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Automatic Parameterization of Rational Curves and Surfaces III: Algebraic Plane Curves

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Abstract

We consider algorithms to compute the genus and rational parametric equations, for implicitly defined irreducible rational plane algebraic curves of arbitrary degree. Rational parameterizations exist for all irreducible algebraic curves of genus 0. The genus is computed by a complete analysis of the singularities of plane algebraic curves, using affine quadratic transformations. The rational parameterization techniques, essentially, reduce to solving symbolically systems of homogeneous linear equations and the computation of resultants.

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1. Introduction

Effective computations with algebraic curves and surfaces are increasingly proving useful in the domain of geometric modeling and computer graphics where current research is involved in increasing the geometric coverage of solids to be modeled and displayed, to include algebraic curves and surfaces of arbitrary degree, see de Montaudoin and Tiller (1984), Sederberg (1984), Hopcroft and Kraft (1985), Farouki (1986). An irreducible algebraic plane curve is implicitly defined by a single prime polynomial equation $f(x,y) = 0$. Rational plane algebraic curves have an alternate representation, namely the rational parametric equations which are given as $(x(t) , y(t))$, where $x(t)$ and $y(t)$ are rational functions in t, i.e., the quotient of polynomials in t. All the polynomials considered here are assumed to be defined over an algebraically closed field of characteristic zero, such as the field of complex numbers.

As both implicit and parametric representations have their inherent advantages it becomes crucial to design algorithms for both these curve representations as well as algorithms to convert efficiently from one to the other, whenever possible. Though all algebraic curves have an implicit representation only irreducible algebraic curves with *genus* $= 0$ are rational, i.e., have a rational parametric representation, see Salmon (1852). The genus of the curve measures the deficiency of singularities on the curve from its maximum allowable limit.

A variety of algorithms have been presented earlier for computing the genus of algebraic curves: by counting the number of linearly independent differentials of the first kind (without poles), Davenport (1979), the computation of the Hilbert function, Mora, Moller (1983), and the computation of ramification indices, Dicrescenzo, Duval (1984). A method of computing the genus of irreducible plane algebraic curves is presented in this paper, which uses affine quadratic transformations and is noteworthy for its simplicity.

Recently, various efficient methods have been given for obtaining the parametric equations for special low degree irreducible rational algebraic curves: degree two and three plane algebraic curves, Abhyankar and Bajaj (1987a,b), the rational space curves arising from the intersection of certain degree two surfaces, Levin (1979), and the rational space curves arising from the intersection of two rational surfaces, Ocken, Schwartz, Sharir (1986). The parameterization algorithms presented in this paper are applicable for implicitly defined irreducible rational algebraic curves of arbitrary degree. The computed rational parameterization is over the traditional power basis, however one may convert this to an equivalent Bernstein form over an arbitrary parameter range, by using the univariate power to Bernstein conversion algorithm of Geisow (1983).

The reverse problem of converting from parametric to implicit equations for algebraic curves, called implicitization is achieved by straightforward elimination methods, i.e., the computation of polynomial resultants, see Rowe (1917), Sederberg, Anderson, Goldman (1984), Bajaj (1987). Efficient computation of polynomial resultants, also known as the Sylvester resultant, see Salmon (1885), van der Waerden (1950) has been considered by various authors: for univariate polynomials, Schwartz (1980), for multivariate polynomials, Collins (1971).

The rest of this paper is as follows. In §2 we examine the intricate relationship of *genus* with the rational parameterization of irreducible plane curves. Examples of rational curves are: conics (degree 2 curves); cubics with a singular (double) point; quartics with three *distinct* double point singularities, etc. In §3 we present an efficient algorithm to construct rational parameterizations for a special class of plane curves. These parameterizations are obtained by taking lines through a *distinct* singular point on the curves, with the slope of the lines being the parameter. This technique suffices for the rational parameterization of conics, cubics with one double point and all irreducible higher degree d curves with a $d-1$ fold *distinct* singularity. In §4 we generalize the algorithm of §3 to provide rational parameterizations for all irreducible rational plane curves. These rational parameterizations are obtained by taking a one parameter family (a *pencil*) of curves of degree $d-2$ through fixed points on the original curve of degree d. Crucial here is the distinction between *distinct* and *infinitely near* singularities of an algebraic plane curve. Various algorithmic techniques are also presented, such as the mapping of points to infinity, the "passing" of a pencil of curves through fixed points, the "blowing up" of singularities by affine quadratic transformations, etc.

2. Genus and Parameterization

An irreducible algebraic curve C_d of degree d in the plane is one which is met by most lines in d points. Lines through a point P meet C_d (outside P) in general at $d - mult_p C_d$ points, where $mult_p C_d = e =$ multiplicity of C_d at P. If $e = 1$ then P is called a simple point. If $e = 2$ then P is called a double point. Similarly we talk about an e-ple point or an e-fold point. If $e = 0$: P is not on C_d. If $e > 1$ we say P is a *singular point* of the curve C_d with multiplicity e. This also leads to the following theorem for curves

Theorem 1: [Bezout] Curves of degree d and curves of degree e, with no common components, meet at $d \cdot e$ points, counting multiplicities and points at infinity. ($C_d \cdot C_e = d \cdot e$ points.)

