1987

The Performance of Numerical Methods for Elliptic Problems with Mixed Boundary Conditions

Wayne R. Dyksen

Calvin J. Ribbens

John R. Rice

Purdue University, jrr@cs.purdue.edu

Report Number:

86-590

Dyksen, Wayne R.; Ribbens, Calvin J.; and Rice, John R., "The Performance of Numerical Methods for Elliptic Problems with Mixed Boundary Conditions" (1987). *Department of Computer Science Technical Reports*. Paper 509.
<https://docs.lib.purdue.edu/cstech/509>

THE PERFORMANCE OF NUMERICAL METHODS
FOR ELLIPTIC PROBLEMS
WITH MIXED BOUNDARY CONDITIONS

Wayne R. Dyksen
Calvin J. Ribbens
John R. Rice

CSD-TR-590
February 1987

THE PERFORMANCE OF NUMERICAL METHODS
FOR ELLIPTIC PROBLEMS
WITH MIXED BOUNDARY CONDITIONS

Wayne R. Dyksen
Calvin J. Ribbens*
John R. Rice*

CSD-TR 590
February 1987

Abstract

We consider solving linear, second order, elliptic partial differential equations with boundary conditions of type Dirichlet (DIR), mixed (MIX) and Neumann (NEU) and using software modules which implement five numerical methods: Hermite collocation plus band Gauss elimination (HC), ordinary finite differences plus band Gauss elimination (5P), ordinary finite differences with Dyakanov iteration (DY), DY with Richardson extrapolation to achieve fourth order convergence (D4), and ordinary finite differences with multigrid iteration (MG). We carry out a performance evaluation in which we measure the grid size and the computer time needed to achieve three significant digits of accuracy in the solution. We compute the changes in these two measures as we change boundary condition types from DIR to MIX and MIX to NEU and then test the following hypotheses: 1) the performance of all the modules is degraded by introducing the derivative terms into the boundary conditions, 2) HC is least affected, 3) the fourth order modules (HC and D4) are less affected than the other second order modules, 4) 5P is most affected. We establish these hypotheses with high levels of confidence using a sample of problems. We also establish with considerable confidence that these modules have the following rankings in absolute comparative time performance: MG (best), HC and D4, DY and 5P (worst).

*Supported in part by Air Force Office of Scientific Research grant AFOSR 84-0385.

1 Introduction

In this paper we investigate the effect of boundary condition types on the performance of several numerical methods for approximately solving elliptic partial differential equations (PDEs). The theoretical performance of classical methods is well-known. For well-behaved problems, one expects a certain rate of error reduction as a function of decreasing grid spacing. In practice however, for realistic problems it is often not enough to know asymptotic rates of convergence. Other considerations may affect the performance of algorithms; the best method in theory may not be the best in practice.

We design an experiment to investigate one of these "other considerations" which can affect the relative performance of numerical methods. We study the effect of mixed and Neumann boundary conditions on various discretization methods; that is whether the presence of derivative terms in the boundary conditions of an elliptic problem changes the relative performance of methods on that problem. Specifically, we study linear elliptic PDEs of the form

$$Lu = au_{xx} + cu_{yy} + du_x + eu_y + fu = g$$

defined on a rectangular domain R , subject to the boundary conditions

$$Mu = ru + su_n = t \quad \text{on } \partial R,$$

where u_n is the derivative in the outward-pointing normal direction, and a, c, d, e, f, g, r, s and t are functions of x and y . In the next section we describe a set of sample problems, the software implementations of the methods studied, and the experiment. In Section 3 we summarize the performance data and its analysis. In Section 4 we discuss our results and draw some conclusions. Some of the raw data is given in the Appendix.

