1986

Quadratic-Spline Collocation Methods for Two Point Boundary Value Problems

Elias N. Houstis
Purdue University, enh@cs.purdue.edu

C. C. Christara

John R. Rice
Purdue University, jrr@cs.purdue.edu

Report Number:
86-584

QUADRATIC-SPLINE COLLOCATION METHODS FOR TWO POINT BOUNDARY VALUE PROBLEMS

E. H. Houstis, C. C. Christara and J. R. Rice
Purdue University
West Lafayette, Indiana 47907

CSD-TR-584
May 1986

QUADRATIC-SPLINE COLLOCATION METHODS FOR TWO POINT BOUNDARY VALUE PROBLEMS

E. N. Houstis(*), *C. C. Christara*(**) *and J. R. Rice*(*)
Purdue University
West Lafayette, Indiana 47907

May 9, 1986

CSD-TR 584

Abstract

A new collocation method based on quadratic splines is presented for second order two point boundary value problems. First, $O(h^4)$ approximations to the first and second derivative of a function are derived using a quadratic spline interpolant of u. Then these approximations are used to define an $O(h^4)$ perturbation of the given boundary value problem. Second, the perturbed problem is used to define a collocation approximation at interval midpoints for which an optimal $O(h^{3-j})$ global estimate for the jth derivative of the error is derived. Further, $O(h^{4-j})$ error bounds for the jth derivative are obtained for certain superconvergence points. It should be observed that standard collocation at midpoints gives $O(h^{2-j})$ bounds. Results from numerical experiments are reported that verify the theoretical behavior of the method.

(*) This research was supported by ARO grant DAA929-83-K-0026 and AFOSR grant 84-0385.

(**) This research was supported by I.K.Y. (National Fellowship Foundation of Greece).

1. INTRODUCTION

In this paper, we consider the numerical solution of a second order two-point boundary value problem

$$Lu = u'' + p(x)u' + g(x)u = f \quad \text{on} \quad I = [a, b], \tag{1.1}$$

subject to mixed boundary conditions

$$Bu = \{\alpha_{0,0}u(a) + \alpha_{0,1}u'(a) = g_0, \ \alpha_{1,0}u(b) + \alpha_{1,1}u'(b) = g_1\}. \tag{1.2}$$

The method considered belongs to the class of finite elements and it is based on collocation by quadratic splines. In order to determine such an approximation a high order expansion of the residual is forced to collocate (interpolate) at certain points. The resulting error in the uniform norm is shown to be $O(h^3)$ globally and $O(h^4)$ at the nodes of a uniform partition. Superconvergence behavior is exhibited for the derivatives at certain points.

Several authors [4], [7], [11] and [12] have studied the approximating properties of the quadratic splines. The results of Marsden [9] show that the quadratic interpolant and its derivatives exhibit superconvergence at specific local points. The paper by Kammener et al [7] studies the projection properties of quadratic interpolatory splines and generalizes the results in [9]. Collocation with quadratic splines for particular instances of the two-point boundary value problem is considered in [7], [8] and [10]. In these studies the convergence obtained is not optimal. In [7] fourth order convergence is obtained by using fourth degree splines. High order collocation residual expansion is used in [1], [3], [5] to obtain optimal cubic spline collocation methods for the same problem. The method considered here can be applied for nonlinear problems and can be extended to two dimensional elliptic problems [2]. Optimal spline collocation methods for higher degree boundary value problems are studied in [6].

2. QUADRATIC SPLINE INTERPOLATION RESULTS

In this section we list and derive a number of quadratic spline interpolation identities. These identities are used in Sections 3 and 4 to formulate and analyze a quadratic spline collocation method for the two-point boundary value problem (1.1), (1.2). Consider the interval $I = [a, b]$ and let $\Delta \equiv \{a = x_0 < x_1 < \cdots < x_N = b\}$ be a uniform partition of I with mesh size h and $T = \{\tau_0 = x_0, \tau_i = (x_i + x_{i-1})/2;\ 1 \le i \le N,\ \tau_{N+1} = x_N\}$ be a set of data points. Throughout, denote by $\mathbf{P}_{2,\Delta}$ the space of piecewise quadratic polynomials and $S_{2\Delta}$ the space of quadratic splines $(\mathbf{P}_{2,\Delta} \cap C^1(I))$ where $C^k(I)$ is the set of functions with k derivatives continuous on I. In this paper we adopt the following notation, S is the quadratic spline interpolant of u such that

$$S(\tau_0) = u(\tau_0) - h^4D^4u(\tau_0)/128,\ S(\tau_i) = u(\tau_i);\ 1 \le i \le N,\ S(\tau_{N+1}) = u(\tau_{N+1}) - h^4D^4u(\tau_{N+1})/128 \tag{2.1}$$

where D^k is the kth derivative operator. Define $S_i = S(\tau_i)$, $u_i = u(\tau_i)$ and let $e(x) = u(x) - S(x)$ be the interpolation error. The following result of Mardsen [9], Kammerer et al [7] is needed to obtain a priori error bounds for the collocation method considered here. We use the max norm unless otherwise indicated.

