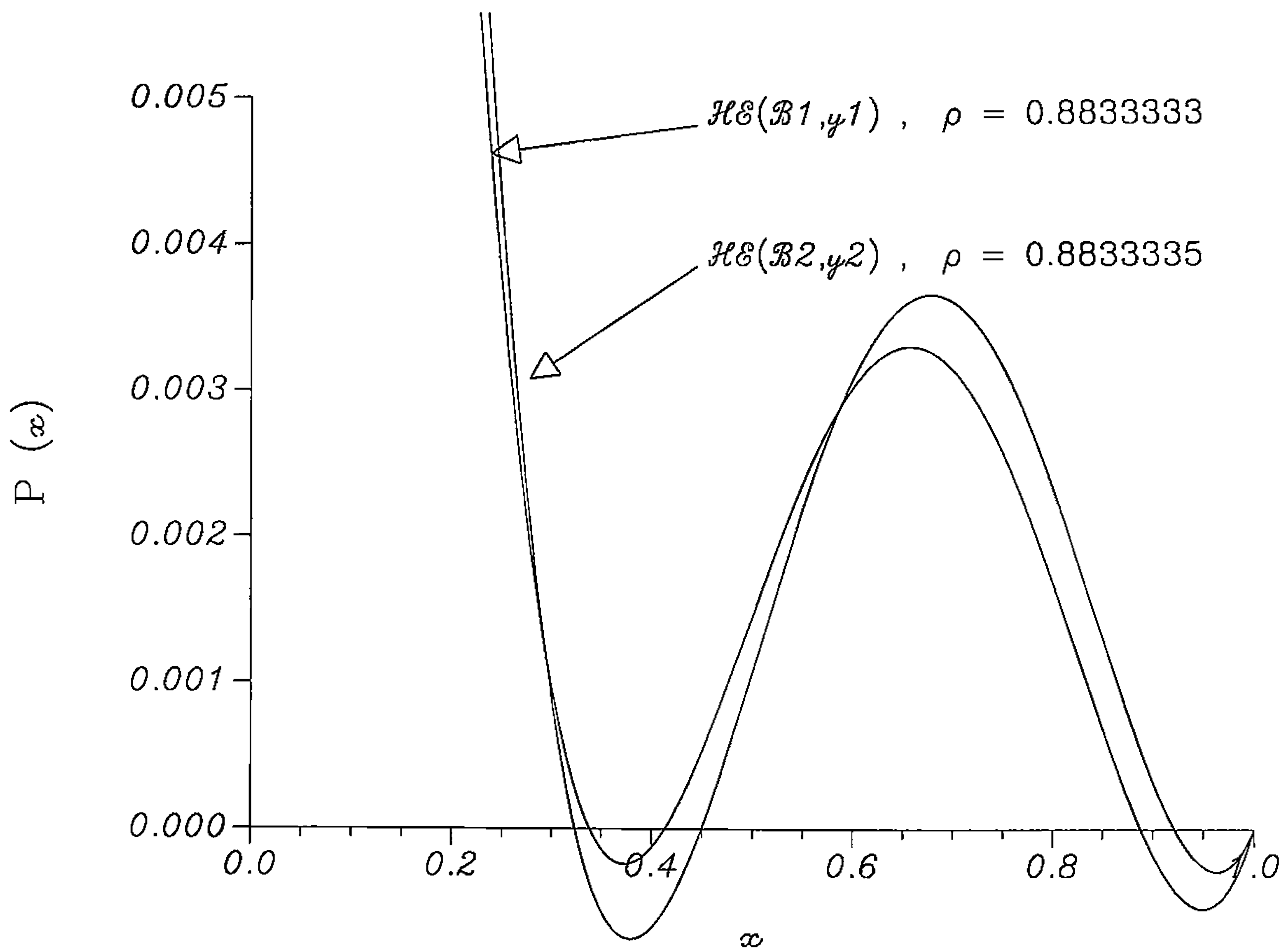

Figure 1a. Location of roots of polynomial

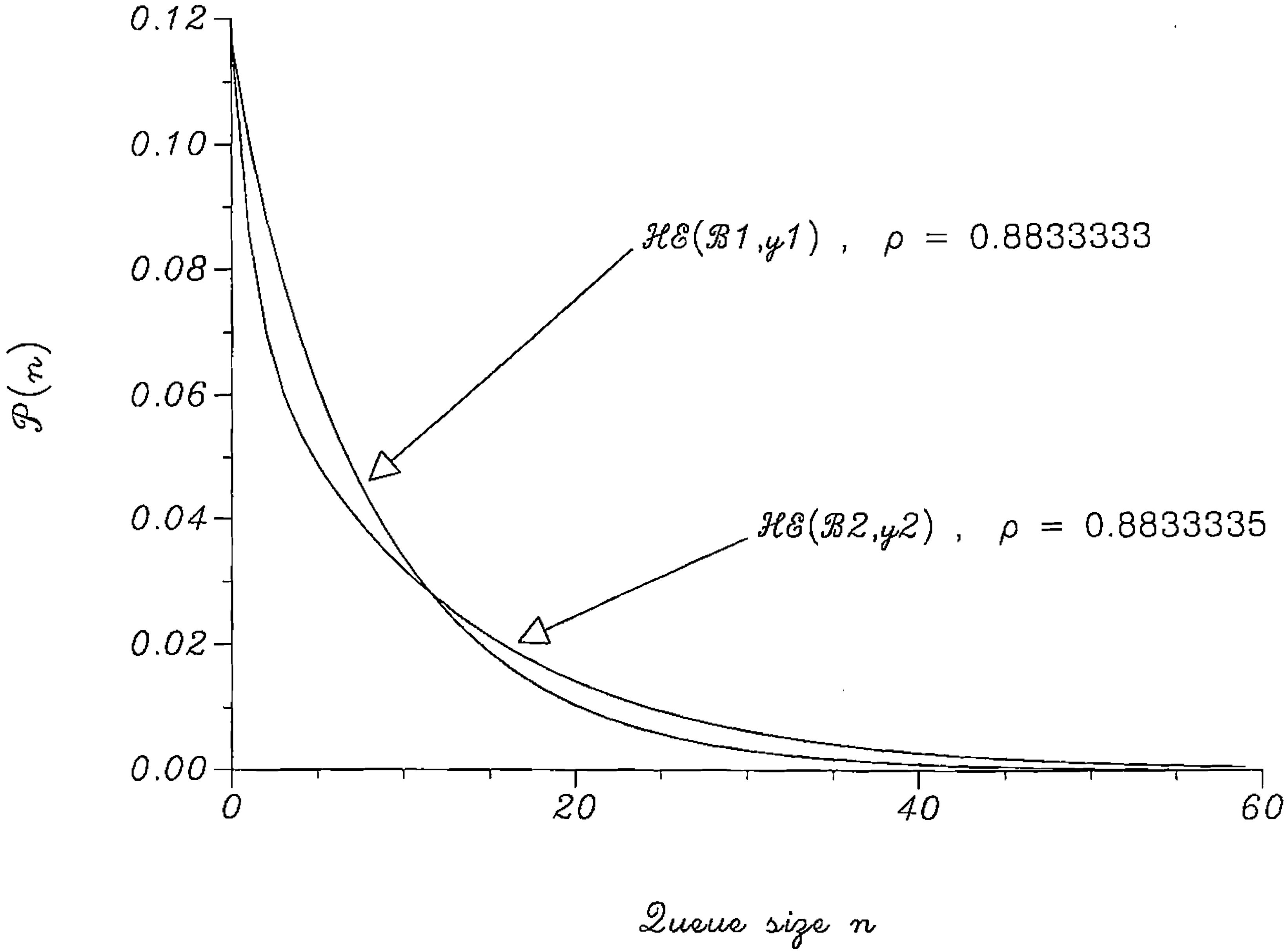

Figure 1b. Queue length distribution

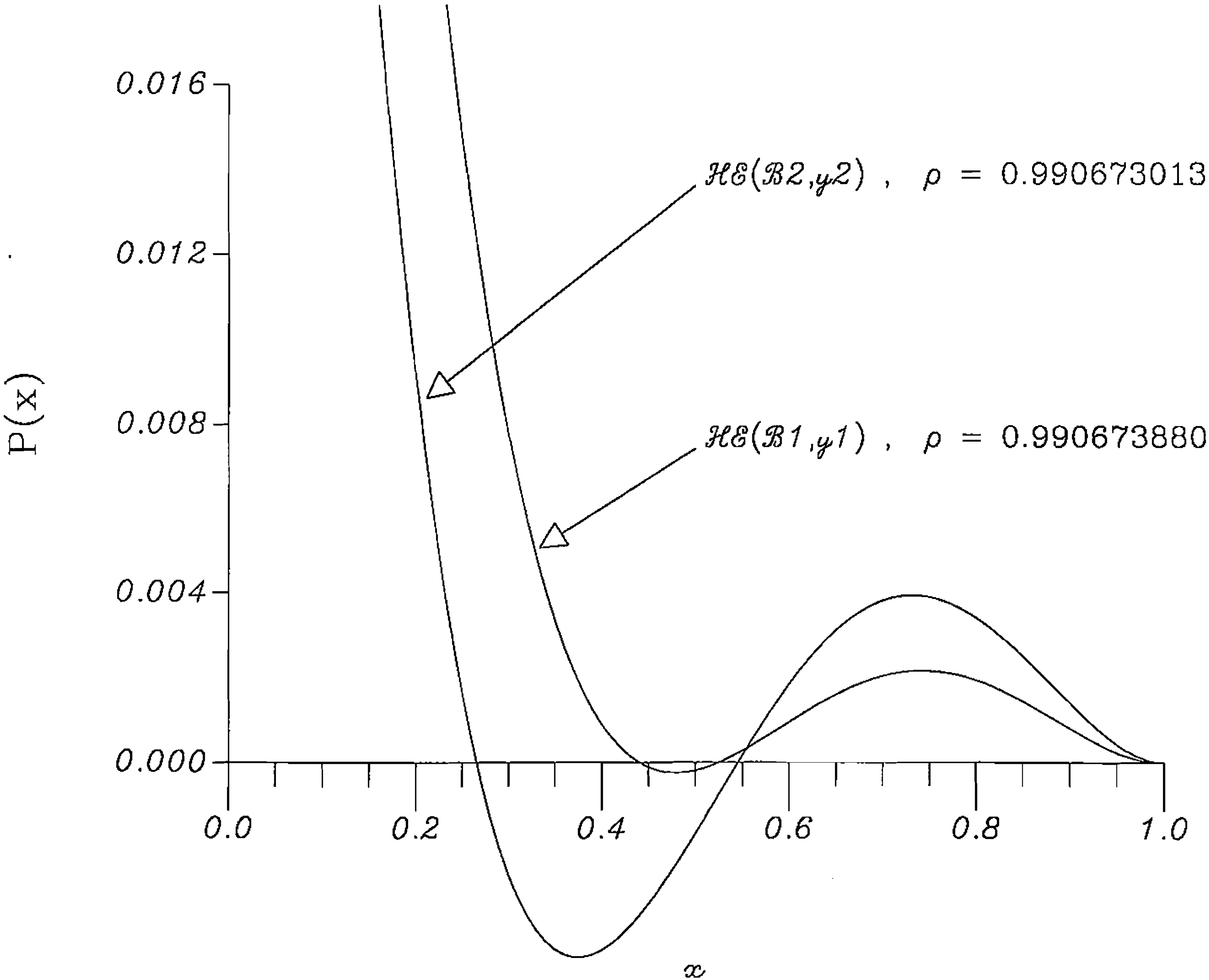

Figure 2a. Location of roots of polynomial

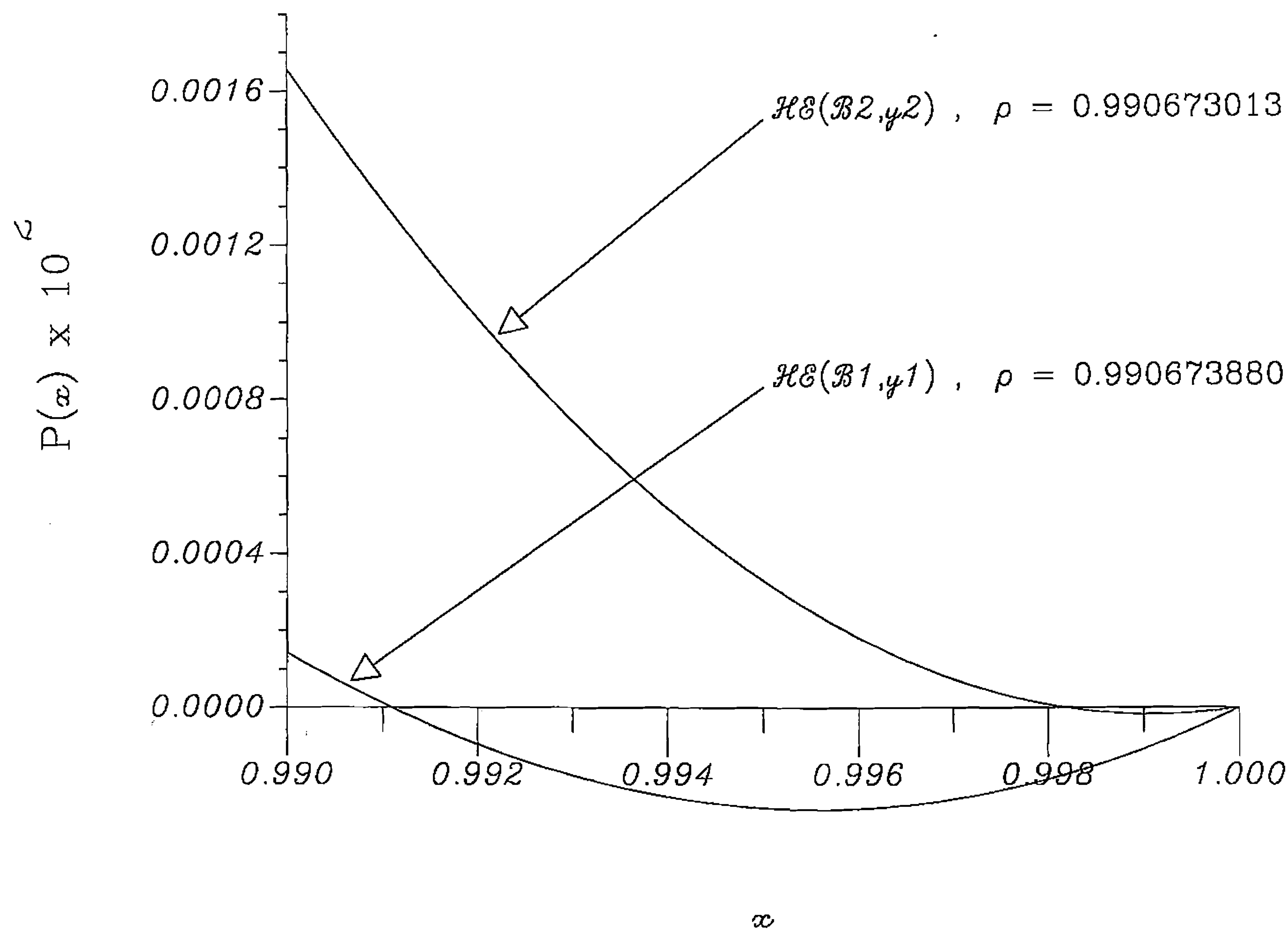

Figure 2b. Location of roots of polynomial

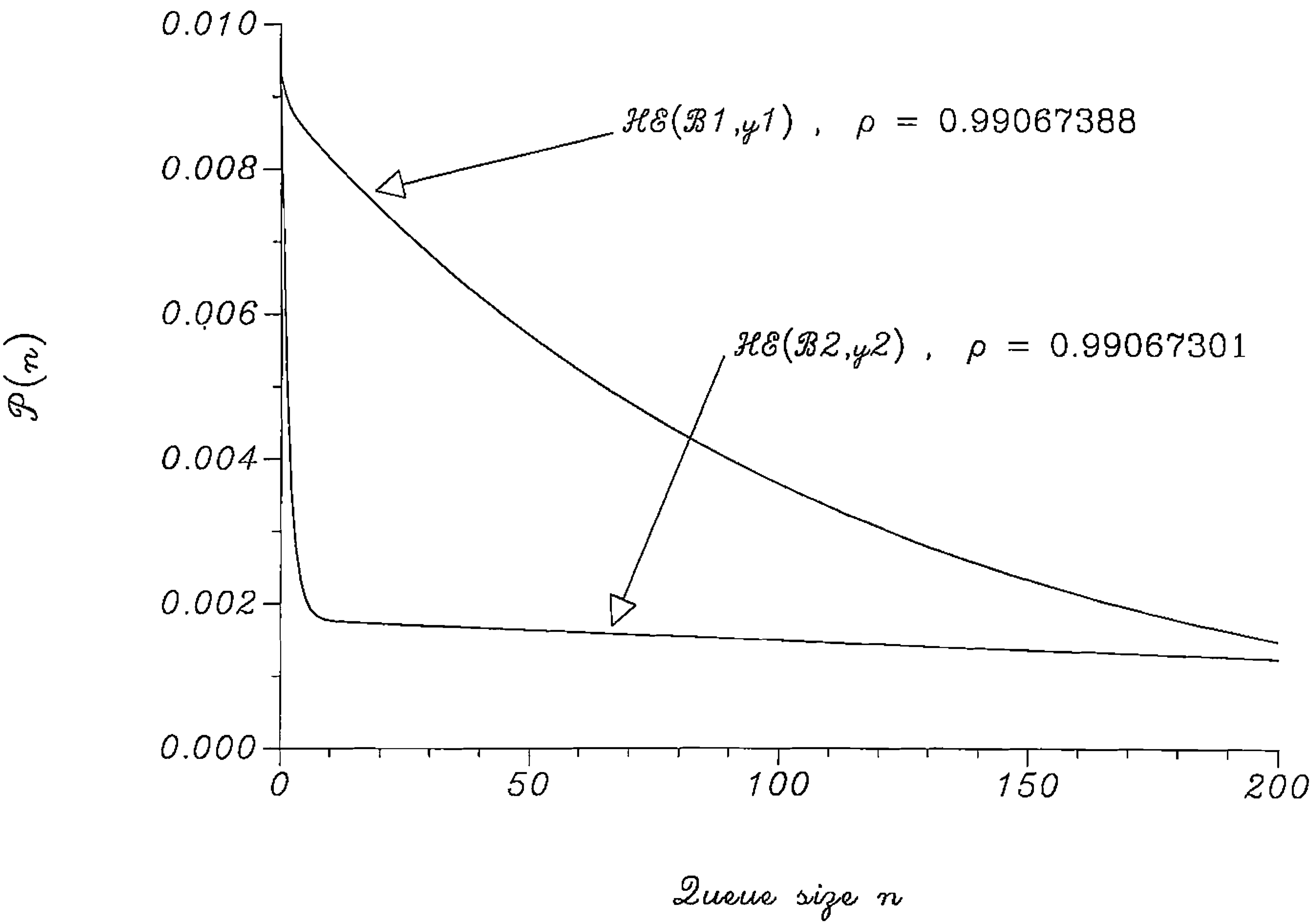

