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GENERATION OF CONFIGURATION SPACES III:
THE CASE OF MOVING ALGEBRAIC CURVES

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Generation of Configuration Spaces III: The Case of Moving Algebraic Curves

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This paper describes the generation of configuration space obstacles in the plane, arising from the motion of objects with algebraic curve boundaries, moving with fixed orientation amongst obstacles bounded by algebraic curves. We show that the boundary of the configuration space obstacles are the *envelopes* of algebraic boundary curves of the reversed object, reversed with respect to a reference point of the object, with the reference point moving on the physical obstacle. Two different algebraic methods are given to generate the boundary of the configuration space obstacles, both of time complexity $O(n \log^2 n)$ for degree n algebraic curves. The task of finding collision free motion is then relatively simple and a number of polynomial time approaches are described.

1. Introduction

Using configuration space, (C-*Space*), to plan collision free motion for a single rigid object amongst physical obstacles, reduces the problem to planning motion for a mathematical point amongst "grown" configuration space obstacles, (the points in C-*Space* which correspond to the object overlapping one or more obstacles). For example, a rigid polygonal object translating and rotating in 2-Dimension space can be represented as a point moving in 3-Dimension C-*Space*. Collision free motion is then simply moving a point in any connected region of free configuration space. The technique thus relies, (and this is the more difficult part), on efficiently generating the configuration space obstacles, (C-*Space obstacles*).

Early uses of the configuration space approach were [10, 17] and more recently [7, 9, 12, 13, 16]. The only efficient algorithm known for generating C-*Space* obstacles has been for polygonal (degree 1) objects and obstacles, using methods for efficiently computing convex hulls [13]. Special cases like moving circles are the only other cases considered in the past [9, 14]. In earlier papers [4], [5], we considered objects and obstacles in three dimensions, bounded by general algebraic surfaces and in particular quadric surfaces.

In this paper we consider objects bounded by general algebraic curves and characterize and generate the boundary of the C-*Space obstacles*, arising from the motion of these objects moving with fixed orientation amongst obstacles also bounded by algebraic curves. In § 2 we first show that the boundary of C-*Space obstacles* are the *envelopes* of algebraic boundary curves of the reversed object, reversed with respect to a reference point of the object, with the reference point moving on the physical obstacle. In § 3 we consider general algebraic boundary curve objects that are moving with fixed orientation, and show that the boundary of the two dimensional C-*Space obstacles* can be efficiently generated by two different algebraic methods, in $O(n\log^2 n)$ time, for degree n algebraic curves. One efficiently produces the parametric equation of the curve while the other efficiently produces the implicit equation. Their time complexity is bound by the time complexity $O(n\log^2 n)$ of computing the resultant of two n degree polynomials, [16].

The unrestricted motion of circular objects deserves special mention in § 5, since not only are there inherent advantages in motion planning, (lower 2-dimension C-*Spaces* as opposed to

3-dimension *C–Spaces* for unrestricted motion of polygonal objects), the configuration spaces generated also provide the blending curves of intersecting conic segments, obtained by the method of circular sweeps. In § 6 we consider how to generate the *C–space* obstacles from general convex objects and obstacles with piecewise smooth boundary curve segments. Finally in § 7 we discuss a number of polynomial time approaches to finding collision free motion for a point in *C–Space*.

2. C-Spaces and Envelopes of Algebraic Curves

Let A be the moving object and B be a fixed obstacle. Let a point of A, say a_0, be designated as the reference point of A. Throughout we consider A to be free to move with fixed orientation. In this case configuration space is 2-dimensional (2–*space*). Let A_p denote the set of points in 2-space covered by A when A is located with a_0 at the point p. A_0 denotes this set of points when a_0 is at the origin. Let B denote the set of points in 2-space occupied by the obstacle B. The *C–space* obstacle corresponding to B is the set of configuration space points $\{p \mid A_p \cap B \neq \emptyset\}$. We define $B–A$ to be the set of points $\{p \mid p = b - a, a \in A, b \in B\}$ in 2-space, where $b - a$ is the vector difference of a and b.

Theorem 1 :[13] Let B' be the *C–space* obstacle corresponding to the obstacle B and the object A, then $B' = B–A_0 = \{p \mid p = b - a, a \in A_0, b \in B\}$.

Proof:

$$\begin{aligned} B' &= \{p \in R^2 \mid B \cap A_p \neq \emptyset\} \\ &= \{p \in R^2 \mid \exists\, b \in B \cap A_p\} \\ &= \{p \in R^2 \mid \exists\, b \in B \text{ such that } b = a + p \text{ for some } a \in A_0\} \\ &= \{p \in R^2 \mid p = b - a \text{ for some } b \in B \text{ and } a \in A_0\} \\ &= B - A_0 \qquad \square \end{aligned}$$

If A and B are convex and have linear (degree 1) boundary curves, that is they are convex polygonal objects then the *boundary* of $B–A_0$ = *Convex Hull* (Vertices (B) – Vertices (A)), [13]. The case of A and B being non-convex can be handled by first decomposing A and B into convex components.

More generally if A and B have algebraic boundary curves of any degree we now show that the *C-Space obstacle* = $Sweep(-A, a_0, B)$ = Sweep area of $-A$ as its reference point a_0 traces through the set B and further the *boundary* of the *C-Space obstacle* = $Envelope(-A, a_0, B)$ = Envelope of $-A$ as its reference point a_0 traces through the set B. We prove this rigorously in *Theorems* 2 and 3 below.

To make our statements rigorous we need to make a few definitions. A point q is said to be *interior* to a set S if all the points sufficiently near to q also belong to S. A point p that is not in a set S is said to be *exterior* to S if all the points sufficiently near to p are also outside the set S. A *boundary* point for a set S is a point that is neither *interior* to S nor *exterior* to S. Let $Interior(S)$ = the set of all interior points of S. $Exterior(S)$ = the set of all exterior points of S. $Bdr(S)$ = the set of all boundary points of S. A set S is *open* if $Interior(S) = S$. A set S is *closed* if $Bdr(S) \subset S$. We assume that A and B are open.

