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Improvement In Pulse Transmission On Coaxial Transmission Lines By Reduction Of Skin Effect

C. M. Evans
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C. M. Evans and L. R. Whicker

August, 1963

Lafayette, Indiana
IMPROVEMENT IN PULSE TRANSMISSION
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I

INTRODUCTION

1.1 General Considerations

The problem of determining the response at a load at one end of a transmission line to a pulse source at the other end can be determined from the response to an impulse source by convolution or by means of Fourier or Laplace transforms. Since, in all but the most trivial cases, the impulse response is most readily obtained by means of transforms, it seems appropriate to sketch briefly this method of attack.

The transfer function may be expressed as a somewhat involved function of the impedances of the source and load, the length of line, and the propagation factor and characteristic impedance of the line. As is shown in Appendix A, the criteria for undistorted transmission of a signal are that the magnitude of the transfer function be independent of frequency and that the phase of the transfer function vary linearly with frequency. Hence, it is desired to design a system by proper choice of the above parameters to approximate these conditions.

It is, however, desirable to consider a much less general problem for an initial investigation. In the following discussion it will be assumed that the source and load are isolated so that multiple reflections may be neglected. In such a case the transfer function is determined primarily by the propagation factor and secondarily by the characteristic impedance and load impedance.
Since the frequency dependence of the propagation factor is usually adversely affected by dielectric losses, this investigation will be restricted to air dielectric lines. For such lines the propagation factor ($\gamma$) and the characteristic impedance ($Z_c$) are given respectively by

$$\gamma = \sqrt{j\omega C (Z_s + j\omega L_e)} ,$$

and

$$Z_c = \sqrt{\frac{Z_s + j\omega L_e}{j\omega C}} ,$$

where $Z_s$ is the surface (internal) impedance per unit length of the conductors, $\omega$ is the frequency, and $L_e$ and $C$ are respectively the external inductance and capacitance per unit length between the conductors. In most cases the surface impedance has a resistive part and an inductive part so that the above expressions may be written as

$$\gamma = \sqrt{j\omega C (R + j\omega L)} ,$$

and

$$Z_c = \sqrt{R + j\omega L \over j\omega C} ,$$

where $R$ is the resistive part of $Z_s$ and $L$ is the sum of the inductive part of $Z_s$ and $L_e$.

In any line of practical interest the real part of the propagation factor must be small which implies that $R$ must be much less than $\omega L$. Hence, the expressions for $\gamma$ and $Z_c$ may be expanded binomially to give

$$\gamma = j\omega \sqrt{L C} \left[ 1 - j {\frac{R}{2 \omega L}} + \frac{1}{8} \left( \frac{R}{\omega L} \right)^2 + \frac{j}{16} \left( \frac{R}{\omega L} \right)^3 + \ldots \right] ,$$
and

\[ Z_c = \sqrt{\frac{L}{C}} [1 - j \frac{R}{\omega L} \ldots ] \quad (1.6) \]

It should be noted that these expansions are valid at high frequencies only and further that the frequency dependence of the various terms in the two series is not obvious unless \( R, L, \) and \( C \) are constant.

1.2 Step Response of Conventional Coaxial Line

In order to demonstrate the use of the method and to illustrate the degradation of wave shape which often occurs, consider a unit step voltage source applied to an infinitely long coaxial line which has a solid inner conductor of radius \( r_1 \) and a thick outer conductor with inner radius \( r_2 \). It is shown by Wigington and Nahmien that for such a line that \( C \) is constant and further that

\[ R = K_1 \sqrt{\omega} \quad (1.7) \]

and

\[ L = L_e + \frac{K_1}{\sqrt{\omega}} \quad (1.8) \]

where

\[ K_1 = \left( \frac{1}{2\pi r_1} + \frac{1}{2\pi r_2} \right) \sqrt{\frac{\mu}{2\sigma}} \quad (1.9) \]

and \( \sigma \) and \( \mu \) are respectively the conductivity and permeability of the conductors. For the infinite line the voltage transfer function between input and any point \( z \) along the line is

\[ \frac{V(z)}{V(0)} = e^{-\gamma z} \quad (1.10) \]
Therefore for a step input the transform of the voltage at any point \( z \) is

\[
V(z) = \frac{e^{-\gamma z}}{j\omega}.
\]  

(1.11)

An approximate expression for the inverse transform may be obtained by keeping only the dominant high frequency terms in the expression for \( \gamma \). This approximation is particularly appropriate for small values of time and hence gives the rise time of the line adequately. Substituting equations 1.7 and 1.8 into equation 1.5 and keeping only the dominant terms gives

\[
\gamma = j\omega \sqrt{L_e C} + K \sqrt{\frac{j\omega C}{2L_e}}.
\]  

(1.12)

Substituting this expression for \( \gamma \) into equation 1.11 and taking the inverse transform gives, for the voltage at any point \( z \) and any time \( t \)

\[
v(z, t) = \left\{ 1 - \text{erf} \left[ \frac{K \sqrt{C} \sqrt{z}}{\sqrt{L_e}} \right] \right\} u(t') J
\]  

(1.13)

where \( t' \) is the retarded time \( t - \sqrt{L_e C} z \), the step function multiplier indicates the delay before the start of the voltage buildup, and the error function is defined by

\[
\text{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-Y^2} dY.
\]  

(1.14)

Figure 1.1 shows a plot of voltage on a normalized time scale where the constant \( K_2 \) is introduced as
Figure 1.1. Normalized Step Response.
Table 1.1
Rise Time Characteristics of Low Loss Styroflex Transmission Line

<table>
<thead>
<tr>
<th>Length of Line</th>
<th>3&quot; Outer Dia.</th>
<th>7/8&quot; Outer Dia.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10 - 90% Rise Time</td>
<td>10 - 90% Rise Time</td>
</tr>
<tr>
<td>100 ft.</td>
<td>$2.5 \times 10^{-11}$ sec.</td>
<td>$1.2 \times 10^{-10}$ sec.</td>
</tr>
<tr>
<td>1,000 ft.</td>
<td>$2.5 \times 10^{-9}$ sec.</td>
<td>$1.2 \times 10^{-8}$ sec.</td>
</tr>
<tr>
<td>10,000 ft.</td>
<td>$2.5 \times 10^{-7}$ sec.</td>
<td>$1.2 \times 10^{-6}$ sec.</td>
</tr>
</tbody>
</table>
In order to read the retarded time directly, it is necessary to multiply the scale reading by $K_2$. Note in particular that the time scale expands as the square of the distance $z$.

