1986

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Report Number:
86-600

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CSD-TR-600
May 1986

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ABSTRACT

Since the appearance of a paper by Carleial and Hellman [2] in 1975, it is known that the bistable behavior of the ALOHA system is associated with a bimodal shape of the backlog steady-state distribution. In this paper, we generalize the problem and ask under what conditions a one-dimensional Markov Chain possesses a multimodal steady-state distribution. We restrict our analysis to uniformly bounded Markov Chains. In this class of Markov Chains we distinguish so called near birth and death processes and prove that under some additional assumptions a shape of the distribution is determined by the transition probabilities located on the principal diagonal, subdiagonal and supdiagonal of the transition matrix. This provides a theoretical explanation for the bistable behavior of the ALOHA system.

1. INTRODUCTION

The evaluation of computer performance is needed during the entire life of a system. However, as computer networks become more and more sophisticated new complications arise which pose highly non-trivial design and analysis problems. The most difficult to treat analytically are pathological behaviors, e.g. congestions, deadlocks, bistability, fairness, hysteresis, and so on. To deal with such problems a new approach is needed which focuses its attention on phenomena (properties) and mutual relationships among them. This approach is called *qualitative analysis* and one advantage follows from the fact that very often the gross behavior of a system is largely independent of detailed quantitative values of system variables, and therefore, does not need detailed quantitative analysis.

The stability problem is a well recognized example of a qualitative approach. In a broad sense stability deals with *a required property* of a system in the presence of *perturbations*. In a stochastic approach to computer systems analysis a source of disturbances (perturbation) is

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usually the input traffic. Here, the definition of stability depends on what one understands by required properties. For example, if a system is described by a Markov Chain with infinite state space, then by stability we can mean ergodicity, that is, whether steady-state probabilities exist or not. When the Markov Chain has a finite state space, then ergodicity is not an interesting property since it is known that every irreducible Markov Chain with finite state space possesses a steady-state solution. However, in such a system a number of new phenomena may occur, which leads to stability considerations (for more details see [11]). The best known example is the ALOHA-type protocol. In this case, the system has a finite Markovian description, but it posesses a bistable behavior if the steady-state solution is a bimodal function [2]. This implies that the shape of steady-state distribution is responsible for the behavior of the system. Stability of such a system is therefore expressed in terms of the shape of the distribution function .

In this paper we consider one-dimensional finite Markov Chains, and study properties of their steady-state distributions. Instead of concentrating on the exact values of the distribution, we would rather investigate the shape of the function. That is, we ask if the distribution function is unimodal, bimodal or multimodal (many maxima of the steady-state distribution). We restrict our analysis to so called uniformly bounded Markov Chains, i.e., such chains that their downward or upward transitions are uniformly bounded. We prove that under some additional assumptions the shape of distribution function depends only on the transition probabilities located on the principal diagonal, subdiagonal and supdiagonal of the transition matrix. Using this approach we give a theoretical explanation for bistable behavior of the ALOHA protocol. Some more examples will also be discussed. Finally, let us point out that to the author's best knowledge, the only paper discussing similar problems is the work by Nelson [8] (see also [9]).

2. PROBLEM FORMULATION AND SOME MOTIVATING EXAMPLES

Let N^t be a one-dimensional Markov Chain with finite state space $C = \{0, 1, \ldots, M\}$. Let $P = \{P_{ij}\}_{i,j \in C}$ and $Q = \{q_{ij}\}_{i,j \in C}$ denote the transition matrix and the infinitesimal generator for N^t when the time t is discrete and continuous, respectively. Then, the steady-state probability vector $\pi = [\pi_0, \pi_1, \ldots, \pi_M]$ is a solution of a system of linear equation $\pi(P - I) = 0$ (discrete time) or $\pi Q = 0$ (continuous time) [1]. Defining a new matrix R which is either $P - I$ (discrete time) or Q (continuous time), the steady-state vector π is a solution of

$$\pi R = 0. \tag{2.1}$$

Note that elements $r_{ij}, i, j \in C$ of R satisfy the following conditions [1]

$$\sum_{j=0}^{M} r_{ij} = 0 \quad \text{for any} \quad i \in C, \tag{2.2a}$$

$$r_{ii} \leq 0, \quad \text{and} \quad r_{ij} \geq 0 \quad \text{if} \quad i \neq j. \tag{2.2b}$$

Throughout the paper we assume that N^t is an irreducible and aperiodic Markov Chain, which implies that for all $k \in C$

$$\pi_k > 0.$$

Consider now the steady-state probabilities π_k, $k \in C$, as a function of k. Some system properties are not dependent on the particular values of the distribution function. However, they may depend on the shape of the function $\pi_{k \in C}$. In particular, it is important to know whether the distribution is a unimodal function (only one maximum), bimodal (two maxima) or n-modal (n maxima) of the probabilities $\pi_{k \in C}$. For example, a bimodal distribution may lead to bistable behavior of a system (see ALOHA protocol [2], [6], [10]). There is a natural way to answer the question by solving the system of linear equations in (2.1). However, this is not acceptable from the qualitative point of view since (2.1) might be too complex to solve, and what is more important, solving (2.1) we restrict our considerations to a particular system and not a class of systems. In a qualitative approach to this problem, we shall investigate properties of steady-state distributions without explicit solution of the system of linear equation (2.1). We illustrate this approach below by some motivating examples.

Example 2.1. *ALOHA Protocol*

Let us consider M geographically distributed users who compete for access to a broadcast channel. The channel is slotted and a slot duration is equal to a packet transmission time. If no central coordination is provided, then packets collision is inevitable: simultaneously transmitted packets colliding and destroy one another. An example is the ALOHA protocol [2], [6], [10], which assumes that colliding users randomly select a future time to retransmit their packets. Under some circumstances, the system possesses a bistable behavior, that is, it oscillates between a "good" state and a "bad" state. It turns out that these two states are approximately identified by the most probable states (i.e., the modes of the distribution function). To be more precisely, let N^t denote the number of active users in the system at the beginning of the t-th slot. Also let X^t and Y^t denote the number of new arrivals to the system, and the number of departures from the system in the t-th slot. Obviously,

$$N^{t+1} = N^t + X^t - Y^t. \tag{2.3}$$

In fact, this stochastic equation is satisfied by the queue length process in any queueing model. In stability analysis, three quantities play an important role: average drift, $d(k)$, average conditional throughput, $S_0(k)$, and average conditional input $S_i(k)$, where

$$d(k) = E\{N^{t+1} - N^t \mid N^t = k\}\ ;\ S_0(k) = E\{Y^t \mid N^t = k\}\ ;\ S_i(k) = E\{X^t \mid N^t = k\}. \tag{2.4}$$

Using (2.3) it is easy to see that [10]

$$d(k) = S_i(k) - S_0(k). \tag{2.5}$$

Carleial and Hellman [2] have shown by numerical analysis that the modes of the steady-state distribution $\pi_{k \in C}$ are approximately equal to the roots of $d(k) = 0$. It also might be checked (it follows from (2.5)) that the above approximation is equivalent to the following condition:

$$\frac{\pi_{k+1}}{\pi_k} \approx \frac{S_i(k)}{S_0(k+1)} \tag{2.6}$$

This implies that the multimodality property of $\pi_{k \in C}$ depends only on the conditional throughput $S_0(k)$ and conditional input $S_i(k)$. In this paper we investigate a possibility to extend this result to a larger class of Markov Chains.