Consider curve C_d of degree d to be also of order e.

$$C_d : f(x, y) = \sum_{e \le i+j \le d} a_{ij} x^i y^j$$

$$= f_d(x, y) + f_{d-1}(x, y) + \cdots + f_e(x, y)$$

(with $f_d(x, y) \neq 0$ and $f_e(x, y) \neq 0$, so that d = degree and e = order). Thus $f_d(x, y)$ is the degree form and $f_e(x, y)$ is the initial or order form. Again, the multiplicity of a point P on C_d is geometrically, the number of points that a line through that point P meets C_d at P. By translation, {if (a, b) is the point P, then $x \to x - a$, $y \to y - b$} we can assume the point P to be the origin. Then the equation of a line through it is $y = mx$. Its intersection with the curve is given by

$$f(x, mx) = f_d(x, mx) + f_{d-1}(x, mx) + \cdots + f_e(x, mx)$$

$$= x^d f_d(1, m) + x^{d-1} f_{d-1}(1, m) + \cdots x^e f_e(1, mx)$$

$$= x^e [f_d(1, m) x^{d-e} + \cdots + f_e(1, m)]$$

Lines through the origin meet the curve, outside the origin, in $d - e$ points. Hence the multiplicity of the origin = e (= *order* of the curve). Thus if the curve C_d has a $d-1$ fold point (origin), then lines through that point meet F at one other point, and thereby parameterizes the curve (rational).

Here we can also note that for most values of m, $f_e(1, m) \neq 0$. The values of m for which it is zero correspond to the tangents $f_e(x, y) = \prod_{i=1}^{e} (y - m_i x)$ to the curve at the origin. (Tangents at P are thus those special lines which meet C_d at P at more than e points, where e = multiplicity of C_d at P.)

Now note for example that there are ∞^5 conics. As an equation of a conic has five independent coefficients and if we take five 'independent' points in the plane and consider a conic passing through these points then this will give five linear homogeneous equations in the five coefficient variables. If the rank of the matrix is 5 then there is a unique conic through these points. In general, the number of independent coefficients of a plane algebraic curve C_d of degree d is $\frac{d(d+3)}{2}$.

One can easily prove by Bezout's theorem that a curve of degree 4, for example, cannot have 4 double points. In general one may see that the number of double points, say DP, of C_d is $\leq \frac{(d-1)(d-2)}{2}$. Assume $DP > \frac{(d-1)(d-2)}{2}$. Then since $\frac{(d-2)(d+1)}{2}$ fixed points determine a C_{d-2} curve and if we choose $\frac{(d-1)(d-2)}{2} + 1$ double points of C_d then to determine C_{d-2} one needs a remaining

$$\frac{(d-2)(d+1)}{2} - (\frac{(d-1)(d-2)}{2} + 1) = (d-2) - 1 = d - 3 \text{ points}$$

So take $(d-3)$ other fixed simple points of C_d. Then we can pass a C_{d-2} curve through the above $\frac{(d-1)(d-2)}{2} + 1$ double points of C_d and $(d-3)$ other simple points of C_d. Then counting the number of points of intersection of C_d and C_{d-2}

$$\begin{aligned} &= (d-1)(d-2) + 2 + d - 3 \\ &= d^2 - 2d + 1 = (d-2)d + 1 \\ &= C_d \cdot C_{d-2} + 1 \end{aligned}$$

which contradicts Bezout. Thus assuming Bezout we see that

$$DP \leq \frac{(d-1)(d-2)}{2}$$

In general, we have Table 1.

degree of curve	1	2	3	4	5	6	...	d
the maximum number of double points	0	0	1	3	6	10	...	$\frac{(d-1)(d-2)}{2}$
the number of curves of the given degree	2	5	9	14	20	27	...	$\frac{d(d+3)}{2}$

Table 1

One definition of the *genus* g of a curve C_d is a measure of how much the curve is deficient from its maximum allowable limit of singularities,

$$g = \frac{(d-1)(d-2)}{2} - DP$$

where DP is a 'proper' counting of the number of double points of C_d (summing over all singularities). In counting the number of double points DP of C_d an e-ple point of C is to be counted as $\frac{1}{2}e(e-1)$ double points. However this counting is not very precise as such is the case only for the so called *distinct* multiple points of C. For a multiple point, that is not *distinct*, one has also to consider *infinitely near* singularities. In general a double point is roughly either a *node* or a *cusp*. If a cusp is given by $y^2 - x^3$ we call it a *distinct* cusp and is counted as a single double point. Cusps other than distinct look like $y^2 - x^{2m+1}$ (an *m−fold* cusp). Though the multiplicity of the origin is two (= *order* of the curve) the origin accounts for m double points when counted properly. The proper counting was achieved by Noether using homogeneous "Cremona quadratic transformations", see Walker (1978). Following Abhyankar (1983) we can achieve the same thing by using "affine quadratic transformations".