Table 1.1 gives some rise time characteristics for a few special cases. It should be noted that neither of the lines considered gives very satisfactory rise times for short pulses for lengths of the order of 1000 feet or more.
II

An Exact Solution for the Constant R Line

2.1 The Semi-Infinite Line

From the high frequency series expression for $\tilde{Y}(\omega)$ given in the previous section it is seen that if $R$, $L$, and $C$ are constants then the first two terms of the series are of the correct form to give distortionless pulse transmission. This result suggests that it may be worthwhile to consider ways of obtaining such a line. However, before this is done a more complete analysis of the pulse distortion associated with such a line is in order. In order to obtain realistic values in the analysis of this hypothetical line the value of $R$ will be chosen to be the value exhibited by corresponding diameter physical lines at 1 Gc.

The problem of finding an exact expression for the response of pulse waveshapes for a constant $R$ line is simplified since the expression for the exact voltage response to a unit impulse input at a distance $\lambda$ from the source has been obtained by G. A. Campbell and is also verified in Reference 2. This solution is valid for a semi-infinite line excited at $\lambda = 0$. The unit impulse response is

$$h(t) = 0 \quad , \quad t < \frac{\lambda}{v}$$

$$h(t) = \exp \left\{ -\left( \frac{R}{2L_C} + \frac{G}{2C} \right) \frac{\lambda}{v} \right\} \delta \left( t - \frac{\lambda}{v} \right)$$

$$+ \frac{1}{vz} \left[ \frac{R}{2L_C} - \frac{G}{2C} \right] \exp \left\{ -\left( \frac{R}{2L_C} + \frac{G}{2C} \right) t \right\}$$

$$I_1 \left\{ \left( \frac{R}{2L_C} - \frac{G}{2C} \right) z \right\} \quad , \quad t \geq \frac{\lambda}{v}$$

(2.1a)
where
\[ v = \frac{1}{\sqrt{L_e C}} \]
\[ z = \sqrt{t^2 - \left(\frac{\lambda}{v}\right)^2} \]

and \( I_1(x) \) denotes the modified Bessel function of the first kind with argument \( x \). When \( G = 0 \) the impulse response reduces to

\[ h(t) = \begin{cases} 0 & , \quad t < \frac{\lambda}{v} \\ \exp \left\{ -\frac{R}{2L_e} \cdot \frac{\lambda}{v} \right\} \delta \left( t - \frac{\lambda}{v} \right) \\ + \frac{1}{v^2} \frac{R}{2L_e} \exp \left\{ -\frac{R}{2L_e} t \right\} I_1 \left\{ \frac{R}{2L_e} z \right\}, \quad t \geq \frac{\lambda}{v} \end{cases} \tag{2.2a} \]

With this impulse response and the convolution integral the voltage response to any pulse waveshape can be obtained by graphical or numerical techniques. Of particular interest is a rectangular pulse of a given width. For a pulse of this type of width \( B \) the convolution integral is

\[ y(t) = \int_0^\infty h(t-\tau) \left\{ u(\tau) - u(\tau-B) \right\} d\tau \tag{2.3a} \]

or

\[ y(t) = \int_0^\infty h(\tau) \left\{ u(t-\tau) - u(t-B-\tau) \right\} d\tau \tag{2.3b} \]

which gives the following when equations 2.2a and 2.2b are substituted into equation 2.3b.
\[
e(t, \lambda) = \int_{-\infty}^{\infty} \exp \left\{ -\frac{R}{2L_e \frac{\lambda}{V}} \right\} \delta \left( \tau - \frac{\lambda}{V} \right) \{ u(t-\tau) - u(t-B-\tau) \} \, d\tau \\
+ \frac{R}{2L_e} \int_{-\infty}^{\infty} \exp \left\{ -\frac{R}{2L_e \frac{\lambda}{V}} \right\} \frac{1}{\sqrt{\tau^2 - \left( \frac{\lambda}{V} \right)^2}} \, d\tau
\]

\[I_1 \left\{ \frac{R L}{2L_e} \sqrt{\tau - \left( \frac{\lambda}{V} \right)^2} \right\} \{ u(t-\tau) - u(t-B-\tau) \} \, d\tau , \quad (2.4a)\]

or in shorthand notation

\[e(t, \lambda) = e_1(t, \lambda) + e_2(t, \lambda) , \quad (2.4b)\]

The first expression is easily integrated to obtain

\[e_1(t, \lambda) = \exp \left\{ -\frac{R}{2L_e \frac{\lambda}{V}} \right\} \{ u(t-\frac{\lambda}{V}) - u(t-B-\frac{\lambda}{V}) \} , \quad (2.5)\]

which represents only an attenuation of the input pulse. The integral of the second term which is proportional to the error cannot readily be evaluated in closed form; however, it is easily evaluated by graphical or numerical techniques.

The distortion of rectangular voltage pulses at various lengths along the line is depicted in Figures 2.1 thru 2.5. These curves give the voltages on semi-infinite lines and no reflection is assumed. These wave shapes would be detected by a high impedance detector placed across the line at the point of interest. Figures 2.1 thru 2.3 give the voltage distortion for a nominal 50 ohm, three inch outer diameter coaxial line which exhibits 0.5 db. attenuation per hundred feet of length. Figure 2.4 indicates the increase in distortion obtained when a 70 ohm, three inch
Figure 2.1. Voltage Waveshapes Obtained on 50 ohm 3-Inch Line. Att. = 0.5 db/100 Ft.
Figure 2.2. Voltage Waveshapes Obtained on 50 OHM 3-Inch Line. Att. = 0.5 db/100 Ft.
Figure 2.3. Voltage Waveshapes Obtained on 50 OHM 3-Inch Line. Att. = 0.5 db/100 Ft.
Figure 2.4. Voltage Waveshapes Obtained on 70 OHM 3-Inch Line. Att. = 0.5 db/100 Ft.
Figure 2.5. Voltage waveshapes obtained on 50 OHM 1-inch line. Alt. = 1.2 3t/100 Ft.
outer diameter coaxial line is considered. Figure 2.5 indicates the increase in distortion obtained when a higher loss smaller diameter line is considered. It should be noted that for pulses which are a few nanoseconds in width the smaller diameter line is satisfactory distortion wise; however, the attenuation associated with a line of this sort limits its application.

