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Abstract

An $O(h^6)$ collocation method based on quintic splines is developed and analyzed for general fourth-order linear two-point boundary value problems. The method determines a quintic spline approximation to the solution by forcing it to satisfy a high order perturbation of the original boundary value problem at the nodal points of the spline. A variation of this method is formulated as a deferred correction method. The error analysis of the new method and its numerical behavior is presented.

1. INTRODUCTION

In this paper we consider the numerical solution of the fourth order two-point boundary value problem

$$Lu \equiv D^4 u(s) + \sum_{j=0}^{3} e^j(s)\, D^j u(s) = f(s), \tag{1.1a}$$

defined in the interval [a,b] and subject to boundary conditions

$$Bu \equiv \sum_{j=0}^{3} (\alpha_{ij} D^i u(a) + \beta_{ij} D^j u(b)) = g_i, \quad i = 0(1)3. \tag{1.1b}$$

The method formulated and applied to (1.1) approximates the solution u by a quintic spline. The approximation is assumed to satisfy a high order perturbation of the problem (1.1) at the nodal points of the quintic spline. Optimal $O(h^6)$ global error estimates are obtained for uniform meshes. Further, it is shown here and in [9] that standard nodal collocation with quintic splines results in $O(h^2)$ method for the class of problems (1.1).

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Nodal collocation methods based on cubic splines were developed and analyzed in [2] and [6] for a restrictive linear case of (1.1). In [7], [8] and [9] second and fourth order collocation methods were considered based on quintic and sextic splines respectively for solving a subclass of linear and nonlinear fourth order problems. The method developed and analyzed here can be applied to general nonlinear two-point boundary value problems and can be extended to two dimensional fourth order problems [3].

2. QUINTIC SPLINE INTERPOLATION RESULTS

In this section we present the error analysis of a special quintic spline interpolant w and derive several asymptotic relations to be used for the formulation of the optimal quintic spline collocation method. For this we consider w to be an element of $Sp_5(\Delta_N) \equiv \{u \mid u \in C^4[a,b]$ and u is a polynomial of degree at most 5 on each subinterval of the partition $\Delta_N\}$ where $\Delta_N \equiv \{a = s_0 \le s_1 \le \cdots \le s_N = b,\ h = s_i - s_{i-1},$ $1 \le i \le N\}$ is the uniform partion of the interval [a,b]. Throughout we denote by $\{B_k\}$ the set of B-splines for $Sp_5(\Delta_N)$ and define $w(s) = \sum_{k=i}^{i+5} a_k\, B_k(s)$, for $s \in [s_i, s_{i+1}]$ to be the quintic spline interpolant of u in $C^{10}[a,b]$, satisfying

(a) the interpolation conditions:

$$w(s_i) = u(s_i) \quad \text{for} \quad 0 \le i \le N \tag{2.1a}$$

and

(b) for $i = 0, 1, N-1, N$ the end conditions:

$$w''''(s_i) = u''''(s_i) - u^{(6)}(s_i)\,\frac{h^2}{12} + u^{(8)}(s_i)\,\frac{h^4}{240}, \tag{2.1b}$$

$$w'''(s_i) = u'''(s_i) - u^{(7)}(s_i)\,\frac{h^4}{240}, \tag{2.1c}$$

$$w''(s_i) = u''(s_i) + u^{(6)}(s_i)\,\frac{h^4}{720}. \tag{2.1d}$$

Let's denote by $w_i \equiv w(s_i)$ and $w_i^{(p)} \equiv w^{(p)}(s_i)$ for all p where $g^{(p)} \equiv D^p g$. Further we define Λ by

$\Lambda g_i \equiv g_{i-2} + 26g_{i-1} + 66g_i + 26g_{i+1} + g_{i+2}$ for any function g evaluated at the nodes of partition Δ_N. Then we have the following recursive relations connecting w and its derivatives [1]:

$$\Lambda\, w' = \frac{120}{24h}\left[\, -w_{i-2} - 10w_{i-1} + 10w_{i+1} + w_{i+2}\right], \tag{2.2a}$$

$$\Lambda\, w'' = \frac{120}{6h^2}\left[\, w_{i-2} + 2w_{i-1} - 6w_i + 2w_{i+1} + w_{i+2}\right], \tag{2.2b}$$

$$\Lambda\, w''' = \frac{120}{2h^3}\left[-w_{i-2} + 2w_{i-1} - 2w_{i+1} + w_{i+2}\right], \tag{2.2c}$$

$$\Lambda\, w'''' = \frac{120}{h^4}\left[\, w_{i-2} - 4w_{i-1} + 6w_i - 4w_{i+1} + w_{i+2}\right] \tag{2.2d}$$

for $i = 2(1)N-2$. Since w interpolates u, after expanding in Taylor series we obtain the following relations

Lemma 2.1. If $u \in C^{10}[a,b]$ and w is the quintic-spline interpolant of u, defined by (2.2) then we have

$$\Lambda\, w' = 120u_i' + 30u_i^{(3)}\, h^2 + \frac{7}{2}\, u_i^{(5)}\, h^4 + O(h^6), \tag{2.3a}$$

$$\Lambda\, w'' = 120u_i'' + 30u_i^{(4)}\, h^2 + \frac{11}{3}\, u_i^{(6)}\, h^4 + O(h^6), \tag{2.3b}$$

$$\Lambda\, w''' = 120u_i''' + 30u_i^{(5)}\, h^2 + 3u_i^{(7)}\, h^4 + O(h^6), \tag{2.3c}$$

$$\Lambda\, w'''' = 120u_i'''' + 20u_i^{(6)}\, h^2 + \frac{3}{2}\, u_i^{(8)}\, h^4 + O(h^6). \tag{2.3d}$$

With the above relations as a basis, we prove the following.

Theorem 2.1. Let w be the unique quintic spline satisfying equations (2.1) for a given function $u \in C^{10}[a,b]$. Then for uniform partitions we have for $i = O(1)N$

$$w_i' = u_i' + O(h^6), \tag{2.4a}$$

$$w_i'' = u_i'' + \frac{h^4}{720} u_i^{(6)} + 0(h^6), \tag{2.4b}$$

$$w_i''' = u_i''' - \frac{h^4}{240} u_i^{(7)} + O(h^6), \tag{2.4c}$$

$$w_i'''' = u_i'''' - \frac{h^2}{12} u_i^{(6)} + \frac{h^4}{240} u_i^{(8)} + O(h^6). \tag{2.4d}$$

Furthermore the following interpolating error estimates hold

$$||(w-u)^{(m)}||_\infty = O(h^{6-m}), \text{ for } m = 0(1)4.$$

Proof. First we prove the relation (2.4d). For this we consider any function $g \in C^6[a,b]$ and easily show that $\Lambda g_i = 120 g_i + 30 g_i'' h^2 + \frac{7}{2} g_i'''' h^4 + O(h^6)$. Letting $g = u'''' - \frac{h^2}{12} u^{(6)} + \frac{h^4}{240} u^{(8)}$ we find

$$\Lambda \left[u_i'''' - \frac{h^2}{12} u_i^{(6)} + \frac{h^4}{240} u_i^{(8)} \right] = 120 u_i^{(4)} + 20 u_i^{(6)} h^2 + \frac{3}{2} u_i^{(8)} h^4. \tag{2.5}$$

If we define $d_i \equiv w_i'''' - u_i'''' + \frac{h^2}{12} u_i^{(6)} - \frac{h^4}{240} u_i^{(8)}$ and subtract equation (2.5) from equation (2.3d) we conclude that

$$\Lambda d_i = O(h^6 || u^{(10)} ||_\infty) \text{ for } 2 \le i \le N-2 \text{ and } d_0 = d_1 = d_{N-1} = d_N = 0. \tag{2.6}$$