2 Numerical Experiment

We select twelve PDEs based on problems from the PDE population of [4]. The elliptic operator Lu and domain R for each problem are given in Table 9 of the Appendix. Four of the problems (Problems 1, 4, 35 and 40) are used exactly as defined in [4]; the remaining eight are problems from [4] modified by introducing parameterized boundary conditions which allow the strength of a derivative term to be varied. Each problem is constructed to have a known true solution. The boundary conditions for each problem are parameterized so that we can easily study three boundary condition "types": Dirichlet (DIR), mixed (MIX), and nearly Neumann (NEU). The general form of the boundary condition for each problem is thus

$$\alpha ru + \beta su_n = t \quad \text{on } \partial R,$$

where r, s and t are fixed functions of x and y . For all but two of the problems r and s are constants. By varying α and β on one or more sides of the domain R we vary the boundary condition type as desired. Thus, for each of twelve elliptic operators we have three boundary condition types, for a total of 36 problems. The choices for α and β are given in Table 1.

We solve these problems using the ELLPACK system [5]. We select five methods of solution, implemented in modules available in ELLPACK:

HC: HERMITE COLLOCATION plus BAND GE. Discretization by collocation with Hermite bicubic basis functions. Linear system solved using band Gauss elimination with scaled partial pivoting.

Table 1: Parameterized boundary conditions, parameter values for each boundary condition type, and sides where these conditions hold for each test problem. Remaining sides have Dirichlet conditions. Problem numbers are from [4].

Problem	Mu	DIR	MIX	NEU	Sides
1	$u + \beta u_n$	$\beta = 0$	$\beta = 1$	$\beta = 1000$	N,S,E,W
3a	$u + \beta u_n$	$\beta = 0$	$\beta = 1$	$\beta = 1000$	N,E
4	$u - \beta(y - y^2)u_n$	$\beta = 0$	$\beta = 1$	$\beta = 1000$	W
4a	$u - \beta(y - y^2)u_n$	$\beta = 0$	$\beta = 1$	$\beta = 1000$	E
6a [†]	$u + \beta u_n$	$\beta = 0$	$\beta = 1$	$\beta = 1000$	N,S,E,W
6b [†]	$u + \beta u_n$	$\beta = 0$	$\beta = 1$	$\beta = 1000$	N,S,E,W
9a	$u + \beta u_n$	$\beta = 0$	$\beta = 1$	$\beta = 1000$	N,E
35 [†]	$\alpha u + \beta u_n$	$\alpha = 1, \beta = 0$	$\alpha = 2, \beta = 1$	$\alpha = 0, \beta = -1$	E,W
35a [†]	$\alpha u + \beta u_n$	$\alpha = 1, \beta = 0$	$\alpha = 2, \beta = 1$	$\alpha = 0, \beta = -1$	E,W
40	$\alpha u + \beta u_n$	$\alpha = 1, \beta = 0$	$\alpha = 0, \beta = 1$	$\alpha = 1, \beta = 1$	E,W
42a	$u + \beta u_n$	$\beta = 0$	$\beta = 1$	$\beta = 1000$	N,S,E,W
53a	$u + \beta u_n$	$\beta = 0$	$\beta = 1$	$\beta = 1000$	N,S,E,W

[†]These pairs of problems are identical except for the (x, y) domain.

5P: 5 POINT STAR plus BAND GE. Classical second order central finite differences plus band Gauss elimination.

DY: DYAKANOV CG. Symmetric 5-point finite differences. Linear system solution by preconditioned conjugate gradient iteration.

D4: DYAKANOV CG4. Symmetric 5-point finite differences. Fourth order accuracy achieved with Richardson extrapolation using a solution on a coarse grid and a fine grid. Linear system solution by preconditioned conjugate gradient iteration.

MG: MULTIGRID MG00. Second order central finite differences. Linear system solution by multigrid scheme.

There is considerable evidence that these implementations are well done and achieve close to the best performance possible using these methods. However, we consistently use the word “module” rather than “method” to emphasize that our study applies only to these implementations.

