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Abstract

A fundamental question arising in queueing system analysis is whether a system is stable or unstable. For systems modelled by infinite Markov chain, we may study ergodicity and nonergodicity of the chains. Foster [6] showed that sufficient conditions for ergodicity are linked with the average drift, however, complications arise when multidimensional Markov chains are analysed. We shall present three methods providing sufficient conditions for ergodicity and nonergodicity of a multidimensional Markov chain. These methods are next applied to two multidimensional queueing systems: buffered contention packet broadcast system and coupled-processor system.

Key words: Markov chains, ergodicity, nonergodicity, queueing systems, packet broadcast system, coupled-processor system.

1. Introduction

A fundamental question arising in systems analysis is whether a system is stable or unstable. In general, it is said that a system *is stable if it has required properties in the presence of some disturbances*. For stochastic systems such as queueing systems, it is assumed that the source of disturbances is the arrival process, while required properties may be as follows: finite average waiting time or queue length, existence of a steady-state solution and so forth. For a queueing system represented by an infinite Markov chain, the stability/in-stability problem may be interpreted in terms of ergodicity/nonergodicity of the Markov chain. For aperiodic, irreducible Markov chains ergodicity means that steady-state distribution exists and all long-run probabilities are positive.

Sufficient conditions for ergodicity of a Markov chain are discussed in many papers [6], [10], [11], [12], [13], [14], [22], however, they are mainly restricted to the one-dimensional case. Since Foster's paper [6] Lyapunov functions (and so called *generalized drift*) found wide applications in classifying Markov chains. Roughly speaking, Foster proved that a Markov chain is ergodic if the generalized drift is negative for all but finitely many states. Pakes [12], Marlin [11], Rosberg [14] and Tweedie [22] extended Foster's condition to either larger classes of Lyapunov functions or wider classes of Markov chains. A problem arises when multidimensional Markov chains are analysed, since in most applications, the negativity of the generalized drift is violated on an infinite number of states. To the author's knowledge only a few papers challenge this problem. Malyshev [10] gave sufficient and necessary condition for ergodicity of a special type (important from a queueing theory point of view) of two-dimensional Markov chains, while Rosberg [13] extended Foster's condition to a class of multidimensional chains, but the criterion has limited applications in queueing models.

Existing sufficient and necessary conditions for ergodicity (e.g. [22]) are very restrictive, since they are too complex to verify. Another approach that establishes necessary conditions for ergodicity, which is adopted in this paper, deals with sufficient conditions for nonergodicity. As Kaplan noted [9] positivity of the drift of a Markov chain for all but finitely many states is *not* sufficient for nonergodicity in a general case and an additional condition known as Kaplan's condition must be added. Kaplan's results were further extended by Sennot et al. [16] and Szpankowski [18], [20]. However, application of these results to multidimensional chains requires some extensions as stated above.

This paper deals with the stability of multidimensional queueing systems described by Markovian models. We restrict our considerations to discrete-space and discrete-time multidimensional Markov chains. As we discussed above the number of papers challenging that problem is very limited, however, there are few papers presenting separate results on ergodicity for individual models, e.g. [2] - [5], [21]. We shall discuss two criteria for ergodicity and three criteria for nonergodicity of multidimensional Markov chains. These results are then applied to two multidimensional queueing systems: buffered contention packet broadcast system (ALOHA system) and coupled processors system.

2. Basic Results

Let $N(t)$ be a Markov chain with denumerable state space C (nonnegative integers) and discrete time $t = 0, 1, \dots$. Throughout the paper we shall assume that $N(t)$ is an *aperiodic and irreducible* chain [1]. Then $N(t)$ is called *ergodic (stable)* iff for all $i, j \in C$ the limit $\lim_{t \rightarrow \infty} \Pr\{N(t) = j \mid N(0) = i\}$ exists, does not depend on i , and is positive for all $j \in C$. Otherwise, we say that the chain is *not ergodic* (we do not distinguish between null recurrent and transient chains).

2.1 Lyapunov functions - generalized drift

Let $V(k)$, $k \in C$ be a lower bounded real-valued function, that is, $V: C \rightarrow \mathbb{R}$

where \mathbb{R} is a set of real numbers, and there exists a constant $\xi \in \mathbb{R}$ such that $V(k) \geq \xi > -\infty$ for all $k \in C$. The function $V(k)$ with the above properties is known as Lyapunov function. For a Markov chain $N(t)$ we define an operator $AV(k)$, $k \in C$ by

$$AV(k) = E[V(N(t+1)) - V(N(t)) \mid N(t) = k],$$

assuming it exists. If $V(k) = k$, then the operator $AV(k)$ becomes a mean drift, $d(k) \triangleq E[N(t+1) - N(t) \mid N(t) = k]$; therefore, we also call $AV(k)$ a *generalized drift*. Then

THEOREM 1.

If there exists a Lyapunov function $V(k)$, $k \in C$, and if for a constant $\epsilon > 0$ and a finite set $H \subset C$

$$\mid AV(k) \mid < \infty \quad \text{for all } k \in H \tag{1a}$$

$$AV(k) \leq -\epsilon \quad \text{for all } k \in C-H \tag{1b}$$

then $N(t)$ is ergodic.

Proof. See [12], [22] if $V(k)$ is nonnegative, otherwise take $\tilde{V}(k) = V(k) - \xi$.

□

Remarks 1.

- (i) Theorem 1 is known as Foster's theorem [6], and its extensions are due to Pakes [12], Marlin [11], Rosberg [14] and Tweedie [22]. The most useful, from the application point of view, is Pakes' lemma which states: *an irreducible aperiodic Markov chain is ergodic if the average drift*
 $d(k) = E[N(t+1) - N(t) \mid N(t) = k]$, $k \in C$ *is finite for all* $k \in C$ *and*
 $\limsup_{k \rightarrow \infty} d(k) < 0$.

- (ii) Conditions (1) may be used to find upper bound for the first re-entry time to a set H which is now assumed to be finite or infinite. Let τ_H denote the first re-entry time to H , and $E_k \tau_H$ the average value of τ_H given $N(0) = k \in C$. Let also $\tau = \min(T, \tau_H)$, $T > 0$. Note that τ is a Markov moment and $Z(t) = V(N(t)) - \sum_{j=0}^{t-1} AV(N(j))$ is a martingale. Then, as a consequence of

Doob's optional stopping theorem for martingales [7] one obtains

$$E_k V(N(\tau)) \triangleq E[V(N(\tau)) \mid N(0) = k] = V(k) + E_k \sum_{t=0}^{\tau-1} AV(N(t))$$

The LHS of the above is lower bounded by ξ under the definition of $V(k)$, while the RHS in the presence of (1b) is upper bounded by: (i) $V(k) - \epsilon E_k \tau$ for $k \in C-H$; (ii) $V(k) + AV(k) - \epsilon E_k(\tau-1)$ for $k \in H$. Thus,

$$E_k \tau \leq \begin{cases} (V(k) - \xi)/\epsilon & \text{for } k \in C-H \\ 1 + (V(k) + AV(k) - \xi)/\epsilon & \text{for } k \in H \end{cases} \quad (2)$$

The above is valid for any $T > 0$, hence by (1a) $E_k \tau_H < \infty$ for $k \in C$. Inequalities (2) are very useful in establishing upper bounds for many quantities, e.g. busy period in queueing models.

(iii) Even though (2) is valid for infinite H the theorem is not true in that case. This limits the applications of it to multidimensional queueing systems, since mostly condition (1b) (or more precisely: condition $d(k) \leq -\epsilon$, $k \in C-H$) is violated on infinitely many states (e.g. on the boundary of the state space C). Rosenberg [13] extended Theorem 1 to a class of multidimensional Markov chains with an infinite set H . He considered multidimensional Lyapunov function $V = (V_1, V_2, \dots, V_M): C \rightarrow R^M$, where $C = \{k = (k_1, \dots, k_M): k_i \geq 0, i = 1, 2, \dots, M\}$. He proved that $AV_i(k) < \infty$, $k \in C$ and $AV_i(k) \leq -\epsilon_i$, $i = 1, 2, \dots, M$ on $k \in C-H_i$ are sufficient for ergodicity for a class of Markov chains, where $H_i = \{k \in C: k_i < N\}$, N is an integer. Unfortunately, the choice of H_i further limits applications of it to queueing problems, as we shall see later. We shall present another approach which overcomes some of the above mentioned restrictions.

In order to present necessary conditions for stability we shall study sufficient conditions for nonergodicity. In general, a simple analogy of Theorem 1 is not valid here, as noted by Kaplan [9]. Let $V_z(k)$, $k \in C$, $z \in [a, b]$ be parametric Lyapunov function, $V_z(\cdot): C \rightarrow R^+$, z is a parameter. We assume that $V_z(k)$,

$k \in C$ is differentiable with respect to $z \in [a,b]$ and let, as before, $AV_z(k)$ be the operator of a Markov chain $N(t)$, assuming it exists. We denote by $A'V_z(k)$ and $AV_z'(k)$, $k \in C$ the derivative of the operator $AV_z(k)$ and the operator of $V_z'(k)$, respectively. We restrict the class of parametric Lyapunov functions assuming:

- (i) $V_z(k)$ is bounded for all $k \in C$, $z \in [a,b]$
- (ii) there is $z_0 \in [a,b]$ such that $V_{z_0}(k) = \text{constant}$ for all $k \in C$
- (iii) let $U \subset [a,b]$ be a neighborhood of z_0 and H a finite subset of C ; then

$$V_z(j) \leq \inf_{i \in H} V_z(i) \text{ for all } j \in C-H$$

- (iv) let $g(z)$ be a nonnegative real-valued function of z defined on U such that $g(z_0) = 0$, $g'(z_0) \neq 0$; then for all $j \in C-H$ and $i \in H$ we assume

$$V_{z_0}'(j)/g'(z_0) \leq V_{z_0}'(i)/g'(z_0)$$

In addition, we adopt one more assumption that is more restrictive than necessary, but simplifies considerations. Namely,

- (v) for all $z \in U$ and $k \in C-H$ $V_z'(k) \leq V_{z_0}'(k)$

Then we prove the following theorem.