Since the coefficient matrix of the equations (2.6) is diagonally dominant and its inverse has L_∞-norm bounded by 1/12, we have that $d_i = O(h^6)$ uniformly in i. This proves relation (2.4d). Equations (2.4b), (2.4c) can be proved following similar arguments. It remains to verify equation (2.4a). For this we observe that w_i' can be written in terms of w, w'' and w''' according to its definition. Specifically, for $i = 0(1)N-1$ we have

$$w_i' = (w_{i+1} - w_i)/h - h(42 w_i'' + 18 w_{i+1}'')/120 + h^2(2 w_{i+1}''' - 3 w_i''')/60$$

and for $i = 1(1)N$

$$w_i' = (w_i - w_{i-1})/h + h(3 w_{i-1}'' + 7 w_i'')/20 + h^2(2 w_{i-1}''' - 3 w_i''')/60.$$

After using the relations (2.1a), (2.4b) and (2.4c) for $w_i - u_i$, $w_i'' - u_i''$, $w_i''' - u_i'''$ and expanding in Taylor series we obtain $w_i' = u_i' + O(h^6)$ for both of the above equations. It is known that the piecewise linear interpolation yields $O(h^2)$ accuracy and since w'''' is the piecewise linear interpolant of an $O(h^2)$ perturbation of u'''', it is clear that $||w'''' - u''''||_\infty = O(h^2)$. Consequently $||w''' - u'''||_\infty = O(h^3)$, after integrating $w'''' - u''''$ from s_i to s for $s_i \le s \le s_{i+1}$ and taking the norms. Similarly, we conclude that $||w'' - u''||_\infty = O(h^4)$, $||w' - u'||_\infty = O(h^5)$, and $||w - u||_\infty = O(h^6)$. This completes the proof of the Theorem.

For later use we derive approximations of high order derivatives of u by a linear combination of values of derivatives of w based on the relations (2.4). Throughout we denote by δ the difference operator such that $\delta w_i \equiv w_{i-1} - 2w_i + w_{i+1}$ and $\delta^2 w_i \equiv w_{i-2} - 4w_{i-1} + 6w_i - 4w_{i+1} + w_{i+2}$.

Corollary 2.1 Under the hypotheses of Theorem 2.1, we have

$$u_i^{(6)} = \delta w_i^{(4)}/h^2 + O(h^4), \quad \text{for} \quad 1 \le i \le N-1, \tag{2.7a}$$

and for $2 \le i \le N-2$

$$u_i^{(8)} = \delta^2 w_i^{(4)}/h^4 + O(h^2), \tag{2.7b}$$

$$u_i^{(7)} = \delta^2 w_i^{(3)}/h^4 + O(h^2), \tag{2.7c}$$

$$u_i^{(6)} = \delta^2 w_i^{(2)}/h^4 + O(h^2). \tag{2.7d}$$

Proof:

From the asymptotic relation (2.4d) we have,

$$\delta w_i^{(4)}/h^2 = \delta u_i^{(4)}/h^2 - \frac{1}{12}\,\delta u_i^{(6)} + \frac{h^2}{240}\,\delta u_i^{(8)} + O(h^4)$$

$$= u_i^{(6)} + \frac{h^2}{12}\,u_i^{(8)} - \frac{1}{12}\left[\,h^2\,u_i^{(8)} + \frac{h^4}{12}\,u_i^{(10)}\right] + \frac{h^4}{240}\,u_i^{(10)} + O(h^4)$$

$$= u_i^{(6)} + O(h^4).$$

The rest of the relations (2.7) can be proved similarly. This completes the proof of the Corollary.

A straightforward application of Corollary 2.1 to the relations (2.4) yields the following important relations.

Corollary 2.2 If $u \in C^{10}[a,b]$, then the following approximations hold at the knots s_i,

$$u_i' = w_i' + O(h^6), \quad \text{for} \quad i = 0(1)N, \tag{2.8a}$$

and for $2 \le i \le N-2$

$$u_i'' = w_i'' - \delta^2 w_i''/720 + O(h^6),$$

$$u_i''' = w_i''' + \delta^2 w_i'''/240 + O(h^6), \tag{2.8b}$$

$$u_i'''' = w_i'''' + \delta w_i^{(4)}/12 - \delta^2 w_i^{(4)}/240 + O(h^6)$$

It turns out that similar asymptotic relations to (2.8b) hold at the points s_0, s_1, s_{N-1}, s_N. For their derivation we obtain approximations of high derivatives of u by a linear combination of derivatives of w evaluated at interior nodes.

Lemma 2.2 If $u \in C^{10}[a,b]$ then we have the following approximations to higher derivatives of u at the end points

$$u_0^{(6)} = \left[2\delta w_1^{(4)} - \delta w_2^{(4)} \right] /h^2 + O(h^2), \tag{2.9a}$$

$$u_0^{(6)} = \left[2\delta w_1^{(4)} - \delta w_2^{(4)} + 2\delta^2 w_2^{(4)} - \delta^2 w_3^{(4)} \right] /h^2 + O(h^4), \tag{2.9b}$$

$$u_N^{(6)} = \left[2\delta w_{N-1}^{(4)} - \delta w_{N-2}^{(4)} \right] /h^2 + O(h^2), \tag{2.9c}$$

$$u_N^{(6)} = \left[2\delta w_{N-1}^{(4)} - \delta w_{N-2}^{(4)} + 2\delta^2 w_{N-2}^{(4)} - \delta^2 w_{N-3}^{(4)} \right] /h^2 + O(h^4), \tag{2.9d}$$

and for $k = 6, 7, 8$

$$u_0^{(k)} = \left[\ 3\delta^2 w_2^{(k-4)} - 2\delta^2 w_3^{(k-4)}\ \right] / h^4 + O(h^2), \tag{2.9e}$$

$$u_1^{(k)} = \left[\ 2\delta^2 w_2^{(k-4)} - \delta^2 w_3^{(k-4)}\ \right] / h^4 + O(h^2), \tag{2.9f}$$

$$u_{N-1}^{(k)} = \left[\ 2\delta^2 w_{N-2}^{(k-4)} - \delta^2 w_{N-3}^{(k-4)}\ \right] / h^4 + O(h^2), \tag{2.9g}$$

$$u_N^{(k)} = \left[\ 3\delta^2 w_{N-2}^{(k-4)} - 2\delta^2 w_{N-3}^{(k-4)}\ \right] / h^4 + O(h^2). \tag{2.9h}$$

Proof: We can first approximate $u_1^{(8)}$, $u_0^{(8)}$, $u_{N-1}^{(8)}$, $u_N^{(8)}$ by $2u_2^{(8)} - u_3^{(8)}$, $2u_1^{(8)} - u_2^{(8)}$, $2u_{N-2}^{(8)} - u_{N-3}^{(8)}$, $2u_{N-1}^{(8)} - u_{N-2}^{(8)}$ respectively and then use Corollary 2.1 to derive (2.9e) to (2.9h) for $k = 8$. Similarly we obtain the relations (2.9e) to (2.9h) for $k = 6,7$. For the derivation of the relations (2.9b), (2.9d) we can approximate $u_0^{(6)}$ and $u_N^{(6)}$ by $2u_1^{(6)} - u_2^{(6)} + u_1^{(8)}\ h^2$ and $2u_{N-1}^{(6)} - u_{N-2}^{(6)} + u_{N-1}^{(8)}\ h^2$ and use the relations of Corollary 2.1 and Lemma 2.2. Finally the relations (2.9a), (2.9c) are obtained by the approximations $2u_1^{(6)} - u_2^{(6)}$ and $2u_{N-1}^{(6)} - u_{N-2}^{(6)}$ to $u_0^{(6)}$ and $u_N^{(6)}$ respectively.