The first two modules are used to solve all 12 problems. The two DYAKANOV modules are applied to the 10 problems that have self-adjoint operators. MULTIGRID MG00 does not allow first derivative terms in an operator so it is only used to solve 9 of the problems. We use the performance evaluation system of [1] and [2] to collect and analyze data on the performance of these modules on the problems of each boundary condition type. The computations are done in double precision on a Ridge 32 under ROS 3.3, using the f77 FORTRAN compiler.

We seek to establish the validity of the following hypotheses:

1. The performance of all the modules is degraded by introducing the derivative terms into the boundary conditions.
2. HC is least affected.

Figure 1: Plot of the log of error at the nodes vs. log of grid size n for Problem 1.

3. The fourth order modules (HC and D4) are less affected than the other second order modules.
4. 5P is most affected.
5. Irrespective of the effect of the derivatives in the boundary conditions, the absolute comparative time performance rankings are: MG (best), HC and D4, DY and 5P (worst).

3 Performance Analysis

Performance is evaluated by the computer time t_3 (seconds) or grid size n_3 needed to achieve a prescribed accuracy—three significant digits is used in this study. The accuracy is measured as the ratio of maximum error at the grid points to maximum true solution at the grid points. To obtain t_3 and n_3 values we use several different grid sizes for each problem-module combination. For example, for 5 POINT STAR we use $n = 5, 9, 17, 33$ and 65 grid lines in each direction. Estimates of n_3 and t_3 are computed by fitting the least squares line through the appropriate data points, and finding the point on that line corresponding to three significant digits. Figure 1 shows an example plot of the logarithm of error versus the logarithm of grid size n for 5 POINT STAR and each of the three boundary condition types. First we compare the modules in terms of the relative performance change as derivatives are introduced into the boundary conditions. In Section 3.2 modules are ranked on the basis of error reduction versus time.

3.1 The effect of derivative boundary conditions on the modules

We first measure the degree to which introducing derivatives into the boundary conditions makes a problem more difficult for each of the modules. The hypothesis is that the change in boundary

Table 2: Percent changes Δn_3 in n_3 as boundary condition type changes from DIR to MIX and from MIX to NEU. A dash indicates that the module is not applicable to the indicated problem.

Problem	From DIR to MIX					From MIX to NEU				
	5P	HC	DY	D4	MG	5P	HC	DY	D4	MG
1	47	3	47	11	—	144	—2	158	41	—
3a	109	6	108	53	114	47	4	44	18	37
4	5	0	7	—3	4	20	0	19	11	18
4a	31	3	32	10	37	49	4	50	25	41
6a	0	0	0	0	0	0	0	0	0	0
6b	29	—2	27	0	22	5	1	5	0	4
9a	84	42	130	89	151	10	3	9	0	5
35	—11	22	—11	26	—16	23	3	22	10	23
35a	22	12	22	26	24	27	2	27	9	22
40	75	4	—	—	—	—15	—11	—	—	—
42a	113	1	—	—	—	41	—2	—	—	—
53a	48	—2	38	—4	26	16	0	15	1	12
Ave	46	7	40	21	40	31	0	35	12	18
Max	113	42	130	89	151	144	4	158	41	41

condition type from DIR to MIX (introduction of derivatives) and from MIX to NEU (increasing the relative size of the derivative terms) will affect all of the modules.

For a given module we compute the percent change in n_3 and t_3 as the boundary condition type changes from DIR to MIX, and from MIX to NEU. For example, if grid size $n_3 = a$ is needed to achieve three digits on a problem of type DIR, and $n_3 = b$ is needed for the corresponding MIX problem, then the percent change Δn_3 is given by $100(b - a)/a$. The percent changes in n_3 from DIR to MIX and from MIX to NEU are listed in Table 2. We see that for nearly all problems n_3 is greater for MIX than for DIR, and for NEU than MIX. This supports the hypothesis that derivatives in the boundary conditions make problems more difficult for these modules, although for HC the increase is not large in most cases, especially from type MIX to NEU. Modules HC and D4 have the smallest percent increase on average. It is not surprising that the fourth order modules (HC and D4) have smaller relative increases in n_3 than the second order methods, since the rate of reduction in error as a function of n is greater for these methods. We also expect the three second order modules to have similar relative increases since they use nearly identical discretizations.