THEOREM 2.

If (i)-(v) hold and for some constant $B \geq 0$

$$| AV_{z_0}'(k) | < \infty \quad \text{for all } k \in C-H \quad (3)$$

$$- AV_{z_0}'(k) / g'(z_0) \geq 0 \quad \text{for all } k \in C-H \quad (4a)$$

$$- AV_z(k) \geq - Bg(z) \quad \text{for all } k \in C-H \text{ and } z \in U \quad (4b)$$

then $N(t)$ is not ergodic, assuming there is a *strict* inequality in (4a) for at least one $k \in C-H$.

Proof. From $N(t)$ construct the Markov chain $\hat{N}(t)$ with state space $C-H$, i.e. the

chain $\hat{N}(t)$ omits all states of H . Note that nonergodicity of $\hat{N}(t)$ implies non-ergodicity of $N(t)$. Moreover, it is easy to prove (for details see [20]) in the presence of (iii) and (iv) that (3), (4) for $N(t)$ imply (3), (4) for $\hat{N}(t)$. Hence, to prove the theorem it is enough to prove that (3), (4) are sufficient for nonergodicity of $\hat{N}(t)$. Suppose to the contrary that $\hat{N}(t)$ satisfying (3) and (4a,b) is ergodic. Then [1] there is a probability measure $\{\hat{\pi}_j\}_{j \in C-H}$ such that $\hat{\pi}_j > 0$ for all $j \in C-H$ and under (i) we find immediately that

$$\sum_{k \in C-H} \hat{\pi}_k \hat{A}V_z(k) = 0, \quad z \in U$$

Hence,

$$\begin{aligned} 0 &= \lim_{z \rightarrow z_0} \sum_{k \in C-H} \hat{\pi}_k [-\hat{A}V_z(k)/g(z)] \geq \sum_{k \in C-H} \hat{\pi}_k \lim_{z \rightarrow z_0} [-\hat{A}V_z(k)/g(z)] = \\ &= \sum_{k \in C-H} \hat{\pi}_k [-\hat{A}'V_{z_0}(k)/g'(z_0)] = \sum_{k \in C-H} \hat{\pi}_k [-\hat{A}V'_{z_0}(k)/g'(z_0)] > 0 \end{aligned} \quad (5)$$

where the first inequality on the RHS is a consequence of Fatou's lemma in the presence of (4b), the ensuing equality derives from l'Hopital's rule together with (ii) and (iv) and the last equality comes from Weierstrasse's criterion in the presence of (v) and (3). The last inequality follows from (4a) under $\hat{\pi}_k > 0$ for $k \in C-H$ and the assumption that in (4a) it is a strict inequality for some $k \in C-H$. This is a desired contradiction. □

Remarks 2.

(i) For $V_z(k) = z^k$, $z \in [0,1]$, $g(z) = 1 - z$, $z_0 = 1$ (see Kaplan [9]), one finds that $-\hat{A}V'_1(k)/g'(1) = E[N(t+1) - N(t) \mid N(t) = k] = d(k)$. Hence, *a Markov chain is not ergodic if $d(k) < \infty$, $d(k) \geq 0$ for all $k \in C-H$ (and $d(k) > 0$ for some $k \in C-H$), and for a constant $B \geq 0$*

$$\sum_{j \in C} P_{kj} (z^k - z^j) \geq -B(1-z), \quad k \in C-H, \quad z \in (0,1) \quad (6)$$

Inequality (6) is known as *Kaplan's condition* [9], while (4b) is called the *generalised Kaplan condition* [16] [20]. Originally, Kaplan [9] assumed $d(k) > 0$, however, $d(k) \geq 0$ with (6) is enough when $d(k) > 0$ for at least one $k \in C-H$, (see also [16]).

- (ii) Kaplan condition (6) and the generalized Kaplan condition (4b) are automatically satisfied if $N(t)$ is *uniformly downward bounded*, i.e., there exists a constant $m > 0$ such that the transition probabilities, $\{P_{kj}\}_{k,j \in C}$, satisfy $P_{kj} = 0$ for $j < k - m$. For details and other extensions see [16] and [20].
- (iii) Condition (3) and assumption (v) are necessary to assure that $A'V_{z_0}(k) = AV'_{z_0}(k)$. They may be replaced by a weaker condition on the function $V_z(k)$, $k \in C$ as proved in [20]. In particular, we do not need condition (3).

Let now $N^M(t) = (N_1(t), \dots, N_M(t))$ be an M -dimensional aperiodic, irreducible Markov chain. We shall establish two useful criteria for ergodicity and nonergodicity of $N^M(t)$ by applying Theorem 1 and 2. We define for $k = (k_1, k_2, \dots, k_M) \in C$

$$V(k) = \sum_{i=1}^M c_i k_i$$

$$V_z(k) = z^i \sum_{i=1}^M c_i k_i \quad z \in [0, 1]$$

where c_i are nonnegative constants, $i = 1, 2, \dots, M$. Noting that

$$AV(k) = AV'_1(k) = \sum_{i=1}^M c_i d_i(k)$$

where $d_i(k)$ is the i th component of a drift-vector of $N^M(t)$ ($d_i(k) = E[N_i(t+1) - N_i(t) \mid N(t) = k]$). Applying Theorem 1 and 2 (assume $g(z) = 1-z$, $z_0 = 1$) we find

COROLLARY 1. A Markov chain $N^M(t)$ is ergodic if for $\epsilon > 0$

$$\sum_{i=1}^M c_i d_i(k) \leq -\epsilon \quad \text{for all } k \in C-H \quad (6)$$

and

$$\left| \sum_{i=1}^M c_i d_i(k) \right| < \infty \quad \text{for all } k \in H$$

where H is a finite subset of C . □

COROLLARY 2. A uniformly downward bounded Markov chain $N^M(t)$ is not ergodic, if for a finite $H \in C$

$$\sum_{i=1}^M c_i d_i(k) \geq 0 \quad \text{for all } k \in C-H \quad (7)$$

with a strict inequality for at least one $k \in C-H$ □

Unfortunately, Corollaries 1 and 2 *do not present* sufficient and necessary condition for ergodicity of $N^M(t)$ since (6), (7) are sets of simultaneous inequalities over $C-H$, and it *is not true* that those values of input parameters which *do not satisfy* (7) automatically satisfy (6). The only known sufficient and necessary conditions for ergodicity are Malyshev's conditions [10] valid for a class of two-dimensional Markov chains. Let us also point out that the choice of constants c_i , $i = 1, \dots, M$ in (6) and (7), is crucial for applications as we shall see in Section 3.

2.2 Comparison tests

The Lyapunov function method discussed above has some limitations when applied to multidimensional Markov chains. Therefore, we present below so called *comparison tests* which overcome some of these restrictions.

The idea of comparison test is as follows: let $N(t)$ be a Markov chain and let there exist Markov chains $\tilde{N}(t)$ and $\underline{N}(t)$ such that $N(t)$ is stochastically smaller than $\tilde{N}(t)$ ($N(t) \underset{st}{\leq} \tilde{N}(t)$) and stochastically greater than $\underline{N}(t)$ ($\underline{N}(t) \underset{st}{\leq} N(t)$). Provided that $\tilde{N}(t)$ is ergodic, then $N(t)$ is ergodic, and if $\underline{N}(t)$ is not ergodic, then $N(t)$ is not ergodic. We shall generalize this idea to the multidimensional case.

We need some new notations. Let $N^M(t) = (N_1(t), \dots, N_M(t))$ be an M -dimensional Markov chain defined on a countable state space $C = I^M$, where I is a set of nonnegative integers. For a set of indices $M = \{1, 2, \dots, M\}$ of $N^M(t)$ we define a *cover* P_n of M as follows

$$P_n = \{\sigma_i = (m_1, \dots, m_L), i = 1, 2, \dots, n: m_k \in M, m_k \neq m_j \text{ if } k \neq j; \sum_{i=1}^n \sigma_i = M\}$$
 For example: if $M = \{1, 2, 3, 4\}$ then one may define $P_3 = \{\sigma_1 = (1, 2, 3), \sigma_2 = (2, 4), \sigma_3 = (3)\}$. We often use P instead of P_n to simplify the notation. Let now for a

given $\sigma = (m_1, \dots, m_L) \in \mathcal{P}_n$ define:

1. so called σ -filtered process of $N^M(t)$ denoted by $N_\sigma(t) \triangleq (N_{m_1}(t), \dots, N_{m_L}(t))$. We sometimes write $N_{m_1} \dots_{m_L}(t)$ instead of $N_\sigma(t)$.

2. let $\hat{N}_\sigma(t) \triangleq \hat{N}_{m_1} \dots_{m_L}(t) = (\hat{N}_{m_1}(t), \dots, \hat{N}_{m_L}(t))$ be an L-dimensional Markov chain defined on the same space as $N_\sigma(t)$, i.e., $C_\sigma \triangleq I^L$

3. for a given $x_\sigma \triangleq (x_{m_1}, \dots, x_{m_L})$ let

$$H(x_{m_1}, \dots, x_{m_L}) \triangleq H(x_\sigma) \triangleq \{y \in I^L : y_{m_i} < x_{m_i}, i = 1, 2, \dots, L\}.$$

Furthermore, we assume that for two vectors $x, y \in I^L$ $x < y$ iff $x_i < y_i$ $i = 1, 2, \dots, L$.