□

There are Markov Chains for which (2.6) is exactly satisfied. An example is birth and death process which finds many applications in the performance evaluation of computer systems.

Example 2.2. *Birth and Death Process*

Let N^t be the birth and death process with $q_{k,k+1} = \lambda_k$ and $q_{k,k-1} = \mu_k$. Then [7]

$$\frac{\pi_{k+1}}{\pi_k} = \frac{\lambda_k}{\mu_{k+1}}. \tag{2.7}$$

It is easy to see that $S_i(k) = \lambda_k$ and $S_0(k) = \mu_k$, so an equality holds in (2.6).

□

A generalization of this result is possible through a diffusion approximation.

Example 2.3. *Diffusion Approximation*

Let us assume for simplicity that N^t represents the queue length in a queueing model. If we treat N^t as a continuous process, and define $p(x)$ as the corresponding density function, then by diffusion approximation $p(x)$ satisfies the following differential equation [7]

$$\frac{d\,p(x)}{dx} = \frac{1}{\beta(x)}\, p(x)[\alpha(x) - \beta'(x)/2]. \tag{2.8}$$

In (2.8) $\alpha(x)$ and $\beta(x)$ are infinitesimal drift and variance, respectively, and $\beta'(x)$ is the derivative of $\beta(x)$. In particular, from (2.8) it is clear that the modes of $p(x)$ are roots of the equation

$$\alpha(x) - \beta'(x)/2 = 0. \tag{2.9}$$

If $\beta'(x) << \alpha(x)$, then (2.9) is reduced to $\alpha(x) = 0$ (fluid approximation [6], [10]). But $\alpha(x)$ is equivalent to the drift function defined in (2.5), so (2.6) is approximately satisfied.

□

In fact, the problem formulated in terms of (2.6) is not interesting, since only a small number of queueing problems fall into this class. The next example gives another formulation of the problem.

Example 2.4. *Reversible process* [5]

Let N^t be reversible Markov process (for details see [5]). Then, in terms of matrices R defined in (2.1), (2.2) it is easy to see that the following holds for reversible Markov Chains [5]:

$$\frac{\pi_{k+1}}{\pi_k} = \frac{r_{k,k+1}}{r_{k+1,k}}. \tag{2.10}$$

If $r_{k,k+1} = S_i(k)$ and $r_{k+1,k} = S_0(k)$, then (2.10) is reduced to (2.7), and if (2.10) is satisfied approximately, then (2.10) is of type (2.6). Eq.(2.10) implies that the shape of $\pi_{k \in C}$ depends only on the subdiagonal and supdiagonal elements of R.

□

In this paper we investigate Markov Chain for which (2.10) is approximately satisfied. However, it must be strongly stressed, that (2.10) or its approximation is a very strong property of the process, and only a restricted class of Markov Chains satisfies this. In the next example we present a Markov Chain for which (2.10) or (2.6) does not hold. Moreover, in the example below, even the roots of the drift equation (2.5) are no longer associated with the modes of steady-state distribution.

Example 2.5. *Counterexample*

Let N^t be Markov Chain with transition probabilities P defined as $p_{k,0} = 1 - p_k$, $p_{k,k+1} = p_k, p_{k,j} = 0$ for $j \neq 0, k+1$, where p_k is a given nonzero probability (nonzero elements are located on the supdiagonal and in the first column of the transition matrix). The average drift $d(k) = (k+1)p_k - k$ and $\pi_{k+1} = p_k\ \pi_k$, that is $\pi_{k+1}/\pi_k = p_k < 1$. The last inequality shows that $\pi_{k \in C}$ is a decreasing function for all k, but we can select p_k such that the drift equation has as many roots as we want. For example, choosing three integers, $k_1, k_2, k_3 \leq M$ and assuming $p_k > k/k{+}1$ for $k < k_1$, $p_k < k/k{+}1$ for $k_1 < k \leq k_2$; $p_k > k/k{+}1$ for $k_2 \leq k \leq k_3$; and $p_k < k/k{+}1$ for $k_3 < k \leq M$, the drift equation has three roots, however, the steady-state distribution is a decreasing function.

□

The last example points out that a Markov Chain with drift property similar to ALOHA-type system does not necessarily imply bistability.

3. UNIFORMLY BOUNDED MARKOV CHAINS

We restrict our considerations to a class of Markov Chains known as uniformly bounded Markov Chains. This section provides definition and some properties of the chains which will be further used to study multimodality.

We say that a Markov Chain N^t is *downward uniformly bounded* (DUB) if there exists an integer $V > 0$ such that $p_{ij} = 0$ or $q_{ij} = 0$ for $j < i - V$. If for all $i \geq V$ $p_{i,i-V} \neq 0$ or $q_{i,i-V} \neq 0$ that the chain is called *strongly downward uniformly bounded* (strongly - DUB). On the other hand, a chain N^t is said to be *upward uniformly bounded* (UUB) if there exists an integer $V > 0$ such that $p_{ij} = 0$ or $q_{ij} = 0$ for $j > i + V$ and the chain is *strongly upward uniformly bounded* if $p_{i,i+V} \neq 0$ or $q_{i,i+V} \neq 0$ for all $i \leq M - V$. Let us associate with matrices P and Q of a uniformly bounded Markov Chain, a new matrix R, which preserves the structured property of P and Q. In the previous section we have assumed $R = P - I$ (discrete case) or $P = Q$ (continuous case). Thus, for DUB - chain the matrix R might be partitioned in the form

$$R = \begin{vmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{vmatrix}, \tag{3.1}$$

where R_{12} is $V \times V$ matrix, then R_{21} is $(M - V+1) \times (M - V+1)$ upper triangular. If R_{21} is non-singular, then the chain (or the matrix R) is strongly-DUB. On the other hand, for UUB-chain the matrix R might be partitioned in the form

$$R = \begin{vmatrix} R'_{11} & R'_{12} \\ R'_{21} & R'_{22} \end{vmatrix}, \tag{3.2}$$

where R'_{21} is $V \times V$ while R'_{12} is non-singular, then the chain (or the matrix R) is strongly-UUB. Furthermore, only strongly uniformly bounded Markov chains are studied under the assumption that $V << M$ which is usually satisfied in practice. For example, if N^t is the queue length in a system , then V may represent the number of servers and M is the capacity of the buffer.

Whenever a system is analyzed by a Markovian model the following set of linear equations are often studied,

$$\mathbf{x}R = -\mathbf{c}, \tag{3.3a}$$

$$\mathbf{t}R^T = -\mathbf{b}, \tag{3.3b}$$

where **x**, **t**, **c**, **b** are one-dimensional row vectors and **c**, **b** are known coefficient-vectors while x and t must be determined. We shall assume that R is strongly-DUB matrix, then the transpose matrix R^T is strongly-UUB matrix, i.e., if R is of type (3.1), then R^T is of type (3.2). For example, the steady-state vector π is a solution of (3.3) if one assumes $R = P - I$ and $\mathbf{c} = [0, 0, \ldots, 0]^T$.