Theorem 2.1. *Let τ_i be the middle points of each subinterval of Δ and $\lambda = (3 \pm \sqrt{3})/6$. If $u \in C^4(I)$, then*

$$|e(x_i)| = O(h^4),\ \ |e'(\tau_i - \lambda h)| = O(h^3),\ \ |e''(x_{i+1/2})| = O(h^2) \tag{2.2}$$

and $||D^k e||_\infty = O(h^{3-k}),\ k = 0,1,2.$

For uniform partitions it can be shown, Mardsen [9], that any $S \in S_{2\Delta}$ satisfies the relation

$$S(x_{i-1}) + 6S(x_i) + S(x_{i+1}) = 4(S(x_{i-1/2}) + S(x_{i+1/2})) \tag{2.3}$$

for $1 \le i \le N - 1$. A direct consequence of this identity is the following theorem.

Theorem 2.2. *If $u \in C^6(I)$, then at the midpoints τ_i of Δ, we have*

$$S_i' = u_i' + \frac{h^2}{24} u_i^{(3)} + O(h^4), \tag{2.4a}$$

and

$$S_i'' = u_i'' - \frac{h^2}{24} u_i^{(4)} + O(h^4) \tag{2.4b}$$

where $u_i^{(k)}$ denotes $D^k u\ (\tau_i)$.

Proof: From (2.1), (2.3) it can be seen that the error function $e(x) = u(x) - S(x)$ at the interior knots x_i, $1 \le i \le N$, satisfies the equations

$$e(x_{i-1}) + 6e(x_i) + e(x_{i+1}) = u(x_{i-1}) - 4u(x_{i-1/2}) + 6u(x_i) - 4u(x_{i+1/2}) + u(x_{i+1}). \tag{2.5}$$

If we denote by $\{b_i\}_1^N$, the right sides of equations (2.5), then by Taylor's expansion we can show that

$$b_i = \frac{h^4}{16} u^{(4)}(x_i) + O(h^6), \quad 1 \le i \le N. \tag{2.6}$$

Further, for any function $g \in C^4$ we have

$$g_{i-1} + 6g_i + g_{i+1} = 8g_i + h^2 g_i^{(2)} + O(h^4). \tag{2.7}$$

In (2.7) we choose $g = -\dfrac{h^4 u^{(4)}}{128}$, add equations (2.5) and (2.7) and then we obtain

$$\alpha_{i-1} + 6\alpha_i + \alpha_{i+1} = O(h^6), \quad 1 \le i \le N-1, \tag{2.8}$$

where $\alpha_i \equiv e(x_i) - \dfrac{h^4 u^{(4)}(x_i)}{128}$. According to the definition of S, $\alpha_0 \equiv u_0 - S_0 - h^4 u_0^{(4)}/128 = 0$, and $\alpha_N \equiv u_N - S_N - h^4 u_N^{(4)}/128 = 0$. The system of equations (2.8) is strictly diagonally dominant. Thus, the inverse of its coefficient matrix exists and its norm is less than 1/4. This implies that, for $0 \le i \le N$,

$$\alpha_i = O(h^6). \tag{2.9}$$

Since S is locally quadratic and τ_i is the midpoint of $[x_i, x_{i-1}]$, we have that

$$S_i' = (S(x_i) - S(x_{i-1}))/h. \tag{2.10}$$

For any function $g \in C^4$ we have by Taylor's expansion

$$g'(\tau_i) = \frac{1}{h}(g(x_i) - g(x_{i-1})) - \frac{h^2 g^{(3)}(\tau_i)}{24} + O(h^4). \tag{2.11}$$

From (2.9), (2.10) and (2.11) we obtain the identity (2.4a). The relation (2.4b) follows easily from the relation

$$D^2u(\tau_i) - D^2S(\tau_i) = 4\{e(x_i) + e(x_{i+1})\}/h^2 - \frac{h^2}{48} u^{(4)}(\tau_i) + O(h^6)$$

and (2.9). This concludes the proof.