Then, we adopt a definition

DEFINITION. We say that $\sigma = (m_1, \dots, m_L)$ - filtered process $N_\sigma(t) = N_{m_1}, \dots, N_{m_L}(t)$ of an M-dimensional Markov chain $N^M(t)$ is stochastically smaller with respect to (w. r. t) distribution than a Markov chain $\hat{N}_\sigma(t) = \hat{N}_{m_1}, \dots, \hat{N}_{m_L}(t)$, denoted

$$N_{m_1}, \dots, N_{m_L}(t) \leq_d \hat{N}_{m_1}, \dots, \hat{N}_{m_L}(t) \quad (N_\sigma(t) \leq_d \hat{N}_\sigma(t)),$$

if for all $x = (x_1, \dots, x_L) \in I^L$ and $k = (k_1, \dots, k_M) \in I^M$, $t = 1, 2, \dots$

$$\Pr\{\hat{N}_{m_i}(t) < x_i, i = 1, 2, \dots, L \mid \hat{N}_{m_1}, \dots, \hat{N}_{m_L}(0) = (k_{m_1}, \dots, k_{m_L})\} \leq$$

$$\Pr\{N_{m_i}(t) < x_i, i = 1, 2, \dots, L \mid N^M(0) = (k_1, \dots, k_M)\}$$

or equivalently

(8)

$$\Pr\{\hat{N}_\sigma(t) \in H(x) \mid \hat{N}_\sigma(0) = k_\sigma\} \leq \Pr\{N_\sigma(t) \in H(x) \mid N^M(0) = k\}$$

□

Then, we prove

LEMMA 1. Let $N^M(t)$ be an aperiodic, irreducible M -dimensional Markov chain and let for all $\sigma \in P_n$ there exist irreducible, aperiodic Markov chains $\tilde{N}_\sigma(t)$ such that $N_\sigma(t) \leq_d \tilde{N}_\sigma(t)$, where $N_\sigma(t)$ is σ -filtered process of $N^M(t)$. Then $N^M(t)$ is ergodic if for all $\sigma \in P_n$, $\tilde{N}_\sigma(t)$ are ergodic.

Proof. Let $\sigma \in P_n$ and $x_\sigma, k_\sigma \in C_\sigma$. By our assumption, $\tilde{N}_\sigma(t)$ is ergodic, then for any $\varepsilon > 0$ there exists such $x_\sigma \in C_\sigma$ that for all $k_\sigma \in C_\sigma$ [1]

$$\lim_{t \rightarrow \infty} \Pr\{\tilde{N}_\sigma(t) \in \bar{H}(x_\sigma) \mid \tilde{N}_\sigma(0) = k_\sigma\} < \varepsilon/n. \quad (9)$$

Hence, there exists $x = (x_1, \dots, x_M) \in C = I^M$ that for all $k = (k_1, \dots, k_M) \in C = I^M$,

$$\begin{aligned} \lim_{t \rightarrow \infty} \Pr\{N^M(t) \in H(x) \mid N^M(0) = k\} &= 1 - \lim_{t \rightarrow \infty} \Pr\{N^M(t) \in \bar{H}(x) \mid N^M(0) = k\} \geq \\ 1 - \lim_{t \rightarrow \infty} \sum_{\sigma \in P_n} \Pr\{N_\sigma(t) \in \bar{H}(x_\sigma) \mid N^M(0) = k\} &\geq \end{aligned} \quad (10)$$

$$1 - \lim_{t \rightarrow \infty} \sum_{\sigma \in P_n} \Pr\{\tilde{N}_\sigma(t) \in \bar{H}(x_\sigma) \mid \tilde{N}_\sigma(0) = k_\sigma\} > 1 - \varepsilon.$$

where the equality on the RHS of (10) is derived from de Morgan's rule, the first inequality follows from a probability law which states that probability of a sum of events is smaller-equal than sum of event probabilities, the ensuing inequality is a consequence of (8) and the last inequality follows from (9).

Let us suppose to the contrary that $N^M(t)$ is not ergodic. Then for a finite number of states, e.g. $H(x)$, $\lim_{t \rightarrow \infty} \Pr\{N^M(t) \in H(x) \mid N^M(0) = k\} = 0$, which is a desired contradiction in the presence of (10).

□

By analogy, we prove the following lemma on nonergodicity of a multidimensional Markov chain:

LEMMA 2. Let $N^M(t)$ be as in Lemma 1 and let there exist $\sigma^* \in P$ and an aperiodic, irreducible Markov chain $\tilde{N}_{\sigma^*}(t)$ such that $\tilde{N}_{\sigma^*}(t) \leq_d N_{\sigma^*}(t)$ where $N_{\sigma^*}(t)$ is σ^* -filtered process of $N^M(t)$. If $\tilde{N}_{\sigma^*}(t)$ is not ergodic then $N^M(t)$ is not ergodic.

Proof. By our assumption, $N_{\sigma^*}(t)$ is not ergodic, then for all $x_{\sigma^*}, k_{\sigma^*} \in C_{\sigma^*}$ and finite $H(x_{\sigma^*})$

$$\lim_{t \rightarrow \infty} \Pr\{N_{\sigma^*}(t) \in H(x_{\sigma^*}) \mid N_{\sigma^*}(0) = k_{\sigma^*}\} = 0 \quad (11)$$

Hence, for all $x, k \in C = I^M$

$$0 \leq \lim_{t \rightarrow \infty} \Pr\{N^M(t) \in H(x) \mid N^M(0) = k\} \leq \lim_{t \rightarrow \infty} \Pr\{N_{\sigma^*}(t) \in H(x_{\sigma^*}) \mid N_{\sigma^*}(0) = k_{\sigma^*}\} = 0. \quad (12)$$

The last inequality is a consequence of the assumption and (8), and the last equality follows from (11).

Suppose to the contrary that $N^M(t)$ is ergodic. Then, since $N^M(t)$ is irreducible, $\lim_{t \rightarrow \infty} \Pr\{N^M(t) \in H(x) \mid N^M(0) = k\} > 0$, which is a desired contradiction to (12). \square

In order to apply Lemma 1 (or 2) we must find a Markov chain $\tilde{N}_{\sigma}(t) = \tilde{N}_{m_1}, \dots, \tilde{N}_{m_L}(t)$ which is stochastically greater w.r.t. distribution than $N_{\sigma}(t) = N_{m_1}, \dots, N_{m_L}(t)$ provided $\tilde{N}_{\sigma}(0) = N_{\sigma}(0)$. It may be difficult in practice to check if (8) is satisfied.

Therefore, we provide below a weaker condition than (8), that is sufficient for Lemma 1 (and 2). Let us notice that for a one-dimensional Markov chains $N(t)$ and $\tilde{N}(t)$ the condition (8) is equivalent to $N(t) \leq_{st} \tilde{N}(t)$ provided $N(0) = \tilde{N}(0)$, where \leq_{st} means *stochastically smaller* [8] [17]. It is well known that $N(t) \leq_{st} \tilde{N}(t)$ is equivalent to a *sample path comparison* which says that there exist stochastic processes $N(t)$ and $\tilde{N}(t)$ in the common probability space such that they have the same distribution functions as $N(t)$ and $\tilde{N}(t)$ respectively and every path of $\{N(t), t \geq 0\}$ lies below the corresponding sample path of $\{\tilde{N}(t), t \geq 0\}$ [8] [17]. In multidimensional state space I^M the ordering " \leq_d " is not equivalent to " \leq_{st} ", but " \leq_{st} " implies " \leq_d " [17]. Thus, assuming in Lemma 1 $N_{\sigma}(t) \leq_{st} \tilde{N}_{\sigma}(t)$, $N_{\sigma}(0) = \tilde{N}_{\sigma}(0)$ for all $\sigma \in P_n$ instead of $N_{\sigma}(t) \leq_d \tilde{N}_{\sigma}(t)$ we have a stronger assumption, but the advantage of the sample path comparison may be taken into account. Hence,

COROLLARY 3. Let hypotheses on $N^M(t)$, $N_\sigma(t)$ and $\tilde{N}_\sigma(t)$ from Lemma 1 hold together with $N_\sigma(t) \leq_{st} \tilde{N}_\sigma(t)$ instead of $N_\sigma(t) \leq_d \tilde{N}_\sigma(t)$ for all $\sigma \in P_n$. Then $N^M(t)$ is ergodic provided $N_\sigma(t)$ are ergodic for all $\sigma \in P_n$. □

COROLLARY 4. Let hypotheses on $N^M(t)$, $N_{\sigma^*}(t)$, $\tilde{N}_{\sigma^*}(t)$ from Lemma 2 hold together with $\tilde{N}_{\sigma^*}(t) \leq_{st} N_{\sigma^*}(t)$ (instead of $\tilde{N}_{\sigma^*}(t) \leq_d N_{\sigma^*}(t)$) for a given $\sigma^* \in P$. Then $N^M(t)$ is not ergodic if $\tilde{N}_{\sigma^*}(t)$ is not ergodic. □

Applications of the above corollaries are given in Section 3.