Further $A_{\bar{p}}$ is *free* (in motion) from B iff $\bar{p}$ is an exterior point of $B' = B - A_0$. $A_{\bar{p}}$ *collides* with B iff $\bar{p}$ is an interior point of $B' = B - A_0$. $A_{\bar{p}}$ *contacts* with B iff $\bar{p}$ is neither an exterior point nor an interior point of $B' = B - A_0$ iff neither $A_{\bar{p}}$ is free from B nor $A_{\bar{p}}$ collides with B.

Theorem 2 : The *boundary* of *C-space* obstacle $B' = Bdr(B - A_0) = \{ \bar{p} \in R^2 \mid A_{\bar{p}}$ *contacts* with $B \}$

Proof : The boundary of *C-space* obstacle B'

$$= \{ \bar{p} \in R^2 \mid \bar{p} \in \mathrm{Bdr}(B') \}$$
$$= \{ \bar{p} \in R^2 \mid \bar{p} \text{ is a boundary point of } B' \}$$
$$= \{ \bar{p} \in R^2 \mid \bar{p} \text{ is neither interior to } B' \text{ nor exterior to } B' \}$$
$$= \{ \bar{p} \in R^2 \mid \bar{p} \text{ is neither free from } B' \text{ nor collides with } B' \}$$
$$= \{ \bar{p} \in R^2 \mid A_{\bar{p}} \text{ contacts with } B \} \quad \square$$

Further using our definitions above of *boundary* and *interior* of sets we have the

$Envelope(-A, a_0, B)$

= Envelope of $-A$ as its reference point a_0 traces through the set B

$= \{ \bar{p} \in R^2 \mid \bar{p} \in \mathrm{Bdr}((-A)_p)$ for some $p \in \mathrm{Bdr}(B)$, but $\bar{p} \notin Interior((-A)_{p'})$ for any $p' \in B \}$

Theorem 3 : $\{ \bar{p} \in R^2 \mid A_{\bar{p}}$ *contacts* with $B \} = Envelope(-A, a_0, B)$

Proof : To see the containment ($\subseteq$) one way : Let $A_{\bar{p}}$ and B make contact at $p \in$ Bdr(B), then $p - \bar{p}$ is a point on the boundary of A_0 and $\bar{p} - p = -(p - \bar{p})$ is a point on the boundary of $(-A)_0$. When we place $(-A)$ with its reference point a_0 at p, $\bar{p}$ is on the boundary of $(-A)_p$.

Further $\bar{p} \notin$ *Interior*$((-A)_{p'})$ for any $p' \in B$. Assuming the contrary, if $\bar{p} \in$ *Interior*$((-A)_{p'})$ for some $p' \in B$, then $\bar{p} \in$ *Interior*$(\{p'\} + (-A)_0)$ for some $p' \in B$. This implies $\bar{p} \in$ Interior($B - A_0$), or equivalently $A_{\bar{p}}$ *collides* with B, (a contradiction).

To see the reverse containment ($\supseteq$) : Let $\bar{p} \in$ *Envelope*$(-A, a_0, B)$, then $\bar{p} \in$ *Bdr*$((-A)_p)$ for some $p \in$ Bdr(B), and $\bar{p} \notin$ *Interior*$((-A)_{p'})$ for any $p' \in B$. Equivalently, $p \in$ Bdr($A_{\bar{p}}$) for some $p \in$ Bdr(B), and $\bar{p} \notin$ *Interior*$(B - A_0)$. Neither $A_{\bar{p}}$ is free from B nor $A_{\bar{p}}$ collides with B. Hence, $A_{\bar{p}}$ contacts with B. □

Hence Theorems 2 and 3 together prove that the *boundary* of the *C–Space obstacle* = *Envelope*$(-A, a_0, B)$. It also follows that the *C–Space* obstacle B' is given by

Sweep$(-A, a_0, B)$

= Sweep area of $-A$ as its reference point a_0 traces through the set B.

$= \{ \bar{p} \in R^2 \mid \bar{p} \in (-A)_p$ for some p $\in B \}$

$= B - A_0$

3. Generating Planar Envelopes

Without loss of generality consider A and B to be convex. Arbitrary A and B are each representable as unions of (not necessarily disjoint) convex pieces. Then the *C–space obstacles* can be generated for each pair of convex pieces separately all with respect to a single reference point. The *C–Space obstacle* for the entire A would then be the union of the separate *C–Space obstacles*. For motion planning it is unimportant whether *C–Space obstacles* overlap, since finding a collision free path which avoids the union of *C–Space obstacles* can be checked for each component of the *C–Space obstacles*. In turn, generating envelopes of convex A and B rids the envelopes of singularities which may occur due to the respective curvatures of the boundaries of A and B, even if the respective boundaries are smooth (non-singular).

We consider the boundary curve of A to have an implicit form while the boundary curve of B consisting of segments of algebraic curves could be in either implicit or parametric form. We present two algebraic methods of generating the envelopes which are the boundary of the *C–Space* obstacles. Given the parametric form of obstacle B, one algorithm obtains the implicit equation of the envelope curve, § 3.1. The other with the implicit equation of obstacle B, obtains both the implicit or the parametric equations of the envelope curves, § 3.2.

3.1. Parametric Method

When the reference point on the object moves on a curve specified by its single independent parameter, the envelopes of the family of object boundary curves whose equation involves only one parameter, $F(x,y,t) = 0$, is given by the simultaneous solution of $F = 0$ and $F_t = 0$, where the latter equation is the first partial derivative of F with respect to t, (*Theorem* 4). Eliminating t between the two equations, by use of resultants[15], gives rise to the implicit equation of the envelope curve.