We now solve Eq. (3.3a) assuming R is a strongly-DUB matrix, i.e. (3.1) holds. We partition $\mathbf{x} = (\mathbf{X}_1, \mathbf{X}_2)$ and $\mathbf{c} = (\mathbf{C}_1, \mathbf{C}_2)$, where $\mathbf{X}_1 = (x_0, x_1, \ldots, x_{V-1})$ and $\mathbf{C}_1 = (c_0, c_1, \ldots, c_{V-1})$. Then (3.3a) can be written as

$$\mathbf{X}_1 R_{11} + \mathbf{X}_2 R_{21} = -\mathbf{C}_1, \tag{3.4a}$$

$$\mathbf{X}_1 R_{12} + \mathbf{X}_2 R_{22} = -\mathbf{C}_2. \tag{3.4b}$$

Since R_{21} is non-singular (3.4a) gives

$$\mathbf{X}_2 = -\mathbf{C}_1 R_{21}^{-1} - \mathbf{X}_1 R_{11} R_{21}^{-1}. \tag{3.5}$$

Let us set

$$H_2 = -R_{11} R_{21}^{-1} \quad \mathbf{d}_2 = C_1 R_{21}^{-1}. \tag{3.6}$$

Then, (3.5) implies

$$\mathbf{X}_2 = \mathbf{X}_1 H_2 + \mathbf{d}_2, \tag{3.7}$$

that is, components of $\mathbf{X}_2$ are expressed by V components of $\mathbf{X}_1$, elements of $V \times (M - V - 1)$ matrix H_2 and $(M - M + 1)$ components of vector $\mathbf{d}_2$. The matrix H_2 and the vector d_2 satisfy equations (see (3.6)).

$$\mathbf{H}_2 R_{21} = -R_{11}, \tag{3.8a}$$

$$\mathbf{d}_2 R_{21} = -\mathbf{C}_1, \tag{3.8b}$$

but since R_{21} is upper triangular Eq. (3.8) can be easily solved. To find an explicit formula on $\mathbf{X}_2$ we must determine $\mathbf{X}_1$. For, by (3.4b) and (3.5)

$$\mathbf{X}_1(R_{12} + H_2 R_{22}) = -\mathbf{C}_2 - \mathbf{d}_2 R_{22}, \tag{3.9}$$

hence, $\mathbf{X}_1$ is a solution of V linear equation (3.9). Since we have assumed $V << M$, (3.9) might be easily solved. The algorithm described by Eqs. (3.4)-(3.9) may be rewritten in a more suitable form. Therefore, let us define matrix H and vector $\mathbf{d}$ as

$$H = (I, H_2), \quad \mathbf{d} = (\mathbf{0}, d_2),$$

where I is $V \times V$ identify matrix while $\mathbf{0}$ is V-elements zero row-vector. We denote an element of H as $h_i(k)$, where $i = 0, 1, \ldots, V - 1, k = 0, 1, \ldots, M$. Then,

Algorithm 1

Step 1. Calculate recursively for each $i = 0, 1, V-1, k = 0, 1, \ldots, M$,

$$h_i(k) = \begin{cases} \delta_{k,1} & k = 0, 1, \ldots, V-1 \\ \dfrac{-1}{r_{k,k-V}} \sum_{j=0}^{k-1} h(j) r_{j,k-V} & k = V, V+1, \ldots, M \end{cases} \tag{3.10a}$$

$$d_k = \begin{cases} 0 & k = 0, 1, \ldots, V-1 \\ \dfrac{-1}{r_{k,k-V}} \left\{ \sum_{j=0}^{k-1} d_j \; r_{j,k-V} + c_{k-V} \right\} & k = V, V+1, \ldots, M, \end{cases} \tag{3.10b}$$

where $\delta_{k,j}$ is Kronecker delta.

Step 2. Solve for $n = M+1, \ldots, M$ the following system of V linear equations

$$\sum_{i=0}^{V-1} x_i \sum_{j=0}^{M} h_i(j) r_{jn} = -c_n - \sum_{j=0}^{M} d_j \; r_{jn}, \tag{3.10c}$$

to obtain $\mathbf{X}_1 = (x_0, x_1, \ldots, x_{V-1})$.

Step 3. Calculate for all $k = V, V+1, \ldots, M$,

$$x_k = \sum_{i=0}^{V-1} x_i \; h_i(k) + d_k. \tag{3.10d}$$

□

The system of linear equations (3.3b) can be solved in a similar way. Now $R' = R^T$ is strongly-UUB and it may be partitioned according to (3.2). Moreover, we partition the vectors $\mathbf{t}$ and $\mathbf{b}$, $\mathbf{t} = [\mathbf{T}_1, \mathbf{T}_2]$, $\mathbf{b} = [\mathbf{B}_1, \mathbf{B}_2]$ where $\mathbf{T}_2$ and $\mathbf{B}_2$ have V components. By imitating the derivations of the Algorithm 1 we find:

Algorithm 2

Step 1. Calculate for all $i = 0, 1, \ldots, V-1$,

$$f_i(k) = \begin{cases} \delta_{ki} & k = M, M-1, \ldots, M-V+1 \\ \dfrac{-1}{r_{k,k+V}} \sum_{j=k+1}^{M} f_i(j) r_{j,k+V} & k = M-V,\ M-V-1, \ldots, 0 \end{cases} \tag{3.11a}$$

$$e_k = \begin{cases} 0 & k = M, M-1, \ldots, M+1 \\ \dfrac{-1}{r_{k,k+V}} \left\{ \sum_{j=k+1}^{M} e_j\, r_{j,k+V} + b_{k+V} \right\} & k = M-V,\ M-V-1, \ldots, 0. \end{cases} \tag{3.11b}$$

Step 2. Solve for $n = 0, 1, \ldots, V-1$ the following system of V linear equations

$$\sum_{i=0}^{V-1} t_{M-i} \sum_{j=0}^{M} r_{n,j}\, f_i(j) = -b_n - \sum_{j=0}^{M} r_{n,j}\, e_j, \tag{3.11c}$$

to obtain $T_2 = (t_M, t_{M-1}, \ldots, t_{M-V+1})$.

Step 3. Calculate for $k = M-V, M-V-1, \ldots, 0$,

$$t_k = \sum_{i=0}^{V-1} f_i(k) t_{M-1} + e_k. \tag{3.11d}$$

□

Remarks

To solve Eq. (3.3) by Gaussian elimination, we need $O(M^2)$ elements to be stored and a total of $O(M^3)$ arithmetic operations is needed [4]. It is easy to see that the algorithms presented above need to store only $O(M)$ elements and to perform $O(M^2)$ arithmetic operations for $V << M$. However, while Gaussian elimination is stable, these algorithms are unstable for $V > 1$. This comes from the fact that the systems of $V \times V$ linear equations (3.10c), (3.11c) are very ill-conditioned if R is psudo-stochastic matrix. For $V = 1$ the algorithm works very well, and it is better in terms of storage and arithmetic operations than Gaussian elimination. It is particularly important if a system of linear equations is to be solved many times, e.g., as in Markov decision process analysis. Nevertheless, for $V>1$ the algorithms have some theoretical importance since they might be used to study multimodality and to derive, in some cases, explicit formulas for the steady state probabilities.