2.3 Unbounded random walk

We shall present one more method giving sufficient condition for nonergodicity of a multidimensional Markov chain. Let us explain the idea on an example. We denote by $N(t)$ the queue length in a system, and by $X(t)$ an arrival process. Let

$$N(t+1) = [N(t) - Y(t)]^+ + X(t)$$

where $a^+ = \max\{0, a\}$ and $Y(t)$ is a stochastic process, $t = 0, 1, 2, \dots$. Assuming $X(t)$, $Y(t)$ are independently, identically distributed (i.i.d.) random variables for all $t = 0, 1, 2, \dots$, $N(t)$ is a Markov chain defined on nonnegative integers I .

Let us define now a new process, $N^*(t)$, $t = 0, 1, 2, \dots$, on all integers I_0 as

$$N^*(t+1) = N^*(t) - Y(t) + X(t).$$

Provided $X(t)$ and $Y(t)$ are statistically independent of $N(t)$ the increments $U(t) = N^*(t+1) - N^*(t) = X(t) - Y(t)$ defined all integers on I_0 is an i.i.d. sequence of random variables. In other words, $N^*(t+1) = N^*(0) + \sum_{\tau=0}^t U(\tau)$ is an unbounded random walk on I_0 . We want to derive some properties of $N(t)$ from the behavior of $N^*(t)$. Note that $N(t)$ and $N^*(t)$ behave in the same way on I_+ , where I_+ is a set of positive integers. In particular, on I_+ , $N(t)$ has i.i.d. increments $U(t)$. We claim that $N(t)$ is not ergodic if $EU = EU(1) > 0$, under some other restrictions mentioned below. This is obviously true, but we shall generalize the above idea to the multidimensional case.

Let I_+ , I , I_0 denote positive, nonnegative and all integers respectively and let $N^M(t) = (N_1(t), \dots, N_M(t))$ be M-dimensional Markov chain defined on I^M . For a $d = (d_1, d_2, \dots, d_M) \in I_+^M$ define

$$A(d) = \{k \in I_+^M : k \geq d\} ; \bar{A}(d) = I_0^M - A(d) = \prod_{i=1}^M \{k \in I_0^M : k_i \leq d_i, i = 1, 2, \dots, M\}$$

Then we prove

THEOREM 3. Let $N^M(t)$, $t = 0, 1, \dots$, be an aperiodic, irreducible M-dimensional Markov chain, and let Ω be a set t_i , $i = 1, 2, \dots$, such that $N^M(t_i) \in A(d)$ for some $d \in I_+^M$. If for all $t_i \in \Omega$, $i = 1, 2, \dots$, the increments $U^M(t_i) = N^M(t_{i+1}) - N^M(t_i) = (U_1(t_i), U_2(t_i), \dots, U_M(t_i))$ satisfy the conditions:

- i) $U^M(t_i)$ are i.i.d. random variables for all $t_i \in \Omega$
- ii) a moment generating functions of $U_m(t_1)$, $m = 1, 2, \dots, M$ exist, i.e., for $h \in \mathbb{R}$ $|Ee^{-h U_m(t_1)}| < \infty$
- iii) for all $m \stackrel{\Delta}{=} 1, 2, \dots, M$ $EU_m = EU_m(t_1) > 0$

then $N^M(t)$ is not ergodic.

Proof. Let us define an M-dimensional process $N^*(t) = (N_1^*(t), \dots, N_M^*(t))$ on I_0^M as follows

$$N^*(t+1) = N^M(0) + \sum_{\tau=1}^t U^M(\tau) \quad (13)$$

where $U^M(t)$ is defined on $A(d)$. In other words, $N^*(t)$ mimics the behaviour of $N(t)$ on $A(d)$, but it is extended to the I_0^M space. Obviously, $N^*(t)$ is unbounded random walk on I_0^M . If $N^*(0) = N^M(0)$ and (13) holds, then for $k \in A(d)$

$$\Pr\{\prod_{t=1}^{\infty} [N(t) \in \bar{A}(d)] \mid N^M(0) = k\} = \Pr\{\prod_{t=1}^{\infty} [N^*(t) \in \bar{A}(d)] \mid N^*(0) = k\} \quad (14)$$

since the event $\prod_{t=1}^{\infty} [N^M(t) \in \bar{A}(d) \mid N^M(0) = k]$ may be interpreted as the probability that $N^M(t)$ starting from $N^M(0) = k \in A(d)$ ever reaches $\bar{A}(d)$, and $N^M(t)$, $N^*(t)$

behave the same in $A(d)$. Then, by (13) and (14)

$$\begin{aligned} \Pr\{\prod_{t=1}^{\infty} [N^M(t) \in \bar{A}(d)] \mid N^M(0) = k\} &= \Pr\{\prod_{t=1}^{\infty} \prod_{m=1}^M [N_m^*(t) \leq d_m] \mid N^M(0) = k\} \\ &\leq \prod_{m=1}^M \Pr\{\prod_{t=1}^{\infty} \sum_{\tau=1}^t U(\tau) \leq d_m - k_m\} \end{aligned} \quad (15)$$

In Appendix A we prove that under assumptions (i) - (iii) for any $\varepsilon > 0$ there exists $k(\varepsilon, d) \in I^M$ such that the RHS of (15) is smaller than ε/M . Hence

$$\Pr\left\{\bigcup_{t=1}^{\infty} [N^M(t) \in \bar{A}(d) \mid N^M(0) = k(\varepsilon, d)]\right\} < \varepsilon. \quad (16)$$

Suppose now to the contrary that $N^M(t)$ is ergodic. Then for any $\varepsilon > 0$ there exists $d_1(\varepsilon) \in I^M$ such that for all $k \in I^M [1]$

$$\lim_{t \rightarrow \infty} \Pr\{N^M(t) \in A(d_1(\varepsilon)) \mid N^M(0) = k\} < \varepsilon$$

which implies that for any $\delta > 0$ exists $T(\delta)$ such that for $t > T(\delta)$

$$\Pr\{N^M(t) \in A(d_1(\varepsilon)) \mid N^M(0) = k\} < \varepsilon + \delta. \quad (17)$$

Assume now in (16) that $t_1 > T(\delta)$ and $d = d_1(\varepsilon)$. Then (16) implies that for $k = k(\varepsilon, d_1(\varepsilon))$

$$\Pr\{N^M(t_1) \in \bar{A}(d_1(\varepsilon)) \mid N^M(0) = k(\varepsilon, d_1(\varepsilon))\} < \varepsilon. \quad (18)$$

For $\bar{A}(d_1(\varepsilon)) \cup A(d_1(\varepsilon)) = I_o^M \supset C$, and summing (17) and (18) for $t = t_1$ and $k = k(\varepsilon, d_1(\varepsilon))$ we prove that $1 < 2\varepsilon + \delta$. This is a desired contradiction as a consequence of arbitrary ε and δ .

□

Remarks 3.

- i) Under the hypotheses of Theorem 3 we may prove that $N^M(t)$ is a transient Markov chain. The proof is more complicated, while from the application point of view nonergodicity is a strong enough property.
- ii) Assumption (i) in the theorem may be relaxed. Instead of identically distributed $U^M(t) = (U_1(t), U_2(t), \dots, U_M(t))$ we must assume that for any $m = 1, 2, \dots, M$ $EU_m = EU_m(t)$ for all $t = 1, 2, \dots$
- iii) It should be noted that the converse of assumption (iii) of the theorem, i.e., $EU_m < 0$ for all $m = 1, 2, \dots, M$, is not sufficient for ergodicity, as we shall see in the next section.

In order to find stability conditions for ALOHA system we shall first apply Lemma 1, 2 and Theorem 3. Let us define for $m \in M$ three stochastic processes:

$$\tilde{N}_m(t+1) = \{\tilde{N}_m(t) - Z_m(t)[1 - \prod_{j \in M - \{m\}} Z_j(t)]^+\}^+ + X_m(t) \quad (21a)$$

$$\underline{N}_m(t+1) = \{\underline{N}_m(t) - Z_m(t)\}^+ + X_m(t) \quad (21b)$$

$$N_m^*(t+1) = N_m^*(t) - Z_m(t)[1 - \prod_{j \in M - \{m\}} Z_j(t)]^+ + X_m(t) \quad (21c)$$

The above processes may be interpreted as: $\tilde{N}_m(t)$ is a queue length in the m -th buffer under the condition that all others queues in the ALOHA system are *never empty*; $\underline{N}_m(t)$ is the queue length in the m -th buffer if all other queues are *always empty*; and $N_m^*(t)$ is an unbounded random walk on I_0 (all integers) under the assumption that all other buffers are never empty. Note that $\tilde{N}_m(t)$, $\underline{N}_m(t)$ and $N_m^*(t)$ are one-dimensional Markov chains, where the first two are defined on I (nonnegative integers), the third one on I_0 . We shall show that for all $m \in M$ the Markov chains $\tilde{N}_m(t)$, $\underline{N}_m(t)$ and $U_m(t) = N_m^*(t+1) - N_m^*(t)$ satisfy the assumptions of Lemma 1, Lemma 2 and Theorem 3, respectively.

