1986

Finding Maximum Cliques on Circular-Arc Graphs

Alberto Apostolico

Susanne E. Hambrusch
Purdue University, seh@cs.purdue.edu

Report Number:
86-643

https://docs.lib.purdue.edu/cstech/559

This document has been made available through Purdue e-Pubs, a service of the Purdue University Libraries. Please contact epubs@purdue.edu for additional information.
FINDING MAXIMUM CLIQUES ON
CIRCULAR-ARC GRAPHS

Alberto Apostolico
Susanne Hambrusch

CSD-TR-643
December 1986
ABSTRACT

An algorithm is presented that, given the intersection model \( S \) of a circular-arc graph \( G \) with \( n \) vertices and \( m \) edges, finds a maximum-sized clique of \( G \) in \( O(n^2 \log \log n) \) time. The previously best time bound for this problem is \( O(n^2 \log n + mn) \).
1. Introduction

Let \( G = (V, E) \) be an undirected graph with \( |V| = n \) vertices and \( |E| = m \) edges. Graph \( G \) is a circular-arc graph if its vertices can be put in a one-to-one correspondence with the elements of some set \( S \) of arcs on the unit circle, in such a way that two vertices are adjacent in \( G \) if and only if the two corresponding arcs in \( S \) have a non-empty intersection [Go]. We call \( S \) the intersection model of \( G \). Circular arc graphs enter diverse applications, for which we refer to [Go, H].

Let an arbitrary clockwise system of abscissae be defined on the unit circle. Then the \( i \)-th arc in \( S \) is identified by the pair \((a_i, b_i)\), where \( a_i \) is the abscissa of the counterclockwise end of the arc and \( b_i \) is the abscissa of the clockwise end. We say that arc segment \((l, r)\) (where \( l \) is the counterclockwise end and \( r \) is the clockwise end) contains position \( x \) if either \( l < x < r \), or \( l > r \) and \( (x > l \) or \( x < r) \). Arc \( i \) contains arc \( j \) if \((a_i, b_i)\) contains both position \( a_j \) and \( b_j \). Arcs \( i \) and \( j \) intersect if either one of the two arcs is contained in the other one, or the two arcs overlap. A subset \( C \) of \( S \) is a clique if each arc in \( C \) intersects all other arcs in \( C \). A clique \( C \) is maximum if its cardinality is not smaller than that of any other clique of \( S \).

In this paper we present an algorithm that, given the intersection model \( S \) of a graph \( G \), finds a maximum clique of \( G \) in \( O(n^2 \log n \log \log n) \) time. The previously best time bound for this problem is \( O(n^2 \log n + mn) \) [H]. Our solution uses some simple properties which we present in the following section. The main construction is then discussed in Section 3. In Section 4, we briefly point to some problems that remain open. For other NP-complete problems that are NP-complete in general, but can be solved efficiently on circular-arc graphs we refer the reader to [J].

2. Preliminaries

We assume without loss of generality that all arcs in \( S \) have length less than 1 and that no two arcs share a common endpoint.

Let \( C \) be a clique of \( S \) and let \( i \in C \) be an arc which does not contain any other arc of \( C \). Then we say that arc \( i \) is a base for \( C \) and we call \( C \) a clique of base \( i \). Clearly, a maximum clique of \( S \) can be obtained by constructing, for every arc \( i \), a clique of base \( i \) and maximum size, and then selecting one largest such clique. Let now \( C \) be a clique of base \( i \) and maximum size. The following two facts hold trivially for \( C \).
Fact 1. All arcs of $S$ that contain arc $i$ are in $C$.

Fact 2. Every arc of $S$ not contained in $i$, but having both endpoints in $(a_i,b_i)$ is also in $C$.