□

4. MULTIMODALITY PROPERTY OF STEADY-STATE DISTRIBUTION

Throughout this section we assume that N^t is a uniformly bounded Markov Chain with $V = 1$. In fact, most of the analysis will be done for downward uniformly bounded Markov Chains, and by analogy we extend the results to upward uniformly bounded Markov Chains.

To study the shape of steady-state distribution $\pi_{k\in C}$ we introduce coefficients a_k defined as

$$a_k = \frac{\pi_{k+1}}{\pi_k} \quad k = 0, 1, \ldots, M-1. \tag{4.1}$$

Note that $a_k > 1$ implies that $\pi_{k\in C}$ increases at k, and $a_k < 1$ suggests that the distribution decreases at k. There is a simple recurrence relationship between the coefficients a_k. Using (3.10) we find that (we drop subscript i in $h_i(k)$ since $V = 1$ is considered),

$$\pi_k = h(k)\pi_0, \quad h(0) = 1, \tag{4.2a}$$

$$r_{k+1,k}\, h(k+1) = -r_{kk}\, h(k) - \sum_{j=0}^{k-1} r_{jk}\, h(j), \quad k = 0, 1, \ldots, M-1, \tag{4.2b}$$

and by (2.2b) $r_{kk} \le 0$. By (4.2b) also $a_k = h(k+1)/h(k)$, hence we obtain

$$a_k = \frac{1}{r_{k+1,k}} \left[-r_{kk} - \sum_{j=0}^{k-1} r_{jk} \prod_{i=j}^{k-1} a_i^{-1}\right] \quad k = 0, 1, \ldots, M-1. \tag{4.3}$$

Note that under the irreducibility assumption on N^t the coefficients a_k and $h(k)$ are positive.

We shall study properties of a_k through some inequalities. The recurrence (4.3) is not very suitable to obtain a lower bound on a_k, therefore we transform (4.3) into another form. Let us start with

LEMMA. For all $n = 1, 2, \ldots, M$ and $k < n$ the following holds

$$\sum_{m=k+1}^{n} [r_{k+1,m} + \sum_{j=0}^{k} r_{jm} \prod_{i=j}^{k} a_i^{-1}] \le 0, \tag{4.4}$$

and the equality in (4.4) holds if and only if $n = M$.

Proof: See Appendix.

□

Assume now $n = M$ in (4.4) and note that the first and the second term of (4.4), by (2.2), are equal to

$$\sum_{m=k+1}^{M} r_{k+1,m} = -r_{k+1,k}$$

$$\sum_{m=k+1}^{M} \sum_{j=0}^{k} r_{jm} \prod_{i=j}^{k} a_i^{-1} = - \sum_{j=0}^{k} \prod_{i=j}^{k} a_i^{-1} \sum_{m=j-1}^{k} r_{jm}.$$

Hence, using the above and computing a_k from (4.4) we finally obtain an alternative form for a_k,

$$a_k = \frac{1}{r_{k+1,k}} \left[-r_{kk} - r_{k,k-1} - \sum_{j=0}^{k} \sum_{m=j-1}^{k} r_{jm} \prod_{i=j}^{k-1} a_i^{-1} \right]. \tag{4.5}$$

The advantage of (4.5) over (4.3) is that in (4.5) the second term is summed over rows of R (see index m) instead of columns of R as in (4.3) (see index j). This is more visible if we rewrite (4.5) as

$$a_k = -\frac{r_{kk} + r_{k,k-1}}{r_{k+1,k}} - \frac{r_{k-1,k-2} + r_{k-1,k-1} + r_{k-1,k}}{r_{k+1,k}\, a_{k-1}} - \frac{r_{k-2,k-3} + r_{k-2,k-2} + r_{k-2,k-1} + r_{k-2,k}}{r_{k+1,k}\, a_{k-2}\, a_{k-1}}$$
$$- \cdots - \frac{r_{00} + r_{01} + \cdots + r_{0k}}{r_{k+1,k}\, a_0\, a_1 \cdots a_{k-1}} \tag{4.5a}$$

Now it is clear that (4.5) is more suitable since the row-wise sum of R is equal to zero (see (2.2)) and very small terms in (4.5a) might be ignored if necessary.

Using (4.3) and (4.5) we prove

THEOREM. (i) For downward uniformly bounded Markov Chains the following holds

$$\underline{a}_k \stackrel{def}{=} -\frac{r_{kk} + r_{k,k+1}}{r_{k+1,k}} \le a_k \le \frac{r_{kk}}{r_{k+1,k}} \stackrel{def}{=} \bar{a}_k. \tag{4.6}$$

Equality in LHS of (4.6) holds iff R is a tridiagonal matrix (i.e. $r_{ij} = 0$ for $|i - j| > 2$). Equality in RHS of (4.6) holds iff $r_{jk} = 0$ for $j = 0, 1, \ldots, k-1, k = 1, 2, \ldots, M-1$.

(ii) For upward uniformly bounded Markov Chain, we have

$$-\frac{r_{k,k+1}}{r_{k+1,\,k+1}} \le a_k \le -\frac{r_{k,k+1}}{r_{k+1,\,k+1} + r_{k+1,\,k+2}}, \tag{4.7}$$

and equality in RHS of (4.7) holds iff R is tridiagonal, while in LHS you can replace the inequality by an equality if $r_{jk} = 0$ $j = k+1, \ldots, M-1, k = 1, 2, \ldots, M$.

Proof. We prove (4.6). The RHS of (4.6) follows immediately from (4.3) if one notes that $r_{jk} \ge 0$ for $j = 0, 1, \ldots, k-1$, and for equality in (4.6) we need $r_{jk} = 0$ for $j = 0, 1, \ldots, k-1$. The LHS of (4.6) is a consequence of (4.5) and the fact that by (2.2) all terms in (4.5) are nonnegative. For equality in (4.6) we need that the second term in (4.5) vanishes (tridiagonal matrix). Inequalities (4.7) are proved in the same way, but Algorithm 2 must be taken into account.

□

From the queueing view point, the lower bound $\underline{a}_k$ is a good bound, and very tight if the matrix R is "similar" to a tridiagonal matrix (details are discussed below). Unfortunately, the upper bound $\bar{a}_k$ is not too tight, but we shall show an example that $\bar{a}_k$ tightly upper bounds a_k. Below we present three improvements of the upper bound.