For Dirichlet boundary conditions (DIR), the modules 5P, DY and MG use exactly the same discretization and hence produce exactly the same errors and have exactly the same values of grid size n_3 . Table 2 shows that these modules use different approximations to derivative boundary conditions as their changes in n_3 are different. Of course, each of these modules has a different value of computer time t_3 even when the errors are identical.

To test the hypotheses about the comparative effect of derivatives in the boundary conditions, for each module we give the percent change Δt_3 in t_3 as the boundary condition type changes (see Table 3). Again we see that modules HC and D4 are affected the least in both situations. Part of the dramatic increase in time for 5P on many problems is explained by the fact that there are

Table 3: Percent changes Δt_3 in t_3 as boundary condition type changes from DIR to MIX and from MIX to NEU.

Problem	From DIR to MIX					From MIX to NEU				
	5P	HC	DY	D4	MG	5P	HC	DY	D4	MG
1	314	127	398	99	—	1513	—5	346	95	—
3a	1123	166	296	159	503	259	15	103	39	91
4	28	134	298	63	16	88	—2	—35	—19	42
4a	175	165	606	117	130	293	13	0	7	106
6a	1	69	1	11	19	0	3	0	0	—1
6b	121	—7	62	15	79	15	5	8	0	7
9a	675	242	372	263	617	35	13	16	0	10
35	—18	95	—20	69	—15	101	5	59	29	36
35a	159	49	54	71	92	120	2	69	26	49
40	664	13	—	—	—	—41	—36	—	—	—
42a	1355	4	—	—	—	202	—4	—	—	—
53a	212	—8	79	7	83	65	4	30	3	25
Ave	401	88	215	87	169	221	1	60	18	41
Max	1355	242	606	263	617	1513	15	346	95	106

simply more unknowns to be solved for if derivative terms are present in the boundary conditions, even if the grid size n remains constant. Further, the band width of the discretization matrix is increased. This is not the case as the boundary condition type changes from MIX to NEU since a fixed grid size n generates a linear system with the same structure in both cases.

We apply a nonparametric analysis to the data to obtain confidence levels on the observed differences between modules. For each problem we rank each module against every other module (pairwise) based on percent increase Δt_3 in t_3 . A ranking of 1 indicates the module has a smaller percent increase; a ranking of 1.5 indicates a tie (whenever the values differ by less than 10 percent). The average ranks over all common problems are used to derive confidence levels using the Friedman, Kendall and Babington-Smith test (see [3]). As an example, Table 4 lists the rankings for 5P versus HC based on percent increase Δt_3 in t_3 from DIR to MIX and from MIX to NEU. The average ranking of 1.92 and 1.08 for a sample of 12 implies that 5P is worse than HC with a confidence of greater than 99%.

Table 5 summarizes the confidence levels on the differences in the data. In terms of statistical significance, most of the perceived differences are not dramatic. The relatively small sample size is the primary cause of this. With 95% or greater confidence level we see that

1. 5P has a greater percent increase in t_3 than the other four modules as the boundary condition type changes from MIX to NEU.
2. HC and D4 both have a smaller increase than MG, and D4 beats DY as well, as the boundary condition type changes from MIX to NEU.
3. MG has a smaller percent increase in t_3 than 5P as the boundary condition type changes from DIR to MIX.

Table 4: Rankings of 5 POINT STAR (5P) and HERMITE COLLOCATION (HC) based on percent increase Δt_3 in t_3 .