3. AN OPTIMAL QUINTIC SPLINE COLLOCATION METHOD

We consider now the linear fourth order equation $Lu = f$ (3.1a) subject to homogeneous boundary conditions $Bu = 0$ (3.1b). In the rest of the paper we assume that $u \in C^{10}[a, b]$ is the solution to the problem (3.1) and w is the quintic spline interpolant of u defined by (2.1). Based on relations (2.8) and Lemma (2.2) we observe that w satisfies,

$$\begin{aligned} & w_0^{(4)} + \frac{1}{240}\left(\ 40\delta w_1^{(4)} - 20\delta w_2^{(4)} + 37\delta^2 w_2^{(4)} - 18\delta^2 w_3^{(4)}\ \right) + e_0^3\left[\ w_0^{(3)} + \frac{1}{240}\,(3\delta^2 w_2^{(3)} - 2\delta^2 w_3^{(3)})\right] + \\ & \qquad e_0^2\left[\ w_0^{(2)} - \frac{1}{720}\,(3\delta^2 w_2^{(2)} - 2\delta^2 w_3^{(2)})\right] + e_0^1\ w_0^{(1)} + e_0^0\ w_0 = f_0 + O(h^6), \end{aligned} \tag{3.2a}$$

$$\begin{aligned} & w_1^{(4)} + \frac{1}{240}\left[\ 20\delta w_1^{(4)} - 2\delta^2 w_2^{(4)} + \delta^2 w_3^{(4)}\ \right] + e_1^3\left[\ w_1^{(3)} + \frac{1}{240}\,(2\delta^2 w_2^{(3)} - \delta^2 w_3^{(3)})\right] + \\ & \qquad e_1^2\left[\ w_1^{(2)} - \frac{1}{720}\,(2\delta^2 w_2^{(2)} - \delta^2 w_3^{(2)})\right] + e_1^1 w_1^{(1)} + e_1^0 w_1 = f_1 + O(h^6), \end{aligned} \tag{3.2b}$$

$$w_i^{(4)} + \frac{1}{240}\left[\, 20\delta w_i^{(4)} - \delta^2 w_i^{(4)} \right] + e_i^3\left[\, w_i^{(3)} + \frac{\delta^2 w_i^{(3)}}{240} \right] + e_i^2\left[\, w_i^{(2)} - \frac{\delta^2 w_i^{(2)}}{720} \right] +$$
$$e_i^1\, w_i^{(1)} + e_i^0\, w_i = f_i + O(h^6), \text{ for } i = 2(1)N-2, \tag{3.2c}$$

$$w_{N-1}^{(4)} + \frac{1}{240}\left[\, 20\delta w_{N-1}^{(4)} - 2\delta^2 w_{N-2}^{(4)} + \delta^2 w_{N-3}^{(4)} \right] + e_{N-1}^3\left[\, w_{N-1}^{(3)} + \frac{1}{240}\,(2\delta^2 w_{N-2}^{(3)} - \delta^2 w_{N-3}^{(3)}) \right] +$$
$$e_{N-1}^2\left[\, w_{N-1}^{(2)} - \frac{1}{720}\,(\, 2\delta^2 w_{N-2}^{(2)} - \delta^2 w_{N-3}^{(2)}) \right] + e_{N-1}^1\; w_{N-1}^{(1)} + e_{N-1}^0\; w_{N-1} = f_{N-1} + O(h^6), \tag{3.2d}$$

$$w_N^{(4)} + \frac{1}{240}\left[\, 40\delta w_{N-1}^{(4)} - 20\delta w_{N-2}^{(4)} + 37\delta^2 w_{N-2}^{(4)} - 18\delta^2 w_{N-3}^{(4)} \right] + e_N^3\left[\, w_N^{(3)} + \frac{1}{240}\,(3\delta^2 w_{N-2}^{(3)} - 2\delta^2 w_{N-3}^{(3)}) \right] +$$
$$e_N^2\left[\, w_N^{(2)} - \frac{1}{720}\,(3\delta^2 w_{N-2}^{(2)} - 2\delta^2 w_{N-3}^{(2)}) \right] + e_N^1\, w_N^{(1)} + e_N^0 w_N = f_N + O(h^6), \tag{3.2e}$$

and

$$B'w \equiv \alpha_{i0} w_0 + \alpha_{i1}\, w_0' + \alpha_{i2}\left[\, w_0'' - \frac{1}{720}\,(3\delta^2 w_2^{(2)} - 2\delta^2 w_3^{(2)}) \right] + \alpha_{i3}\left[\, w_0^{(3)} + \frac{1}{240}\,(3\delta^2 w_2^{(3)} - 2\delta^2 w_3^{(3)}) \right] +$$
$$\beta_{i0}\, w_N + \beta_{i1} w_N' + \beta_{i2}\left[\, w_N^{(2)} - \frac{1}{720}\,(3\delta^2 w_{N-2}^{(2)} - 2\delta^2 w_{N-3}^{(2)}) \right] + \tag{3.2f}$$
$$\beta_{i3}\left[\, w_N^{(3)} + \frac{1}{240}(3\delta^2 w_{N-2}^{(3)} - 2\delta^2 w_{N-3}^{(3)}) \right] = O(h^6) \text{ for } i = 0(1)3.$$

Let L' denote the perturbation of L defined by the left side of equations (3.2a) to (3.2e). Then we see that w satisfies the relations

$$L'w_i = f_i + O(h^6) \text{ for}, \; 0 \le i \le N \tag{3.3a}$$
$$B'w = O(h^6), \text{ and } Bw = O(h^4). \tag{3.3b}$$

Notice that L' is defined by

$$L'g_i = Lg_i + \frac{1}{240}\left[\, 20\delta g_i^{(4)} - \delta^2 g_i^{(4)} \right] + e_i^3\,\frac{\delta^2 g_i^{(3)}}{240} - e_i^2\,\frac{\delta^2 g_i^{(2)}}{720}, \text{ for } i = 2(1)N-2.$$

The above observations are summarized in the following Lemma.

Lemma 3.1. Let w be the quintic spline interpolant of the solution u to problem (3.1). If $u \in C^{(10)}[a,b]$, then w satisfies the relations

$$[Lw - f]_{s_i} = O(h^2), \text{ for } 0 \le i \le N \quad Bw = O(h^4), \tag{3.4a}$$

$$[L'w - f]_{s_i} = O(h^6) \text{ for } 0 \le i \le N \quad B'w = O(h^6), \tag{3.4b}$$

and

$$Bw = O(h^6) \text{ if } \alpha_{ij} = \beta_{ij} = 0 \text{ for } j = 2, 3. \tag{3.4c}$$

3.1 Formulation of the Quintic Spline Collocation Method

This method is based on determining a quintic spline $z \in Sp_5(\Delta_N)$ which satisfies

$$[L'z - f]_{s_i} = 0 \text{ for } 0 \le i \le N \text{ and } B'z = 0. \tag{3.5}$$

Before we proceed to the main analysis of this method, we must verify the solvability of the equations (3.2) for the case $e^j \equiv 0$, $j = 0(1)3$. For this we denote by Q_4 the coefficient matrix of the system (3.2) and easily prove the following Lemma.

Lemma 3.2 Let $e^j \equiv 0$, for $j = 0(1)3$. Then the system of equations (3.2) is uniquely solvable for $\{z_i^{(4)}\}_1^N$, and the solution vector satisfies $\max\{|z_0^{(4)}|, \ldots, |z_N^{(4)}|\} \le \max\{|f_0|, \ldots, |f_N|\}$. That is, the matrix Q_4 satisfies $||Q_4^{-1}||_\infty \le 1.66$.