2.2 The Terminated Line

The analysis of the terminated line is greatly simplified if the effects of reflections from the source can be neglected. Little generality is lost since, for matched inputs, these reflections do not arise. Moreover, for situations in which the pulse width is small compared to the delay time, source reflections make little contribution to the received signal. Thus, neglecting source effects, a Thevenin equivalent circuit can be obtained to indicate the voltage received across the terminating element.

The equivalent circuit is shown in Figure 2.6. In this circuit the lines characteristic impedance is not simply $\sqrt{L_e/C}$ but is given by

$$Z_c = \sqrt{\frac{R + j\omega L_e}{j\omega C}}$$

$$= \sqrt{\frac{L_e}{C}} \sqrt{1 + \frac{R}{j\omega L_e}}$$

which can be approximated at high frequencies by

$$Z_c \approx \sqrt{\frac{L_e}{C}} - j \frac{R}{2\omega \sqrt{L_e C}}$$

(2.6)

Thus it is seen that $Z_c$ can be replaced by a lumped resistance, $R_c$, in series with a lumped capacity, $C_c$, with values...
Figure 2.6. Equivalent Circuit at Output of Transmission Line.
\[ R_c = \sqrt{\frac{L_e}{C}} \]  \hspace{1cm} (2.8a)

and

\[ C_c = \frac{2\sqrt{L_e C}}{R} \] \hspace{1cm} (2.8b)

The values of \( R_c \) and \( C_c \) for the three lines under consideration are tabulated in Table 2.1 and comparison of frequency responses of \( Z_c \) obtained from lumped approximations and the exact expressions for one of the lines is given in Figure 2.7. From this figure and the table it is seen that for short pulses the approximation is very good.

By terminating the line in its characteristic impedance the voltage waveshapes depicted in the previous figures can be obtained without additional distortion. Thus it is seen from Figures 2.3 that for a 3 inch 50 ohm line a 10^{-8} second pulse can be received on a 10,000 feet line with less than 2\% distortion. Figure 2.4 indicates that a further improvement is obtained when a 70 ohm line is utilized.

The results which have been formulated in this section indicate that, if the effects of overall attenuation can be neglected, a coaxial line in which \( R \) is a constant completely eliminates the rise time problem associated with existing transmission lines. Moreover, these results indicate that, as long as pulse widths are restricted to a few nanoseconds, nearly perfect pulse reproduction can be obtained for transmission systems which are one or two miles in length.
Table 2.1
Approximate Characteristic Impedances for Constant R Transmission Lines

<table>
<thead>
<tr>
<th>Type Line</th>
<th>Attenuation</th>
<th>Characteristic Impedance</th>
<th>Frequency Below Which Exact and Approximate Expressions Differ by more than 10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 inch - 50 OHM</td>
<td>0.5db/100ft.</td>
<td>Rc = 50 ( \times ) ( \frac{3.45 \times 10^{-8}}{f} )</td>
<td>1.4 ( \times ) 10^5 cycles/sec.</td>
</tr>
<tr>
<td>1 inch - 50 OHM</td>
<td>1.2db/100ft.</td>
<td>50 ( \times ) ( \frac{1.44 \times 10^{-8}}{f} )</td>
<td>6.0 ( \times ) 10^6 cycles/sec.</td>
</tr>
<tr>
<td>3 inch - 70 OHM</td>
<td>0.5db/10ft.</td>
<td>70 ( \times ) ( \frac{3.52 \times 10^{-8}}{f} )</td>
<td>1.2 ( \times ) 10^6 cycles/sec.</td>
</tr>
</tbody>
</table>
Figure 2.7. Comparison of Exact and Approximate Characteristic Impedance for a 50 OHM 3 Inch Line. Att. = 0.5 db/100 Ft.
Figure 3.1. Plane Solid Conductor.
III

The Effect of Laminated Conductors on Surface Impedance

3.1 The Skin Effect Analysis

Before considering how the propagation constant of a coaxial line is modified by using laminated conductors, it is well to review the usual skin effect relations.

For the plane configuration depicted in Figure 3.1 with current flow in the z direction it is shown in the literature that, when displacement current is neglected, the equation giving the current density in the material is

$$J_z(x) = J_0 e^{-\gamma x},$$

(3.1)

where

$$J_z(0) = J_0$$

$$\gamma = (1+j) \sqrt{\pi f \mu \sigma}.$$  

If the usual definition is then assumed

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}},$$

equation 3.1 becomes

$$J_z(x) = J_0 e^{-x/\delta} e^{-j\kappa \delta}.$$  

(3.2)

A surface impedance can then be defined as
where \( K \) is defined as the linear current density per meter and is given by

\[
K = \int_0^\infty J_z(x) \, dx = \int_0^\infty J_0 \, e^{-x/(l+j)} \, dx ,
\]

which then gives after some algebra

\[
Z_s = \sqrt{\frac{\pi \frac{\tau \mu}{\sigma}}{1+j}} .
\]

For a circular conductor of radius \( r_0 \) the limiting values of \( Z_s \) for high and low frequencies are respectively

\[
Z_{s\text{-high}} = \frac{(1+j)}{2\pi r_0} \sqrt{\frac{\pi \frac{\tau \mu}{\sigma}}{1+j}}
\]

and

\[
Z_{s\text{-low}} = \frac{1}{\pi r_0 \sigma} .
\]

For intermediate frequencies an approximate expression for the resistive part of \( Z_s \) is

\[
\text{Re} (Z_s) = \frac{1}{\pi \sigma} \left\{ 1 + \frac{1}{40} \left( \frac{r_0}{\frac{\tau \mu}{\sigma}} \right)^4 \right\} .
\]

For completeness these relations are plotted in Figure 3.2.