Theorem 4 : Let a reversed object $-A$ be defined by $-A : f(x,y) < 0$ with Bdr($-A$) : $f(x,y) = 0$ for some smooth function $f : R^2 \rightarrow R$, and $C : R \rightarrow R^2$ be a smooth curve in R^2 with $C(t) = (c_1(t), c_2(t))$. Let $F(x,y,t) = f(x-c_1(t), y-c_2(t))$ for $t \in R$. Then for $(x,y) \in$ Bdr($\{(x,y) \in R^2 \mid F(x,y,t) < 0,\ t \in R\}$), $F(x,y,t) = F_t(x,y,t) = 0$ for some $t \in R$, where F_t is the first partial derivative of F with respect to t.

Proof : If $(x,y) \in Bdr\{(x,y) \in R^2 \mid F(x,y,t) < 0\} = Bdr(Sweep(-A, a_0, C))$, then $(x,y) \in Envelope(-A, a_0, C)$. This implies $(x,y) \in$ Bdr($(-A)_{C_{(t)}}$) for some $t \in R$, and $(x,y) \notin Interior((-A)_{C_{(t)}})$ for any $t \in R$. Equivalently, $F(x,y,t_0) = 0$ for some $t_0 \in R$, and $F(x,y,t) \geq 0$ for any $t \in R$.

Let $g : R \rightarrow R$ be defined by $g_{(x,y)}(t) = F(x,y,t)$ for $t \in R$, then g has its minimum at t_0. And so, $g'_{(x,y)}(t_0) = 0$ and $g_{(x,y)}(t_0) = 0$. Hence we get $F(x,y,t_0) = F_t(x,y,t_0) = 0$. □

Hence, given the parametric equations of the boundary of the physical two dimensional obstacles one can generate the corresponding implicit envelope curve boundary of the configuration space obstacle. Further if parametric equations are not known we can quickly generate them. For conics there always exist a rational parameterization, [1]. *Cubics*, degree three curves, do not always have a rational parameterization. However they always have a parameterization of the type which allows a single square root of rational functions, [2]. The parametrization of higher degree plane curves and space curves are discussed in [3].

3.2. Implicit Method

When object A moves by making contact with the algebraic curve, given by $g(\alpha,\beta) = 0$ the envelope of the family of boundary curves of the moving object satisfy the simultaneous equations of *Theorem* 5.

Theorem 5 : Let a reversed object $-A$ be given by $-A : f(x,y) < 0$ with $\mathrm{Bdr}(-A)$: $f(x,y) = 0$ for some smooth function $f : R^2 \to R$, and an obstacle B be given by B : $g(\alpha,\beta) < 0$ with $\mathrm{Bdr}(B) : g(\alpha,\beta) = 0$ for some smooth function $g : R^2 \to R$. Also let

$$F(x,y,\alpha,\beta) = f(\alpha-x,\beta-y) \quad \text{for } (x,y), (\alpha,\beta) \in R^2.$$

Then for $(x,y) \in \mathrm{Bdr}(\{(x,y) \in R^2 \mid F(x,y,\alpha,\beta) < 0,\ g(\alpha,\beta) = 0\})$,

$$\begin{cases} F(x, y, \alpha, \beta) = 0 & (1) \\ g(\alpha, \beta) = 0 & (2) \\ F_\alpha \cdot g_\beta - g_\alpha \cdot F_\beta = 0 & (3) \end{cases}$$

Proof : $(-A)_p$ is given by $(-A)_p : f(x-\alpha,y-\beta) < 0$ for $p = (\alpha,\beta) \in R^2$, and $A_{\bar{p}}$ is given by $A_{\bar{p}} : f(\alpha-x,\beta-y) < 0$ for $\bar{p} = (x,y) \in R^2$. When $A_{\bar{p}}$ makes a contact with B at $p = (\alpha,\beta)$, $A_{\bar{p}}$ and B have a common tangent line. And so, $\nabla g = (g_\alpha, g_\beta)$ and $(f_\alpha, f_\beta) = (F_\alpha, F_\beta)$ have opposite directions. Hence, for some $k < 0$,

$$g_\alpha = k\, F_\alpha$$

$$g_\beta = k\, F_\beta$$

And

$$F_\alpha \cdot g_\beta - g_\alpha \cdot F_\beta = F_\alpha \cdot k\, F_\beta - k\, F_\alpha \cdot F_\beta = 0.$$

Equations (1) and (2) are trivial. □

Eliminating y (and x) from the equations (1), (3) above, gives rise to the envelope curve equations x (and y) in terms of α and β. To obtain parametric equations of the envelope curve all we need to do is to obtain the rational parametric equations of α and β, [1,2,3].

One could also generate the implicit equations of the envelope curve. We could implicitize the parametric equations obtained above, by use of resultants, or proceed more directly as follows. Eliminate β (and α) from the equations (1) and (3) above which gives rise to equations α and β in terms of x and y. To obtain implicit equations of the envelope curve all we need to do is to substitute these equations into (2).