Problem	From DIR to MIX		From MIX to NEU	
	5P	HC	5P	HC
1	2	1	2	1
3a	2	1	2	1
4	1	2	2	1
4a	2	1	2	1
6a	1	2	1.5	1.5
6b	2	1	2	1
9a	2	1	2	1
35	1	2	2	1
35a	2	1	2	1
40	2	1	1.5	1.5
42a	2	1	2	1
53a	2	1	2	1
Ave	1.75	1.25	1.92	1.08

Table 5: Confidence levels for the significance of the differences in average ranks for percent change in time t_3 to achieve three digits of accuracy. The entry in row i and column j is the confidence level that module j has a smaller percent increase in t_3 than module i .

From DIR to MIX						From MIX to NEU					
HC	-					HC	-				
D4	<80	-				D4	<80	-			
MG	<80	80	-			MG	95	95	-		
DY	90	<80	<80	-		DY	<80	95	<80	-	
5P	90	<80	95	<80	-	5P	99	99	97.5	99	-
	HC	D4	MG	DY	5P		HC	D4	MG	DY	5P

Table 6: Confidence levels for the significance of the differences in average ranks for percent change in time t_3 to achieve three digits of accuracy, without using Problems 4, 6a and 35.

From DIR to MIX						From MIX to NEU					
HC	-					HC	-				
D4	<80	-				D4	<80	-			
MG	90	97.5	-			MG	90	<80	-		
DY	99	90	<80	-		DY	<80	90	<80	-	
5P	99	99	97.5	<80	-	5P	99	99	95	99	-
	HC	D4	MG	DY	5P		HC	D4	MG	DY	5P

A closer look shows more about the influence of the derivative terms in the boundary conditions. The data in Table 3 and 4 indicate that the results for three of the test problems go against the general trend. We consider each of these problems along with a second problem in the study which is identical except for one important feature.

Problem 4 The only side with a derivative boundary condition for this problem is the west side. The solution and its first derivative with respect to x are both comparatively small there. Hence, it may be that the effect of the derivative boundary condition on a module is small if the terms involved are small. Problem 4a is the same as Problem 4 except the derivative boundary condition is on the east side, where the solution is changing more rapidly. The results for Problem 4a are similar to the typical effect observed on the remainder of the test problems.

Problem 6a Two characteristics of this problem may be significant for this study: a) the large coefficient of u in the PDE may reduce the relative importance of terms in the boundary conditions, and b) most of the terms in the true solution and its derivatives are zero on the boundary of the domain. In Problem 6b the second characteristic is removed by using a slightly smaller region and the results show more of a difference in performance among the various modules, although the large coefficient of u may still minimize the perceived effect of the derivative boundary conditions.

Problem 35 The time t_3 needed by the second order finite difference module to obtain three digits of accuracy for this problem actually decreases when the derivative term is introduced (see Table 3). An important feature of this problem is the symmetry of the solution with respect to the domain. When this symmetry is removed, as in Problem 35a, by changing the domain slightly, the results are more typical. It may be that the symmetry cancels out some of the error that would otherwise be introduced in the finite difference approximations.

Table 6 gives the confidence levels on the differences in average rank when these three problems (4, 6a and 35) are deleted from the study. The even smaller sample sizes make statistical significance more difficult to achieve, but the following are supported with a high level of confidence (significance 95% or higher).

1. HC, D4 and MG are affected less than 5P as the boundary condition type changes from DIR to MIX.

Table 7: Confidence levels on the differences in average ranks for time t_3 to achieve three digits of accuracy. The entry in row i and column j is the confidence level that t_3 for module j is smaller than for module i .

Boundary condition type DIR						Boundary condition type MIX					
MG	–					MG	–				
HC	95	–				HC	97.5	–			
D4	99	<80	–			D4	90	<80	–		
DY	99	99	99	–		DY	99	99	99	–	
5P	99	99	99	99	–	5P	99	99	99	99	–
	MG	HC	D4	DY	5P		MG	HC	D4	DY	5P

Boundary condition type NEU					
MG	–				
HC	90	–			
D4	90	<80	–		
DY	99	99	99	–	
5P	99	99	99	99	–
	MG	HC	D4	DY	5P

2. HC is affected less than DY as the boundary condition type changes from DIR to MIX.
3. D4 is affected less than MG as the boundary condition type changes from DIR to MIX.
4. All four other modules are affected less than 5P as the boundary condition type changes from MIX to NEU.