We use the above results to prove the following:

Theorem 2.3. *Let S be the quadratic spline interpolant of $u \in C^6(I)$ defined by (2.1). Then at $\{\tau_i\}_2^{N-1}$ the following relations hold:*

$$u_i^{(4)} = (S_{i-1}'' - 2S_i'' + S_{i+1}'')/h^2 + O(h^2), \tag{2.12a}$$

$$u_i^{(3)} = (S_{i+1}'' - S_{i-1}'')/2h + O(h^2), \tag{2.12b}$$

$$= (S_{i-1}' - 2S_i' + S_{i+1}')/h^2 + O(h^2), \tag{2.12c}$$

$$u_i^{(2)} = (S_{i-1}'' + 22S_i'' + S_{i+1}'')/24 + O(h^4), \tag{2.12d}$$

$$u_i' = -(S_{i-1}' - 26S_i' + S_{i+1}')/24 + O(h^4). \tag{2.12e}$$

Proof: Equation (2.12a) follows from the relation $u_i^{(4)} = (u_{i-1}'' - 2u_i'' + u_{i+1}'')/h^2 + O(h^2)$ and (2.4b). The relation (2.12d) is a direct consequence of (2.12a) and (2.4b). Similarly, we have $u_i^{(3)} = (u_{i-1}' - 2u_i' + u_{i+1}')/h^2 + O(h^2)$, which, using relation (2.4a), implies (2.12c). From (2.12c) and (2.4a) we obtain (2.12e). In order to show (2.12b) we use the relation

$$(S_{i+1}'' - S_{i-1}'')/2h = (u_{i+1}'' - u_{i-1}'')/2h - \frac{h}{48}(u_{i+1}^{(4)} - u_{i-1}^{(4)}) + O(h^4) = u_i^{(3)} + O(h^2).$$

This concludes the proof.

If we define the difference operator Λ by $\Lambda\, g_i \equiv (g_{i-1} - 2g_i + g_{i+1})/h^2$ then the relations (2.12a), (2.12c) appear as

$$u_i^{(4)} = \Lambda\, S_i'' + O(h^2),\ \ u_i^{(3)} = \Lambda\, S_i' + O(h^2), \quad 2 \le i \le N-1. \tag{2.13}$$

In order to approximate the higher derivatives of u at $\{x_0, \tau_1, \tau_N, x_N\}$ we make use of the relations

$$\begin{aligned} u_0^{(k)} &= \left[3u_1^{(k)} - u_2^{(k)}\right]/2 + O(h^2),\ \ u_1^{(k)} = 2u_2^{(k)} - u_3^{(k)} + O(h^2), \\ u_N^{(k)} &= 2u_{N-1}^{(k)} - u_{N-2}^{(k)} + O(h^2),\ \ u_{N+1}^{(k)} = \left[3u_N^{(k)} - u_{N-1}^{(k)}\right]/2 + O(h^2), \end{aligned} \tag{2.14}$$

for $k = 3, 4$. Using (2.13) we obtain the following approximations:

$$\begin{aligned} u_0^{(k)} &= \left[5\,\Lambda\, S_2^{(k-2)} - 3\,\Lambda\, S_3^{(k-2)}\right]/2 + O(h^2),\ \ u_{N+1}^{(k)} = \left[5\,\Lambda\, S_{N-1}^{(k-2)} - 3\,\Lambda\, S_{N-2}^{(k-2)}\right]/2 + O(h^2), \\ u_1^{(k)} &= 2\,\Lambda\, S_2^{(k-2)} - \Lambda\, S_3^{(k-2)} + O(h^2) \ \text{ and } \ u_N^{(k)} = 2\,\Lambda\, S_{N-1}^{(k-2)} - \Lambda\, S_{N-2}^{(k-2)} + O(h^2), \end{aligned} \tag{2.15}$$

for $k = 3, 4$. From the relations (2.12) to (2.15) we conclude easily the following corollary.

Corollary 2.1. *Under the hypotheses of Theorem 2.3 and (2.14), (2.15) we have at τ_1 and τ_N the following relations:*

$$\begin{aligned} u_1'' &= (26S_1'' - 5S_2'' + 4S_3'' - S_4'')/24 + O(h^4), \\ u_N'' &= (26S_N'' - 5S_{N-1}'' + 4S_{N-2}'' - S_{N-3}'')/24 + O(h^4), \\ u_1' &= (22S_1' + 5S_2' - 4S_3' + S_4')/24 + O(h^4), \\ u_N' &= (22S_N' + 5S_{N-1}' - 4S_{N-2}' + S_{N-3}')/24 + O(h^4). \end{aligned} \tag{2.16}$$