4. Examples

I. When we have a family of circles of radius r whose centers run through a plane curve given by $g(\alpha, \beta) = 0$, the family of circles can be given by

$$\begin{cases} F(x, y, \alpha, \beta) = (x - \alpha)^2 + (y - \beta)^2 - r^2 = 0 \\ g(\alpha, \beta) = 0 \end{cases}$$

The envelope of this family of circles satisfy the following simultaneous equations

$$\begin{cases} F(x, y, \alpha, \beta) = (x - \alpha)^2 + (y - \beta)^2 - r^2 = 0 & (1) \\ g(\alpha, \beta) = 0 & (2) \\ F_\alpha \cdot g_\beta - g_\alpha \cdot F_\beta = 0 & (3) \end{cases}$$

From 1, we have

$$x^2 - 2\alpha x + \alpha^2 - r^2 + (y - \beta)^2 = 0 \qquad (1')$$

$$y^2 - 2\beta y + \beta^2 - r^2 + (x - \alpha)^2 = 0 \qquad (2')$$

From 3, we have

$$-2(x - \alpha) \cdot g_\beta - g_\alpha \cdot (-2(y - \beta)) = 0$$

$$(x - \alpha) \cdot g_\beta - g_\alpha (y - \beta) = 0$$

$$g_\beta \cdot x - \alpha g_\beta - g_\alpha (y - \beta) = 0 \qquad (3')$$

and similarly

$$g_\alpha \cdot y - \beta g_\alpha - g_\beta (x - \alpha) = 0 \qquad (3'')$$

By 1' and 3', we have

$$\begin{vmatrix} 1 & -2\alpha & \alpha^2 - r^2 + (y-\beta)^2 \\ g_\beta & -\alpha g_\beta - g_\alpha(y-\beta) & 0 \\ 0 & g_\beta & -\alpha g_\beta - g_\alpha(y-\beta) \end{vmatrix} = 0$$

$$(\alpha g_\beta + g_\alpha(y-\beta))^2 - g_\beta\left\{2\alpha(\alpha g_\beta + g_\alpha(y-\beta)) - g_\beta(\alpha^2 - r^2 + (y-\beta)^2)\right\} = 0$$

$$(g_\alpha^2 + g_\beta^2)(y-\beta)^2 = g_\beta^2 r^2$$

$$\therefore y = \beta \pm \frac{g_\beta r}{\sqrt{g_\alpha^2 + g_\beta^2}}$$

Similarly

$$x = \alpha \pm \frac{g_\alpha \cdot r}{\sqrt{g_\alpha^2 + g_\beta^2}}$$

where $g(\alpha, \beta) = 0$

1. When the underlying curve is a straight line

$$g(\alpha, \beta) = A\alpha + B\beta + C = 0$$

$$\Rightarrow g_\alpha = A, \quad g_\beta = B$$

$$\sqrt{g_\alpha^2 + g_\beta^2} = \sqrt{A^2 + B^2}$$

$$\therefore \begin{cases} x = \alpha \pm \dfrac{A \cdot r}{\sqrt{A^2 + B^2}} \\ y = \beta \pm \dfrac{B \cdot r}{\sqrt{A^2 + B^2}} \end{cases}$$

$$\begin{cases} \alpha = x \pm \dfrac{A \cdot r}{\sqrt{A^2 + B^2}} \\ \beta = y \pm \dfrac{B \cdot r}{\sqrt{A^2 + B^2}} \end{cases}$$

$$\Rightarrow A \cdot (x \pm \frac{A \cdot r}{\sqrt{A^2+B^2}}) + B \cdot (y \pm \frac{B \cdot r}{\sqrt{A^2+B^2}}) + C = 0$$

$$Ax + By + C \pm \frac{(A^2+B^2)\cdot r}{\sqrt{A^2+B^2}} = 0$$

$$\therefore Ax + By + C \pm r\sqrt{A^2+B^2} = 0$$

Hence, the envelopes are two straight lines.

2. When the underlying curve is a circle of radius R.

We may assume this underlying circle is centered at the origin.

$$g(\alpha, \beta) = \alpha^2 + \beta^2 - R^2 = 0$$

$$\Rightarrow g_\alpha = 2\alpha, \quad g_\beta = 2\beta,$$

$$\sqrt{g_\alpha^2 + g_\beta^2} = \sqrt{4\alpha^2 + 4\beta^2} = 2R.$$

$$\begin{cases} x = \alpha \pm \frac{2\alpha \cdot r}{2R} = \alpha \pm \frac{\alpha r}{R} = \alpha(1 \pm \frac{r}{R}) \\ y = \beta \pm \frac{2\beta \cdot r}{2R} = \beta(1 \pm \frac{r}{R}) \end{cases}$$

$$\begin{cases} \alpha = (\frac{R}{R \pm r})x \\ \beta = (\frac{R}{R \pm r})y \end{cases}$$

$$\Rightarrow (\frac{R}{R \pm r})^2 x^2 + (\frac{R}{R \pm r})^2 y^2 - R^2 = 0$$

$$\therefore x^2 + y^2 = (R \pm r)^2$$

Hence, the envelopes are two circles of radii $(R + r)$ and $(R - r)$.

3. When the underlying curve is an ellipse.

We may assume the underlying ellipse is centered at origin and its axes are parallel to x and y axes

$$\therefore\ g(\alpha, \beta) = B^2\alpha^2 + A^2\beta^2 - A^2B^2 = 0$$

$$\Rightarrow g_\alpha = 2B^2\alpha, \quad g_\beta = 2A^2\beta$$

$$\sqrt{g_\alpha^2 + g_\beta^2} = 2\sqrt{B^4\alpha^2 + A^4\beta^2}$$

$$\therefore \begin{cases} x = \alpha \pm \dfrac{B^2\alpha r}{\sqrt{B^4\alpha^2 + A^4\beta^2}} \\ y = \beta \pm \dfrac{A^2\beta r}{\sqrt{B^4\alpha^2 + A^4\beta^2}} \end{cases}$$

Let $\alpha = A\cos\theta$, $\beta = B\sin\theta$, $0 \le \theta \le 2\pi$, then

$$\begin{cases} x = A\cos\theta \pm \dfrac{B^2A\cos\theta \cdot r}{\sqrt{B^4A^2\cos^2\theta + A^4B^2\sin^2\theta}} \\ = \mathrm{A}\cos\theta \pm \dfrac{rB\cos\theta}{\sqrt{A^2\sin^2\theta + B^2\cos^2\theta}} \\ y = B\sin\theta \pm \dfrac{r\,A\sin\theta}{\sqrt{A^2\sin^2\theta + B^2\cos^2\theta}} \end{cases}$$

where $0 \le \theta \le 2\pi$

II. When we have a family of ellipses of radius r whose centers run through a plane curve given by $g(\alpha, \beta) = 0$, the family of ellipses can be given by

$$\begin{cases} F(x, y, \alpha, \beta) = b^2(x - \alpha)^2 + a^2(y - \beta)^2 - a^2b^2 = 0 \\ g(\alpha, \beta) = 0 \end{cases}$$