Consider for example, the cusp $y^2 - x^3 = 0$ which has a double point at the origin. The quadratic transformation† (or substitution) $\bar{q}$ given by

$$x = \bar{x} \quad \text{and} \quad y = \bar{x}\,\bar{y} \qquad (1)$$

yields

$$0 = y^2 - x^3 = \bar{x}^2\bar{y}^2 - \bar{x}^3 = \bar{x}^2(\bar{y}^2 - \bar{x})$$

and cancelling out the extraneous factor $\bar{x}^2$ we get the nonsingular parabola $\bar{y}^2 - \bar{x} = 0$. So the origin in this case was a *distinct* singular point and counted as a single double point. To desingularize the *m−fold* cusp one has to make a succession of m transformations of the type (1). Only the m^{th} successive application of (1) changes the multiplicity of the origin from two to one. Hence in this case, counting properly, we say that the cusp has one *distinct* double point and $(m-1)$ *infinitely near* double points, giving a total *DP* count of m.

In a general procedure for counting double points, given an e-fold point P of a plane curve C, we choose our coordinates to bring P to the origin and then apply (1). If now $C: f(x, y) = 0$, then the substitution (1) transforms C into the curve $\bar{C}: \bar{f}(\bar{x}, \bar{y}) = 0$ given by

$$f(\bar{x}, \bar{x}\,\bar{y}) = \bar{x}^e\, \bar{f}(\bar{x}, \bar{y}).$$

$\bar{C}$ will meet the line $E: \bar{x} = 0$ in the points $P^1, \ldots, P^m$, the roots of $\bar{f}(0, y) = 0$ which corresponds to the tangents to C at P. If P^i is a e_i-fold point of $\bar{C}$, then we shall have $e_1 + \cdots + e_m \le e$. We say that $P^1, \ldots, P^m$ are the points of C in the first neighborhood of P, and the multiplicity of C at P_i is e_i. Now iterate this procedure. The points of C *infinitely near* P can be diagrammed by the *singularity tree* of C at P: (see Figure 1).

† The quadratic transformation $\bar{q}$ maps the origin to the line $\bar{x} = 0$, and is one-one for all points (x, y) with $x \neq 0$. Viewed alternatively, $\bar{q}$ maps tangent directions to f at the origin to different points on the exceptional line $\bar{x} = 0$. This may be seen by noting that the lines $y = mx$ are mapped to parallel lines $\bar{y} = m$ which intersect the exceptional line at points $(0, m)$. But $\bar{q}$ does not map the line $x = 0$ properly, so we must make sure that $x = 0$ is not a tangent direction to the curve at the origin. This is done by a nonsingular linear transformation $x = u\hat{x} + v\hat{y}$ and $y = r\hat{x} + s\hat{y}$ where neither $u\hat{x} + v\hat{y}$ nor $r\hat{x} + s\hat{y}$ are tangents to f at the origin.

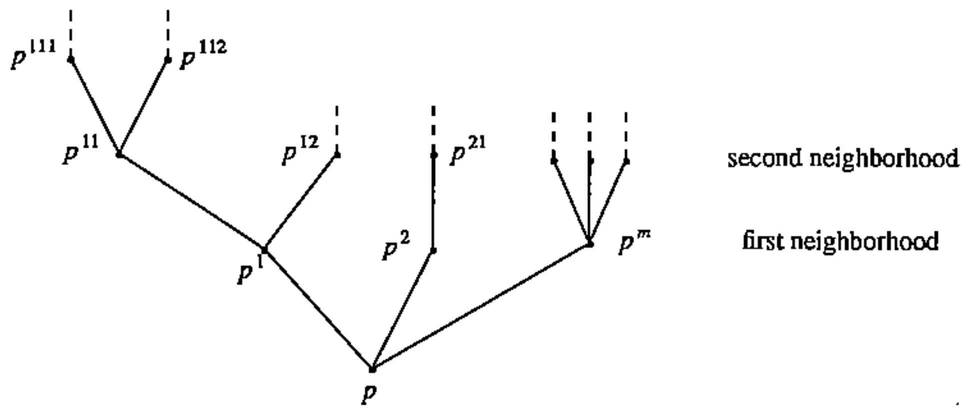

Figure 1: Singularity Tree

At every node of this tree (including the root) we keep a count equal to the multiplicity of C at that point which will then be $\geq$ the number of branches arising at that node. It follows that every node higher than a certain level will be unforked, that is have a single branch. The desingularization theorem for algebraic plane curves, see Abhyankar (1983), or Walker (1978), says that at every node higher than a certain level, the count equals one; in other words, C has only a finite number of singularities infinitely near P. Thus, since C has only finitely many *distinct* singularities, it follows that C has only a finite number of singular points, *distinct* as well as *infinitely near*.

Thus, by summing the counts of each node and counting $\frac{1}{2}e(e-1)$ double points for a count e and additionally summing over all singularities of C and their corresponding singularity trees, we obtain a precise count of the total number of double points DP of C. With this proper counting of double points one then has the following

Theorem 2 : [Cayley-Reimann] $g = 0$ *iff* C has a rational parametrization.

In other words if the given plane curve has its maximum allowable limit of singularities, then it is rational.