Let us define a new sequence $\bar{a}_k^{(1)}$ as

$$\begin{aligned} \bar{a}_0^{(1)} &= a_0 \\ \bar{a}_k^{(1)} &= \bar{a}_k - \frac{r_{k-1,k}}{r_{k+1,k}} \, \frac{1}{\bar{a}_{k-1}^{(1)}}, \quad k = 1, \ldots, M-1. \end{aligned} \tag{4.8}$$

Then, by (4.5) $a_k \le \bar{a}_k^{(1)} < \bar{a}_k$. An improvement over (4.8) might be achieved if we take more terms of the sum in (4.3) into account. For example, let us introduce

$$\bar{a}_0^{(2)} = a_0 \quad a_1^{(2)} = a_1$$

$$\bar{a}_k^{(2)} = \bar{a}_k - \frac{r_{k-2,k}}{r_{k+1,k}} \; \frac{1}{\bar{a}_{k-2}^{(2)} \; \bar{a}_{k-1}^{(2)}} - \frac{r_{k-1,k}}{r_{k+1,k}} \; \frac{1}{\bar{a}_{k-1}^{(2)}} . \tag{4.9}$$

Then, obviously $a_k < \bar{a}_k^{(2)} \le \bar{a}_k^{(1)} \le \bar{a}_k$. The disadvantage of (4.8) and (4.9) is that these upper bounds are computed by a recurrence. This might be relaxed if $\bar{a}_{k-1}^{(1)}$ is replaced by $\bar{a}_k$. For example, (4.8) implies

$$\bar{a}_k^{(3)} = \bar{a}_k - \frac{r_{k-1,k}}{r_{k+1,k} \; \bar{a}_k}, \tag{4.10}$$

and $a_k \le a_k^{(1)} \le \bar{a}_k^{(3)} \le \bar{a}_k$.

Approximation for almost tridiagonal matrices R.

It was mentioned before (and it will be numerically verified in the next section) that the lower bound $\underline{a}_k$ is a tight bound for a_k if the matrix R is "similar" to a tridiagonal matrix. This means, roughly speaking, that the elements on the principal diagonal, subdiagonal and supdiagonal of the matrix R are relatively much larger than the other elements of R. Such a property is satisfied by a large class of queueing models, as we shall see in the next section.

More precisely, let us define a class of Markov Chains with the above property as:

DEFINITION. A downward bounded Markov Chain N^t is called *near birth and death process* (in short: near-BD process) if there exist a small positive number ε such that for any n, $k \in C = \{0, 1, \ldots, M\}$ the following holds

$$\frac{r_{n,n+l}}{\max\{0, \min\{r_{k,k+1}, -r_{kk}, r_{k,k-1}\}\}} = \Theta(\varepsilon^{l-1})^*, \quad l = 2, \ldots, M-n. \tag{4.11}$$

If (4.10) is satisfied, then the matrix R is called near tridiagonal.

* We say that $x = \Theta(\varepsilon)$ if there exist constants K_1 and K_2 such that $K_1\varepsilon < x < K_2\,\varepsilon$ [4].

□

Roughly speaking, equation (4.11) assures that the elements of R out of the diagonal, subdiagonal and supdiagonal are of at least ε-order of magnitude smaller than the ones of the diagonals.

Let us now consider the lower bound $\underline{a}_k$ for near BD Markov Chains. This bound is tight if in (4.5) (equivalently (4.5a)) the second, third and other terms are much smaller than $\underline{a}_k$. To assure it let us choose a small number β ($\beta << 1$) such that the second term of (4.5a) is β times smaller than $\underline{a}_k$, that is

$$\underline{a}_k \geq -\frac{1}{\beta}\,\frac{r_{k-1,\,k-2} + r_{k-1,\,k-1} + r_{k-1,k}}{r_{k+1,k}\;a_{k-1}}.$$

Taking into account (4.6), the above is equivalent to

$$a_{k-1} > \frac{1}{\beta}\,\frac{r_{k-1,\,k-2} + r_{k-1,\,k-1} + r_{k-1,k}}{r_{kk} + r_{k,k-1}}. \tag{4.12}$$

But by (4.11) the RHS of (4.12) is of order $\Theta(\varepsilon/\beta)$, hence

$$a_{k-1} = \Theta(\varepsilon/\beta) \tag{4.13}$$

and it implies that the second term in (4.5a) is β times smaller than $\underline{a}_k$.

Condition (4.13) is not easy to verify in practice, since one must know a_{k-1}. But since $\underline{a}_{k-1} \geq a_{k-1}$ we can replace (4.13) by

$$\underline{a}_{k-1} = \Theta(\varepsilon/\beta).$$

We prove

THEOREM. Let N^t be a near BD process, and ε, β be two small positive numbers. If for all $l = 0, 1, \ldots, k-1$

$$\underline{a}_l = \Theta(\varepsilon/\beta), \tag{4.14}$$

then

$$a_k = \underline{a}_k + \Theta(\beta). \tag{4.15}$$

Proof. Using (4.14) for $l = k - 1$ and (4.11) for $l = 2$ we find that the second term in (4.5a) is

$$\frac{r_{k-1,\,k-2} + r_{k-1,\,k-1} + r_{k-1,k}}{r_{k+1,k}\; a_{k-1}} = \Theta(\beta).$$

Similarly, using (4.14) for $l = k-1,\ k-2$ and (4.11) for $l = 3$ we show that the third term is of order $\Theta(\beta^2)$ and so on. Since $\Theta(\beta) + \Theta(\beta^2) + \cdots + \Theta(\beta^{k-1}) = \Theta(\beta)$ we prove (4.15).

□

Finally, let us relate the coefficients a_k and the drift $d(k)$ (see (2.4)), conditional throughput $S_0(k)$ and conditional input $S_i(k)$. We now assume that N^t satisfies stochastic equation (2.3), hence also $d(k) = S_i(k) - S_0(k)$ as (2.5) shows. Note that the average drift might be expressed as

$$d(k) = \sum_{m=k-1}^{M} (m-k) r_{k,m} = S_i(k) - S_0(k). \tag{4.16}$$

Let us now assume that $S_0(k) = r_{k,k-1}$ (or more generally $S_0(k) = r_{k,k-1} + \Theta(\varepsilon)$) which is satisfied by a relatively large class of queueing models. Then, using the above (4.16), (4.11) and noting that

$$S_i(k) = \sum_{m=k+1}^{M} (m-k) r_{k,m}, \tag{4.17}$$

we find

$$\underline{a}_k = -\frac{r_{kk} + r_{k,k-1}}{r_{k+1,k}} = \frac{r_{k,k+1} + r_{k,k+2} + \cdots + r_{k,M}}{r_{k+1,k}} =$$

$$= \frac{S_i(k)}{S_0(k)} - \frac{r_{k,k+1} + 2r_{k,k+3} + \cdots + (M - k - 1)\, r_{k,M}}{r_{k+1,k}}$$

$$= \frac{S_i(k)}{S_0(k)} + \Theta(\varepsilon). \tag{4.18}$$

Using (4.15), (4.18) and assuming $\varepsilon = \beta$ we obtain

COROLLARY. Let N^t be near BD process satisfying stochastic equation (2.3). If $r_{k,k+1} = S_0(k)$, then

$$a_k = \frac{S_i(k)}{S_0(k+1)} + \Theta(\varepsilon). \tag{4.19}$$

□

This indicates also that the modes of $\pi_{k \in C}$ are associated with the zeros of the drift function.