The envelope of this family of ellipses satisfy the following simultaneous equations

$$\begin{cases} F(x, y, \alpha, \beta) = b^2(x-\alpha)^2 + a^2(y-\beta)^2 - a^2b^2 = 0 & (1) \\ g(\alpha, \beta) = 0 & (2) \\ F_\alpha \cdot g_\beta - g_\alpha \cdot F_\beta = 0 & (3) \end{cases}$$

From 1, we have

$$b^2x^2 - 2b^2\alpha x + b^2\alpha^2 - a^2b^2 + a^2(y-\beta)^2 = 0 \qquad (1')$$

$$a^2y^2 - 2a^2\beta y + a^2\beta^2 - a^2b^2 + b^2(x-\alpha)^2 = 0 \qquad (1'')$$

From 3, we have

$$-2b^2(x-\alpha) \cdot g_\beta - g_\alpha \cdot (-2a^2(y-\beta)) = 0$$

$$b^2(x-\alpha)g_\beta - g_\alpha \cdot (a^2(y-\beta)) = 0$$

$$b^2 g_\beta x - b^2\alpha g_\beta - g_\alpha \cdot a^2(y-\beta) = 0 \qquad (3')$$

$$a^2 g_\alpha y - a^2\beta g_\alpha - g_\beta \cdot b^2(x-\alpha) = 0 \qquad (3'')$$

By 1' and 3', we have

$$\begin{vmatrix} b^2 & -2b^2\alpha & b^2\alpha^2 - a^2b^2 + a^2(y-\beta)^2 \\ b^2 g_\beta & -b^2\alpha g_\beta - g_\alpha \cdot a^2(y-\beta) & 0 \\ 0 & b^2 g_\beta & -b^2\alpha g_\beta - g_\alpha \cdot a^2(y-\beta) \end{vmatrix} = 0$$

$$(b^2\alpha g_\beta + g_\alpha a^2(y-\beta))^2 - g_\beta \cdot \{2b^2\alpha(b^2\alpha g_\beta + g_\alpha a^2(y-\beta)) - b^2 g_\beta \cdot (b^2\alpha^2 - a^2b^2 + a^2(y-\beta)^2)\} = 0$$

$$a^4 g_\alpha^2 (y-\beta)^2 - a^2\, b^4 g_\beta^2 + a^2b^2 g_\beta^2 (y-\beta)^2 = 0$$

$$(a^4 g_\alpha^2 + a^2b^2 g_\beta^2)(y-\beta)^2 = a^2b^4 g_\beta^2$$

$$(a^2 g_\alpha^2 + b^2 g_\beta^2)(y-\beta)^2 = b^4 g_\beta^2$$

$$\therefore\ y = \beta \pm \frac{b^2 g_\beta}{\sqrt{a^2 g_\alpha^2 + b^2 g_\beta^2}}$$

Similarly

$$x = \alpha \pm \frac{a^2 g_\alpha}{\sqrt{a^2 g_\alpha^2 + b^2 g_\beta^2}}$$

where $g(\alpha, \beta) = 0$

1. When the underlying curve is a straight line

$$g(\alpha, \beta) = A\alpha + B\beta + C = 0$$

$$\Rightarrow g_\alpha = A,\ \ g_\beta = B$$

$$\sqrt{a^2 g_\alpha^2 + b^2 g_\beta^2} = \sqrt{a^2 A^2 + b^2 B^2}$$

$$\therefore \begin{cases} x = \alpha \pm \dfrac{a^2 \cdot A}{\sqrt{a^2 A^2 + b^2 B^2}} \\ y = \beta \pm \dfrac{b^2 \cdot B}{\sqrt{a^2 A^2 + b^2 B^2}} \end{cases}$$

$$\begin{cases} \alpha = x \pm \dfrac{a^2 A}{\sqrt{a^2 A^2 + b^2 B^2}} \\ \beta = y \pm \dfrac{b^2 \cdot B}{\sqrt{a^2 A^2 + b^2 B^2}} \end{cases}$$

$$A(x \pm \frac{a^2 \cdot A}{\sqrt{a^2 A^2 + b^2 B^2}}) + B \cdot (y \pm \frac{b^2 \cdot B}{\sqrt{a^2 A^2 + b^2 B^2}}) + C = 0$$

$$Ax + By + C \pm \frac{a^2 \cdot A^2 + b^2 \cdot B^2}{\sqrt{a^2 A^2 + b^2 B^2}} = 0$$

$$\therefore\ Ax + By + C \pm \sqrt{a^2 A^2 + b^2 B^2} = 0$$

Hence, the envelopes are two straight lines.