To apply Lemma 1 and 2 (more precisely: Corollary 3 and 4) we first prove that $\underline{N}_m(t) \leq_{st} \tilde{N}_m(t)$ and $\underline{N}_m(t) \leq_{st} N_m(t)$ under the assumption that $\underline{N}_m(0) = \tilde{N}_m(0) = N_m(0)$, where $N_m(t)$ is an m -filtered process of $N^M(t)$. In order to take advantage of the sample path theorem, as the first step we construct three queueing systems on the same probability space, modelled by (20), (21a) and (21b) respectively, which have identical arrival processes $X_m(t)$ and identical control-vectors $Z(t) = (Z_1(t), \dots, Z_M(t))$ for all t . *This not only means that the arrival processes and control-vector have the same joint distributions, but that they have the same sample paths.* Let us denote the queue lengths in these new constructed systems by $N_m(t)$, $\tilde{N}_m(t)$ and $\underline{N}_m(t)$, respectively. They satisfy (20), (21a) and (21b) with $X_m(t)$ and $Z(t)$ replace by $X_m(t)$ and $Z(t)$. Assume now $\underline{N}_m(0) = \tilde{N}_m(0) = N_m(0)$. Then, comparing these three processes or equivalently (20), (21a) and (21b), one shows immediately that for all $t \geq 0$: $\{\underline{N}_m(t), t \geq 0\} \leq \{N_m(t), t \geq 0\}$, $\{N_m(t), t \geq 0\} \leq \{\tilde{N}_m(t), t \geq 0\}$, which by the sample path theorem proves that $\underline{N}_m(t) \leq_{st} \tilde{N}_m(t)$ and

$N_m(t) \stackrel{st}{\leq} N_m(t)$. On the other hand, applying Theorem 1 to $\tilde{N}_m(t)$ we show that $\tilde{N}_m(t)$ is ergodic if $\lambda_m < q_m$, where

$$q_m \triangleq r_m \prod_{\substack{j=1 \\ j \neq m}}^M \bar{r}_j$$

By Theorem 2 we also show that $N_m(t)$ is not ergodic if $\lambda_m \geq r_m$. Hence, we have proved Property 1. (a) The ALOHA system is ergodic if for all $m \in M$

$$\lambda_m < q_m \tag{22}$$

(b) The ALOHA system is not ergodic if for an $m \in M$

$$\lambda_m \geq r_m \tag{23}$$

We denote by E_1^C and J_1^C the subsets of the ergodicity region E , and nonergodicity region, J , such that $\lambda \in E_1^C$ if (22) is satisfied and $\lambda \in J_1^C = \bigcup_{m=1}^M \{\lambda \in R_+^M : \lambda_m \geq r_m\}$.

To apply Theorem 3, note that $U_m(t) = N_m^*(t+1) - N_m^*(t)$ satisfies conditions (i) and (ii), and $EU_m > 0$, i.e., $\lambda_m > q_m$. Hence,

Property 2. The ALOHA system is not ergodic if for all $m \in M$

$$\lambda_m > q_m \tag{24}$$

We denote by J^R a subset of J such that (24) is satisfied for all $m \in M$. Note that neither $J^R \subset J_1^C$ nor $J_1^C \subset J^R$ for $M > 1$.

As a consequence of Property 1a and 2 we find sufficient and necessary conditions for ergodicity of a buffered symmetric ALOHA system:

Property 3. Let $\lambda = \lambda_m$ and $r = r_m$ for all $m \in M$. Then for $N(t)$ to be ergodic it is necessary to have

$$\lambda \leq r \bar{r}^{M-1} \tag{25a}$$

and sufficient to have

$$\lambda < r \bar{r}^{M-1} \tag{25b}$$

□

We shall enhance further the conditions (22) and (23), but now we shall deal with Lyapunov function method, i.e., we apply Corollaries 1 and 2 to the ALOHA system. Note that the conditional throughput $S_o^m(n)$, $n \in I^M$, $m \in M$ in the system is

$$S_o^m(n_1, \dots, n_M) = r_m \chi(n_m) \prod_{\substack{j=1 \\ j \neq m}}^M \frac{r_j}{r_j} \chi(n_j) \quad (26)$$

Then, letting $H = (0, 0, \dots, 0)$ in Corollaries 1 and 2, we know that $N^M(t)$ is ergodic if

$$\sum_{i=1}^M c_i \lambda_i < \sum_{i=1}^M c_i S_o^i(n) \triangleq S_o^\Sigma(n) \quad (27a)$$

and nonergodic if

$$\sum_{i=1}^M c_i \lambda_i \geq \sum_{i=1}^M c_i S_o^i(n) \triangleq S_o^\Sigma(n) \quad (27b)$$

for all $n \in C-H$. But, by the property of the function $\chi(n)$, we may restrict a set of $n \in C-H$ satisfying (27) to $C-H \supset D = \{(n_1, \dots, n_M) : n_i = 1 \text{ or } n_i = 0, i = 1, 2, \dots, M\}$. Hence, there are $2^M - 1$ inequalities in (27). Since the LHS of (27) is the same for all $n \in D$, then for (27a) we must find the smallest value of the RHS of (27a), while for (27b) the greatest value of (27b) must be determined. These values depend on the constant $c_m \geq 0$ $m \in M$, as the following shows

Property 4. Let $\sum_{i=1}^M r_i \leq 1$. Then,

a) $N^M(t)$ is *ergodic* if

$$\sum_{i=1}^M \frac{\lambda_i}{r_i} < 1 \quad (28a)$$

b) $N^M(t)$ is *not ergodic* if

$$\sum_{i=1}^M \frac{\lambda_i}{r_i} \geq \sum_{i=1}^M \frac{1}{r_i} \prod_{j=1}^M r_j \quad (28b)$$

or if for an $m \in M$

$$\lambda_m + \frac{r_m}{r_m} \prod_{\substack{j=1 \\ j \neq m}}^M r_j \geq r_m \quad (28c)$$

Property 5. For all $n, m \in M$ $r_n + r_m \geq 1$. Then $N^M(t)$ is not ergodic if

$$\sum_{i=1}^M \frac{\lambda_i}{r_i} \geq 1 \quad (28d)$$

Proof of (28a). Assume in (27a) $c_i = 1/r_i$ for all $i \in M$. We have to prove that the smallest value of the RHS of (27a) is equal to 1. Let us adopt a notation: an element $n \in D$ we denote by $n = (1^{i_1}, 1^{i_2}, \dots, 1^{i_k}, 0^{M-k})$ if at positions $i_1, i_2, \dots, i_k \in M$ of n there are 1, otherwise zeros. Then denoting by $S_o^\Sigma(n)$ the RHS of (27) we have $S_o^\Sigma(1^{i_1}, 0^{M-k}) = 1$, while for $k \geq 2$

$$S_o^\Sigma(1^{i_1}, \dots, 1^{i_k}, 0^{M-k}) = \prod_{i=1}^k \prod_{\substack{j=1 \\ j \neq i}}^k \bar{r}_{i_j} \quad (29)$$

By induction we may easily prove that for all $k \geq 1$

$$\prod_{i=1}^k \bar{r}_{i_j} \geq 1 - \prod_{i=1}^k r_i \quad (30)$$

Then, applying (30) to (29) we obtain

$$S_o^\Sigma(1^{i_1}, \dots, 1^{i_k}, 0^{M-k}) \geq \prod_{i=1}^k (1 - \prod_{\substack{j=1 \\ j \neq i}}^k r_{i_j}) = k - (k-1) \prod_{i=1}^k r_{i_j} \geq 1$$

where the last inequality follows from $\sum_{i=1}^M r_i \geq 1$.

Proof of (28b). For nonergodicity we must consider (27b) and choose the greatest value of $S_o^\Sigma(n)$ for $n \in D$. Let us put $c_i = \bar{r}_i$, $i \in M$ and note that

$$S_o^\Sigma(1, 1, \dots, 1) = \prod_{i=1}^M \bar{r}_i \prod_{j=1}^M r_j$$

i.e., $S_o^\Sigma(1, \dots, 1)$ is equal to the RHS of (28b). Let now $(i_1, \dots, i_{M-1}) \in M^{M-1}$. Then, show that $S_o^\Sigma(1, 1, \dots, 1) \leq S_o^\Sigma(1^{i_1}, \dots, 1^{i_{M-1}}, 0)$ iff $\sum_{i=1}^M r_i \leq 1$. Use induction to prove that $S_o^\Sigma(1^{i_1}, \dots, 1^{i_k}, 0^{M-k}) \leq S_o^\Sigma(1^{i_1}, \dots, 1^{i_{k-1}}, 0^{M-1+k})$ if $\sum_{i=1}^M r_i \leq 1$, for all $k = M-1, M-2, \dots, 2$. This shows (28b).

Proof of (28a). Choose for a given $m \in M$ $c_m = 1$, $c_i = \frac{r_m}{r_i}$ for $i \in M - \{m\}$. Obviously, $S_o^\Sigma(1^m, 0^{M-1}) = r_m$. Let $(i_1, \dots, i_k) \in M^k$, and consider two cases:

a) for some $1 \leq j \leq k$ $i_j = m$. For simplicity assume $j = 1$. Then

$$S_o^\Sigma(1^m, 1^{i_2}, \dots, 1^{i_k}, 0^{M-k}) = r_m \left(\prod_{j=2}^M \bar{r}_{i_j} + \sum_{n=2}^k r_{i_n} \prod_{\substack{j=2 \\ j \neq n}}^k \bar{r}_{i_j} \right) \leq r_m$$

since the expression in the brackets is not a complete Bernoulli distribution, it is smaller than 1.

b) for all $1 \leq j \leq k$ $i_j \neq m$. Then

$$S_o^\Sigma(1^{i_1}, \dots, 1^{i_k}, 0^{M-k}) = \frac{r_m}{r_m} \sum_{n=1}^k r_{i_n} \prod_{\substack{j=1 \\ j \neq n}}^k \bar{r}_{i_j} \leq \frac{r_m}{r_m} \bar{r}_m = r_m$$

Thus, we proved that $S_o^\Sigma(1^m, 0^{M-1})$ is the greatest among all values of RHS of (27b).