3.2 The Two Conductor Laminate

As a first attempt to consider the effects on the propagation constant by using laminated conductors consider the configuration shown in Figure 3.3.
Figure 3.2. Plot of Surface Resistance Versus Frequency for a Solid Copper Conductor.

\[ \sigma = 5.8 \cdot 10^6 \text{ mhos/Meter} \]
\[ \mu = 4 \pi \cdot 10^7 \text{ Henries/Meter} \]
\[ r_0 = 0.25 \text{ inches} \]
Figure 3.3. Two Laminate Plane Solid Conductor.
For this situation material 1 is of finite thickness, thus the current density expression in material 2 is of the form

\[ J_{z2} = C e^{-\gamma_2 (x-d)} \]  

(3.9)

and the current density expression in material 1 is given by

\[ J_{z1} = A \sinh \gamma_1 x + B \cosh \gamma_1 x \]  

(3.10)

For both materials

\[ E_z = \frac{J_z}{\sigma} \quad (3.11) \]

and

\[ H_y = \frac{1}{j\omega \mu} \frac{dE_z}{dx} = \frac{\sigma}{\gamma_2} \frac{dE_x}{dx} \]  

(3.12)

Hence the following relations are obtained:

\[ E_{z1} = \frac{1}{\sigma_1} \left[ A \sinh \gamma_1 x + B \cosh \gamma_1 x \right] \]  

(3.13)

\[ E_{z2} = \frac{C}{\sigma_2} \frac{\sigma_1}{\sigma_2} e^{-\gamma_2 (x-d)} \]  

(3.14)

\[ H_{y1} = \frac{1}{\gamma_1} \left[ A \cosh \gamma_1 x + B \sinh \gamma_1 x \right] \]  

(3.15)

\[ H_{y2} = -\frac{C}{\gamma_2} \frac{\sigma_1}{\gamma_2} e^{-\gamma_2 (x-d)} \]  

(3.16)

At the boundary between the two materials the tangential components of the fields must be continuous. When the corresponding equations are equated, the following relations are obtained:

\[ A = -C \left[ \frac{\sigma_1}{\sigma_2} \sinh \gamma_1 d + \frac{\gamma_1}{\gamma_2} \cosh \gamma_1 d \right] \]  

(3.17)
\[ B = C \left[ \frac{\sigma_1}{\sigma_2} \cosh \gamma_1 d + \frac{\gamma_1}{\gamma_2} \sinh \gamma_1 d \right] . \]  

(3.18)

The linear current density is given by

\[ \hat{K} = \hat{n} \times \hat{H} , \]  

(3.19)

or

\[ K_z = - \frac{\partial E}{\partial x} \bigg|_{x=0} . \]  

(3.20)

From equation 3.15 it is seen that, when \( x = 0 \),

\[ - \frac{\partial E}{\partial x} = - \frac{1}{\gamma_1} A . \]  

(3.21)

The electric field at \( x = 0 \) is obtained from equation 3.13 and is

\[ E_{z_1}(0) = \frac{B}{\sigma_1} . \]  

(3.22)

The expression for the surface impedance then becomes

\[ Z_s = \frac{E_z}{K_z} = - \frac{B}{\sigma_1 A} . \]  

(3.23)

When the appropriate expressions for \( A \) and \( B \) are substituted into this expression, the equation for \( Z_s \) becomes

\[ Z_s = \frac{\gamma_1}{\sigma_1} \left( \frac{\sinh \gamma_1 d + \sigma_1 \gamma_2}{\sigma_2 \gamma_1} \cosh \gamma_1 d \right) = \frac{\sigma_1}{\gamma_2} \left( \frac{\cosh \gamma_1 d + \sigma_1 \gamma_2}{\sigma_2 \gamma_1} \sinh \gamma_1 d \right) . \]  

(3.24)

This is in agreement with similar expressions found in the literature.
The evaluation of equation 3.24 is not as straightforward as it might at first seem since $\gamma_1$, which has been defined in the last section, is a complex quantity. After substituting for $\gamma_1$ and utilizing some algebra a more useful form is obtained as

$$Z_s = (1+j)\sqrt{\frac{\pi f \mu_1}{\sigma_1}}.$$ 

$$\frac{\tanh a + g}{1 + g \tan a} = \frac{\tan a (\tanh a + g)}{(1 + g \tan a) + j \tan a (\tanh a + g)},$$

where

$$a = \sqrt{\pi} \mu_1 \sigma_1 \frac{f}{d}$$

and

$$g = \frac{\sqrt{\sigma_1 \mu_2 / \sigma_2 \mu_1}}{\sigma_1 \mu_1}.$$

Using the above expression for $Z_s$ a family of curves giving values of surface resistance and reactance have been obtained as functions of frequency for various values of $\sqrt{\sigma_1 \mu_2 / \sigma_2 \mu_1}$.

A thickness of material 1 has been chosen as 1.55 at a frequency of 100c. These curves are shown in Figures 3.4 and 3.5. From these it can be seen that for values of $g = \frac{\sqrt{\sigma_1 \mu_2 / \sigma_2 \mu_1}}{500}$ more the value of the surface resistance is nearly constant while the value of the surface reactance varies considerably with frequency.

As indicated by equation 3.6 these curves also give a good approximation of the surface impedance of curved surfaces at high frequencies.
Figure 3.4. Surface Resistance of a Two Conductor Laminate.
Figure 3.5. Surface Reactance of a Two Conductor Laminate.
The shapes of these carves will receive further consideration in a later section when particular transmission lines are considered.

3.3 The Three Conductor Laminate

In the preceding analysis material 2 was considered to be of infinite thickness. For very high frequencies material 2 is only slightly penetrated by the current and this material does indeed behave as if it were of infinite extent. However, for lower frequencies the finite thickness of material 2 must be considered.

The structure of the three material laminate is given in Figure 3.6. The current density expressions for this situation are as follows:

For material 1 \((0 \leq x \leq d_1)\).

\[
\begin{align*}
J_{z_1} &= A \sinh \gamma_1 x + B \cosh \gamma_1 x \\
E_{z_1} &= \frac{A}{\sigma_1} \sinh \gamma_1 x + \frac{B}{\sigma_1} \cosh \gamma_1 x \\
H_{z_1} &= \frac{A}{\delta_1} \cosh \gamma_1 x + \frac{B}{\delta_1} \sinh \gamma_1 x
\end{align*}
\]

For material 2 \((0 \leq x' \leq d_2)\).

\[
\begin{align*}
J_{z_2} &= C \sinh \gamma_2 x' + D \cosh \gamma_2 x' \\
E_{z_2} &= \frac{C}{\sigma_2} \sinh \gamma_2 x' + \frac{D}{\sigma_2} \cosh \gamma_2 x' \\
H_{z_2} &= \frac{C}{\delta_2} \cosh \gamma_2 x' + \frac{D}{\delta_2} \sinh \gamma_2 x'
\end{align*}
\]

For material 3 \((0 \leq x'')\).

\[
J_{z_3} = E e^{-\gamma_3 x''}
\]

\[(3.32)\]
Figure 3.6. Three Laminate Plane Solid Conductor.
\[ E_{z3} = \frac{E}{\sigma_3} e^{-\gamma_3 x'} \]  
\[ H_{z3} = -\frac{E}{\gamma_3} e^{-\gamma_3 x'} \]  

(3.33)  

(3.34)

where

\[ x' = (x - d_1) \]

\[ x'' = [x - (d_1 + d_2)] \]