3.2 Formulation of a Deferred Correction Quintic Spline Collocation Method

An alternative formulation of the method is to view the determination of z as a three step collocation method. Before we formulate this method we write equations (3.2) in the following form:

$$\begin{aligned} Lw_0 = f_0 + \Big[60\Big(2\delta w_1^{(4)} - \delta w_2^{(4)} + 2\delta^2 w_2^{(4)} - \delta^2 w_3^{(4)} \Big] - 3\Big(3\delta^2 w_2^{(4)} - 2\delta^2 w_3^{(4)} - 3\delta^2 w_2^{(3)} + 2\delta^2 w_3^{(3)} \Big) \\ - 3\delta^2 w_2^{(2)} + 2\delta^2 w_3^{(2)} \Big] /720 + O(h^6), \end{aligned} \tag{3.6a}$$

$$Lw_1 = f_1 + \Big[60\delta w_1^{(4)} - 3\Big(2\delta^2 w_2^{(4)} - \delta^2 w_3^{(4)} - 2\delta^2 w_2^{(3)} + \delta^2 w_3^{(3)} \Big) - 2\delta^2 w_2^{(2)} + \delta^2 w_3^{(2)} \Big] /720 + O(h^6), \tag{3.6b}$$

$$Lw_i = f_i + \Big[60\delta w_i^{(4)} - 3\Big(\delta^2 w_i^{(4)} - \delta^2 w_i^{(3)} \Big) - \delta^2 w_i^{(2)} \Big] /720 + O(h^6), \text{ for } i = 2(1)N-2, \tag{3.6c}$$

$$Lw_{N-1} = f_{N-1} + \Big[\, 60\delta w_{N-1}^{(4)} - 3\Big[\, 2\delta^2 w_{N-2}^{(4)} - \delta^2 w_{N-3}^{(4)} - 2\delta^2 w_{N-2}^{(3)} + \delta^2 w_{N-3}^{(3)} \Big] - 2\delta^2 w_{N-2}^{(2)} + \delta^2 w_{N-3}^{(2)} \Big] /720 + O(h^6), \tag{3.6d}$$

$$Lw_N = f_N + \Big[\, 60\Big(\, 2\delta w_{N-1}^{(4)} - \delta w_{N-2}^{(4)} + 2\delta^2 w_{N-2}^{(4)} - \delta^2 w_{N-3}^{(4)} \Big] - 3\Big(\, 3\delta^2 w_{N-2}^{(4)} - 2\delta^2 w_{N-3}^{(4)} - 3\delta^2 w_{N-2}^{(3)} + 2\delta^2 w_{N-3}^{(3)} \Big] - 3\delta^2 w_{N-2}^{(2)} + 2\delta^2 w_{N-3}^{(2)} \Big] /720 + O(h^6). \tag{3.6e}$$

Now we define the *three step collocation method* as follows:

Step 1: Determine a $\overline{z} \in Sp_5(\Delta_N)$ such that it satisfies

$$[L\overline{z} - f\,]_{s_i} = 0, \text{ for } i = 0(1)N, \quad B\overline{z} = 0. \tag{3.7}$$

Step 2: Find a $\overline{\overline{z}} \in Sp_5(\Delta_N)$ such that it satisfies

$$[L\overline{\overline{z}} - \overline{f}\,]_{s_i} = 0, \text{ for } i = 0(1)N, \quad B\overline{\overline{z}} = 0 \tag{3.8}$$

where $\overline{f}_i$ are defned as follows:

$$\overline{f}_0 = f_0 + \frac{1}{12}\Big(\, 2\delta\overline{z}_1^{(4)} - \delta\overline{z}_2^{(4)} \Big],$$

$$\overline{f}_i = f_i + \frac{1}{12}\,\delta\overline{z}_i^{(4)} \text{ for } i = 1(1)N{-}1, \tag{3.9}$$

and

$$\overline{f}_N = f_N + \frac{1}{12}\Big(\, 2\delta\overline{z}_{N-1}^{(4)} - \delta\overline{z}_{N-2}^{(4)} \Big].$$

In the above formulation we have assumed that the higher derivatives of u can be estimated at $\{s_i\}_0^N$ by

$$u_i^{(6)} = \delta\overline{z}_i^{(4)}/h^2 + O(h^2) \text{ for } i = 1(1)N{-}1,$$
$$u_0^{(6)} = \Big[\, 2\delta\overline{z}_1^{(4)} - \delta\overline{z}_2^{(4)} \Big]/h^2 + O(h^2),$$

and

$$u_N^{(6)} = \Big[\, 2\delta\overline{z}_{N-1}^{(4)} - \delta\overline{z}_{N-2}^{(4)} \Big]/h^2 + O(h^2). \tag{3.10}$$

Step 3: Determine a $z \in Sp_5(\Delta_N)$ such that it satisfies

$$[Lz - \bar{\bar{f}}]_{s_i} = 0, \quad i = 0(1)N, \text{ and } Bz = \bar{\bar{g}} \tag{3.11}$$

where $\bar{\bar{f}}$ and $\bar{\bar{g}}$ are

$$\bar{\bar{f}}_0 = \bar{f}_0 + \left[60\left(2\delta^2\overline{\overline{z}}_2^{(4)} - \delta^2\overline{\overline{z}}_3^{(4)} \right) - 3\left(3\delta^2\overline{\overline{z}}_2^{(4)} - 2\delta^2\overline{\overline{z}}_3^{(4)} - 3\delta^2\overline{\overline{z}}_2^{(3)} + 2\delta^2\overline{\overline{z}}_3^{(3)} \right) - 3\delta^2\overline{\overline{z}}_2^{(2)} + 2\delta^2\overline{\overline{z}}_3^{(2)} \right] / 720,$$

$$\bar{\bar{f}}_1 = \bar{f}_1 + \left[-3\left(2\delta^2\overline{\overline{z}}_2^{(4)} - \delta^2\overline{\overline{z}}_3^{(4)} - 2\delta^2\overline{\overline{z}}_2^{(3)} + \delta^2\overline{\overline{z}}_3^{(3)} \right) - 2\delta^2\overline{\overline{z}}_2^{(2)} + \delta^2\overline{\overline{z}}_3^{(2)} \right] / 720,$$

$$\bar{\bar{f}}_i = \bar{f}_i + \left[-3\left(\delta^2\overline{\overline{z}}_i^{(4)} - \delta^2\overline{\overline{z}}_i^{(3)} \right) - \delta^2\overline{\overline{z}}_i^{(2)} \right] / 720 \quad \text{for } i = 2(1)N-2,$$

$$\bar{\bar{f}}_{N-1} = \bar{f}_{N-1} + \left[-3\left(2\delta^2\overline{\overline{z}}_{N-2}^{(4)} - \delta^2\overline{\overline{z}}_{N-3}^{(4)} - 2\delta^2\overline{\overline{z}}_{N-2}^{(3)} + \delta^2\overline{\overline{z}}_{N-3}^{(3)} \right) - 2\delta^2\overline{\overline{z}}_{N-2}^{(2)} + \delta^2\overline{\overline{z}}_{N-3}^{(2)} \right] / 720, \tag{3.12a}$$

$$\bar{\bar{f}}_N = \bar{f}_N + \left[60\left[2\delta^2\overline{\overline{z}}_{N-2}^{(4)} - \delta^2\overline{\overline{z}}_{N-3}^{(4)} \right] - 3\left(3\delta^2\overline{\overline{z}}_{N-2}^{(4)} - 2\delta^2\overline{\overline{z}}_{N-3}^{(4)} - 3\delta^2\overline{\overline{z}}_{N-2}^{(3)} + 2\delta^2\overline{\overline{z}}_{N-3}^{(3)} \right) - 3\delta^2\overline{\overline{z}}_{N-2}^{(2)} + 2\delta^2\overline{\overline{z}}_{N-3}^{(2)} \right] / 720,$$

$$\bar{\bar{g}}_0 = \frac{1}{720}\left(3\delta^2\overline{\overline{z}}_2^{(2)} - 2\delta^2\overline{\overline{z}}_3^{(2)} \right) a_{i2} + a_{i3}\frac{1}{240}\left(3\delta^2\overline{\overline{z}}_2^{(3)} - 2\delta^2\overline{\overline{z}}_3^{(3)} \right], \quad i = 0(1)3$$

and

$$\bar{\bar{g}}_N = \frac{1}{720}\left(3\delta^2\overline{\overline{z}}_{N-2}^{(2)} - 2\delta^2\overline{\overline{z}}_{N-3}^{(2)} \right) b_{i2} + b_{i3}\frac{1}{240}\left(3\delta^2\overline{\overline{z}}_{N-2}^{(3)} - 2\delta^2\overline{\overline{z}}_{N-3}^{(3)} \right] \quad i = 0(1)3 .$$