Proof of (28d). This follows the proof of (28a). In particular, we must show that (29) is greater than 1 under the assumption of Property 5. Note that for $k = 2$

$$S_o^\Sigma(1^{i_1}, 1^{i_2}, 0^{M-2}) = \bar{r}_{i_1} + \bar{r}_{i_2} = 2 - (r_{i_1} + r_{i_2}) \leq 1$$

since for all $i_1, i_2 \in M$ $r_{i_1} + r_{i_2} \leq 1$. Then use induction with respect to k to prove (28d). □

We denote by E_1^L a set of $\lambda \in \mathcal{R}_+^n$ satisfying (28a), and by J_1^L, J_2^L, J_3^L subsets of nonergodicity region J satisfying (28b), (28c) and (28d) respectively, where L stands for the Lyapunov function method. Note that $J_2^L = \bigcup_{m=1}^M \{\lambda \in \mathcal{R}_+^M : (28c) \text{ holds for } m \in M\}$. Moreover, applying (28b) to the symmetric case we may show for $r \leq 1/M$ that a necessary condition for ergodicity is $\lambda < rr^{-M-1}$ instead of $\lambda \leq rr^{-M-1}$ as in (25a).

Before we present some remarks about the above proven results we first enhance the conditions given in Property 1 by applying Lemma 1 and 2 to a more sophisticated case. Proving Property 1 we found that one dimensional Markov chains $\tilde{N}_m(t)$ and $N(t)$ upper and lower bounded m -filtered process $N_m(t)$ of $N^M(t)$. In other words, as a cover of M we chose 1-tuples $\sigma_1 = (1), \sigma_2 = (2), \dots, \sigma_M = (M)$, $P_M = \{\sigma_1, \sigma_2, \dots, \sigma_M\}$, since ergodicity and nonergodicity conditions of $N_m(t), \tilde{N}_m(t), m = 1, 2, \dots, M$ were

relatively easy to find. However, due to Malyshev [10] we also know sufficient and necessary conditions for ergodicity of a class of two-dimensional Markov chains. Let us, as an example, consider a two-queue ALOHA system, $N^2(t) = (N_1(t), N_2(t))$. Naturally, the average drift $d(k) = (d_1(k), d_2(k))$, $k \in I^2$ is constant on positive integers, $I_+^2 = \{(k_1, k_2) : k_1 > 0, k_2 > 0, k_i \text{-integer}\}$ and equal to $d_1 \stackrel{\Delta}{=} \lambda_1 - r_1 \bar{r}_2$, $d_2 \stackrel{\Delta}{=} \lambda_2 - \bar{r}_1 r_2$ (this condition is required in Malyshev's theorem). Introducing

$$a_1(2) = \bar{r}_1 d_1 + r_1 d_2 ; a_2(1) = r_2 d_1 + \bar{r}_2 d_2$$

Malyshev's conditions imply that [19]:

the ALOHA system with $M = 2$ is ergodic if and only if

$$i) \quad a_1(2) < 0 \text{ and } a_2(1) < 0 \text{ if } r_1 + r_2 \leq 1 \quad (31a)$$

$$ii) \quad a_1(2) < 0 \text{ or } a_2(1) < 0 \text{ if } r_1 + r_2 > 1 \quad (31b)$$

(see Fig. 1)

We generalize conditions (31) to an M -dimensional ALOHA system. Therefore, let us choose two queues, $n, m \in M$ and define

$$\tilde{N}_m(t+1) = \{\tilde{N}_m(t) - Z_m(t)[1 - \chi(N_n(t))_{j \in M - \{n, m\}} Z_j]^+ \}^+ + X_m(t) \quad (32)$$

$$\tilde{N}_n(t+1) = \{\tilde{N}_n(t) - Z_n(t)[1 - \chi(N_m(t))_{j \in M - \{n, m\}} Z_j]^+ \}^+ + X_n(t)$$

as well as

$$\tilde{N}_m(t) = \{\tilde{N}_m(t) - Z_m(t)[1 - Z_n(t)\chi(N_n(t))]\}^+ + X_m(t) \quad (33)$$

$$\tilde{N}_n(t) = \{\tilde{N}_n(t) - Z_n(t)[1 - Z_m(t)\chi(N_m(t))]\}^+ + X_n(t)$$

Under the assumptions mentioned above $\tilde{N}_{mn}(t) = (\tilde{N}_m(t), \tilde{N}_n(t))$ and $N_{mn}(t) = (N_m(t), N_n(t))$ are two-dimensional Markov chains. Moreover, $\tilde{N}_{nm}(t)$ may be interpreted as queue lengths in the n -th and m -th queues under the condition that all other queues in the ALOHA system are never empty; $N_{mn}(t)$ models the queue lengths in the system under the condition that all other queues are always empty (or in other words, $N_{nm}(t)$ represents the ALOHA system with two queues, n and m).

Now let for $n, m \in M$

$$d_n = \lambda_n - r_n \prod_{\substack{j=1 \\ j \neq n}}^M \bar{r}_j \quad d'_n = \lambda_n - r_n \bar{r}_m$$

$$a_n(m) = d_n \bar{r}_n + d'_m r_n \quad a'_n(m) = d'_n \bar{r}_n + d'_m r_n$$

Then, using (31a) we may prove that [19]

(a) N_{mn} is ergodic if and only if ($r_j \neq 0$ for $j \in M - \{n, m\}$)

i) $a_n(m) < 0$ and $a_m(n) < 0$ if $r_n + r_m \leq 1$

ii) $a_n(m) < 0$ or $a_m(n) < 0$ if $r_n + r_m > 1$

(34)

(b) N_{mn} is not ergodic if and only if

i) $a'_n(m) \geq 0$ and $a'_m(n) \geq 0$ if $r_n + r_m \leq 1$

ii) $a'_n(m) \geq 0$ or $a'_m(n) \geq 0$ if $r_n + r_m > 1$

(35)

Moreover, using the same sample path arguments as in the proof of Property 1 we show that the (m, n) -filtered process of $N^M(t)$, $N_{mn}(t) = (N_m(t), N_n(t))$ satisfies: $N_{mn}(t) \leq_{st} \tilde{N}_{mn}(t)$ and $N_{mn}(t) \leq_{st} N_{mn}(t)$. Hence, Lemma 1 and 2 may be applied to determine stability conditions of $N^M(t)$.

According to Lemma 1 and 2 we must define a cover P_n of M . Let $P_n = \{\sigma_i; \text{ for } i = 1, 2, \dots, n; \sigma_i = (n) \text{ or } \sigma_i = (n, m), n, m \in M\}$. Then

$$\Lambda_{\sigma}^E \triangleq \begin{cases} \{\lambda \in R_+^M : \lambda_n < q_n\} & \text{if } \sigma = (n) \\ \{\lambda \in R_+^M : \lambda_n \text{ and } \lambda_m \text{ satisfy (34)}\} & \text{if } \sigma = (n, m) \end{cases}$$

$$\Lambda_{\sigma}^J \triangleq \begin{cases} \{\lambda \in R_+^M : \lambda_n \geq r_n\} & \text{if } \sigma = n \\ \{\lambda \in R_+^M : \lambda_n \text{ and } \lambda_m \text{ satisfy (35)}\} & \text{if } \sigma = (n, m) \end{cases}$$

and finally,

$$\bar{E}_2^C = \bigcap_{\sigma \in P_n} \Lambda_{\sigma}^E \quad J_3^C = \bigcup_{\sigma \in P_n} \Lambda_{\sigma}^J \quad (36)$$

where P_n is a cover of M such that $\sigma \in P_n$ is either a 1-tuple $\sigma = (n)$ or a 2-tuple $\sigma = (n,m)$. Hence, we proved

Property 6. (a) The ALOHA system is ergodic if $\lambda \in E_2^C$

(b) The ALOHA system is not ergodic if $\lambda \in J_3^C$. □

Comparing E_1^C and J_1^C found in Property 1 and E_2^C, J_3^C it is obvious that $E_1^C \subset E_2^C$ and $J_1^C \subset J_3^C$, however, inequalities (22), (23) determining E_1^C and J_1^C are given explicitly, while E_2^C, J_3^C are defined using slightly more complicated procedures.

We illustrate these results by two examples. First, we consider the two-queue ALOHA system. Then, (31) gives sufficient and necessary conditions for ergodicity of $N^2(t)$. In Fig. 1 we plotted ergodicity region $E = \langle OABC \rangle$ for $r_1 + r_2 \leq 1$ (Fig. 1b) and for $r_1 + r_2 > 1$ (Fig. 1a). On the other hand, by Property 1 (P. 1) and Property 4a (P. 4a) we find subsets of E , namely $E_1^C = \langle OB_2B_1 \rangle$ (P. 1), $E_1^L = \langle OAC \rangle$ (P. 4a) for $r_1 + r_2 \leq 1$, and $E_1^C = \langle OB_2B_1 \rangle$ for $r_1 + r_2 > 1$. Applying Property 4b to $m = 1$ and $m = 2$ in formula (28c) ($r_1 + r_2 \leq 1$) we recognize the whole nonergodicity region J which consists of points $(\lambda_1, \lambda_2) \in R_+^2$ outside $\langle OABC \rangle$. We denote such a region by $\langle ABC\infty \rangle$, where ∞ indicates the point in the infinity which belongs to the outside of $\langle OABC \rangle$ in R_+^2 . Let us point out that $J_2^L = J = \langle ABC\infty \rangle = \langle AA'\infty \rangle \cup \langle C'C\infty \rangle$ and $\langle AA'\infty \rangle$ is determined by (28c) for $m = 1$, and $\langle C'C\infty \rangle$ for $m = 2$. In the case $r_1 + r_2 > 1$ we apply Property 2 and Property 5 to find subsets of nonergodicity region $J_1^R = \langle B''BB'\infty \rangle$ (P. 2) and $J_3^L = \langle AC\infty \rangle$ (P. 5). In fact, in that case we may determine exactly the ergodicity region $E = \langle OABC \rangle$ by Lyapunov method substituting in (27a) $c_1 = 1, c_2 = \frac{1}{r_1}$ or $c_1 = \frac{1}{r_2}, c_2 = 1$, but this cannot be extended to $M > 2$.