2. When the underlying curve is a circle of radius R.

We may assume this underlying circle is centered at the origin.

$$g(\alpha, \beta) = \alpha^2 + \beta^2 - R^2 = 0$$

$$\Rightarrow g_\alpha = 2\alpha, \quad g_\beta = 2\beta$$

$$\sqrt{a^2 g_\alpha^2 + b^2 g_\beta^2} = 2\sqrt{a^2\alpha^2 + b^2\beta^2}$$

$$\begin{cases} x = \alpha \pm \dfrac{a^2\alpha}{\sqrt{a^2\alpha^2 + b^2\beta^2}} \\ y = \beta \pm \dfrac{b^2 \cdot \beta}{\sqrt{a^2\alpha^2 + b^2\beta^2}} \end{cases}$$

Let $\alpha = R\cos\theta$, $\beta = R\sin\theta$, $0 \le \theta \le 2\pi$, then

$$\begin{cases} x = R\cos\theta \pm \dfrac{a^2 \cdot R\cos\theta}{\sqrt{a^2R^2\cos^2\theta + b^2R^2\sin^2\theta}} \\ \quad = R\cos\theta \pm \dfrac{a^2\cos\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \\ y = R\sin\theta \pm \dfrac{b^2\sin\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \end{cases}$$

where $0 \le \theta \le 2\pi$

3. When the underlying curve is an ellipse whose axes are parallel to the axes of the ellipses in the family.

We may assume the underlying ellipse is centered at the origin and its axes are parallel to the x & y axes.

$$\therefore\ g(\alpha, \beta) = B^2\alpha^2 + A^2\beta^2 - A^2B^2 = 0$$

$$\Rightarrow g_\alpha = 2B^2\alpha, \quad g_\beta = 2A^2\beta$$

$$\sqrt{a^2 g_\alpha^2 + b^2 g_\beta^2} = 2\sqrt{a^2B^4\alpha^2 + b^2A^4\beta^2}$$

$$\begin{cases} x = \alpha \pm \dfrac{a^2 \cdot B^2\alpha}{\sqrt{a^2B^4\alpha^2 + b^2A^4\beta^2}} \\ y = \beta \pm \dfrac{b^2 \cdot A^2\beta}{\sqrt{a^2B^4\alpha^2 + b^2A^4\beta^2}} \end{cases}$$

Let $\alpha = A\cos\theta$, $\beta = B\sin\theta$, $0 \le \theta \le 2\pi$, then

$$\begin{cases} x = A\cos\theta \pm \dfrac{a^2 \cdot B^2 \cdot A\cos\theta}{\sqrt{a^2B^4 \cdot A^2\cos^2\theta + b^2A^4B^2\sin^2\theta}} \\ \quad = A\cos\theta \pm \dfrac{a^2B\cos\theta}{\sqrt{a^2B^2\cos^2\theta + b^2A^2\sin^2\theta}} \\ y = B\sin\theta \pm \dfrac{b^2A\sin\theta}{\sqrt{a^2B^2\cos^2\theta + b^2A^2\sin^2\theta}} \end{cases}$$

where $0 \le \theta \le 2\pi$

4. When the underlying curve is an ellipse whose axes are not parallel to the axes of the ellipses in the family.

We may assume the underlying ellipse is centered at the origin even though its axes are not parallel to the x & y axes, and we may assume that the underlying ellipse has its major axes making angle ϕ with x & y axes.

$$\alpha' = r\cos(\theta - \phi) = r\cos\theta\cos\phi + r\sin\theta\sin\phi = \alpha\cos\phi + \beta\sin\phi$$

$$\beta' = r\sin(\theta - \phi) = r\sin\theta\cos\phi - r\cos\theta\sin\phi = \beta\cos\phi - \alpha\sin\phi$$

$$g(\alpha, \beta) = B^2 \cdot (\alpha\cos\phi + \beta\sin\phi)^2 + A^2(\beta\cos\phi - \alpha\sin\phi)^2 - A^2B^2 = 0$$

$$\begin{cases} g_\alpha = 2B^2(\alpha\cos\phi + \beta\sin\phi)\cdot\cos\phi \;+2A^2(\beta\cos\phi - \alpha\sin\phi)\cdot(-\sin\phi) \\ g_\beta = 2B^2(\alpha\cos\phi + \beta\sin\phi)\cdot\sin\phi \;+2A^2\cdot(\beta\cos\phi - \alpha\sin\phi)\cdot\cos\phi \end{cases}$$

$$\begin{cases} g_\alpha = 2B^2\cdot\alpha'\cdot\cos\phi - 2A^2\cdot\beta'\cdot\sin\phi \\ g_\beta = 2B^2\cdot\alpha'\cdot\sin\phi + 2A^2\cdot\beta'\cdot\cos\phi \end{cases}$$

$$g_\alpha^2 = 4B^4\cdot(\alpha')^2\cdot\cos^2\phi + 4A^4\cdot(\beta')^2\sin^2\phi - 8A^2B^2(\alpha'\beta')\sin\phi\cos\phi$$

$$g_\beta^2 = 4B^4\cdot(\alpha')^2\sin^2\phi + 4A^4\cdot(\beta')^2\cos^2\phi + 8A^2B^2(\alpha'\beta')\sin\phi\cos\phi$$

$$a^2g_\alpha^2 + b^2g_\beta^2 = 4B^4(\alpha')^2\cdot(a^2\cos^2\phi + b^2\sin^2\phi) + 4A^4(\beta')^2\cdot(a^2\sin^2\phi + b^2\cos^2\phi)$$
$$+ 8A^2B^2(\alpha'\beta')(\sin\phi\cos\phi)\cdot(b^2 - a^2)$$

$$\begin{cases} x = \alpha \pm \dfrac{a^2 g_\alpha}{\sqrt{a^2g_\alpha^2 + b^2g_\beta^2}} \\ y = \beta \pm \dfrac{b^2 g_\beta}{\sqrt{a^2g_\alpha^2 + b^2g_\beta^2}} \end{cases}$$

where g_α, g_β, and $\sqrt{a^2g_\alpha^2 + b^2g_\beta^2}$ are given above

5. Moving Circles

A practical methodology that is increasingly gaining ground in robot task planning, is that of hierarchical representations, [14]. The notion of hierarchical representations involves attempting to solve problems concerning physical objects by starting with very simple representations of the properties involved, and introducing more complex representations only as they are required to solve the problem. A system thus could initially approximate all objects by interior and exterior enclosing circles. If the exterior circles do not intersect at all during a planned motion, the motion is known to be safe. If the interior circles intersect during such a motion, a collision free motion is impossible. If neither of these conditions are met the system then could proceed to a finer level of detail. In most industrial applications, the workspace environment of the robot is sparsely cluttered and finding collision free paths for general objects by considering the motion of enclosing circles of the objects would suffice. Possibly, though at high computational cost, exact high degree representations could be used.