For the formulation of (3.11) we use the following relations for $k = 6, 7, 8$

$$u_i^{(k)} = \delta^2\overline{\overline{z}}_i^{(k-4)}/h^4 + O(h^2) \quad i = 2(1)N-2,$$

$$u_0^{(k)} = \left[3\delta^2\overline{\overline{z}}_2^{(k-4)} - 2\delta^2\overline{\overline{z}}_3^{(k-4)} \right] / h^4 + O(h^2),$$

$$u_1^{(k)} = \left[2\delta^2\overline{\overline{z}}_2^{(k-4)} - \delta^2\overline{\overline{z}}_3^{(k-4)} \right] / h^4 + O(h^2), \tag{3.12b}$$

$$u_{N-1}^{(k)} = \left[2\delta^2\overline{\overline{z}}_{N-2}^{(k-4)} - \delta^2\overline{\overline{z}}_{N-3}^{(k-4)} \right] / h^4 + O(h^2),$$

and

$$u_N^{(k)} = \left[3\delta^2\overline{\overline{z}}_{N-2}^{(k-4)} - 2\delta^2\overline{\overline{z}}_{N-3}^{(k-4)} \right] / h^4 + O(h^2).$$

4. CONVERGENCE ANALYSIS AND ERROR BOUNDS

Before we proceed to analyze the quintic spline collocation method, we have to introduce some notation in order to represent the equations (1.1), (3.5) in an integral form. First we assume that the boundary value problem $u^{(4)} = 0$, $Bu = 0$ is uniquely solvable. This means that there is a Green's function $G(s,t)$ for this problem. If we denote by $v_N = z^{(4)}$, $v = u^{(4)}$ and assume that z and v satisfy the boundary conditions, then v_N and v can be obtained via the Green's function. Specifically we have that

$$z^{(m)}(s) = \int_a^b \frac{\partial^m G(s,t)}{\partial s^m} \, v_N(t)dt, \quad u^{(m)}(s) = \int_a^b \frac{\partial^m G(s,t)}{\partial s^m} \, v(t)dt \quad m = 0(1)3.$$

Furthermore we define the following operators:

D_N : $C[a,b] \to R^{N+1}$, $(D_N g)_i = g(s_i)$ for $0 \le i \le N$,

M_N : $R^{N+1} \to C[a,b]$ via piecewise linear interpolation at $\{s_i\}_0^N$,

K : $C[a,b] \to C[a,b]$ such that $Kg(s) \equiv \sum_{i=0}^{3} e^i(s) \int_a^b \frac{\partial^i G(s,t)}{\partial s^i} \, g(t)dt.$

4.1 Convergence Analysis of Quintic Spline Collocation Method

In this section, we prove the convergence of the first formulation of the quintic spline collocation method. For this, we introduce the operator R defined as

$$Rg = E_3 Q_3 D_N \int_a^b \frac{\partial^3 G(s,t)}{\partial s^3} \, g(t)dt + E_2 Q_2 D_N \int_a^b \frac{\partial^2 G(s,t)}{\partial s^2} \, g(t)dt +$$

$$E_1 D_N \int_a^b \frac{\partial G(s,t)}{\partial s} \, g(t)dt + E_0 D_N \int_a^b G(s,t) g(t)dt$$

where $E_i = \mathrm{diag}(e^i(s_m))$ is an $(N+1) \times (N+1)$ diagonal matrix for $i = 0(1)3$ and Q_i is a 5-diagonal matrix $(N+1) \times (N+1)$ with respect to $\underline{z}^{(i)} = \left[\, z_0^{(i)}, z_1^{(i)}, \ldots, z_N^{(i)} \right]^T$ for $i = 2, 3$.

According to this notation equations (3.1) and (3.5) can be written in the form

$$(I + K)v = f \tag{4.1a}$$

and

$$Q_4 D_N v_N + R v_N = D_N f \tag{4.1b}$$

respectively.

Since Q_4 is non-singular then equation (4.1b) can be written equivalently as

$$(I + M_N Q_4^{-1} R) v_N = P_N f \tag{4.1c}$$

where $P_N \equiv M_N Q_4^{-1} D_N$.

Notice thet P_N is an operator that maps $C[a,b]$ onto the continuous piecewise linear functions with breakpoints $\{s_i\}_0^N$. Further we have from [1] that the sequence of operators P_N converges strongly to the identity operator $I: C[a,b] \to C[a,b]$, that is $||P_N g - g||_\infty \to 0$ for each fixed $g \in C[a,b]$. In order to prove convergence we need the following result.

Lemma 4.1. If g is a continuous function, then we have that $||M_N Q_4^{-1} Rg - Kg||_\infty \to 0$ that is $M_N Q_4^{-1} R$ converges to K.

Proof: According to the definition of R and K we easily obtain

$$\begin{aligned} ||M_N Q_4^{-1} Rg - Kg||_\infty &\le ||M_N Q_4^{-1} Rg - M_N D_N Kg||_\infty + ||M_N D_N Kg - Kg||_\infty \\ &\le ||M_N||_\infty \, ||Q_4^{-1}||_\infty \, ||Rg - Q_4 D_N Kg||_\infty + ||M_N D_N Kg - Kg||_\infty = \\ &\quad 1.66 \, ||Rg - Q_4 D_N Kg||_\infty + O(h^2) \end{aligned} \tag{4.2}$$

since $||M_N||_\infty = 1$ and $||Q_4^{-1}||$ is bounded. If we evaluate the term $(Rg - Q_4 D_N Kg)$ at the points $\{s_i\}_0^N$ we can show that the equation at each point s_i can be dominated by the modules of continuity of the continuous functions of $r^{(m)} \equiv \int_a^b \frac{\partial^m G(\cdot,t)}{\partial s^m} g(t)dt$, $m = 0(1)3$ over a width of $5h$, so $||Rg - Q_4 D_N Kg||_\infty$ tends to zero. That is

$$(Rg - Q_4 D_N Kg)_0 = \frac{e_0^3}{240}\left[-74 r_0^{(3)} + 252 r_1^{(3)} - 348 r_2^{(3)} + 252 r_3^{(3)} - 98 r_4^{(3)} + 16 r_5^{(3)} \right] + O(h)$$

$$+ \frac{e_0^2}{720} \left[-234 r_0^{(2)} + 812 r_1^{(2)} - 1148 r_2^{(2)} + 852 r_3^{(2)} - 338 r_4^{(2)} + 56 r_5^{(2)} \right] + O(h)$$

$$+ \frac{e_0^1}{240} \left[-77 r_0^{(1)} + 266 r_1^{(1)} - 374 r_2^{(1)} + 276 r_3^{(1)} - 109 r_4^{(1)} + 18 r_5^{(1)} \right]$$

$$+ \frac{e_0^0}{240} \left[-77 r_0 + 266 r_1 - 374 r_2 + 276 r_3 - 109 r_4 + 18 r_5 \right] = O(5h)$$

$$(R_g - Q_4 D_N Kg)_1 = \frac{e_1^3}{240} \left[-16 r_0^{(3)} + 22 r_1^{(3)} + 12 r_2^{(3)} - 28 r_3^{(3)} + 12 r_4^{(3)} - 2 r_5^{(3)} \right] + O(h)$$