Let us now consider a more sophisticated example in the three-dimensional state space to illustrate applications of Property 6. In Fig. 2 we plotted subsets of ergodicity region for $r_1 = 0.3, r_2 = 0.2$ and $r_3 = 0.1$. Property 1a shows that $E \supset E_1^C = \langle O\alpha\beta\gamma\sigma\theta\phi\omega \rangle$, and application of Property (4a) gives us $E \supset E_1^L = \langle O\Sigma\Psi\Gamma \rangle$.

Applying Property 6a to the following three covers: $P_2^{(1)} = \{\sigma_1 = (1,2), \sigma_2 = (3)\}$, $P_2^{(2)} = \{\sigma_1 = (1), \sigma_2 = (2,3)\}$ and $P_2^{(3)} = \{\sigma_1 = (1,3), \sigma_2 = (2)\}$, we finally obtain $E_2^C = \langle OAB\delta CDE\delta FGH\phi I\omega \rangle$. Obviously, $E_1^C \subset E_2^C$, but neither $E_1^L \subset E_2^C$ nor $E_2^C \subset E_1^L$. It should be also clear, by Property 1 and 2, that the point $\omega = (q_1, q_2, \dots, q_M)$ (for $M = 3, \omega = (r_1 \bar{r}_2 \bar{r}_3, \bar{r}_1 r_2 \bar{r}_3, \bar{r}_1 \bar{r}_2 r_3)$) is the boundary point of E , that is, in any neighborhood of ω there are points $\lambda \in \mathbb{R}_+^M$ belonging to E and J . On the other hand, by Property 1b and (4a), we find that $\Psi = (r_1, 0, 0)$, $\Sigma = (0, r_2, 0)$ and $\Gamma = (0, 0, r_3)$ are boundary points (more generally: any point $(0, 0, \dots, r_k, 0, \dots, 0)$ where r_k is in the k -th position of M -tuple is a boundary point). This confirms Fig. 3, where subsets of nonergodicity region J are plotted, namely: $J_2^L = \langle O\Xi\Psi\Gamma\Phi\Omega\omega \rangle$ (Property 4b, formula (28c) for $m = 1, 2, 3$) and $J^R = \langle \omega XYZ \rangle$ (Property 2). In Fig. 3 we mark ergodicity region found in Fig. 2.

Remarks 4.

- (i) It might be interesting to find such constants c_1, c_2, \dots, c_M in the Lyapunov method (formula (27)) which determine the largest ergodicity or nonergodicity region. By a simple geometrical consideration under the assumption $c_i \geq 0$ $i = 1, 2, \dots, M$ it is easy to notice that such optimal values of the constants are $c_i = \frac{1}{r_i}$ for $\sum_{i=1}^M r_i \leq 1$, as it was done in Property 4a.
- (ii) We conjecture that for $M = 3$ the ergodicity region E is $\langle O\Xi\Psi\Gamma\Phi\Omega\omega \rangle$, where the points $\Xi = (r_1 \bar{r}_2, \bar{r}_1 r_2, 0)$, $\Phi = (r_1 \bar{r}_3, 0, \bar{r}_1 r_3)$, $\Gamma = (0, r_2 \bar{r}_3, \bar{r}_2 r_3)$ lie on the planes $\lambda_3 = 0$, $\lambda_2 = 0$ and $\lambda_1 = 0$ respectively. They are determined as boundary points of the ergodicity region of the two-dimensional Markov chains assuming the third queue is not fed, i.e., $\lambda_1 = 0$, or $\lambda_2 = 0$, or $\lambda_3 = 0$ (the points Ξ, Φ, Γ are equivalent to point B in Fig. 1).

We shall study stability conditions of $N^M(t)$ through the results proven before.

In particular, to apply Lemma 1, 2 and Theorem 3 let us define

$$V_m^{\min} = \min \{V_m^{(0)}, V_m^{(1)}\} \quad V_m^{\max} = \max \{V_m^{(0)}, V_m^{(1)}\}, \quad m \in M$$

and

$$\tilde{N}_m(t+1) = [\tilde{N}_m(t) - V_m^{\min}]^+ + X_m(t) \quad (38a)$$

$$\underline{N}_m(t+1) = [\underline{N}_m(t) - V_m^{\max}]^+ + X_m(t) \quad (38b)$$

$$N_m^*(t) = N_m^*(t) - V_m^{(1)} + X_m(t) \quad (38c)$$

Using sample path arguments, as in the proof of Property 1 in Sec. 3.1, we show that m -filtered process of $N^M(t)$, $N_m(t)$, satisfies $N_m(t) \leq_{st} \tilde{N}_m(t)$ and $\underline{N}_m(t) \leq N_m(t)$.

Hence, by Lemmas 1 and 2,

Property 7. a) The coupled-processors system is *ergodic* if for all $m \in M$

$$\lambda_m < V_m^{\min} = \min \{V_m^{(0)}, V_m^{(1)}\} \quad (39a)$$

b) The system is *not ergodic* if for an $m \in M$

$$\lambda_m \geq V_m^{\max} = \max \{V_m^{(0)}, V_m^{(1)}\} \quad (39b)$$

□

To apply Theorem 3 let us define $A(V^{(1)}) = \{n \in I_+^M : n_i \geq V_i^{(1)}, i = 1, 2, \dots, M\}$.

Then, $N^M(t)$ in $A(V^{(1)})$ satisfies the conditions of Theorem 3, that is, $U_m(t) = N_m^M(t+1) - N_m(t)$ in $A(V^{(1)})$ are i.i.d., and possess moment generating functions (since $U_m(t)$ is downward uniformly bounded), and $EU_m > 0$ if $\lambda_m > V_m^{(1)}$. In fact, $N^*(t)$ defined by (38c) is an extension of $N^M(t)$ from $A(V^{(1)})$ to the state space I_0^M , as was done in (13). Hence,

Property 8. The coupled-processors system is *not ergodic* if for all $m \in M$

$$\lambda_m > V_m^{(1)} \quad (40)$$

□

It should be pointed out that the converse condition to (40), i.e., $\lambda_m < V_m^{(1)}$ for $m \in M$, does not assure ergodicity of $N^M(t)$, as it may be checked by Malyshev's conditions or by comparison with Fayolle's results [5] for $M = 2$. Instead of

$\lambda_m < V_m^{(1)}$ we must assume (39a) for stability.

Finally, let us apply Corollaries 1 and 2. By (6), (7) and (20) we must first determine conditional throughputs $S_o^k(n)$ and define a finite subset H , where conditions (6) and (7) may be violated. It is easy to notice that

$$S_o^k(n) = \min \{ n_k, v_k^{(\chi(n_{k+1}, v_{k+1}^{(1)}))} \}, \quad k \in M, \quad n \in I^M \quad (41)$$

where $v_k^{(\chi(n_{k+1}, v_{k+1}^{(1)}))}$ is $v_k^{(0)}$ or $v_k^{(1)}$ depends on the value of $\chi(n_{k+1}, v_{k+1}^{(1)})$. Let us also define $H = \{n \in I^M: n_i < v_i^{\max}, i \in M\}$. Then, by Corollary 1 $N^M(t)$ is ergodic if for $n \in I^M - H$

$$\sum_{i=1}^M c_i \lambda_i < \sum_{i=1}^M c_i \min \{ n_i, v_i^{(\chi(n_{i+1}, v_{i+1}^{(1)}))} \} \quad (42a)$$

and by Corollary 2 the system is not ergodic if for $n \in I^M - H$

$$\sum_{i=1}^M c_i \lambda_i \geq \sum_{i=1}^M c_i \min \{ n_i, v_i^{(\chi(n_{i+1}, v_{i+1}^{(1)}))} \}. \quad (42b)$$

As before, to get explicit results for stability we must find the minimum of the RHS of (42a) over $n \in I^M - H$, while for instability the maximum of the RHS of (42b) is needed over $n \in I^M - H$. We present one result of this type:

Property 9. If $\sum_{i=1}^M v_i^{(1)} / v_i^{(0)} \geq 1$, then $N^M(t)$ is *ergodic* if

$$\sum_{i=1}^M \lambda_i / v_i^{(0)} < 1$$

Proof. Let us denote the RHS of (42a) by $S_o^\Sigma(n)$ and let $c_i = 1/v_i^{(0)}$, $i = 1, 2, \dots, M$. Note that $S_o^\Sigma(0, 0, \dots, v_i^{(0)}, 0, \dots, 0) = 1$. Moreover, for $n \in I^M - H$ there exists $m \in M$ such that $S_o^m(n) = v_m^{(0)}$ (it is enough to assume $n_m \geq v_m^{(0)}$ $n_{m+1} < v_{m+1}^{(1)}$) or for all $m \in M$ $S_o^m(n) = v_m^{(1)}$. In the former case $S_o^\Sigma(n) \geq 1$ since $c_m = 1/v_m^{(0)}$, and in the latter case $S_o^\Sigma(n) = \sum_{i=1}^M v_i^{(1)} / v_i^{(0)}$ is at least 1 by the assumption. \square

The same type of results may be obtained for nonergodicity by considering (42b) and choosing appropriate constants c_i , $i \in M$.