Note also that in counting singularities we consider all the singularities of the projective curve. That is we consider the singularities at both finite distance as well as at infinity. The process of considering singularities at infinity is no different than that at finite distance. With regard to homogeneous coordinates let us consider $Z = 0$ to be the line at infinity. By swapping one of the axis lines $x = 0$ or $y = 0$ with the line at infinity we can bring the points at infinity to the affine plane. We illustrate this as well as *Theorem* 2 by means of an example. Consider again the *m–fold* cusp $y^2 - x^{2m+1}$. We have seen earlier that the origin accounts for m double points when counted properly. Now consider the singularity at infinity. We swap the Z=0 line with the Y=0 line by homogenizing and then setting $Y = 1$.

$$Y^2Z^{2m-1} - X^{2m+1} \Rightarrow z^{2m-1} - x^{2m+1}$$

The singularity at infinity is again at the origin and of multiplicity $2m - 1$ accounting for $\frac{(2m-1)(2m-2)}{2}$ double points. On applying an appropriate quadratic transformation $x = \bar{x}$ and $z = \bar{x}\,\bar{z}$

$$\bar{z}^{2m-1} - \bar{x}^2$$

with the multiplicity at the origin reduced to 2. After a sequence of $m-1$ additional quadratic transformations the multiplicity at the origin finally reduces to one. These *infinitely near* singularities then account for totally $m-1$ additional double points, resulting in a total DP count for the curve to be equal to

$$m + \frac{(2m-1)(2m-2)}{2} + m - 1 = \frac{(2m)(2m-1)}{2}$$

which is exactly the maximum number of allowable double points for a curve of degree $2m + 1$. Hence the m-*fold* cusp has genus 0 and is rational with a parameterization given by

$$x = t^2 \qquad y = t^{2m+1}$$

3. Parameterizing with Lines

The geometric idea of parametrizing a circle or a conic is to fix a point and take lines through that point which meet the conic at one additional point. Hence conics always have a rational parameterization, with the slope of the line being the single parameter. Next, consider a cubic curve, C_3. A cubic curve is a curve to which most lines intersect in three points. If we consider a *singular* cubic curve then lines through the singular (double) point meet the curve at one additional point and hence rationally parametrize the cubic curve. If C_3 has no singular points, then C_3 cannot be parametrized by rational functions. Now intersecting a curve C with a pencil of lines through a fixed point P on it, can be achieved by sending the point P on C to infinity. To understand this, let us first consider an irreducible conic which is represented by the equation

$$g(x, y) = ax^2 + by^2 + cxy + dx + ey + f$$

From the genus formula of §2 we note that all conics are rational. Further Bezout confirms that the irreducible conic cannot contain a double point for otherwise the conic consists of two lines. We observe that the trivial parameterizable cases are the parabola $y^2 = x$ which has no term in x^2; the parabola $x^2 = y$ which has no term in y^2; and the hyperbola $xy = 1$ which has no terms in x^2 and y^2. The non-trivial case arises when a and b are both non-zero, e.g. ellipse. This then suggests that to obtain a rational parameterization all we need to do is to kill the term in y^2 say, by a suitable linear transformation resulting in the equation

$$(rx + s)y + (ux^2 + vx + w) = 0.$$

Then one could obtain a rational parametrization

$$x = t$$
$$y = \frac{-(ut^2 + vt + w)}{(rt + s)}$$

The elimination of the x^2 or the y^2 term through a coordinate transformation is said to make the conic irregular in x or y respectively. Geometrically speaking, a conic being irregular in x or y means that most lines parallel to the x or y axis respectively, intersect the conic in one point. Note that most lines through a fixed point on the conic meet the conic in one additional varying point. By sending the fixed point to infinity we make all these lines parallel to some axis and the curve irregular in one of the variables (x, or y) and hence amenable to parameterization. The coordinate transformation we select is thus one which sends any point on the conic to infinity along either of the coordinate axis x or y.

As an example consider the unit circle and fix a simple point $P\,(-1, 0)$ on it

x, y affine coordinates (−1, 0)

X, Y, Z homogeneous coordinates (−1, 0, 1)

and send P to a point at infinity along the y-axis. That is, send (−1, 0, 1) to (0, 1, 0). (Explanation: A point on y-axis is like $(0, p, 1)$ divide by p $(\frac{0}{p}, \frac{p}{p}, \frac{1}{p})$ now let $p \to \infty$ and thus we obtain (0, 1, 0)).

This we achieve by a homogeneous linear transformation which transforms (−1, 0, 1) to (0, 1, 0)

$$X \to \alpha\overline{X} + \beta\overline{Y} + \gamma\overline{Z}$$
$$Y \to \hat{\alpha}\overline{X} + \hat{\beta}\overline{Y} + \hat{\gamma}\overline{Z}$$
$$Z \to \alpha^*\overline{X} + \beta^*\overline{Y} + \gamma^*\overline{Z}$$

The chosen point on the circle (−1, 0, 1) determines

$$-1 = \beta$$
$$0 = \hat{\beta}$$
$$1 = \beta^*$$

and the α's and γ's are chosen such that the det $|\,\alpha's\,,\,\beta's\,,\,\gamma's\,| \neq 0$, yielding a well defined invertible transformation. So let us take as our homogeneous linear transformation

$$X \to -\overline{Y}$$
$$Y \to \overline{Z}$$
$$Z \to \overline{X} + \overline{Y}$$