Using these equations and the continuity of the tangential components of the electric and magnetic fields at the two boundaries, the expression for the surface impedance is obtained as

\[ Z_s = \frac{\gamma_1}{\sigma_1} \left\{ \frac{\frac{\sigma_1}{\gamma_1} \left( \frac{\sigma_2}{\gamma_2} + \frac{\gamma_2}{\sigma_3} \tanh \gamma_2 d_2 \right)}{\frac{\sigma_1}{\gamma_1} \tanh \gamma_1 d_1 \left( \frac{\sigma_2}{\gamma_2} + \frac{\gamma_2}{\sigma_3} \tanh \gamma_2 d_2 \right) + \frac{\gamma_1}{\gamma_2} \left( \frac{\gamma_2}{\sigma_3} + \frac{\sigma_2}{\gamma_3} \tanh \gamma_2 d_2 \right)} \right\} \]

(3.35)

Further, if material 3 is assumed to be air or other dielectric material with \( e_r \) small and \( \sigma_3 = 0 \), equation 3.35 becomes, after simplifying algebra,

\[ Z_s = \frac{\gamma_1}{\sigma_1} \left\{ 1 + \frac{\gamma_1}{\gamma_2} \frac{\sigma_2}{\sigma_1} \left( \tanh \gamma_1 t_1 \right) \left( \tanh \gamma_2 t_2 \right) \right\} \]

\[ \left\{ \tanh \gamma_1 t_1 + \frac{\gamma_1}{\gamma_2} \frac{\sigma_2}{\sigma_1} \tanh \gamma_2 t_2 \right\} \]

(3.36)

The simple appearance of this expression is again somewhat deceiving as \( \gamma_1 \) and \( \gamma_2 \) are complex quantities. When these quantities are substituted
into equation 3.36 it becomes

\[ Z_s = (1+j) \sqrt{\pi f \frac{\mu_1}{\sigma_1}} \]

\[
\left\{ \begin{array}{l}
(g - g \tanh b \tan b \tanh c \tan c \\
+ \tanh b \tanh c - \tan b \tan c)
\end{array} \right.
\]

\[
+ j \left( g \tanh b \tan b + g \tanh c \tan c \\
+ \tanh c \tan b + \tan c \tanh b \right)
\]

\[
\left[ g \tanh b + \tanh c - \tan b \tan c \left( g \tanh c \\
+ \tanh b \right) \right] + j \left[ g \tan b + \tan c \\
+ \tanh c \tan b \left( g \tan c + \tan b \right) \right]
\]

where

\[ g = \frac{\gamma_2 \sigma_1}{\gamma_1 \sigma_2} = \sqrt{\frac{\mu_2 \sigma_1}{\mu_1 \sigma_2}} \]

\[ b = \sqrt{\pi \mu_1 \sigma_1 f d_1} \]

\[ c = \sqrt{\pi \mu_2 \sigma_2 f d_2} \]

Using equation 3.37 with \( g = 500 \), the surface resistance and reactance have been calculated for two values of \( d_2 \). These results are given in Figures 3.7 and 3.8. From these curves it is seen that at high frequencies, for the values of thickness used, there is little change from the previous results. However, at lower frequencies where the thickness of material 2 is of the order of a skin-depth or less the resistance is flattened and the reactance decreases rapidly. These curves will receive further consideration in a following section.
Figure 3.7. Surface Resistance of a Three Conductor Laminate.
Figure 3.8. Surface Reactance of a Three Conductor Laminate.
IV

Designs of Pulse Transmission Lines

4.1 Line Requirements

It was shown in the last section that, for a plane conductor made up of 2 materials with the first laminate of $1.25 \times 10^{-4}$ inch thickness and with $g = \sqrt{\frac{\sigma_1 \mu_2}{\sigma_2 \mu_1} \approx \infty}$, nearly constant skin resistance is obtained from less than 1 Kc. to 1 Gc. It was also shown that the reactance for $g \approx \infty$ has an approximate linear slope on the log-log plot. Thus, in accordance with the results of the last section, it seems that a transmission line with these characteristics would indeed have good pulse transmission characteristics. The next question which must be answered is how can a situation in which $g \approx \infty$ be achieved.

4.2 The Very Thin Wall Line

If material 2 is air then $\sigma_2 = 0$ and $g = \infty$. However, due to the thickness of material 1 this is not practical. A dielectric material other than air could be used provided that $\varepsilon_r$ is not large. A possible coaxial configuration for a nominal 3 inch 50-ohm line is shown in Figure 4.1 and its characteristics are given in Figure 4.2. The low frequency resistance is found by use of the relation

$$Z_{\text{dc}}^x = \frac{2\pi t_1}{\sigma_1} (r_1 + r_2), \quad (4.1)$$
Figure 4.1. Design of 50-Ohm Coaxial Line.
Figure 4.2. Approximate Frequency Response for $Z_g$ of a Conductor-Dielectric Laminate Coaxial Line.
where

\[ r_1 = \text{radius of inner conductor} \]

\[ r_2 = \text{radius of outer conductor} \]

and the high frequency resistance and reactance are found by substituting the values of surface impedance, \( Z_s \) (plane), for a plane conductor as given by Figures 3.4 and 3.5 into the relation

\[ Z_s^{\text{high freq}} = Z_s \text{ (plane)} \frac{1}{2\pi} \left( \frac{1}{r_1} + \frac{1}{r_2} \right). \] (4.2)

For this fairly simple case the exact solution can also be obtained with a medium amount of effort. The exact solution is given in Appendix B, and it is seen that there is excellent agreement with the values obtained from the approximate relationships.

From the characteristics given in Figure 4.2, it is seen that the resistance is constant and the total inductive reactance is essentially that for a lossless line. Thus the mathematics, as well as experimental results for thin walled coaxial lines\(^{25,6,22}\), at first seem to indicate that this type of line is indeed the solution. However, for long lengths of line this type of structure may not be practical. First it would be difficult to hold reasonable manufacturing tolerance on the outer conductor-dielectric laminate for long lengths. Secondly the bonding of the thin layer of conductor to the dielectric might prove difficult, and finally, the physical strength and rigidity of this type of structure might not be adequate for use in the field. However, a strip transmission line or a parallel plane configuration certainly merits consideration.
4.3 The Two Conductor Laminate Line

A coaxial structure which appears to have many of the desirable qualities of the line just discussed but with fewer of the associated manufacturing problems is shown in Figure 4.3. In this structure material 1 would be silver or copper and material 2 would be a magnetic alloy selected so as to obtain the largest possible $\mu_r$. For typical alloys it may be possible to obtain values of $g$ greater than 500. The characteristics of a line in which $g = 500$ will now be considered.