$$+ \frac{e_1^2}{720} \left[-56 r_0^{(2)} + 102 r_1^{(2)} - 28 r_2^{(2)} - 28 r_3^{(2)} + 12 r_4^{(2)} - 2 r_5^{(2)} \right] + O(h)$$

$$+ \frac{e_1^1}{240} \left[-18 r_0^{(1)} + 31 r_1^{(1)} - 4 r_2^{(1)} - 14 r_3^{(1)}) + 6 r_4^{(1)} - r_5^{(1)} \right]$$

$$+ \frac{e_1^0}{240} \left[-18 r_0 + 31 r_1 - 4 r_2 - 14 r_3 + 6 r_4 - r_5 \right] = O(5h)$$

$$(R_g - Q_4 D_N Kg)_i = \frac{e_i^3}{240} \left[2 r_{i-2}^{(3)} - 28 r_{i-1}^{(3)} + 52 r_i^{(3)} - 28 r_{i+1}^{(3)} + 2 r_{i+2}^{(3)} \right] + O(h)$$

$$+ \frac{e_i^2}{720} \left[2 r_{i-2}^{(2)} - 68 r_{i-1}^{(2)} + 132 r_i^{(2)} - 68 r_{i+1}^{(2)} + 2 r_{i+2}^{(2)} \right] + O(h)$$

$$+ \frac{e_i^1}{240} \left[r_{i-2}^{(1)} - 24 r_{i-1}^{(1)} + 46 r_i^{(1)} - 24 r_{i+1}^{(1)} + r_{i+2}^{(1)} \right]$$

$$+ \frac{e_i^0}{240} \left[r_{i-2} - 24 r_{i-1} + 46 r_i - 24 r_{i+1} + r_{i+2} \right] = O(5h), \;\; i = 2(1)N-2$$

$$(R_g\ Q_4 D_N Kg)_{N-1} = \frac{e_{N-1}^3}{240} \left[-2 r_{N-5}^{(3)} + 12 r_{N-4}^{(3)} - 28 r_{N-3}^{(3)} + 12 r_{N-2}^{(3)} + 22 r_{N-1}^{(3)} - 16 r_N^{(3)} \right] + O(h)$$

$$+ \frac{e_{N-1}^2}{720} \left[-2 r_{N-5}^{(2)} + 12 r_{N-4}^{(2)} - 28 r_{N-3}^{(2)} - 28 r_{N-2}^{(2)} + 102 r_{N-1}^{(2)} - 56 r_N^{(2)} \right] + O(h)$$

$$+ \frac{e_{N-1}^1}{240} \left[-r_{N-5}^{(1)} + 6 r_{N-4}^{(1)} - 14 r_{N-3}^{(1)} - 4 r_{N-2}^{(1)} + 31 r_{N-1}^{(1)} - 18 r_N^{(1)} \right]$$

$$+\frac{e_{N-1}^{0}}{240}\left[\,-r_{N-5}+6r_{N-4}-14r_{N-3}-4r_{N-2}+31r_{N-1}-18r_{N}\,\right]=O(5h)$$

$$(R_g-Q_4 D_N\, Kg)_N=\frac{e_N^3}{240}\left[\,16r_{N-5}^{(3)}-98r_{N-4}^{(3)}+252r_{N-3}^{(3)}-348r_{N-2}^{(3)}+252r_{N-1}^{(3)}-74r_{N}^{(3)}\right]+O(h)$$

$$+\frac{e_N^2}{720}\left[\,56r_{N-5}^{(2)}-338r_{N-4}^{(2)}+852r_{N-3}^{(2)}-1148r_{N-2}^{(2)}+812r_{N-1}^{(2)}-234r_{N}^{(2)}\right]+O(h)$$

$$+\frac{e_N^1}{240}\left[\,18r_{N-5}^{(1)}-109r_{N-4}^{(1)}+276r_{N-3}^{(1)}-374r_{N-2}^{(1)}+266r_{N-1}^{(1)}-77r_{N}^{(1)}\right]$$

$$+\frac{e_N^0}{240}\left[\,18r_{N-5}-109r_{N-4}+276r_{N-3}-374r_{N-2}+266r_{N-1}-77r_{N}\,\right]=O(5h)$$

This completes the proof of the Lemma.

Next we present the main convergence Theorem.

Theorem 4.1. If we assume that

(a1) the coefficients e^i, $i=0(1)3$ and f are continuous in $I=[a,b]$,

(a2) the boundary value problem (1.1) has a unique solution u in $C^{10}[a,b]$,

(a3) the problem $u^{(4)}=0$, $Bu=0$, is uniquely solvable,

then

(r1) the collocation approximation $z\in Sp_5(\Delta_N)$ defined by equations (3.5) exists,

(r2) (case 1) If a_{ij}, b_{ij}, $i=0(1)3$, $j=2,3$ differ from zero, then we have $||(u-z)^{(m)}||_\infty=O(h^4)$ $m=0,1,2$, $||(u-z)^{(k)}||_\infty=O(h^{6-k})$, $k=3,4$ and $|(u-z)_i^{(m)}|=O(h^4)$ $m=0(1)3$ and $|(u-z)_i^{(4)}|=O(h^2)$.

(r3) (case 2) If $a_{ij}\equiv b_{ij}=0$, $i=0(1)3$ $j=2,3$ then we have the optimal estimates $||(u-z)^{(m)}||_\infty=O(h^{6-m})$, $m=0(1)4$ $|(u-z)_i^{(m)}|=O(h^6)$, $m=0,1$, $|(u-z)_i^{(m)}|=O(h^4)$, $m=2,3$ and $|(u-z)_i^{(4)}|=O(h^2)$.

Proof: The assumption (a3) implies that $(I+K)^{-1}$ exists and it is a bounded linear operator. From Lemma 4.1 we have that $(I+M_N Q_4^{-1}R)^{-1}$ exists and it is bounded. This concludes (r1). Now, we consider the problem $w^{(4)}=z_N$, $Bw=O(h^4)$ for (case 1), and the problem $w^{(4)}=z_N$ $Bw=O(h^6)$ for (case 2). From (a3) it follows that there is a cubic polynomial $\bar{z}$ such that

$$B\bar{z}=Bw=O(h^4),\ \ ||\bar{z}^{(k)}||_\infty=O(h^4)\ \ k=0(1)3\quad (case\ 1) \tag{4.3a}$$

and

$$B\bar{z} = Bw = O(h^6), \ \ ||\bar{z}^{(k)}||_\infty = O(h^6) \ \ k = 0(1)3 \quad (case 2). \tag{4.3b}$$

From the solvability of $(w - \bar{z})^{(4)} = z_N$, $B(w - \bar{z}) = 0$, (a3), and equations (3.4 b,c) we have, for both cases, that

$$(I + M_N Q_4^{-1} R)(w^{(4)} - \bar{z}^{(4)}) = P_N f + O(h^6). \tag{4.4}$$

Subtracting (4.4) and (4.1c) we have

$$(I + M_N Q_4^{-1} R)(w^{(4)} - \bar{z}^{(4)} - z^{(4)}) = O(h^6). \tag{4.5}$$

This implies that

$$||w^{(4)} - \bar{z}^{(4)} - z^{(4)}||_\infty = O(h^6) \tag{4.6}$$

because of the boundedness of $(I + M_N Q_4^{-1} R)^{-1}$. Since $(w - \bar{z} - z)^{(4)} = r_N$, $B(w - \bar{z} - z) = 0$ is uniquely solvable (a3) we have

$$(w - \bar{z} - z)(s) = \int_a^b G(s,t)(w^{(4)} - \bar{z}^{(4)} - z^{(4)})(t)dt \tag{4.7}$$

Using equations (4.6) and (4.7) we obtain

$$||(w - \bar{z} - z)^{(m)}||_\infty = O(h^6) \quad m = 0(1)3$$

From (4.3a,b), Theorem 2.1 and the use of triangle inequality, we obtain results (r2) and (r3). This completes the proof.