Note we transformed the circle $x^2+y^2-1=0$ to $X^2+Y^2-Z^2=0$ by homogenizing. On applying this transformation we eliminate the $\bar{Y}^2$ term

$$\bar{Y}^2+\bar{Z}^2-(\bar{X}+\bar{Y})^2=0$$

$$-2\bar{X}\bar{Y}=\bar{X}^2-\bar{Z}^2$$

$$\bar{Y}=\frac{\bar{Z}^2-\bar{X}^2}{2\bar{X}}$$

Then dehomogenizing $\bar{Z}=1$ and using the linear transformation to obtain the original affine coordinates

$$x=\frac{X}{Z}=\frac{-\bar{Y}}{\bar{X}+\bar{Y}}$$
$$y=\frac{Y}{Z}=\frac{1}{\bar{X}+\bar{Y}}$$

and setting $\bar{X}=t$ we obtain the rational parametrization of the circle

$$\begin{array}{l} \bar{X}=t \\ \bar{Y}=\dfrac{1-t^2}{2t} \end{array} \rightarrow \begin{array}{l} x=\dfrac{-(1-t^2)/2t}{t+\dfrac{1-t^2}{2t}}=\dfrac{1-t^2}{1+t^2} \\ y=\dfrac{1}{t+\dfrac{1-t^2}{2t}}=\dfrac{2t}{1+t^2} \end{array}$$

In general, curves of degree d with a *distinct* $d-1$ fold point can be rationally parameterized by sending the $d-1$ fold point to infinity. Consider $f(x,y)$ a polynomial of degree d in x and y representing a plane algebraic curve C_d of degree d with a *distinct* $d-1$ fold singularity. Singularities of a plane curve can computationally be obtained by simultaneously solving the equations $f=f_x=f_y=0$ where f_x and f_y are the x and y partial derivatives of f, respectively. One way of obtaining the common solutions is to find those roots of $Res_x(f_x,f_y)=0$ which are also the roots of $f=0$. Here $Res_x(f_x,f_y)$ is the resultant of f_x and f_y treating them as polynomials in x. Note singularities at infinity can be obtained the same way after replacing the line at infinity with one of the coordinate axes. In particular on homogenizing a plane curve $f(x,y)$ to $F(X,Y,Z)$ we can set $Y=1$ to obtain $\bar{f}(x,z)$ thereby swapping the line at infinity $Z=0$ with the line $Y=0$. Now the above procedure can be applied to $\bar{f}(x,z)$ to find the singularities at infinity.

Let us obtain the $d-1$ fold singularity of the curve C_d and translate it to the origin. Then we can write

$$f(x,y)=f_d(x,y)+f_{d-1}(x,y)$$

where f_d, (degree form), consists of the terms of degree d and f_{d-1} consists of terms of degree $d-1$.

Alternatively on homogenizing this curve we obtain

$$F(X, Y, Z) = a_0Y^d + a_1Y^{d-1}X + \cdots + a_dX^d$$

$$+ b_0Y^{d-1}Z + b_1Y^{d-2}X\,Z + \cdots + b_dX^{d-1}Z$$

Now by sending the singular point (0,0,1) to infinity along the Y axis we can eliminate the Y^d term. This as before by a homogeneous linear transformation which maps the point (0,0,1) to the point (0,1,0) and given by

$$X = \overline{X} \qquad Y = \overline{Z} \qquad Z = \overline{Y}$$

which yields

$$F(\overline{X}, \overline{Y}, \overline{Z}) = a_0\overline{Z}^d + a_1\overline{Z}^{d-1}\overline{X} + \cdots + a_d\overline{X}^d$$

$$+ b_0\overline{Z}^{d-1}\overline{Y} + b_1\overline{Z}^{d-2}\overline{X}\,\overline{Y} + \cdots + b_d\overline{X}^{d-1}\overline{Y}$$

$$\overline{Y} = -\frac{a_0\overline{Z}^d + a_1\overline{Z}^{d-1}\overline{X} + \cdots + a_d\overline{X}^d}{b_0\overline{Z}^{d-1} + b_1\overline{Z}^{d-2}\overline{X} + \cdots + b_d\overline{X}^{d-1}}$$

Then dehomogenizing, $\overline{Z} = 1$ and using the linear transformation to obtain the original affine coordinates

$$x = \frac{X}{Z} = \frac{\overline{X}}{\overline{Y}}$$

$$y = \frac{Y}{Z} = \frac{\overline{Z}}{\overline{Y}}$$

and setting $\overline{X} = t$ we obtain the rational parametrization of the curve.

Alternatively we could have symbolically intersected a single parameter family (*pencil*) of lines through the $d-1$ fold singularity with C_d and obtained a rational parameterization with respect to this parameter. This concept of passing a pencil of curves through singularities is generalized in the next section.