Finally, let us consider an example of a Markov Chain N^t which is *not* a near birth and death process (however it is uniformly bounded) such that multimodality is neither associated with a_k nor with the average drift.

Example 4.1

Let the transition matrix P for a Markov Chain N^t be defined as: $p_{k,M} = p_k$, $p_{k,k-1} = 1 - p_k$, $p_{k,j} = 0$, $j \neq M, k-1$. It is easy to see that the drift is $d(k) = (M + k - 1)p_k - 1$, that is, nonzero elements are on the subdiagonal and in the last column of the transition matrix). Note also that the steady-state probabilities satisfy $\pi_{k+1}/\pi_k = 1/(1 - p_k) > 1$, so the function $\pi_{k \in C}$ increases over $C = \{0, 1, \ldots, M\}$. On the other hand, we may select p_k in such a way that $d(k)$ is negative or positive in different ranges of k, i.e., $d(k) = 0$ might have as many zeros as we want. Note also that in that case a_k is equal to the upper bound $\bar{a}_k$. In particular, (4.15) does not hold.

□

5. APPLICATIONS AND NUMERICAL RESULTS

We apply the above analysis to a qualitative study of some computer system models. We mainly concentrate on the ALOHA-type protocols, since it was our motivating example for the above investigations.

Let us assume that in a slotted ALOHA system [6], [10] M users compete for an access to a single channel. Two random access protocols are usually considered:

(a) *Random access with retransmission discrimination discipline* [6] (WRD-discipline), where a user with a newly generated packet transmits it with a probability one in the nearest slot, while users with collided packets send them with a probability r in the next slots.

(b) *Random access without retransmission discrimination discipline* [10] (ORD-discipline), that is, newly generated and retransmitted packets are treated the same and they access the channel with probability r.

To specify the model, let p represent the probability of generating a new packet within a slot. Then the number of blocked users, N^t, at the beginning of the t-th slot is a downward uniformly bounded Markov Chain with $V = 1$ (single channel). Transition probabilities are easy to compute, and the reader may consult [6] for WRD protocol and [10] for ORD protocol. It also turns out that the transition matrix P is near tridiagonal with $\varepsilon = 0.1$ for a wide range of input parameters M, r and $p \ll r$. The conditional throughput, $S_0(k)$, and conditional input $S_i(n)$ are given by [6], [10].

for ORD protocol

$$S_i(k) = (M-k)p \qquad S_0(k) = kr(1-r)^{k-1} \tag{5.1}$$

for WRD protocol

$$S_i(k) = (M - k)p \ ; \quad S_0(k) = kr(1-r)^{k-1}(1-p)^{M-k} + (M-k)p(1-p)^{M-k-1}(1-r)^k \tag{5.2}$$

Moreover, for WRD protocol $p_{k,k-1} = S_0(k)$, while for ORD protocol $p_{k,k-1} = kr(1-r)^{k-1}(1-p)^{M-k} = kr(1-r)^{k-1} + \Theta(p)$. But, in most applications p is very small ($p << r < 1$), hence we can safely ignore $\Theta(p)$ and assume that $p_{k,k-1} = S_0(k)$. Hence, the corollary proved above may be applied to these cases.

In Fig.1 and 2 we plot the steady-state distribution and a_k, lower bound $\underline{a}_k$ and four upper bounds $\overline{a}_k$, $a_k^{(1)}$, $\overline{a}_k^{(2)}$, $\overline{a}_k^{(3)}$ (see (4.8)-(4.10)) for WRD protocol with $M = 30$, $r = 0.2$ $p = 0.005$ and $p = 0.008$. The graphs show that the lower bound is very tight, while none of the upper bounds is a very tight bound. However, note that the upper bounds at least preserves the shape of a_k. Numerical values confirm our approximation (4.15) with $\varepsilon = \beta = 0.1$. It is easy to see that the analysis of the lower bounds $\underline{a}_k$ is sufficient for multimodality analysis of the steady-state distribution. For multimodality we only need to check if a_k ($\underline{a}_k$) is smaller or greater than one.

Table 1 provides numerical values for the coefficient a_k and its bounds for ORD protocol with $M = 30$ $r = 0.2$, $p = 0.008$. The last column of the table contains also the ratio $S_i(k)/S_0(k+1)$. The above conclusions about the bounds are confirmed, and in addition we find out that the coefficients a_k is tightly approximated by the ratio $S_i(k)/S_0(k+1)$ as (4.20) predicts. This suggests also a simple approximation of the ALOHA-system, namely, instead of solving a large system of linear equations, we may apply (4.20). In fact, (4.20) provides an analytical explanation for the birth and death approximation of the system investigated by the authors of [9].

Table 1. ORD ALOHA protocol. $M = 30, r = 0.2, p = 0.008$.

k	$\underline{a}_k$	a_k	$\bar{a}_k$	$\bar{a}_k^{(1)}$	$\bar{a}_k^{(2)}$	$\bar{a}_k^{(3)}$	$S_i(k)/S_0(k+1)$
0	1.351	1.351	1.351	1.351	1.351	1.351	1.200
1	0.668	0.737	1.288	0.737	0.737	0.737	0.725
2	0.464	0.551	1.292	0.623	0.551	0.808	0.583
3	0.384	0.474	1.314	0.692	0.484	1.014	0.527
4	0.354	0.446	1.346	0.850	0.483	1.085	0.508
5	0.352	0.444	1.386	0.996	0.544	1.140	0.508
6	0.369	0.459	1.432	1.090	0.674	1.186	0.523
7	0.398	0.487	1.484	1.149	0.846	1.229	0.548
8	0.439	0.526	1.542	1.191	1.002	1.270	0.582
9	0.490	0.575	1.606	1.228	1.110	1.314	0.625
10	0.550	0.632	1.677	1.262	1.180	1.360	0.677
11	0.619	0.699	1.756	1.297	1.230	1.411	0.737
12	0.697	0.774	1.842	1.334	1.272	1.467	0.806
13	0.785	0.858	1.936	1.374	1.312	1.529	0.883
14	0.881	0.952	2.039	1.419	1.355	1.599	0.970
15	0.987	1.054	2.150	1.470	1.402	1.677	1.065
16	1.102	1.165	2.269	1.528	1.457	1.762	1.170
17	1.224	1.283	2.395	1.594	1.519	1.856	1.283
18	1.353	1.407	2.527	1.668	1.592	1.955	1.402
19	1.485	1.536	2.663	1.749	1.674	2.060	1.526
20	1.618	1.664	2.799	1.836	1.763	2.167	1.652
21	1.747	1.788	2.930	1.924	1.855	2.270	1.774
22	1.863	1.901	3.049	2.007	1.944	2.365	1.888
23	1.958	1.991	3.146	2.074	2.018	2.440	1.976
24	2.017	2.045	3.207	2.110	2.061	2.484	2.032
25	2.021	2.045	3.213	2.097	2.054	2.479	2.036
26	1.946	1.965	3.140	2.007	1.971	2.401	1.960
27	1.758	1.772	2.954	1.808	1.776	2.222	1.772
28	1.413	1.422	2.610	1.455	1.425	1.903	1.426
29	0.852	0.856	2.051	0.8919	0.858	1.404	0.861