4.2 Convergence Analysis and Error Bounds For The Three Step Collocation Method

Before we present the convergence of the three steps defined in Section 3.2, we must introduce an integral representation of equations (3.7), (3.8) and (3.11). For this we assume that the boundary value problem $u^{(4)} = 0$, $Bu = 0$ is uniquely solvable. That is there is a Green's function $G(s,t)$ for this problem.

If we denote by $\bar{z}^{(4)} = p_N$, $\bar{\bar{z}}^{(4)} = q_N$, $z^{(4)} = r_N$ and assume that $\bar{z}$, $\bar{\bar{z}}$ and z satisfy the boundary conditions (1.1b), then we can obtain p_N, q_N, and r_N via the Green's function.

Using the notation introduced in Section 4, we can write equations (3.7), (3.8) and (3.11) as

$$(I + P_N K)p_N = P_N f, \tag{4.8}$$

$$(I + P_N K)q_N = P_N \bar{f}, \tag{4.9}$$

and

$$(I + P_N K)r_N = P_N \bar{\bar{f}}, \tag{4.10}$$

respectively. This is possible since $P_N = M_N Q_N^{-1} D_N$ is the linear projection which maps continuous functions to $Sp_1(\Delta_N)$ via piecewise interpolation at $\{s_i\}_0^N$, and $P_N p_N = p_N$, $P_N q_N = q_N$ and $P_N r_N = r_N$.

From the definition of P_N it is clear that $||P_N g - g||_\infty$ tends to zero as h converges to zero for any $g \in C[a,b]$. Thus $||P_N K - K||_\infty$ tends to zero since K is a completely continuous operator. This implies that $(I + P_N K)^{-1}$ exists and it is uniformly bounded for sufficiently small h.

First we proceed to present the convergence of Step 1.

Theorem 4.2. Under the assumptions of Theorem 4.1, we have

(r1) the collocation approximation $\bar{z} \in Sp_5(\Delta_N)$ defined by equations (3.7) exists,

(r2) the global error estimates
$||(u - \bar{z})^{(m)}||_\infty = O(h^2)$ $m = 0(1)4$,

and

(r3) the local error estimates
$|(u - \bar{z})^{(m)}|_* = O(h^2)$ $m = 0(1)4$.

Proof: The result (r1) is a direct consequence of (a3) and the uniform boundedness of $(I + P_N K)^{-1}$. Next we consider the problem $w^{(4)} = y_N$, $Bw = O(h^4)$. From (a3) it follows that there is a cubic function c such that

$$Bc = Bw = O(h^4) \quad ||c^{(m)}||_\infty = O(h^4) \quad m = 0(1)3. \tag{4.11}$$

From the solvability of $(w - c)^{(4)} = y_N$, $B(w - c) = 0$ and the equations (3.4a) we conclude that

$$(I + P_N K)(w^{(4)} - c^{(4)}) = P_N f + O(h^2) \tag{4.12}$$

Subtracting (4.12) and (4.8) we have

$$(I + P_N K)(w^{(4)} - c^{(4)} - \bar{z}^{(4)}) = O(h^2). \tag{4.13}$$

This implies that

$$||w^{(4)} - c^{(4)} - \bar{z}^{(4)}||_\infty = O(h^2).$$

Since $(w - c - \bar{z})^{(4)} = x_N$, $B(w - c - \bar{z}) = 0$ is uniquely solvable we have

$$(w - c - \bar{z})(s) = \int_a^b G(s,t)(w^{(4)} - c^{(4)} - \bar{z}^{(4)})(t)dt.$$

Therefore we have

$$||(w - c - \bar{z})^{(m)}||_\infty = O(h^2) \qquad m = 0(1)3. \tag{4.14}$$

Thus, (r2) and (r3) follow from Theorem 2.1 and equations (4.11). This completes the proof of Theorem.

From relations (4.13), (4.14) we obtain $\delta w_i^{(4)} = \delta \bar{z}_i^{(4)} + O(h^4)$ provided the asymptotic terms are smooth enough. So equations (3.6) become

$$[Lw - \bar{f}]_i = O(h^4), \quad 0 \le i \le N, \text{ with } Bw = O(h^4). \tag{4.15}$$

Next, we subtract relations (3.8) and (4.15) to obtain

$$L(w - \bar{\bar{z}})(s_i) = O(h^4), \quad 0 \le i \le N \text{ and } B(w - \bar{\bar{z}}) = O(h^4). \tag{4.16a}$$

We consider the problem $(w - \bar{\bar{z}})^{(4)} = r_N$, $B(w - \bar{\bar{z}}) = O(h^4)$. Notice that there is a cubic function c such that

$$B(w - \bar{\bar{z}}) = Bc = O(h^4), \quad ||c^{(m)}||_\infty = O(h^4) \quad m = 0(1)3.$$

From (a3) and $B(w - \bar{\bar{z}} - c) = 0$ we can write equations (4.16a) as

$$(I + P_N K)(w^{(4)} - \bar{\bar{z}}^{(4)} - c^{(4)}) = O(h^4) \text{ and } (w - \bar{\bar{z}} - c)^{(m)} = O(h^4), \ m = 0(1)3 \tag{4.16b}$$

which imply the estimates

$$||(w - \bar{\bar{z}} - c)^{(m)}|| = O(h^4), \quad m = 0(1)4. \tag{4.16c}$$

Continuing as in Theorem 4.2 we have the following results

Theorem 4.3. Under the hypotheses of Theorem 4.1, we conclude that,

(r1) the collocation approximation $\bar{\bar{z}}$ exists,

(r2) $||(u - \bar{\bar{z}})^{(m)}||_\infty = O(h^4)$ $m = 0(1)2$ and $||(u - \bar{\bar{z}})^{(m)}|| = O(h^{6-m})$ $m = 3,4$,

(r3) $|(u - \bar{\bar{z}})^{(m)}|_* = O(h^4)$ $m = 0(1)4$.

Now we proceed to show convergence of the third step of this method. For this, we use the relations (4.16b), assuming that the asymptotic terms are sufficiently smooth, to conclude that

$$\delta^2 w_i^{(4)} = \delta^2 \bar{\bar{z}}_i^{(4)} + O(h^6), \quad \delta^2 w_i^{(3)} = \delta^2 \bar{\bar{z}}_i^{(3)} + O(h^6), \quad \delta^2 w_i^{(2)} = \delta^2 \bar{\bar{z}}_i^{(2)} + O(h^6).$$

So we can write equations (3.6) as

$$[Lw - \bar{\bar{f}}]_* = O(h^6), \text{ for } i = 0(1)N, \text{ with } [Bw - \bar{\bar{g}}] = O(h^6). \tag{4.17}$$

Subtracting equations (3.11) and (4.17) we obtain

$$L(w - z)_* = O(h^6), \text{ for } i = 0(1)N, \; B(w - z) = O(h^6). \tag{4.18}$$

Notice that there is a cubic function c such that

$$B(w - z) = Bc = O(h^6) \text{ and } ||c^{(m)}||_\infty = O(h^4) \quad m = 0(1)3. \tag{4.19}$$

Since $B(w - z - c) = 0$ and (a3) holds, equations (4.18) can be written as

$$(I + P_N K)(w^{(4)} - z^{(4)} - c^{(4)}) = O(h^6).$$

Thus, we have the relation

$$||(w - z - c)^{(4)}||_\infty = O(h^6).$$

Applying the same arguments as before, we obtain the following optimal results.