4. Parameterizing with Higher Degree Curves

From the genus formula and Bezout's theorem we note that an irreducible rational quartic curve in the plane has either a *distinct* triple point or three *distinct* double points. The rational parameterization of the quartic with a *distinct* triple point is handled by the method of §3. Let us then consider an irreducible quartic curve C_4 with three *distinct* double points. From the table of §2 we know that through 5-points a conic can be passed. Choose three double points and a simple point on the curve C_4, yielding a one parameter family (pencil) of conics, $C_2(t)$. Now $C_4 \cdot C_2(t) = 8$ points. Since the fixed points (3 double points and a simple point) account for $2 + 2 + 2 + 1 = 7$ points, the remaining point on C_4 is the variable point, giving us a rational parametrization of C_4, in terms of parameter t.

Computationally we proceed as follows. Consider first C_4 with three distinct double points. We first obtain the three double point singularities of the homogeneous quartic $F(X,Y,Z)$ as well as a simple point on it. Let them be given by (X_1,Y_1,Z_1), (X_2,Y_2,Z_2), (X_3,Y_3,Z_3) and (X_4,Y_4,Z_4) respectively. Consider next the general equation of a homogeneous conic C_2 given by

$$G(X,Y,Z) = aX^2 + bY^2 + cXY + dXZ + eYZ + fZ^2 = 0$$

which has six coefficients however five independent unknowns as we can always divide out by one of the nonzero coefficients. We now try to determine these unknowns to yield a one parameter family of curves, $C_2(t)$. We pass C_2 simply through the singular double points and the simple point of C_4. (In general we shall pass a curve through an *m−fold* singularity with multiplicity $m-1$). In other words we equate for $i=1,\cdots,4$,

$$F(X_i,Y_i,Z_i) = G(X_i,Y_i,Z_i) = 0$$

This yields a linear system of 4 equations in five unknowns. Set one of the unknowns to be t and solve for the remaining unknowns in terms of t.

Next compute the intersection of C_4 and $C_2(t)$, by computing $Res_Y(F,G)$ which is a polynomial in X, Z and t. On dehomegenizing this polynomial by setting $Z=1$, (since resultants of homogeneous polynomials are homogeneous) and dividing by the common factors $(x - x_i)^2$ for $i=1..3$ and $(x - x_4)$ we obtain a polynomial linear in x which yields the rational parameterization. The process when repeated for y by taking the $Res_X(F,G)$ and dividing by the common factors $(y - y_i)^2$ for $i=1..3$ and $(y - y_4)$ yields a polynomial in y and t and linear in y which yields the rational parameterization.

Next consider an example of a quintic curve with *infinitely near* singularities. In particular, the homogenized quintic cusp $C_5 : F(X,Y,Z) = Y^2Z^3 - X^5$ has a *distinct* double point and an *infinitely near* double point (in the first neighborhood) at $(0\,,0\,,1)$, and a *distinct* triple point and an *infinitely near* double point at $(0\,,1\,,0)$. Counting all the double points, properly, we see that C_5 has 6 double points and hence is of genus 0 and rational. To obtain the parameterization we pass a one parameter family of cubics $C_3(t)$ given by $G(X,Y,Z) = aX^3 + bY^3 + cX^2Y + dXY^2 + eX^2Z + fY^2Z + gXYZ + hXZ^2 + iYZ^2 + jZ^3$ through the singularities of C_5. Passing $C_3(t)$ through the *distinct* double point (with multiplicity $2 - 1 = 1$) is obtained as before by equating

$$F(0, 0, 1) = G(0, 0, 1) = 0\ldots \qquad (1)$$

and the *distinct* triple point, (with multiplicity $3 - 1 = 2$) by equating

$$F(0, 1, 0) = G(0, 1, 0) = 0 \ldots \qquad (2)$$

$$F_X(0, 1, 0) = G_X(0, 1, 0) = 0 \ldots \qquad (3)$$

$$F_Z(0, 1, 0) = G_Z(0, 1, 0) = 0 \ldots \qquad (4)$$

These conditions for our example curve C_5 makes $j = 0$, $b = 0$, $d = 0$ and $f = 0$ in $C_3(t)$ yielding the curve $\overline{G}(X, Y, Z) = aX^3 + cX^2Y + eX^2Z + gXYZ + hXZ^2 + iYZ^2$.

We now wish to pass $C_3(t)$ through the *infinitely near* double point in the first neighborhood of the singularity at (0, 0, 1) of C_5. To achieve this we apply the quadratic transformation $X = \overline{X}$, $Y = \overline{XY}$, $Z = \overline{Z}$ centered at (0, 0, 1) to both $F(X, Y, Z)$ and $\overline{G}(X, Y, Z)$. The transformed equation $F_T = \overline{Y}^2\overline{Z}^3 - \overline{X}^3$ has a double point at (0, 0, 1) and we pass the curve of the transformed equation $\overline{G}_T = a\overline{X}^2 + c\overline{X}^2Y + e\overline{XZ} + g\overline{XYZ} + h\overline{Z}^2 + i\overline{YZ}^2$ through the double point as before by equating

$$F_T(0, 0, 1) = \overline{G}_T(0, 0, 1) = 0 \ldots \qquad (5)$$

This condition makes $h = 0$ in $C_3(t)$ yielding $\hat{G}(x, y, z) = aX^3 + cX^2Y + eX^2Z + gXYZ + iYZ^2$.