Note that these comparisons do not apply to the general mix of PDE problems, but rather to problems with no “special features”. The special features seen in Problems 4, 35 and 6b occur naturally in practice and thus the confidence levels of Table 5 should be used unless one has additional information.

By and large, these results support our hypotheses regarding the effect of the derivatives in the boundary conditions. The most noticeable exception is the claim that HC suffers less than D4 as the type changes from MIX to NEU. On two problems module D4 suffers no adverse effect at all from the change, and on one problem there is even a positive effect (see Table 3). This may be due to the improved characteristics of the discretization matrix for the iterative solution scheme. Even so, for most of the problems, and especially when the “Ave” and “Max” values in Table 3 for HC and D4 are compared, it seems that HC suffers less than D4.

3.2 Overall performance of the modules

We now rank the five modules on the basis of computer time t_3 to achieve three digits of accuracy. Here we are no longer concerned with the relative influence of the boundary condition types but concerned with the “bottom-line” performance of the modules. For each type of boundary condition—DIR, MIX and NEU—we rank the modules pairwise. The hypothesis is that for problems to which it applies, MG is best, followed by D4 and HC, then D4, and 5P last. The confidence levels on the differences in average ranks are summarized in Table 7 and we can rank the modules

Table 8: Overall computing time performance: average over 9 problems of ratios of t_3 for four modules over t_3 for MG module.

Module	Boundary Condition		
	DIR	MIX	NEU
D4	2.43	2.26	1.75
HC	6.05	4.60	4.11
DY	18.79	25.27	19.03
5P	263.43	331.35	435.78

as claimed:

1. MULTIGRID MG00
2. HERMITE COLLOCATION and DYAKANOV CG4
3. DYAKANOV CG
4. 5 POINT STAR

For boundary conditions of type DIR the differences indicated by these rankings hold at a significance level of 95% or greater. For boundary conditions of type MIX, however, the level of confidence for the difference between MG and D4 is only 90%; and for NEU boundary conditions we have only a 90% significance on the differences between MG and each of HC and D4. This suggests that the confidence with which one would choose MG over HC or D4 decreases as the derivative boundary conditions are introduced into the problem. The results in the previous section also suggest this. Although most of the differences in average rank between MG, HC and D4 in Section 3.1 are not statistically significant, the data in Table 3 suggest that on average MG is affected relatively more by the derivatives in the boundary conditions than is either HC or D4.

One final way to evaluate the overall performance of the five modules is to compare the performance of the “best” (MG) against the other four. The values in Table 8 indicate that on the average D4 comes closer to the performance of MG than do the other three modules. The relative superiority of MG over D4 and HC is again seen to be decreasing as boundary condition type changes from DIR to MIX to NEU. Module 5P gets consistently worse relative to MG as the strength of the derivative term is increased. DY gets worse from DIR to MIX, but improves from MIX to NEU. This may be due to the improved convergence behavior of the linear system resulting from the NEU boundary conditions which DY solves by conjugate gradient iteration.

4 Discussion and Conclusions

1. Based on change in the grid size n_3 , all modules suffer to some degree by the introduction of derivatives into the boundary conditions. As the coefficient of the derivative term is increased (change from type MIX to NEU) HC suffers very little, while the others continue to show some effect.
2. Based on the change in solution time t_3 , the modules fall into three groups (from least affected to most): 1) D4 and HC, 2) DY and MG, 3) 5P. Most of the effect is seen as the boundary condition type changes from DIR to MIX. The strengthening of the derivative term as the

boundary conditions change from MIX to NEU affects 5P significantly, but the other modules less so. Modules HC and D4 seem to be especially insensitive to the size of the derivative term in the boundary condition.