Thus the arcs of $C$ can be partitioned into three sets $C_i^1$, $C_i^2$, and $C_i^3$, where

$$C_i^1 = \{ j \mid \text{arc } j \text{ contains arc } i \}$$

$$C_i^2 = \{ j \mid \text{arc } i \text{ does not contain arc } j, \text{ but both } a_j \text{ and } b_j \text{ are in } (a_i,b_i) \}$$

$$C_i^3 = \{ j \mid \text{either } a_j \text{ or } b_j, \text{ but not both, are in } (a_i,b_i) \}$$

Note that, given base $i$, the arcs in $C_i^1$ and $C_i^2$ are uniquely determined, but the same is not true of $C_i^3$. However, the arcs in $C_i^3$ must form a clique of maximum size among all cliques that contain only arcs with exactly one endpoint in $(a_i,b_i)$. Henceforth, we will use $C_i^3$ to denote any such maximum-sized clique.

3. Finding a Maximum Clique of Base $i$.

In this section we show that a maximum clique of base $i$ can be constructed in $O(n \log \log n)$ time. The non-trivial part of our strategy is in its way of constructing set $C_i^3$, since sets $C_i^1$ and $C_i^2$ can be trivially extracted from $S$ in linear time.

Once $C_i^1$ and $C_i^2$ are removed from $S$, our algorithm forms set $S_i$, where

$$S_i = S - C_i^1 - C_i^2 - \{ j \mid \text{arc } i \text{ contains arc } j \} - \{ j \mid \text{arc } i \text{ and arc } j \text{ do not intersect} \}.$$ 

Thus set $S_i$ contains all the arcs with exactly one endpoint in $(a_i,b_i)$. Every arc in $S_i$ is a candidate for $C_i^3$. An optimal selection of arcs can be performed conveniently on the rollout of the circle associated with arc $i$. This rollout is obtained by opening the circle at endpoint $a_i$, and then by mapping each circular arc of $S_i$ into a suitable rectilinear segment. With reference to Figure 1, an arc $j$ in $S_i$ with $a_j$ contained in $(a_i,b_i)$ is mapped in the $A$-segment $(a_j,b_j)$, while an arc with $b_j$ contained in $(a_i,b_i)$ is mapped in the $B$-segment $(b_j,a_j)$ (in a sense, a B-segment complements the circular arc associated with it).

Without loss of generality, let $b_i = 0$ and $k = |S_i|$, and let $(l_1,r_1), \ldots , (l_k,r_k)$ be the list of the segments of the rollout of $i$ in order of increasing right endpoint (i.e., $l_j < 0 < r_j$ and $r_j = j$). We observe that the subset of $S_i$ represented by the collection of all the $A$-segments in the rollout is a clique, and so is the union of this subset and $\{ C_i^1 \cup C_i^2 \}$. The same properties hold for the subset of $S_i$ which is mapped into the set of B-segment. We say that $A$-segment $x$ and $B$-segment $y$ are
in conflict if \( l_y < l_z < r_x < r_y \). In Figure 1, for example, A-segment 2 is in conflict with B-segments 3 and 6. Obviously, set \( C_i^3 \) cannot contain any pair of arcs such that the two corresponding segments are mutually conflicting.

The procedure TRADE described below performs an optimal selection of A-segments and B-segments in the rollout of \( i \). Through a left-to-right scan of endpoints in \([1,k]\), TRADE constructs a set \( C_3 \) containing the maximum number of non-conflicting segments of the rollout. We refer to \( C_3 \) as a maximum rollout clique of base \( i \). The procedure constructs then \( C_i^3 \) as the set of all arcs which correspond to segments in \( C_3 \). By our discussion, taking the union of this set and \( \{C_j \cup C_i^2\} \) yields a maximum clique of base \( i \).