There is a further advantage in considering circles. Approximations of the moving object by the lowest degree (planar) curves, i.e., polygonal objects amongst polygonal obstacles lead to an immediate computational difficulty. The unrestricted motion of a polygonal rigid object reduces to the motion of a point in 3-dimensional configuration space where both finding connected regions for collision free motion and characterizing 3-dimensional *C–space* obstacles is more difficult, [16]. The unrestricted motion of a circle on the other hand is transformed to the motion of a point amongst 2-dimensional *C–space* obstacles, [9].

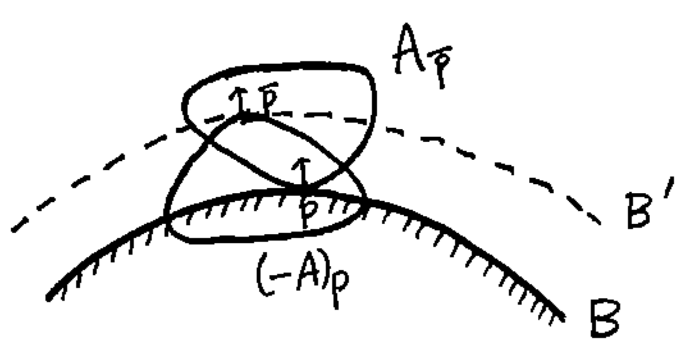
Generating the 'grown' *C–Space obstacle* for moving circular objects and obstacles whose boundary consists of segments of algebraic curves can be separated into growing curves and vertices. Curves are grown by generating envelopes for the circle moving along the obstacle boundary curve. For lines and circles the grown curve is of the same kind and thus of degree 2. However for other curves the envelopes may be of higher degrees, (e.g. degree 4 for ellipse). The vertices of the physical obstacle B give rise to pieces of circular envelope curves.

Vertices of the obstacle give rise to circular segments as follows. At a common vertex v of two different edges the vertex v has 2 normal directions. These 2 normal vectors determine 2 points on the circle centered at v. These 2 points determine a geodesic circular arc on the circle of radius r. One can show that the circular edge defined by this geodesic circular arc is on the *C–space* obstacle B'. When the boundary of a convex obstacle B is a piecewise smooth curve, then the *C–space* obstacle B' of B due to a moving circle has a smooth boundary curve. This is because the edges of the *C–space* obstacle B', due to the way they are constructed, fill out the discontinuities of the directions of the curve normals on the vertices of B. The normal directions of the smooth blending circular edges over vertices of B, of the constructed *C–space* obstacle B' give all the missing outward normal directions between two adjacent non-smoothly connecting boundary edges of B.

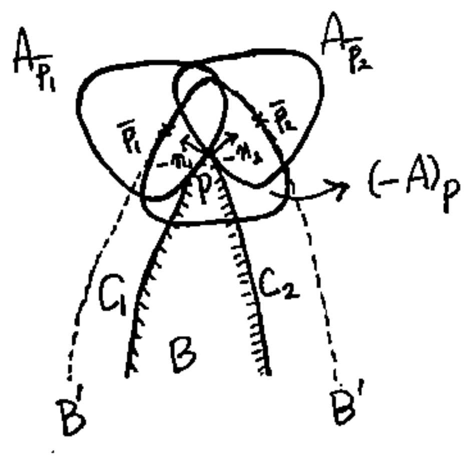
6. Objects and obstacles with piecewise smooth boundary curve segments

When we move a convex object A around a convex obstacle B, both having piecewise smooth boundary curves, the difficulty in generating *C–space* obstacle lies in dealing with the singularities on the moving object and the obstacle. When A and B have convex smooth boundary curves, the *C–space* boundary generation can be done by the envelope method described in § 3. But, in the general case of A and/or B having piecewise smooth boundary curves, we can apply the envelope method only in part (that is, for the contacts between the smooth boundary curve segments of A and B) and we have to consider how to generate *C–space* boundary curve segments resulting from the contacts between a singular point and other parts of the boundary (smooth curve segment or another singular point). We will consider each of these special cases in the following sections.

Before that we need to make the following simple observation. In the case of moving circles (with its center taken to be its reference point), a boundary point on the obstacle is 'grown' into a point at distance r, (the radius of a moving circle), in the outward normal direction. For circles then this gives quite a simple generation operation for *C–space* obstacles. In the case of a moving general convex object, a point p on the obstacle B's boundary will be 'grown' into the point $\bar{p}$ at which the reference point of the moving object is located at the moment the object A contacts with the obstacle B at p. When we place the reversed object $-A$ with it's reference point at the point p, the point $\bar{p}$ will be on the boundary of $(-A)_p$ and furthermore $(-A)_p$ has a outward normal direction at $\bar{p}$ which is parallel to the outward normal direction of B at the point p. Since A and $-A$ are convex, we can uniquely determine a point $\bar{p}$ on the boundary of $-A$ from the outward normal direction of B at p.



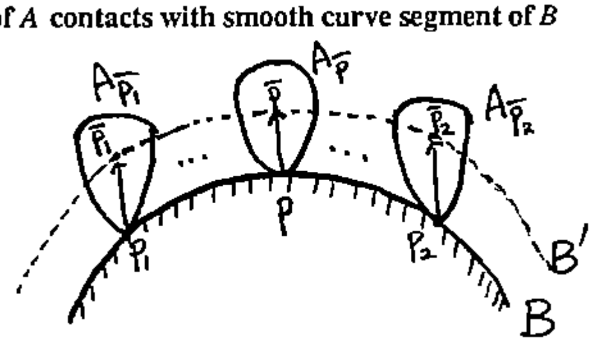
6.1. Smooth curve segment of A contacts with singular point of B



Suppose A moves from left to the right by sliding along C_1, contacting at p, and then sliding along C_2. When A first comes into contact with B at the point p, the corresponding boundary contact point of A has its outward normal direction similar to the direction $-n_1$. While A moves contacting at the point p, the outward normal directions of A at the corresponding contact boundary points of A change smoothly from $-n_1$ to $-n_2$. After contacting at p with the outward normal direction similar to the direction $-n_2$, A will slide along some other points on the curve C_2.