3. Statistically, we cannot conclude that HC is affected less than D4. On the average it seems that this is the case for the change from MIX to NEU but that the effect is very similar for the change from DIR to MIX.
4. The statistical analysis on the basis of overall performance suggests that the modules should be ranked as follows: 1) MG, 2) D4 and HC, 3) D4, 4) 5P. For problems to which it applies, MG is significantly better than the others. The advantage of MG over D4 and HC does seem to shrink when derivatives are introduced into the boundary conditions. This also supports the claim that MG is affected more by the derivatives in the boundary conditions than either D4 or HC. For problems to which it applies, D4 seems to be better than HC, although our data is not sufficient to establish this with any degree of statistical significance.
5. Other characteristics of a problem may minimize or mask the effect of derivatives in the boundary conditions. We see examples where the size of the solution or its derivative, the symmetry of the solution with respect to the domain, or the size of a coefficient of the PDE causes the "usual" effect to be absent.
6. We do not give the data for changes from DIR to NEU. They show that the effects are essentially the sum of the effects from DIR to MIX and from MIX to NEU. Thus the various conclusions stated above hold with even higher levels of confidence as the boundary condition type changes from DIR to NEU.

References

- [1] R. F. Boisvert, E. N. Houstis and J. R. Rice, "A system for performance evaluation of partial differential equation software", *IEEE Trans. Software Engineering*, 5(1979), pp 418-425.
- [2] J. P. Bonomo, W. R. Dyksen and J. R. Rice, "The ELLPACK performance evaluation system", Purdue University, Computer Science Department Report CSD-TR 569, 1986.
- [3] M. Hollander and D. A. Wolfe, *Nonparametric Statistical Methods*, John Wiley, New York, 1975.
- [4] J. R. Rice, E. N. Houstis and W. R. Dyksen, "A population of linear, second order, elliptic partial differential equations on rectangular domains", *Math Comp.*, 36(1981), pp 475-484.
- [5] J. R. Rice and R. F. Boisvert, *Solving Elliptic Problems Using ELLPACK*, Springer-Verlag, New York, 1985.

Appendix

Table 9: Elliptic operator Lu and domain R for each of the sample problems.

Problem	Lu	R
1	$(e^{xy}u_x)_x + (e^{-xy}u_y)_y - \frac{1}{1+x+y}u$	$[0, 1] \times [0, 1]$
3a	$u_{xx} + u_{yy}$	$[0, 1] \times [0, 1]$
4	$u_{xx} + u_{yy}$	$[0, 1] \times [0, 1]$
4a	$u_{xx} + u_{yy}$	$[0, 1] \times [0, 1]$
6a	$u_{xx} + u_{yy} - (100 + \cos(2\pi x) + \sin(3\pi y))u$	$[0, 1] \times [0, 1]$
6b	$u_{xx} + u_{yy} - (100 + \cos(2\pi x) + \sin(3\pi y))u$	$[.1, .9] \times [.1, .9]$
9a	$u_{xx} + u_{yy} - 100u$	$[0, 1] \times [0, 1]$
35	$u_{xx} + u_{yy}$	$[-1, 1] \times [-1, 1]$
35a	$u_{xx} + u_{yy}$	$[-.25, 1] \times [-.25, 1]$
40	$u_{xx} + (1 + x^2)u_{yy} - yu_x$	$[0, 1] \times [0, 1]$
42a	$u_{xx} + u_{yy} + u_y - u$	$[-1, 2] \times [0, 1]$
53a	$u_{xx} + u_{yy} - 10u$	$[0, 1] \times [0, 1]$

Table 10: Time t_3 (seconds) to obtain three digits of accuracy for each module on each problem to which that module applies. Data is rounded to two digits.