TRADE initializes set \( C_3 \) to contain all the A-segments of the rollout. Subsequently, the procedure may "match-up" A-segments with conflicting B-segments according to a criterion described shortly. These match-ups enable TRADE to recognize situations in which an equal number of A-segments can be "exchanged" for an equal number of B-segments without creating any conflicts. At the \( j \)-th iteration, segment \((l_j, r_j)\) is handled as follows. If \((l_j, r_j)\) is a B-segment not in conflict with any A-segment currently in \( C_3 \), then this B-segment is added to \( C_3 \). Otherwise, let \( p \) be the number of A-segments in \( C_3 \) that (1) are in conflict with the B-segment \((l_j, r_j)\) and (2) have not been matched-up with a conflicting B-segment. If \( p = 1 \), TRADE exchanges the conflicting A-segment with the current B-segment, and possibly performs other exchanges as well, as specified later. The overall operation yields a consistent (i.e., with no conflicting segments) set \( C_3 \) having the same cardinality as the original set, and it may have the effect of reducing the number of A-segments that could be in conflict with B-segments in the future. If \( p > 1 \), TRADE cannot perform an exchange without decreasing the current cardinality of \( C_3 \). However, such an exchange might prove profitable at some later stage, since the point could be reached where removing a certain number of A-segments would make room for a larger number of B-segments. In view of this possibility, the procedure matches up B-segment \( j \) with the conflicting and currently unmatched A-segment \( x \) having the smallest left endpoint. The matched pair \((x, j)\) is then recorded in a list associated with the unmatched A-segment \( y \) such that \( l_y > l_x \) and \( l_y \) is as small as possible.

The management of A-segments is quite simpler. Whenever an A-segment is met during the
scan, TRADE adds it to the list AVAIL. This list keeps track of all the already scanned A-
segments in C3 that are yet to be matched. In addition, every A-segment y stored in AVAIL is
provided with an initially empty list $L_y$. At each iteration for which y is kept in AVAIL, $L_y$ will
contain zero or more pairs representing currently unexchanged matches. AVAIL is used also to
determine how many A-segments are in conflict with a given B-segment. A more formal descrip-
tion of TRADE is as follows.

Procedure TRADE
Input: The segments $(l_1, r_1), \ldots, (l_k, r_k)$ of the rollout of $i$.
Output: A set $C_i^3$.

begin
(1) Initialize: $C_3 = \{ j \mid (l_j, r_j) \text{ is an A-segment} \}; \text{ AVAIL} = \emptyset$

(2) Scan:
   for $j=1$ to $k$ do
     if $(l_j, r_j)$ is an A-segment then $\text{AVAIL} = \text{AVAIL} \cup \{ j \}; \text{ } L_j = \emptyset$
     else (* $(l_j, r_j)$ is a B-segment *)
       $\text{conflict}# = | \{ h \mid h \in \text{AVAIL} \text{ and } l_j < l_h \} |$

(2.1) if $\text{conflict}# = 0$ then $C_3 = C_3 \cup \{ j \}$

(2.2) if $\text{conflict}# = 1$ then (* an exchange is performed *)
     let $x$ be the A-segment in AVAIL with $l_x > l_j$
     let the elements in list $L_x$ be $\{ \alpha_1, \beta_1, \ldots, \alpha_s, \beta_s \}, s \geq 0$
     $\text{AVAIL} = \text{AVAIL} - \{ x \}$;
     $C_3 = C_3 \cup \{ j, \alpha_1, \ldots, \beta_s \} - \{ x, \alpha_1, \ldots, \beta_s \}$

(2.3) if $\text{conflict}# > 1$ then (* a match-up is recorded *)
     let $x$ be the A-segment in AVAIL with $l_x > j$ and $l_x$ as small as possible;
     let $y$ be the A-segment in AVAIL with $l_y > l_x$ and $l_y$ as small as possible;
     $\text{AVAIL} = \text{AVAIL} - \{ x \}$;
     append list $L_x$ together with the entry $(x, j)$ to list $L_y$
   endfor

(3) Output:
   $C_i^3 = \{ j \mid (l, r) \text{ is an A-segment of } C_3, j \in S_i, \text{ and } l=a_j \text{ and } r=b_j \} \cup$
   $\{ j \mid (l, r) \text{ is a B-segment of } C_3, j \in S_i, \text{ and } l=b_j \text{ and } r=a_j \}$
end.