Similarly we pass C_3 through the *infinitely near* double point in the first neighborhood of the singularity at (0, 1, 0) of C_5. To achieve this we apply the quadratic transformation $X = \hat{X}$, $Y = \hat{Y}$, $Z = \hat{X}\hat{Z}$ centered at (0, 1, 0) to both $F(X, Y, Z)$ and $\hat{G}(X, Y, Z)$. The transformed equation $F_T = \hat{Y}^2\hat{Z}^3 - \hat{X}^2$ has a double point at (0, 1, 0) and we pass the curve of the transformed equation $\hat{G}_T = a\hat{X} + c\hat{Y} + e\hat{X}\hat{Z} + g\hat{Y}\hat{Z} + i\hat{Y}\hat{Z}^2$ through the double point as before by equating

$$F_T(0, 1, 0) = \hat{G}_T(0, 1, 0) = 0 \ldots \qquad (6)$$

This condition makes $c = 0$ in C_3 yielding $\vec{G}(x, y, z) = aX^3 + eX^2Z + gXYZ + iYZ^2$.

Our final condition to determine pencil of cubics $C_3(t)$ is to choose two simple points on C_5, say (1, 1, 1) and (1, −1, 1) and pass C_3 through it by equating.

$$F(1, 1, 1) = \vec{G}(1, 1, 1) = 0 \ldots \qquad (7)$$

$$F(1, -1, 1) = \vec{G}(1, -1, 1) = 0 \ldots \qquad (8)$$

Note that in total we applied eight conditions to determine the pencil, since nine conditions completely determine the cubic. The last two conditions yield the equations

$$a + e + g + i = 0$$

$$a + e - g - i = 0 \quad .$$

In choosing the pencil $C_3(t)$ we allow one of the coefficients to be t and we may divide out by another coefficient (or choose it to be 1). The above equations yield $a + e = 0$ and $g + i = 0$ and on choosing $a = t$ and $g = 1$ we obtain $e = -t$ and $i = -1$. Hence our homogeneous cubic pencil is given by $G_3(X, Y, Z, t) = tX^3 - tX^2Z + XYZ - YZ^2$ or the dehomogenized pencil $G_3(x, y, t) = tx^3 - tx^2 + xy - y = 0$. This yields $y = -tx^2$. Intersecting it with the dehomogenized quintic $C_5 : y_2 - x^5$ yields $t^2x^4 - x^5 = 0$ or $x = t^2$ on dividing out by the common factor x^4. Finally the parametric equations of the rational quintic C_5 are given by $x = t^2$ and $y = -t^5$.

In the general case we consider an irreducible curve C_d with the appropriate number of *distinct* and *infinitely near* singularities which make C_d rational (*genus* 0). We pass a curve C_{d-2} through these singular points and $d-3$ additional simple points of C_d. Consider again $F(X,Y,Z)$ and $G(X,Y,Z)$ as the homogeneous equations of curves C_d and C_{d-2} respectively. For a distinct singular point of multiplicity m of C_d at the point (X_i,Y_i,Z_i) we pass the curve C_{d-2} through it with a multiplicity of $m-1$. To achieve this we equate

$$F(X_i,Y_i,Z_i) = G(X_i,Y_i,Z_i)$$

$$F_X(X_i,Y_i,Z_i) = G_X(X_i,Y_i,Z_i)$$

$$F_Y(X_i,Y_i,Z_i) = G_Y(X_i,Y_i,Z_i)$$

$$F_{XX}(X_i,Y_i,Z_i) = G_{XX}(X_i,Y_i,Z_i)$$

$$F_{XY}(X_i,Y_i,Z_i) = G_{XY}(X_i,Y_i,Z_i)$$

$$F_{YY}(X_i,Y_i,Z_i) = G_{YY}(X_i,Y_i,Z_i)$$

$$\cdots\cdots\cdots$$

$$F_{X^jY^k}(X_i,Y_i,Z_i) = G_{X^jY^k}(X_i,Y_i,Z_i) \qquad 0 \le j + k \le m-2$$

For an *infinitely near* singular point of C_d with its associated singularity tree we pass the curve C_{d-2} with multiplicity $r-1$ through each of the points of multiplicity r in the first, second, third, ..., neighborhoods. To achieve this we apply quadratic transformations T_i to both $F(X,Y,Z)$ and $G(X,Y,Z)$ centered around the *infinitely near* singular points corresponding to the singularity tree. The appropriate multiplicity of passing is achieved by equating the transformed equations F_{T_i} and G_{T_i} and their partial derivatives as above.