4. Conclusions

We have considered stability conditions for multidimensional Markov chains. Three methods have been investigated: Lyapunov functions, comparison tests and unbounded random walk. The constraints of the first method in the multidimensional environment lie in a *finite set* $H \subset C$, which may violate stability conditions. Therefore, the comparison tests have been derived. Nevertheless, in these tests we require the existence of ergodic (nonergodic) Markov chains upper (lower) bounding a filtered process of the analysed Markov chain. The simplest method is unbounded random walk, however, this method is restricted to a special class of Markov chains.

The methods have been applied to stability analysis of multidimensional queueing systems. At first, a buffered asymmetric ALOHA system was studied and we found a number of stability conditions. Then an M coupled-processors system was considered from the ergodicity point of view. Most of the results are new or presented in a new way. In fact, to the author's knowledge, Properties 1 and 2 was previously established by Tsybakov and Mikhaikov [21]. However, they used much more complex analysis. Moreover, Falin in [3] found an ergodicity condition for the ALOHA system by the Lyapunov method assuming $c_1 = c_2 = \dots = c_M = 1$, which is not the optimal choice for c_i , as we have shown.

Appendix A

Let $U(t)$ be an i.i.d. random variable, possessing a moment generating function $\phi(h) = \mathbb{E}e^{-hU(1)}$, $h \geq 0$ and $EU = EU(1) > 0$. We prove that for any $\epsilon > 0$ there exists $k(\epsilon) > 0$ such that

$$\Pr\left\{\sum_{t=1}^{\infty} \sum_{\tau=1}^{\infty} U(\tau) \leq -k(\epsilon)\right\} < \epsilon \quad (\text{A1})$$

what implies (16) in the proof of Theorem 3

First of all, note that for any t and $k \geq 0$

$$\Pr\{U(t) \leq -k\} \leq e^{-kh} \phi(h) \quad , h > 0 \quad (\text{A2})$$

Indeed, for $h > 0$

$$\begin{aligned} e^{-kh} \phi(h) &= \sum_{j=-\infty}^{-k} \Pr\{U(t) = j\} e^{-h(j+k)} + \sum_{j=-k+1}^{\infty} \Pr\{U(t) = j\} e^{-h(j+k)} \geq \\ &\geq \sum_{j=-\infty}^{-k} \Pr\{U(t) = j\} e^{-h(j-k)} \geq \sum_{j=-\infty}^{-k} \Pr\{U(t) = j\} = \Pr\{U(t) \leq -k\} \end{aligned}$$

Note now that $\phi(0) = 1$ and $\phi'(0) = -EU(t) < 0$. Hence, there exists $h^* > 0$ and $\delta > 0$ such that $\phi(h^*) = e^{-\delta}$, and by (A2)

$$\Pr\{U(t) \leq -k\} \leq \exp[-h^*k - \delta] \quad (\text{A3})$$

By the i.i.d. assumption of $U(t)$, $t = 0, 1, \dots$, and (A3), we find that

$$\begin{aligned} \Pr\left\{\sum_{t=1}^{\infty} \sum_{\tau=1}^t U(\tau) \leq -k\right\} &\leq \sum_{t=1}^{\infty} \Pr\left\{\sum_{\tau=1}^t U(\tau) \leq -k\right\} \leq e^{-h^*k} \sum_{t=1}^{\infty} e^{-\delta t} = \\ &= e^{-h^*k} e^{-\delta} / (1 - e^{-\delta}) \end{aligned} \quad (\text{A4})$$

Since δ does not depend on k , hence (A1) follows from (A4).

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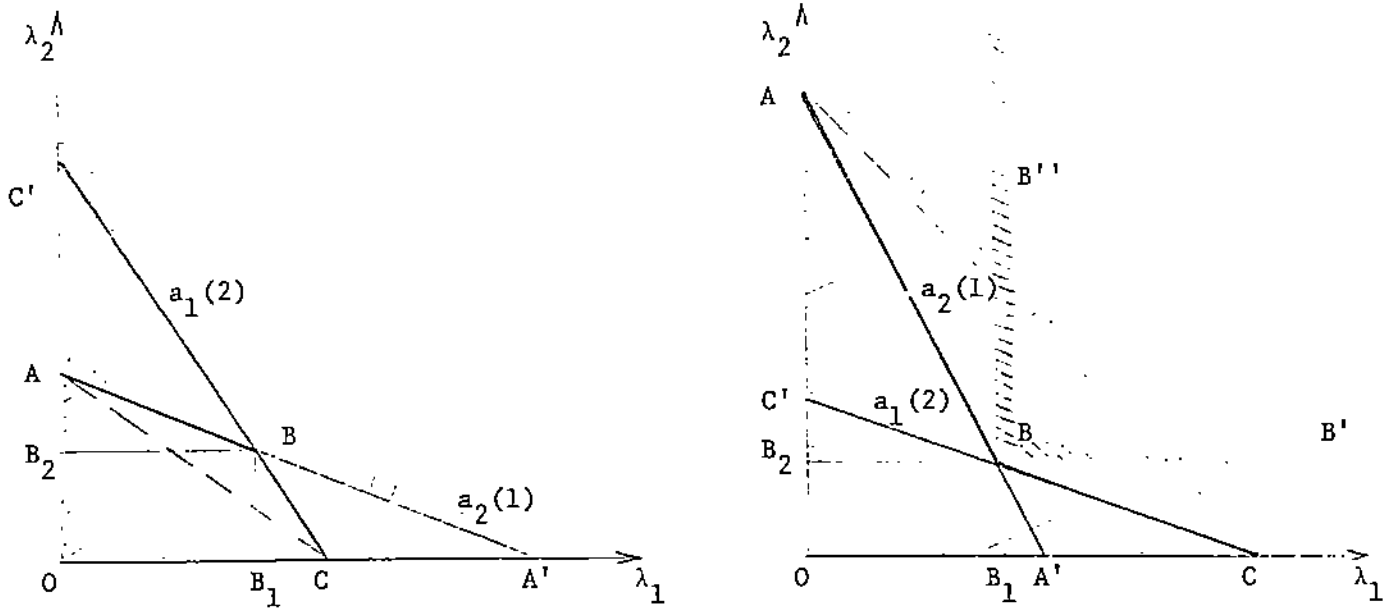


Figure 1. Ergodicity region $\langle OABC \rangle$ and nonergodicity region $\langle ABC \infty \rangle$ in the ALOHA system for $M=2$

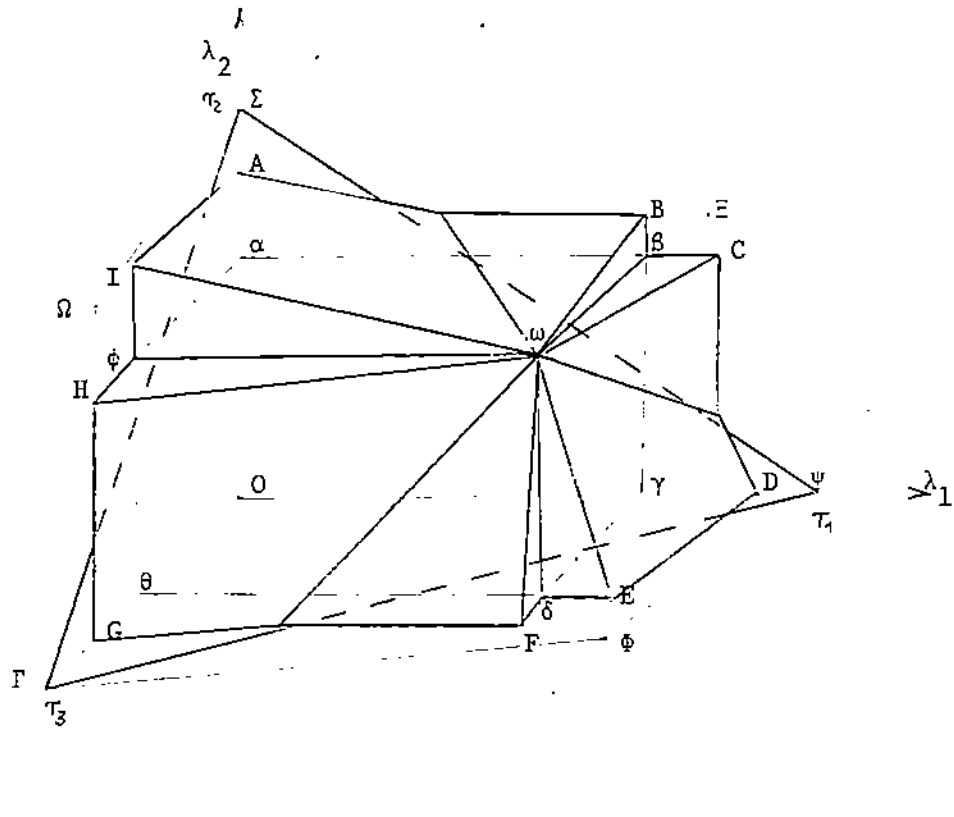


Figure 2. Subsets of ergodicity region in the ALOHA system with $M=3$, $r_1=0.3$
 $r_2=0.2$, $r_3=0.1$; $E_1^C = \langle O\alpha\beta\gamma\delta\theta\phi\omega \rangle$, $E_1^L = \langle O\Sigma\Psi\Gamma \rangle$, $E_2^C = \langle OABBCDE\delta FGH\phi I\omega \rangle$