We consider the quadratic spline function ϕ defined by

$$\phi(x) \equiv x^2,\ 0 \le x \le 1\ ; -3 + 6x - 2x^2,\ 1 \le x \le 2\ ; 9 - 6x + x^2,\ 2 \le x \le 3$$

and 0 elsewhere. Then a set of basis functions for $S_{2,\Delta}$ are the functions $B_i(x) = 2\phi((x-a)/h - i + 2)/3$ for $0 \le i \le N+1$. In order to obtain a high order approximation of the first derivative of u at the end points $\{x_0, x_N\}$ and at the nodal points we consider the identities

$$S'(x_{i-1}) + 6S'(x_i) + S'(x_{i+1}) = 8(S(\tau_{i+1}) - S(\tau_i))/h, \tag{2.16a}$$

and

$$3S'(x_0) + S'(x_1) = 8(S(\tau_1) - S(x_0))/h\ ; 3S'(x_N) + S'(x_{N-1}) = 8(S(x_N) - S(\tau_N))/h. \tag{2.16b}$$

These relations are true for any $S \in S_{2,\Delta}$ and can be viewed as direct consequence of the definition of the basis functions B_i. Based on the identities (2.16) we prove the following:

Theorem 2.4. *If S is the interpolant of u in $S_{2,\Delta}$ defined by (2.1) and $u \in C^6(I)$, then*

$$S'(x_i) = u'(x_i) - \frac{h^2 u^{(3)}(x_i)}{12} + O(h^4), \tag{2.17}$$

for $0 \le i \le N$.

Proof: For the proof, we denote $S'(x_i)$ by S_i', $S'(\tau_i)$ by $S'_{i-1/2}$, $u(x_i)$ by u_i and $u(\tau_i)$ by $u_{i-1/2}$. According to the definition of the interpolant S the relations (2.16) become

$$S_{i-1}' + 6S_i' + S_{i+1}' = 8(u_{i+1/2} - u_{i-1/2})/h, \tag{2.18a}$$

and

$$3S_0' + S_1' = 8(u_{1/2} - u_0 + h^4 u_0^{(4)}/128),\ 3S_N' + S_{N-1}' = 8(u_{N-1/2} - u_N + h^4 u_N^{(4)}/128). \tag{2.18b}$$

By Taylor's expansion, we have

$$\frac{8}{h}(u_{i+1/2} - u_{i-1/2}) = 8u_i' + \frac{h^2}{3}u_i^{(3)} + O(h^4), \tag{2.19}$$

and

$$u_{i+1}' + 6u_i' + u_{i-1}' = 8u_i' + h^2 u_i^{(3)} + O(h^4). \tag{2.20}$$

From (2.18a), (2.19) and (2.20) we obtain

$$e_{i-1}' + 6e_i' + e_{i+1}' = \frac{2}{3}h^2 u_i^{(3)} + O(h^4), \tag{2.21}$$

where $e_i = u(x_i) - S(x_i)$. If we apply (2.20) for the function $-\frac{1}{12}h^2 u^{(2)}(x)$, add it to (2.21) and denote by $d_i = e_i' - \frac{h^2}{12}u_i^{(3)}$, then we obtain

$$d_{i-1} + 6d_i + d_{i+1} = O(h^4). \tag{2.22}$$

Similarly, we can establish the expansions

$$\frac{8}{h}\left(u_{1/2} - u_0 + h^4 u_0^{(4)}/128\right) = 4u_0' + hu_0'' + h^2 u_0^{(3)}/6 + h^3 u_0^{(4)}/12 + O(h^4),$$

and

$$3u_0' + u_1' = 4u_0' + hu_0'' + h^2 u_0^{(3)}/2 + h^3 u_0^{(4)}/6 + O(h^4). \tag{2.23}$$

From the above relation we obtain that

$$3e_0' + e_1' = h^2 u_0^{(3)}/3 + h^3 u_0^{(4)}/12 + O(h^4). \tag{2.24}$$

If we apply the relation (2.23) to the function $-\dfrac{h^2 u^{(2)}(x)}{12}$ and add it to (2.24) we obtain

$$3d_0 + d_1 = O(h^4). \tag{2.25a}$$

In the same way we obtain

$$3d_N + d_{N-1} = O(h^4). \tag{2.25b}$$

The system of equations (2.22), (2.25) is strictly diagonal dominant and the inverse of its coefficient matrix has norm less than ½. Thus, for $0 \le i \le N$, $d_i = O(h^4)$ which proves relation (2.17) and concludes this proof.

This result allows us to obtain high accuracy approximations to the derivatives of u at the boundary nodes.