Observe that exchanging $s$ A-segments for $s$ B-segments as done in step (2.2) is not only at
least as good as keeping those A-segments in the current version of C3. In fact the number of B-
segments that could be added to C3 in the future cannot ever be decreased by the exchange, and
may be actually increased. Before describing the implementation of TRADE, we prove that the
segments in C3 at the outset correspond to a maximum rollout clique of base i. This immediately
implies that the corresponding set C_i^3 as generated in step (3) represents a maximum clique
among the arcs in $S_i$.

Let $C_3(0)$ be the set of all the A-segments in the rollout of $i$, and let $C_3(j)$, $1 \leq j \leq k$ be the set
of segments in C3 at the end of the $j$-th iteration of TRADE. Clearly, $|C_3(0)| \leq |C_3(1)| \leq \ldots \leq |C_3(k)|$. The following lemma characterizes the sets $C_3(j)$, $0 \leq j \leq k$ and
also establishes the basic invariants of TRADE.

Lemma 1. By the end of the $j$-th iteration of TRADE ($0 \leq j \leq k$), the following properties hold.

(1) For every A-segment $\alpha$ such that $\alpha < j$ and $\alpha \notin C_3(j)$, there exists a distinct B-segment $\beta$ in
$C_3(j)$ such that $\beta \leq j$ and $\alpha$ and $\beta$ are in conflict.

(2) Every B-segment $\beta$ such that $\beta < j$ and $\beta \notin C_3(j)$ is in conflict with at least two A-segments
in $C_3(j)$.

(3) For any set $B$ of B-segments not in $C_3(j)$ and having endpoints as in (2) above, there exists
a set $A$ of A-segments in $C_3(j)$ such that $|A| > |B|$, and every segment in $A$ is in conflict
with at least one segment in $B$.

Proof: Property 1 follows from straightforward induction on $j$: the matched pairs traded in the
course of all executions of step (2.2) up to the $j$-th iteration represent the relation of the claim.

As for Property 2, observe first that in order for some B-segment $\beta < j$ not to be included in
$C_3(j)$, the $\beta$-th iteration of TRADE must have resulted in case (2.3). This entails the existence of
at least two A-segments in $C_3(\beta - 1)$ which were unmatched and also in conflict with $\beta$ at the time
of the $\beta$-th iteration. Let then $U = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be the set, sorted in order of increasing left end-
point, of all the unmatched A-segments in conflict with $\beta$ at the beginning of the $\beta$-th iteration.
Step (2.3) of TRADE puts the pair $(\alpha_1, \beta)$ in the list $L_{\alpha_i}$. Assume for a contradiction that the
number of A-segments in $U$ that are also in $C_3(j)$ is less than two. Then, since $\alpha_1 \in C_3(j)$, $\alpha_2$ cannot be in $C_3(j)$. By the control structure of TRADE, this implies that there is some B-segment
$\beta'$, $\beta < \beta' \leq j$ such that either (Case 1) $\alpha_2$ is exchanged with $\beta'$ at the $\beta'$-th iteration, or else (Case 2)
\( \alpha_2 \) is matched up with \( \beta' \) at that iteration and then exchanged with it at some later iteration \( j' \), \( \beta' < j' \leq j \). With reference to Case 2, observe that the pair \((\alpha_2, \beta')\) is created and added to some \( L \)-list precisely when \( \beta' \) is handled by TRADE, and the pair \((\alpha_1, \beta)\) is added to that same list at that point. It is easy to check that, from this moment on, any list containing the pair \((\alpha_2, \beta')\) must also contain \((\alpha_1, \beta)\). Since lists are traded in an exhaustive fashion in case (2.2), any trade that exchanges \( \alpha_2 \) with \( \beta' \) must also exchange \( \alpha_1 \) with \( \beta \). But this contradicts the hypothesis that \( \beta \notin C^3(\ell) \). The same contradiction is easily derived for Case 1.