The thickness of material 2 has arbitrarily been chosen as 0.032 inches as this value would be suitable for manufacturing reasons and would provide mechanical rigidity; and from Figure 3.6 it is seen that only a small improvement at low frequencies is obtained by reducing the thickness by a factor of 2.

The dc resistance of the structure is obtained by considering both the inner conductor and the outer conductor to be made up of two resistors in parallel. Thus the dc resistance is given for the inner conductor as

$$R_{1\text{-dc}} = \left( \frac{1}{2\pi r_1 t_1 \alpha_1} \right) \parallel \left( \frac{1}{2\pi r_1 t_2 \alpha_2} \right), \quad (4.3)$$

and the dc resistance of the outer conductor is given by

$$R_{2\text{dc}} = \left( \frac{1}{2\pi r_2 t_1 \alpha_1} \right) \parallel \left( \frac{1}{2\pi r_2 t_2 \alpha_2} \right), \quad (4.4)$$

with the total dc resistance given by

$$Z_{s\text{-dc}} = R_{1\text{dc}} + R_{2\text{dc}}. \quad (4.5)$$
Figure 4.3. Design of 50 OHM Laminate Coaxial Line.

\[
\begin{align*}
\text{\(t_1\)} &= 1.25 \cdot 10^{-4} \text{ in.} \\
\text{\(t_2\)} &= 0.032 \text{ in.} \\
\text{\(r_1\)} &= 0.652 \text{ in.} \\
\text{\(r_2\)} &= 1.500 \text{ in.}
\end{align*}
\]
In these expressions, it is assumed that both $t_1$ and $t_2$ are so small in comparison to $r_1$ and $r_2$ that $r_1$ and $r_2$ may be considered constant over the particular laminate under consideration.

Again, the high frequency approximation for the surface impedance is obtained using the expression

$$Z_{s-hf.} = \frac{Z_s(\text{plane})}{2\pi} \left( \frac{1}{r_1} + \frac{1}{r_2} \right), \quad (4.6)$$

where in this the values of $Z_s(\text{plane})$ are obtained from Figures 3.7 and 3.8. Also, it should be mentioned that although this equation is a high frequency approximation it is fairly good even at low frequencies since the ratio $r_1/(t_1 + t_2)$ is large. The approximate frequency response for this configuration is given in Figure 4.4.

The reactance $\omega L_e$ arising from the geometry of the line is also shown in this figure for comparison to the surface reactance. It is seen that at frequencies in excess of $10^5$ cycles the total inductive reactance is essentially just $\omega L_e$ whereas at frequencies below $10^4$ cycles it is essentially the surface reactance. Thus it is seen that at frequencies between $10^5$ and $10^9$ cycles the desired response essentially has been obtained.

Although the desired response has not been obtained at low frequencies, it is believed that this configuration should provide considerable improvement in the fidelity of nanosecond pulse transmission as the high frequency requirements have been satisfied.
Figure 4.4. Approximate Frequency Response of the Surface Impedance of a 2-Laminate Coaxial Transmission Line.
In the last section a design for an improved line was conceived; however, it was found that at low frequencies and frequencies above $10^9$ cycles the response of this line deviated from that which was desired. In this section the transient response of the designed line will be considered.

To determine the exact response of the designed line would be very difficult if not impossible. However, a somewhat pessimistic approximation of the line's performance can be obtained by considering the line to pass only the middle range of frequencies where the line has the desired characteristics and neglect the high and low frequency contributions. In other words the transient response of a band pass filter will be investigated. The investigation of a band pass filter's transient response is further supported by the filter like characteristics of the other components used in a transmission system such as a terminating oscilloscope. At present an oscilloscope with a high frequency response extending to around 5 Gc, is considered very broadband.

As a beginning, consider passing a narrow rectangular pulse through an ideal filter with a passband from $10^4$ to $10^9$ cycles. That is the characteristics of the filter are

\begin{align}
F(\omega) &= 0 & 0 \leq \omega \leq 2\pi \times 10^4 \\
F(\omega) &= 1 & 2\pi \times 10^4 \leq \omega \leq 2\pi \times 10^9 \\
F(\omega) &= 0 & 2\pi \times 10^9 < \omega \quad (5.1)
\end{align}
A linear phase delay term could also be included which would only delay the output; however, it has been deleted for simplicity.

The transient response of the filter is easiest found by using Fourier Transforms. Doing this the frequency spectrum of the input pulse shown in Figure 5.1 is found to be

\[ g(\omega) = \frac{1}{2\pi} \int_{t_0}^{t_0/2} E_0 e^{-j\omega t} \, dt, \quad (5.2) \]

which gives

\[ g(\omega) = \frac{E_0 t_0}{2\pi} \left( \frac{\sin \omega t_0}{\omega t_0} \right). \quad (5.3) \]

The output of the filter then becomes

\[ e_0(t) = \int_{-\omega_1}^{\omega_2} \left( \frac{E_0 t_0}{2\pi} \right) \left( \frac{\sin \omega t_0}{\omega t_0} \right) e^{j\omega t} \, d\omega 
+ \int_{\omega_1}^{\omega_2} \left( \frac{E_0 t_0}{2\pi} \right) \left( \frac{\sin \omega t_0}{\omega t_0} \right) e^{j\omega t} \, d\omega, \quad (5.4) \]

where

\[ \omega_1 = \text{low frequency cutoff} \]
\[ \omega_2 = \text{high frequency cutoff}. \]

In this form these integrals are not readily suitable for integration; however, they may be rewritten in the form
Figure 5.1. Input Voltage Waveform.
which implies that the response of a band pass filter is the difference of the responses of two low pass filters. This is a particularly useful fact in that the contribution of the low frequency components can readily be seen.