A simple counting argument now shows us that this method generates the correct number of conditions which specifies C_{d-2} and furthermore the total intersection count between C_d and C_{d-2} satisfies

Bezout. A curve C_d of *genus* = 0 has the equivalent of exactly $\frac{(d-1)(d-2)}{2}$ double points. Then to pass a curve C_{d-2} through these double points and $d-3$ other fixed simple points of C_d and one variable point specified by t, the total number of conditions (= to the total number of linear equations) is given by

$$\frac{(d-1)(d-2)}{2} + (d-3) + 1 = \frac{(d-2)(d+1)}{2}$$

which is exactly the number of independent unknowns to determine C_{d-2} (see table of §2). Next, counting the number of points of intersection of C_d and C_{d-2}

$$\begin{aligned} &= (d-1)(d-2) + d - 3 + 1 \\ &= (d-2)d \\ &= C_{d-2} \cdot C_d \end{aligned}$$

satisfying Bezout. For further details of the applicability of Bezout's theorem with respect to *infinitely near* singularities, see Abhyankar (1973). Then computing the $\mathrm{Res}_x(C_d, C_{d-2})$ which yields a polynomial of degree $d(d-2)$ in y and dividing by the common factors corresponding to the $(d-3)$ simple points (a polynomial of degree $(d-3)$ in y) and $\frac{(d-2)(d-1)}{2}$ double points (a polynomial of degree $(d-2)(d-1)$ in y) yields a polynomial in y and t which is linear in y, (for the single variable point) and thus gives a rational parameterization of y in terms of t. Similarly repeating with $\mathrm{Res}_y(C_d, C_{d-2})$ yields a rational parameterization of x in terms of t.

As an example consider the m*-fold* cusp $y^2 - x^{2m+1}$ once again (for the last time). We know from §2 that it is a rational curve with genus 0 and with a distinct double point and $m-1$ infinitely near double points at the origin (0,0,1) and a distinct $(2m-1)$-fold singularity and $m-1$ infinitely near double points at infinity (0,1,0). Now we pass a pencil of curve C_{2m-1} of degree $2m-1$ appropriately (as explained above) through these singularities and also through $2m+1-3 = 2m-2$ simple points of the m-fold cusp C_{2m+1}.

In the following let $F(X, Y, Z) = 0$ be the equation of C_{2m+1} and $G(X, Y, Z)$ the equation of C_{2m-1}. Now the conditions available to specify a pencil of curves C_{2m-1} is given as follows. A total of $2m-2$ conditions are given by equating F and G at the $2m-2$ simple points of C_{2m+1}. Further by equating F and G and the corresponding transformed F_{T_i} and G_{T_i} (transformed by a sequence of quadratic transformations) at the *distinct* and *infinitely near* double points of the origin (0,0,1) and *infinitely near* double points of infinity (0,1,0). This totally accounts for $m + m-1 = 2m-1$ additional conditions. Finally through the $(2m-1)$ fold singularity at infinity of C_{2m+1} the pencil C_{2m-1} is passed with multiplicity $2m-2$ which is obtained by equating the equations and the partial derivatives $F_{X^jY^k} = G_{X^jY^k}$ for all $0 \le j+k < 2m-2$ which yields $\frac{(2m-2)(2m-1)}{2}$ conditions. One final condition is achieved by equating one of the coefficients of C_{2m-1} to 't'. Hence totally the conditions available to specify the pencil of curves C_{2m-1} is given by $1 + 2m-2 + 2m-1 + \frac{(2m-2)(2m-1)}{2} = \frac{(2m-1)(2m+2)}{2}$ which is exactly the number of conditions required to specify a pencil of curve C_{2m-1} as given by the table in §2. This then yields a linear system of $(2m-1)(m+1)$ equations in the same number of unknowns and can be easily solved.

Finally, note that the total number of intersections (counting multiplicities) between C_{2m-1} are given by 1 { single variable point} + $(2m-2)$ {fixed simple points} + $2(2m-1)$ {double points} + $(2m-1)(2m-2)$ {$2m-2$ multiplicity of C_{2m-1} at the $(2m-1)$-fold singularity of C_{2m+1}} = $(2m-1)(2m+1)$ satisfying Bezout. Hence on computing the $\mathrm{Res}_x(C_{2m+1}, C_{2m-1})$ and dividing by the common factors corresponding to the $(2m-2)$ simple points, $(2m-1)$ double points and the $2m-2$ multiplicity of C_{2m-1} at the $(2m-1)$-fold singularity of C_{2m+1} yields a polynomial in y and t which is linear in y, (for the single variable point) and thus gives a rational parameterization of y in terms of t. Similarly repeating with $\mathrm{Res}_y(C_{2m+1}, C_{2m-1})$ yields a rational parameterization of x in terms of t.

5. Conclusion

In this paper we presented algorithms to obtain rational parameterizations of irreducible algebraic curves. These methods also apply to all irreducible *planar* algebraic curves, where planar curves are either specified by a single polynomial equation in the plane, $f(x,y) = 0$ or may be specified by two polynomial equations in space, $f(x,y,z) = 0$ and $g(x,y,z) = 0$ (defining an irreducible space curve) where one of the two equations is rational. In the latter case the two equations specifying the space curve are easily mapped to a single polynomial equation $h(s,t) = 0$ describing the curve in the parametric plane $s-t$ of the rational surface. This mapping between the (x,y,z) points of the space curve and the (s,t) points of the plane curve is birational (one to one and onto) and hence a rational parameterization of this plane curve gives a rational parameterization of the space curve. Automatic rational parameterization algorithms provide this birational mapping for intersection curves of low degree rational surfaces, Abhyankar and Bajaj (1987a, b), Sederberg (1987). Rational parameterization techniques for irreducible algebraic space curves which are specified by two polynomial equations in space, without conditions on the rationality of the defining surfaces, are considered in Abhyankar and Bajaj (1987c).

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