Theorem 4.4 Under the hypotheses of Theorem (4.1) we have

(r1) the collocation approximation z exists,

(r2) $||(u-z)^{(4)}||_\infty = O(h^{6-m}) \quad m = 0(1)4,$

(r3) $|(u-z)^{(m)}|_\pi = O(h^6) \;\; m=0,1, \;\; |(u-z)^{(m)}|_\pi = O(h^4) \;\; m=2,3, \;\; |(u-z)^{(4)}|_\pi = O(h^2)$

5. NUMERICAL RESULTS

In this section, we present the computational behavior of the optimal quintic spline collocation method formulated in section §3.1. The numerical realization of this method is referred throughout as P5C4COL (order = 6) and it is applicable for non homogeneous boundary conditions. For comparison purposes, we have implemented the standard nodal quintic spline collocation which we refer as P5C4COL (order = 2). The software P5C4COL is tested for a number of fourth order problems that have been used to test other methods. Table 1 to 5 present errors and estimated order of convergence for five problems. These data indicate agreement with the theoretical analysis of the method. The timing of the method is presented in Table 6. All computations were carried out on a VAX 8600 in double precision.

PROBLEM 1:

This example was used in [8] to verify the convergence of collocation methods based on quintic and sextic splines. The equation is

$$u^{(4)} + su = -(8 + 7s + x^3)e^s, \quad s \in [0,1]$$

subject to boundary conditions

$$u(0) = u(1) = 0, \;\; u'(0) = 1 \;\; \text{and} \;\; u'(1) = -e \; .$$

Its exact solution is $u(s) = s(1-s)e^s$. Table 1 indicates the convergence of P5C4COL and the methods developed in [8]. The error is the maximum of the absolute errors at the nodes

Mesh	P5C4COL (order = 6)		P5C4COL (order = 2)	Sextic [8]
h	Error	Convergence	Error	Error
1/8	3.31–8		1.72–4	2.22–7
		5.94		
1/16	2.13–10		4.29–5	1.37–8
		5.99		
1/32	3.34–12		1.07–5	8.54–10
		3.33		
1/64	3.33–13			

Table 1. Errors and estimated order of convergence for P5C4COL. The errors of the method reported in [8] are included for comparison.

PROBLEM 2:

This example was obtained from a nonlinear problem considered in [8]. The equation is

$$u^{(4)} + 24(1+s)^{-3}u' - 72(1+s)^{-4}u = 0, \quad s \in [0,1]$$

subject to boundary conditions

$$u(0) = g_1,\ u'(0) = g_2,\ u(1) = g_3 \text{ and } u'(1) = g_4 \tag{5.1a}$$

or

$$u(0) = g_1,\ u''(0) = g_2,\ u(1) = g_3 \text{ and } u''(1) = g_4, \tag{5.1b}$$

where the g_i's are determined by the exact solution of the differential equation $u(s) = 4/(1+s)^2$. Table 2 indicates the convergence of P5C4COL under the two sets of boundary conditions.

Mesh	P5C4COL (order = 6)				P5C4COL (order = 2)	
	Boundary Conditions (5.1a)		Boundary Conditions (5.1b)		Boundary Conditions (5.1b)	
h	Error	Order of Convergence	Error	Order of Convergence	Error	Order of Convergence
1/8	2.06–5		8.61–5		3.36–2	
		5.34		5.90		2
1/16	5.10–7		1.44–6		8.25–3	
		5.77		5.92		2
1/32	9.33–9		2.38–8		2.06–3	
		5.97		5.20		2
1/64	1.49–10*		6.47–10*		5.13–4	

* Round-off error effect

Table 2. Error of P5C4COL and estimated order of convergence for Problem 2 under boundary conditions (5.1a) and (5.1b).

PROBLEM 3:

This is an example of a full fourth order operator with constant and non constant coefficients. The operators are:

$$L_1 u \equiv u^{(4)} + u^{(3)} + u^{(2)} + u^{(1)} + u$$

and

$$L_2 u \equiv u^{(4)} + 3s^3 u^{(3)} + 2e^s u^{(2)} + u^{(1)} + s^{1/2} u$$

with boundary conditions

$$u(0) = g_1, \;\; u'(0) = g_2, \;\; u''(1) = g_3 \;\text{ and }\; u'''(1) = g_4 \,.$$

The right sides of the differential equations and boundary conditions have been selected so that $u = \exp(x)$.

Table 3 indicates the convergence of P5C4COL.

Mesh	P5C4COL (order = 6)				P5C4COL (order = 2)	
	L_1 operator		L_2 operator		L_1 operator	L_2 operator
h	Error	Order of Convergence	Error	Order of Convergence	Error	Error
1/8	6.67–9		6.22–9		2.79–4	2.40–4
		6.48		6.34		
1/16	7.46–11		7.66–11		7.70–5	6.69–5
		2.95		3.66		
1/32	9.68–12*		6.08–12*		2.01–5	1.76–5

* Round off error effect

Table 3. Errors and estimated order of convergence of P5C4COL for Problem 3.

PROBLEM 4:

This example was considered in [7]. The equation is:

$$u^{(4)} = (s^4 + 14s^3 + 49s^2 + 32s - 12)e^s \tag{5.2a}$$

subject to boundary conditions

$$u(0) = u'(0) = u(1) = u'(1) = 0. \tag{5.2b}$$

Its true solution is $u(s) = s^2(1-s)^2e^s$. The problem corresponds to the bending of a thin beam clamped at both ends. Table 4 indicates the convergence of P5C4COL. Notice that P5C4COL (order = 6) can not run with $h \geq 1/8$, because of the length of various stencils involved.

Mesh	P5C4COL (order = 6)		P5C4COL (order = 2)
h	Error	Order of Convergence	Error
1/8	3.80–7		2.09–3
		5.92	
1/16	6.26–9		5.22–4
		5.99	
1/32	9.86–11		1.31–4
		6.04	
1/64	1.50–12		

Table 4. Errors and estimated order of convergence of P5C4COL for Problem 4.

PROBLEM 5:

This example was adopted from Problem 4 in [7], which describes the symmetrical bending of a laterally loaded circular plate. The governing equation is

$$u^{(4)} + u^{(3)} + \frac{1}{s}\, u^{(2)} - \frac{1}{s^2}\, u^{(1)} = f, \quad s \in [1,2]$$

subject to boundary conditions

$$u(1) = g_1,\ u'(1) = g_2,\ u(2) = g_3, \text{ and } u'(2) = g_4 .$$

The right sides of the equation and boundary conditions are determined so that the exact solution is

$$u(s) = \frac{s^2}{4}\left[\ln \frac{s}{2} - 1 \right] - \frac{s^2}{8}\left[\frac{0.7}{1.3} + \frac{2}{3}\ln 2 \right] - \frac{2.6}{2.1}\ln 2\, \ln\frac{s}{2} + \frac{3.3}{2.6} + \frac{\ln 2}{3} .$$

Table 5 presents the convergence of P5C4COL and Table 6 indicates the time requirements of P5C4COL (order = 6) on a VAX 8600 for the same problem.

Mesh	P5C4COL (order = 6)		P5C4COL (order = 2)
h	Error	Order of Convergence	Error
1/8	8.30–8		4.00–5
		5.90	
1/16	1.93–9		1.02–5
		5.37	
1/32	3.37–11		2.54–6
		8.00	
1/64	1.32–13		

Table 5. Errors and estimates of order of convergence of method P5C4COL for Problem 5.

h	Equation Generation Time (sec)	Solution Time (sec)	Total Time (sec)
1/32	0_0	.0333	.0666
1/64	.0167	.0667	.1
1/128	.0333	.1167	.1833
1/256	.0666	.25	.3667
1/512	.15	.483	.7
1/1024	.25	.9833	1.333

Table 6. Breakdown timing of P5C4COL (order = 6) software for generating the quintic spline collocation equations and their direct solution using a nonsymmetric band solver from LINPACK.

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