The evaluation of equation 5.5 is obtained by first replacing \( e^{j\omega t} \) by \( \cos \omega t + j \sin \omega t \) which gives

\[
e_0(t) = E_0 \left\{ \int_{-\omega_2}^{\omega_2} \left( \sin \frac{\omega t}{2} \right) \cos \omega t \, d\omega \right\}
\]

\[
- \int_{-\omega_1}^{\omega_1} \left( \sin \frac{\omega t}{2} \right) \cos \omega t \, d\omega + j \int_{-\omega_2}^{\omega_2} \left( \sin \frac{\omega t}{2} \right) \sin \omega t \, d\omega
\]

\[
- j \int_{-\omega_1}^{\omega_1} \left( \sin \frac{\omega t}{2} \right) \sin \omega t \, d\omega \}
\]

The third and fourth integrals have integrands which are odd functions of frequency, and these integrals are zero for all values of \( t \). The first and second integrals, however, have integrands which are even functions of frequency and need be integrated only from 0 to the upper limits. After using a suitable trigonometric identity, equation 5.5 becomes
These integrals cannot be evaluated directly in terms of elementary functions; however, the integrals have been calculated and tabulated\(^{20}\). The usual tabulation is for

\[
\text{Si}(x) = \int_{0}^{x} \frac{\sin u}{u} \, du \quad (5.8)
\]

After utilizing a change of variables equation (5.7) becomes

\[
e_o(t) = \frac{E_o}{\pi} \left\{ \text{Si} \left[ \frac{\omega_2 (t + t_o)}{2} \right] - \text{Si} \left[ \frac{\omega_2 (t - t_o)}{2} \right] \right\}
- \frac{E_o}{\pi} \left\{ \text{Si} \left[ \frac{\omega_1 (t + t_o)}{2} \right] - \text{Si} \left[ \frac{\omega_1 (t - t_o)}{2} \right] \right\} \quad (5.9)
\]

Equation 5.9 has been evaluated for a pulse width of 1x10\(^{-8}\) second with \(\omega_1 = 2\times10^4\) and \(\omega_2 = 2\times10^9\). The results are given in Figure 5.2. Several interesting things may be observed by considering this figure. First, it is noticed that due to the non realizability of an ideal filter there is seen to be an output before the input pulse starts. Second, it is seen that there is oscillation across the entire top of the output pulse. Third, it is seen that the response is essentially the same as for a low pass filter of bandwidth \(\omega_2\). That is, the magnitude of the second group
Figure 5.2. Transient Response for a $10^{-8}$ Second Pulse Passed Through an Ideal Bandpass Filter.
of terms in equation 5.9 is very small for a $10^{-8}$ second pulse. These observations require further consideration.

In order to see just how large a pulse can be transmitted through the filter with only a small amount of distortion resulting from the blocking of the low frequency components consider the second group of terms of equation 5.9

$$e_{o_2}(t) = \frac{E_0}{\pi} \left\{ \text{Si} \left[ m_1(t + \frac{t_0}{2}) \right] - \text{Si} \left[ m_1(t - \frac{t_0}{2}) \right] \right\}, \quad (5.10)$$

where

$$e_o(t) = e_{o_1}(t) + e_{o_2}(t).$$

For $t = 0$ equation 5.10 has its maximum value which reduces to

$$e_{o_2}(t) = \frac{E_0}{\pi} \left(2\right) \text{Si} \left( \frac{m_1 t_0}{2} \right), \quad (5.11)$$

which is approximately for small arguments

$$e_{o_2}(0) \approx \frac{E_0}{\pi} m_1 t_0. \quad (5.12)$$

Thus if $e_{o_2}(0)$ is equal to say $0.05 \ E_0$ a value of $t = 2.5 \times 10^{-6}$ second is obtained. It is seen that any low frequency distortion to pulses of width less than $10^{-7}$ second can indeed by neglected.

The question of the non realizibility of the ideal filter introducing non physical oscillations into the pulse response can be solved by replacing the ideal filter by a physical filter which might approximate the response of the oscilloscope which terminates the transmission line.
The filter which has been chosen is the four pole maximally flat filter whose frequency characteristics are shown in Figure 5.3. An $f_o = 2 \times 10^9$ cycles has been chosen. The filters response to a $10^{-8}$ second width pulse is shown in Figure 5.4. It is seen that the oscillations at the top of the pulse are damped much more rapidly than in the case of the ideal filter; although, the initial overshoot is still present. A 10-90% rise time of about $2 \times 10^{-10}$ second is obtained. The effect of either increasing or decreasing $f_o$ is displayed in Figure 5.5.

From the preceding analysis and subject to limitations concerning length of line as indicated in Section 2, it appears that pulses in the range from $1 \times 10^{-9}$ to $1 \times 10^{-7}$ second can be transmitted without appreciable distortion. Also it appears that the deviation of the surface resistance at frequencies in excess of $10^9$ cycles will produce effects which are secondary in comparison to the bandwidth effects of the oscilloscope.

In general it appears that in the complete system the designed line does as well as the idealized constant resistance line considered in Section 2.
Figure 5.3. Frequency Characteristics of a Four Pole Maximally Flat Low Pass Filter.
Figure 5.4. Transient Response for a $10^{-8}$ Second Pulse Through a Maximumly Flat Filter.
REFERENCES


Figure 5.5. Step Function Response for Maximally Flat Four Pole Filter.


20. Mathematical Tables Project, National Bureau of Standards, Tables of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments, Columbia University Press, 1947.


Consider the configuration shown in Figure A.1. If the signal is undistorted, then it can only be scaled in amplitude and shifted in time and hence can be written as
\[ y(t) = e^{-\alpha t} f(t-\tau), \]
where \( \tau \) is a constant delay.

It is then desirable to know the form of the transfer function \( G(\omega) \). This is easiest obtained by considering first the expression for \( Y(\omega) \) which is
\[ Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt \]
\[ = \int_{-\infty}^{\infty} e^{-\alpha t} f(t-\tau) e^{-j\omega t} dt. \]  

This becomes then
\[ Y(\omega) = e^{-\alpha \omega} \int_{-\infty}^{\infty} f(\lambda) e^{-j(\omega+\tau) \lambda} d\lambda, \]
or
\[ Y(\omega) = e^{-\alpha -j\omega\tau} \int_{-\infty}^{\infty} f(\lambda) e^{-j(\omega+\tau) \lambda} d\lambda, \]  
which is
\[ X(\omega) = e^{-\alpha -j\omega\tau} F(\omega). \]
Figure A.1. The System Under Consideration.
Thus if the network has a transfer function of the form

\[ G(\omega) = e^{-\alpha - j\omega \tau} \]

(A.5)

then the desired form is obtained.

For the case in which \( P(\omega) \) is limited to a certain band of frequencies then equation A.4 indicates that equation A.5 need be satisfied only over the same band.
APPENDIX B

The Exact Expression for the Surface Impedance of a Thin Walled Coaxial Structure

For the very thin walled coaxial structure shown in Figure B.1a, the exact expression for the surface impedance is obtained by considering the surface impedance of each conductor separately.