Property 3 is an easy consequence of the preceding two. Indeed, let \( \beta_1, \cdots, \beta_s \) be the segments in set \( B \), with \( r_{\beta_1} < \cdots < r_{\beta_s} \). At the time TRADE handles \( \beta_q \) \((1 \leq q \leq s)\) this B-segment is matched to a corresponding and distinct A-segment \( \alpha_q \). By Property 2, there must be another A-segment \( y \) in \( C^3(\ell) \) which is distinct from any \( \alpha_q \) and is also in conflict with \( \beta_q \). Setting then \( A = \{ \alpha_1, \alpha_2, \cdots, \alpha_q, y \} \) yields \(|A| = s+1 > s = |B| \). \( \square \)

We now show that the set \( C^3 = C^3(\ell) \) is a maximum rollout clique of base \( i \). Let \( C^3 \) be a maximum rollout clique of base \( i \) that has as many arcs as possible in common with clique \( C_3 \).

Lemma 2. \( C_3 = C^3 \).

Proof: Recall that the segments of the rollout of \( i \) are \((l_1, r_1), \cdots, (l_k, r_k) \) with \( r_j = j \). For \( 0 \leq j \leq k \), let \( RE_j \) (resp. \( RE_j^* \)) be the set of segments in \( C^3 \) (resp. \( C^3* \)) having right endpoints not larger than \( j \). Observe that \( RE_0 = RE_0^* = \{ i \} \). Let \( p > 0 \) be the smallest integer for which an "adversary" who knows the composition of set \( C^3* \) claims that \( RE_p \neq RE_p^* \). We will force this adversary in a contradiction, thus showing that \( C_3 = C^3* \). We distinguish two cases, depending on whether segment \( p \) is a B-segment or an A-segment.

Case 1. Segment \( p \) is a B-segment.

Assume first that \( p \in C^3 \) but \( p \notin C^3* \). It is easy to see then that \( p \) is not in conflict with any segment having right endpoint larger than \( p \). Thus, whatever the remaining selections in \( C^3* \), adding segment \( p \) to \( C^3* \) yields a set of nonconflicting segments having larger size than \( C^3* \), a contradiction. Assume now \( p \notin C^3 \). By the second property of Lemma 1, there exist two A-segments in \( RE_{p-1} \) that are in conflict with segment \( p \). Since both A-segments are also in \( C^3* \) by hypothesis, \( p \notin C^3* \) follows. In conclusion, \( p \) cannot be a B-segment, for otherwise we would have \( RE_p = RE_p^* \).
Case 2. Segment \( p \) is an A-segment.

Assume first that \( p \in C3, p \notin C3' \). Then \( p \) must be in conflict with at least one B-segment of \( C3' \), otherwise it could be added consistently to this set, thus generating a contradiction. Let \( j_1 \) be a B-segment in \( C3' \) that is in conflict with \( p \). Then \( j_1 \in C3 \). By the second property of Lemma 1, there exists another A-segment in \( C3 \) that is in conflict with \( j_1 \). Let \( p_1 \) be this A-segment. Obviously, \( p_1 > p \) and \( p_1 \notin C3' \). The adversary is now offered to trade-in segment \( j_1 \) for the two segments \( p \) and \( p_1 \). Clearly, such an exchange cannot be accepted without violating the hypothesis that \( C3' \) has maximum size. Thus, there must be a B-segment \( j_2 \) in \( C3' \) that is in conflict with either \( p \) or \( p_1 \), or both. But then A-segments \( p \) and \( p_1 \) in \( C3 \) and B-segments \( j_1 \) and \( j_2 \) not in \( C3 \), and Lemma 1 imply the existence of a third A-segment \( p_2 \in C3 \) such that \( p_2 \) is in conflict with \( j_1 \), or \( j_2 \), or both. Again, the adversary cannot accept to exchange the B-segments \( j_1 \) and \( j_2 \) for the A-segments \( p, p_1 \) and \( p_2 \) without falling in contradiction. Thus, \( C3' \) must exhibit a new B-segment \( j_3 \) that is in conflict with one or more among \( p, p_1, p_2 \). Continuing with this argument, the point must be reached where the adversary is forced to accept the trade at the time \( C3' \) runs out of new conflicting B-segments. This leads to a contradiction since \( C3' \) is shown to be not of maximum size. Hence, if \( p \notin C3 \), we have \( p \in C3' \) and \( RE_p = RE_p' \).