Figure 2c. Queue length distribution

traffic intensity. Since it has a greater largest root, the $M/HE_3(B_2, y_2)/1$ system will have a longer tail, and this is shown in Figure 2c. Thus we see that, a larger value of traffic intensity does not guarantee a longer tail in queue length distribution. Additionally, even though the difference in traffic intensity is of the order of 10^{-7}, the two systems exhibit vastly different queue length distributions.

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APPENDIX

For $m=3$, the leading coefficients of Sturm's remainders [15] are given by R_1, and R_2, which are defined as follows. We first define the functions

$$f_1=243\ A^3\ D^2 \tag{46}$$

$$f_2=((-162\ A^2\ B-324\ A^3)\ C+36\ A\ \ B^3+54\ A^2\ B^2+270\ A^3\ B-360\ A^4)\ D \tag{47}$$

$$f_3=36\ A^2\ C^3 \tag{48}$$

$$f_4=(180\ A^3+72\ A^2\ B-9\ A\ \ B^2)\ C^2 \tag{49}$$

$$f_5=(-18\ A\ \ B^3-162\ A^2\ B^2-108\ A^3\ B+288\ A^4)\ C \tag{50}$$

$$f_6=27\ A\ \ B^4 \tag{51}$$

$$f_7=18\ A^2\ B^3 \tag{52}$$

$$f_8=27\ A^3\ B^2 \tag{53}$$

$$f_8=-216\ A^4\ B \tag{54}$$

$$f_9=144\ A^5 \tag{55}$$

$$h_1=36\ A^2\ C^2 \tag{56}$$

$$h_2=(-24\ A\ \ B^2-24\ A^2\ B+48\ A^3)\ C \tag{57}$$

$$h_3=4\ B^4 \tag{58}$$

$$h_4=8\ A\ \ B^3 \tag{59}$$

$$h_5=-12\ A^2\ B^2 \tag{60}$$

$$h_6 = -16\,A^3\,b \tag{61}$$

$$h_7 = 16\,A^4 \tag{62}$$

Now R_2 can be computed as

$$R_2 = \frac{\sum_{i=1}^{10} f_i}{\sum_{j=1}^{7} h_j} \tag{63}$$

and R_1 can be computed as

$$R_1 = 2B^2 + 2A\,B - 4A^2 - 6A\,C \tag{64}$$

where the quantities A, B, C, and D are given in (19).