Problem	Boundary condition type DIR				
	Module				
	5P	HC	DY	D4	MG
1	6.3E+01	8.7E-01	1.8E+01	2.1E+00	--
3a	7.9E+01	7.5E-01	1.4E+01	1.3E+00	6.5E-01
4	3.4E+01	3.7E-01	7.4E+00	9.3E-01	4.2E-01
4a	3.4E+01	3.7E-01	7.4E+00	9.3E-01	3.8E-01
6a	9.4E+02	4.3E+01	9.6E+01	1.5E+01	4.1E+00
6b	5.9E+02	4.0E+01	7.0E+01	9.9E+00	2.8E+00
9a	4.9E+03	8.4E+01	9.1E+01	1.6E+01	4.7E+00
35	3.1E+01	1.2E+00	6.6E+00	1.4E+00	4.0E-01
35a	1.9E+00	4.0E-01	1.3E+00	6.2E-01	7.2E-02
40	1.9E+01	3.6E+00	--	--	--
42a	3.8E+01	1.0E+01	--	--	--
53a	3.8E+03	6.8E+01	1.2E+02	1.0E+01	7.5E+00
Ave	8.7E+02	2.1E+01	4.3E+01	5.8E+00	2.3E+00
Max	4.9E+03	8.4E+01	1.2E+02	1.6E+01	7.5E+00

Table 10: (Continued)

Boundary condition type MIX					
Problem	Module				
	5P	HC	DY	D4	MG
1	2.6E+02	2.0E+00	9.0E+01	4.1E+00	--
3a	9.7E+02	2.0E+00	5.7E+01	3.5E+00	3.9E+00
4	4.3E+01	8.7E-01	3.0E+01	1.5E+00	4.9E-01
4a	9.3E+01	9.8E-01	5.2E+01	2.0E+00	8.8E-01
6a	9.5E+02	7.2E+01	9.7E+01	1.6E+01	4.8E+00
6b	1.3E+03	3.7E+01	1.1E+02	1.1E+01	4.9E+00
9a	3.8E+04	2.9E+02	4.3E+02	5.9E+01	3.4E+01
35	2.5E+01	2.4E+00	5.3E+00	2.4E+00	3.4E-01
35a	4.9E+00	5.9E-01	2.1E+00	1.1E+00	1.4E-01
40	1.5E+02	4.1E+00	--	--	--
42a	5.6E+02	1.1E+01	--	--	--
53a	1.2E+04	6.2E+01	2.1E+02	1.1E+01	1.4E+01
Ave	4.5E+03	4.0E+01	1.1E+02	1.1E+01	7.0E+00
Max	3.8E+04	2.9E+02	4.3E+02	5.9E+01	3.4E+01

Boundary condition type NEU					
Problem	Module				
	5P	HC	DY	D4	MG
1	4.2E+03	1.9E+00	4.0E+02	8.1E+00	--
3a	3.5E+03	2.3E+00	1.2E+02	4.8E+00	7.5E+00
4	8.1E+01	8.5E-01	1.9E+01	1.2E+00	6.9E-01
4a	3.7E+02	1.1E+00	5.2E+01	2.2E+00	1.8E+00
6a	9.5E+02	7.4E+01	9.7E+01	1.6E+01	4.8E+00
6b	1.5E+03	3.9E+01	1.2E+02	1.1E+01	5.3E+00
9a	5.1E+04	3.2E+02	5.0E+02	5.9E+01	3.7E+01
35	5.0E+01	2.5E+00	8.3E+00	3.1E+00	4.6E-01
35a	1.1E+01	6.0E-01	3.5E+00	1.3E+00	2.0E-01
40	8.6E+01	2.6E+00	--	--	--
42a	1.7E+03	1.0E+01	--	--	--
53a	1.9E+04	6.5E+01	2.7E+02	1.1E+01	1.7E+01
Ave	6.9E+03	4.3E+01	1.6E+02	1.2E+01	8.3E+00
Max	5.1E+04	3.2E+02	5.0E+02	5.9E+01	3.7E+01