Consider now the case when \( p \notin C3 \) and \( p \in C3' \). Then there must be a B-segment in \( C3 \), say, \( j_1 \), that is in conflict with \( p \) and is matched to \( p \) during the \( j_1 \)-th iteration of TRADE. The adversary is now offered to trade-in segment \( p \) in exchange for B-segment \( j_1 \). Accepting the exchange would contradict the assumption that \( C3' \) agrees with \( C3 \) on a maximum number of segments. Thus \( C3' \) must contain another A-segment, say, \( p_1 \), such that \( p_1 \) conflicts with \( j_1 \) and \( p_1 > p \). Obviously, \( p_1 \notin C3 \) and, by the operation of TRADE, \( C3 \) contains a B-segment that was used at some point to replace \( p_1 \). Let this B-segment be \( j_2 \). Then again, we offer \( j_1 \) and \( j_2 \) in exchange for \( p \) and \( p_1 \), thus forcing the adversary to exhibit a third A-segment. This argument continues until the adversary runs out of A-segments. In fact, \( C3 \) is always able to provide a new matching B-segment for any A-segment supplied for \( C3' \), in force of the first property of Lemma 1. Hence, if \( p \notin C3 \), we have \( p \in C3' \) and \( RE_p = RE_p' \). □

We now turn to the implementation of our algorithm. We can assume that the set \( S \) is stored in a circular array of \( 2n \) cells, with a convenient format. This can be obtained through a straight-
forward $O(n \log n)$ preprocessing, the details of which are omitted. The construction of the set of segments when handling base $i$ can be performed trivially in linear time. An $O(k \log k)$ time implementation for TRADE is then easily obtained by using a bitvector to represent set $C3$ and a balanced tree for set AVAIL [AHU]. Every leaf $x$ of this balanced tree corresponds to an A-segment and it has a linked list representing list $L_x$ associated with it. In such an implementation conflict# can be computed in $O(\log k)$ time, step (2.1) requires constant time, and step (2.2) and (2.3) can each be done in $O(\log k)$ time. Thus the $O(k \log k)$ overall time follows. This time bound can be reduced to $O(k \log \log k)$ using the data structures described in [vE] and the references therein.

**Theorem.** A maximum rollout clique of base $i$ can be constructed in $O(n \log \log n)$ time.

**Proof:** By the discussion of this section and Lemmas 1 and 2.

**Corollary.** Given $n$ arcs $(a_1,b_1), \ldots, (a_n,b_n)$ of a circular-arc graph $G$ a maximum clique of $G$ can be determined in $O(n^2 \log \log n)$ time.


A number of problems remain open. First, the techniques presented in this paper do not seem to extend straightforwardly to the more general problem of finding a clique of maximum weight for the intersection model of a weighted circular arc graph. By contrast, the $O(n^2 \log n + nm)$ technique in [H] applies to the weighted case as well. Thus the question is open of whether the weighted case can be dealt with more efficiently. For the case of unitary weights, it would be interesting to eliminate the logarithmic factor from the upper bound. Finally, lower bounds for both versions of the problem are still to be derived.

**References**


(a) A circular-arc graph. Taking base \( i=7 \) yields \( S_i = \{1,2,3,4,5,6\} \).
Arc 8 is included into set \( S_{i+1} \), arc 9 is discarded when constructing a clique of base 7.

(b) The rollout of base \( i=7 \) consisting of A-segments 2, 4 and 5 and
B-segments 1, 3, and 6 (shown with dashed vertical lines and a heavy horizontal line).

FIGURE 1