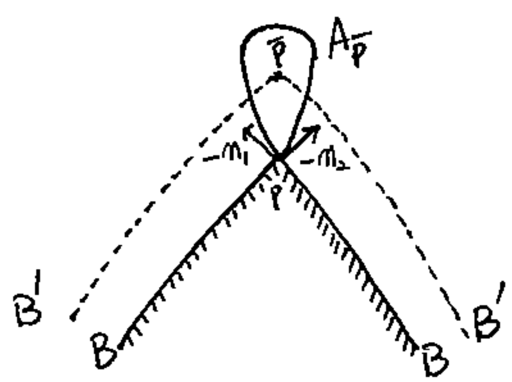
Now, let's interpret this motion in terms of $-A$. While A moves contacting at the point p, the reference point of A moves along a boundary curve of $(-A)_p$ from a point where $-A$ has outward normal direction similar to the direction n_1 to a point where $-A$ has outward normal direction similar to the direction n_2. Hence, the growing of a singular boundary point of B can be done by finding the boundary curve segment on which the reversed object $-A$ has outward normal directions between n_1 and n_2.

6.2. Singular point of A contacts with smooth curve segment of B



When A first comes into contact with B at a point p_1, A and B have opposite outward normal directions. Point p_1 will be grown into the point $\bar{p}_1$ which is the current position of the reference point of A. While A slides with its singular point contacting with a smooth boundary curve of the obstacle B, the differences $\bar{p}-p$ of the reference point positions and the contact points are the same. Hence, the curve segment of B contacting with a singular point of A will be grown into a *C−space* obstacle curve segment by translating it by $\bar{p}-p$.

6.3. Singular point of A contacts with singular point of B



Two different cases can occur right before the singular point of A comes into contact with the singular point of B. In the first case, the singular point of A slides along a smooth boundary curve of B and it comes into contact with a singular point of B. In the second case, a smooth boundary curve of A slides at the singular point of B until singular points of A and B come into contact. The *C−space* generation for the first case can be done by the method of § 6.2 and the second case can be done by the method of § 6.1. Similarly after the singular-singular contact, *C−space* generation can again be done by the methods of the previous two sections. Thus singular-singular contacts are the only sources to result in singular points on the generated *C−space* obstacle boundary.

6.4. Generation of C-space obstacle boundary curve segments

First, locate singular points $p_1, \ldots, p_m$ on the object A and at each singular point p_i ($i = 1, \ldots, m$) compute two normal directions $n_{i,1}$ & $n_{i,2}$ ($i = 1, \ldots, m$) corresponding to the two adjacent curve segments non-smoothly intersecting at p_i. Similarly, locate singular points $q_1, \ldots, q_n$ of the obstacle B and compute normal directions $n'_{j,1}$ & $n'_{j,2}$ ($j = 1, \ldots, n$) corresponding to each q_j (j

= 1, ..., n). Now, locate the boundary points (or straight line curve segments) of B at which B has outward normal directions same as the directions $n_{i,1}$ & $n_{i,2}$ (i = 1, ..., m). Using the points obtained in this way and the original singular points as the end points of boundary curve segments, we can divide the boundary of B into a sequence of curve segments of shorter length. We can do similar segmentation to the boundary of A. Now, depending on the range of outward normal directions on each segment or end point of a segment, we can apply growing operations on pairs consisting of a segment of A and a segment of B, pairs consisting of a segment of A and a singular point of B, or pairs consisting of a singular point of A and a segment of B. From the way the segments are constructed, the segments of A and B on the same pair have the same range of normal directions, and the segment of A and the point of B on the same pair have the same range of normal directions, ... and so on. On each segment-segment pair we can use the envelope method developed in § 3. Further on each segment-point pair we can use the methods described in this section.

7. Planning Point Motion

Known algorithms for planning paths between an initial point and a final point in *C–Space* can be classed into two approaches; one which reduces the problem to *graph searching* and the other which tries to find a path directly. Of the methods which reduces to graph searching we have the Tarski-Collins cell decomposition based method, [16] and the generalized Voronoi based (retraction) approaches [9]. In the cell decomposition approach, the free *C–Space* is partitioned into a finite collection of connected subsets called cells and the *graph* represents the adjacency relation between these cells. In the Voronoi based approach the free *C–Space* is again partitioned into a finite collection of connected subsets (a tesselation of *C–Space*), called Voronoi cells, whose intersections are at a maximum clearance from the *C–Space* obstacles. The *graph* is the 1-dimension network in *C–Space* which arises from the successive intersections of higher dimension cells, [9].

In the more direct approach, the path between the two points is taken to be a (rational) algebraic curve. This is intersected with the boundary curves of the *C–Space obstacles* and the intersection points are computed. The collision free path is then piecewise segments of the polynomial curve in free space and geodesic paths on the boundary curves of the *C–Space obstacles*. The intersection points can be obtained as the roots of $p(t)$, a polynomial function of the parameter t of the curve, and can be computed by standard applications of numerical or symbolic polynomial root solving techniques, [8]. The simplest algebraic curve is the straight line and with this choice of initial path, the intersection points on the *C−Space obstacles* with curve degree up to degree 4, can in fact be directly computed in closed form by the methods of Bhaskara (quadratic), Cardan (cubic) and Ferrari (quartic).

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