Finally, we must stress that the above approximations of the ALOHA system work quite well because the condition (4.14) holds, that is $a_k = \Theta(1)$. Without that condition, the approximation might be quite poor. To see it, let us consider an M|D|1 synchronized queueing system. Then the transition matrix P is downward uniformly bounded with $p_{k,k+l} = b_{l+1}$, $k \neq 0$, $l = -1, 0, 1, \ldots,$ and $p_{0,l} = b_l$ where $b_l = \frac{\lambda^l}{l!} e^{-\lambda}$ (Poisson arrival process) [7]. The P matrix looks like below:

$$P = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & \cdots \\ b_0 & b_1 & b_2 & b_3 & \cdots \\ 0 & b_0 & b_1 & b_2 & \cdots \\ 0 & 0 & b_0 & b_1 & \cdots \\ . & . & . & . & \cdots \end{bmatrix}.$$

The probabilities b_k exponentially decreases with k, hence the matrix P is near tridiagonal and (4.11) is satisfied with $\varepsilon = 0.1$. The lower bound $\underline{a}_k$ is equal to

$$\underline{a}_k = \frac{1 - b_0 - b_1 - b_2}{b_0} \tag{5.3}$$

and for $\lambda < 0.5$ $\underline{a}_k << 1$. For example, for $\lambda = 0.1$ $\underline{a}_k = 0.0052$. Hence, (4.14) is not satisfied for $\beta = \varepsilon = 0.1$. In that case, we expect that (4.15) is not a tight approximation for a_k. Table 2 contains some values of $\underline{a}_k$ and a_k for $\lambda = 0.2$. It is easy to see that $\underline{a}_k$ does not tightly evaluate a_k, however, for larger values of k the approximation improves. This confirms our observation that condition (4.4) is quite essential for good approximation of a_k by the lower bound $\underline{a}_k$.

Table 2. M|D|1 system with $\lambda = 0.2$

k	$\underline{a}_k$	a_k
0	0.221	0.221
1	0.021	0.118
2	0.021	0.082
3	0.021	0.075
4	0.021	0.975
5	0.021	0.070
6	0.021	0.070
7	0.021	0.070
8	0.021	0.070
9	0.021	0.070
10	0.021	0.070

6. CONCLUSIONS

Noting that quantitative analysis of queueing models becomes increasingly complicated, we have argued that a qualitative approach might be a remedy. We focused our attention on the multimodality property of steady-state distributions. The importance of such a study comes from the fact that the behavior of some systems may depend on the number of maxima of the corresponding steady-state solution. The analysis was restricted to one-dimensional uniformly bounded Markov Chains, however, we have presented sufficient conditions for the multimodality. We have also shown a relationship between multimodality and the average drift (more precisely, conditional throughput and conditional input). This enables us to give a theoretical explanation for the bistable behavior of ALOHA type systems. Finally, we intend that this study begins a trend towards computer network performance through qualitative analyses.

APPENDIX. *Proof of the lemma.*

We prove (4.4), that is, we show that for all $k = 1, 2, \ldots,$ and $n < k$

$$\sum_{m=n+1}^{k} \left[r_{nn} + \sum_{j=0}^{n-1} r_{jm} \prod_{i=j}^{n-1} a_i^{-1} \right] \le -r_{nm} - \sum_{j=0}^{n-1} r_{jn} \prod_{i=j}^{n-1} a_i^{-1}, \tag{A1}$$

and equality in (A1) holds if and only if $k = M$. Let us notice first that by (2.2) the following holds

$$\sum_{i=t}^{k} r_{ji} \le 0 \quad \text{for all} \quad t \le j \le k, \tag{A2}$$

where $0 \le t < k \le M$. Having that we prove (A1) by induction with respect to n. For $n = 1$, by (A2) with $t = 0$ and 1, we find

$$0 \geq \sum_{i=0}^{k} r_{1i} = r_{10} + \sum_{i=1}^{k} r_{1i} \geq r_{10} \sum_{i=1}^{k} r_{0i}/(-r_{00}) + \sum_{i=1}^{k} r_{1i} =$$

$$\sum_{i=1}^{k} (r_{0i}/a_0 + r_{jk}) = r_{01}/a_0 + r_{11} + \sum_{i=2}^{k} (r_{0i}/a_0 + r_{1i}), \tag{A3}$$

which proves (A1) for $n = 1$ if one notes that $-r_{11} - r_{01}/a_0 > 0$. Let now (A1) be true for $n - 1$ and note the RHS of (A1) for $n - 1$ is equal to $a_{n-1}/r_{n,n-1} > 0$. Hence, by (A2) for $t = n$

$$0 \geq \sum_{i=n}^{k} r_{ni} + r_{n,n-1} \geq \sum_{i=n}^{k} r_{ni} + r_{n,n-1} \frac{\sum_{m=n}^{k} \left[r_{n-1,m} + \sum_{j=0}^{n-2} r_{jm} \prod_{i=j}^{n-2} a_i^{-1} \right]}{-r_{n-1,n-1} - \sum_{j=0}^{n-2} r_{j,n-1} \prod_{i=j}^{n-2} a_i^{-1}} =$$

$$\sum_{i=n}^{k} r_{n,i} + \frac{1}{a_{n-1}} \left\{ \sum_{m=n}^{k} \left[r_{n-1,m} + \sum_{j=0}^{n-2} r_{jm} \prod_{i=j}^{n-2} a_i^{-1} \right] \right\} = \tag{A4}$$

$$r_{nn} + \sum_{j=0}^{n-1} r_{jn} \prod_{i=j}^{n-1} a_i^{-1} + \sum_{i=n+1}^{k} \left[r_{ni} + \sum_{j=0}^{k} \sum_{j=0}^{n-1} r_{jm} \prod_{i=j}^{n-1} a_i^{-1} \right],$$

and (A4) proves (A1) noting that the first two terms of (A4) represents RHS of (A1). Equality in (A1) holds if and only if there is equality in (A3) and (A4). By (2.2) this implies that k=M, which proves the lemma.