For the inner conductor as shown in Figure B.1b the surface impedance is obtained by satisfying the boundary conditions for an inward propagating cylindrical wave. For region 1 the current density must remain finite for \( r=0 \). Thus the equations for current density, electric, and magnetic fields are of the form:

\[ J_{z1} = A J_0(\chi_1, r) \] (B.1)

\[ E_{z1} = \frac{A}{\sigma_1} J_0(\chi_1, r) \] (B.2)

\[ H_{\varphi 1} = \frac{A}{j\omega \mu \sigma_1} \frac{\chi_1}{J_0(\chi_1, r)} \] (B.3)

For region 2 the more general relations hold:

\[ J_{z2} = B J_0(\chi_2 r) + C H_0^{(1)}(\chi_2 r) \] (B.4)

\[ E_{z2} = \frac{B}{\sigma_2} J_0(\chi_2 r) + \frac{C}{\sigma_2} H_0^{(1)}(\chi_2 r) \] (B.5)
Figure B.1. The Very Thin Walled Coaxial Transmission Line.
\[ H_{\Omega 2} = \frac{1}{j\omega \mu_2 \sigma_2} \left\{ B \, J'_0 (\gamma_2 r) + C \, H'_0 (\gamma_2 r) \right\} \]  
\[ (B.6) \]

Equating the tangential components of the electric and magnetic fields at the boundary of the two regions the relationship of \( B \) to \( C \) is obtained as

\[ \frac{B}{C} = -\frac{\gamma_2 H'_0 (\gamma_2 r_1) J_0 (\gamma_1 r_1) - \gamma_1 H'_0 (\gamma_1 r_1) J'_0 (\gamma_2 r_1)}{J'_0 (\gamma_2 r_1) J_0 (\gamma_1 r_1) - J_1 J_0 (\gamma_1 r_1) J'_0 (\gamma_2 r_1)} . \]
\[ (B.7) \]

For the special case of interest \( \sigma_1 = 0 \), and even if dielectric effects are considered the product \( \gamma_1 r_1 \) is small in comparison to \( \gamma_2 r_1 \) thus in the limit the ratio reduces to

\[ \frac{B}{C} = -\frac{H'_0 (\gamma_2 r_1)}{J'_0 (\gamma_2 r_1)} . \]
\[ (B.8) \]

The surface impedance is obtained by considering the ratio of the electric field to the total current at \( r_2 \) which is

\[ Z_s = -\frac{1}{2\pi r_2} \frac{E_{z_2} (r_2)}{H_{\Omega 2} (r_2)} . \]
\[ (B.9) \]

After utilizing equation \( B.4 \) and substituting for \( E_{z_2} (r_2) \) and \( H_{\Omega 2} (r_2) \), equation \( B.9 \) becomes

\[ Z_s = -\frac{j^{-1/2}}{2\pi r_2} \sqrt{\frac{\omega \mu}{\sigma_2}} \times \]
\[ \left\{ \frac{J_0 (\gamma_2 r_2) H'_0 (\gamma_2 r_1) - J'_0 (\gamma_2 r_1) H'_0 (\gamma_2 r_2)}{J'_0 (\gamma_2 r_2) H_0 (\gamma_2 r_1) - J'_0 (\gamma_2 r_1) H'_0 (\gamma_2 r_2)} \right\} . \]
\[ (B.10) \]
For frequencies in excess of say 1,000 cycles the arguments of the Bessel functions become large and they can be replaced by their large argument approximations. Doing this equation B.10 becomes

\[ Z_s = \frac{4}{2\pi r_2} \left\{ \frac{1 + e^{-2(\alpha+j\alpha)(r_2-r_1)}}{1 - e^{-2(\alpha+j\alpha)(r_2-r_1)}} \right\} , \quad (B.11) \]

where \( \alpha = \sqrt{\omega \mu_2 \sigma_2} \).

For the outer conductor as shown in Figure B.1c the surface impedance is obtained in a similar fashion for an outward propagating cylindrical wave. In this case the pertinent equations are:

\[ J_{z_3} = D J_o (\gamma_3 r) + E H_o^{(2)} (\gamma_3 r) \quad (B.12) \]

\[ E_{z_3} = \frac{D}{\sigma_3} J_o (\gamma_3 r) + \frac{H}{\sigma_3} H_o^{(2)} (\gamma_3 r) \quad (B.13) \]

\[ H_{\gamma_3} = \frac{D}{j\omega \mu_3} \frac{\gamma_3}{\sigma_3} J'_o (\gamma_3 r) + \frac{E}{j\omega \mu_3 \sigma_3} H_o^{(2)'} (\gamma_3 r) \quad (B.14) \]

\[ J_{z_4} = F H_o^{(2)} (\gamma_4 r) \quad (B.15) \]

\[ E_{z_4} = F H_o^{(2)} (\gamma_4 r) \quad (B.16) \]

\[ H_{\gamma_4} = \frac{F}{j\omega \sigma_4} \gamma_4 H_o^{(2)'} (\gamma_4 r) \quad (B.17) \]

After satisfying the boundary conditions between regions 3 and 4 and imposing the restriction that \( \gamma_4 r \) be small, the skin effect relation is given by
Again for large arguments of the Bessel functions the equation simplifies considerably. It becomes

\[ Z_s = \frac{j^{1/2}}{2\pi \sigma_3} \sqrt{\frac{\omega \mu_3}{\sigma_3}} \left\{ \frac{-2(\tau + j\tau) (r_4 - r_3)}{1 - e^{-2(\tau + j\tau)(r_4 - r_3)}} \right\} \]  

where

\[ \tau = \sqrt{\frac{\omega \mu_3}{\sigma_3}} \]  

For the special case of interest \( \tau = \alpha \) and \( (r_2 - r_1) = (r_4 - r_3) \). Since the total surface impedance is the sum of the surface impedances of the two conductors, it is simply

\[ Z_s^{\text{total}} = \frac{j^{1/2}}{2\pi} \sqrt{\frac{\omega \mu_3}{\sigma}} \left( \frac{1}{r_2} + \frac{1}{r_3} \right) \left\{ \frac{-2(\alpha + j\alpha)(r_2 - r_1)}{1 - e^{-2(\alpha + j\alpha)(r_2 - r_1)}} \right\} \]  

The results obtained using this expression are given in Figure B.2.
Figure B.2. Exact Solution for Surface Impedance for a Very Thin Walled Line.