□

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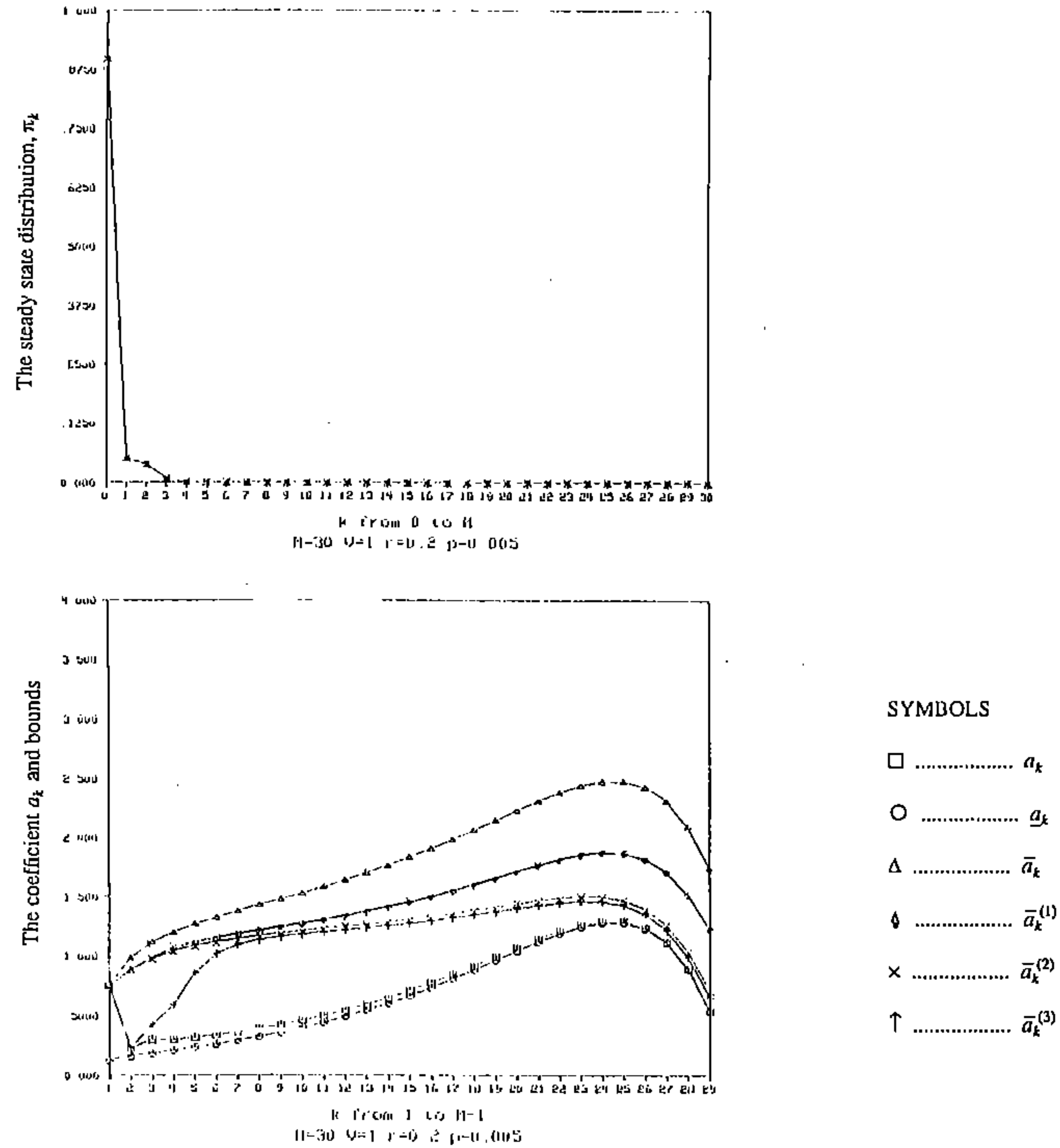

Figure 1. The steady state distribution , the coefficient a_k and its bounds for WRD ALOHA protocol with $p = 0.005$.

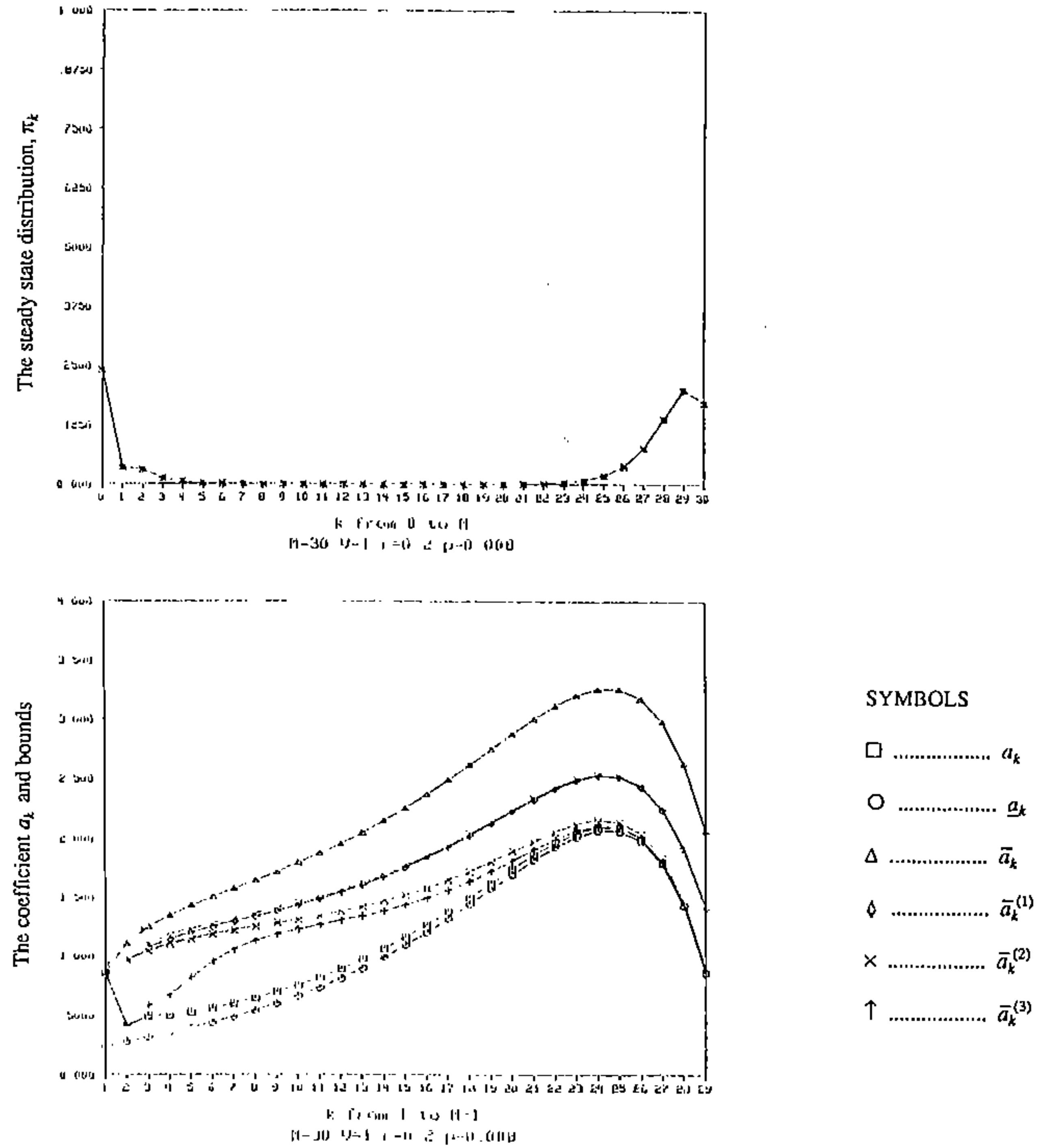

Figure 2. The steady state distribution, the coefficient a_k and its bounds for WRD ALOHA protocol with $p = 0.008$.