Corollary 2.2. *Under the assumptions of Theorem 2.4, we have the following relations at the boundary nodes x_0 and x_N:*

$$u'(x_0) = (24S_0' + 5S_1' - 13S_2' + 11S_3' - 3S_4')/24 + O(h^4) \tag{2.26a}$$

$$u'(x_N) = (24S_{N+1}' + 5S_N' - 13S_{N-1}' + 11S_{N-2}' - 3S_{N-3}')/24 + O(h^4) \tag{2.26b}$$

3. THE METHOD OF QUADRATIC-SPLINE COLLOCATION

We consider the linear second order equation $Lu = f$ subject to homogeneous boundary conditions $Bu = 0$. Based on the relations (2.4a), (2.4b) and (2.17) we observe that the interpolant S of u satisfies the relations

$$LS_i = f_i - \frac{h^2}{24} u_i^{(4)} + \frac{h^2}{24} p_i\, u_i^{(3)} + O(h^4), \quad 1 \le i \le N, \tag{3.1a}$$

and

$$BS_j = -\frac{h^2}{12} \alpha_{k,1} u_j^{(3)} + O(h^4), \quad (k,j) = (0,0), (1, N+1). \tag{3.1b}$$

Notice that due to (2.13) and (2.15) the relations (3.1) can be written as

$$LS_1 = f_1 - \frac{h^2}{24}\left[2\Lambda S_2'' - \Lambda S_3''\right] + \frac{h^2}{24} p_1 \left[2\Lambda S_2' - \Lambda S_3'\right] + O(h^4), \tag{3.2a}$$

$$LS_i = f_i - \frac{h^2}{24}\Lambda S_i'' + \frac{h^2}{24} p_i \Lambda S_i' + O(h^4), \; 2 \le i \le N-1, \tag{3.2b}$$

$$LS_N = f_N - \frac{h^2}{24}\left[2\Lambda S_{N-1}'' - \Lambda S_{N-2}''\right] + \frac{h^2}{24} p_N \left[2\Lambda S_{N-1}' - \Lambda S_{N-2}'\right] + O(h^4). \tag{3.2c}$$

and the boundary relations:

$$BS_0 = \frac{h^2}{24}\, \alpha_{0,1} \left[5 \Lambda S_2' - 3 \Lambda S_3'\right] + O(h^4),$$
$$BS_{N+1} = \frac{h^2}{24}\, \alpha_{1,1} \left[5 \Lambda S_{N-1}' - 3 \Lambda S_{N-2}'\right] + O(h^4). \tag{3.2d}$$

If in (3.2) we move the terms involving the approximations of the high order derivatives of u to the left side, we obtain the relations

$$L' S_i = f_i + O(h^4), \quad 1 \le i \le N, \tag{3.3a}$$

and

$$B' S_j = O(h^4), \quad j = 0, N+1, \tag{3.3b}$$

where L' and B' denote perturbations of L and B respectively, defined as

$$L'u_i \equiv Lu_i + \frac{h^2}{24}\, u_i^{(4)} - \frac{h^2}{24}\, p_i\, u_i^{(3)}, \ 1 \le i \le N,$$

and

$$B'u_j \equiv Bu_j + \frac{h^2}{12}\, \alpha_{k,1}\, u_j^{(3)}, \ (k,j) = (0,0), (1, N+1).$$

The above observations are summarized in the following lemma:

Lemma 3.1. *Let S be the quadratic spline interpolant of the solution u of (1.1), (1.2) at the data points* T. *If $u \in C^{(6)}(I)$, then S satisfies the relations*

$$[LS - f]_{x=\tau_i} = O(h^2), \text{ for } i = 1 \text{ to } N,$$
$$[BS]_{x = x_0, x_N} = O(h^2),$$
$$[L'S - f]_{x=\tau_i} = O(h^4), \text{ for } i = 1 \text{ to } N,$$
$$[B'S]_{x = x_0, x_N} = O(h^4). \tag{3.4}$$

3.1. Formulation of the Quadratic-Spline Collocation Method

We now define the *quadratic-spline collocation method* as determining the approximation z_Δ in $S_{2,\Delta}$ that satisfies

$$\left[L'z_\Delta - f\right]_{x=\tau_i} = 0, \text{ for } i = 1 \text{ to } N, \tag{3.5}$$

and the boundary conditions

$$B'z_\Delta|_{x = x_0, x_N} = 0. \tag{3.6}$$

Throughout we refer to this formulation as *one step spline collocation* method.

An alternative formulation of the method is to view the determination of an approximation u_Δ in $S_{2,\Delta}$ as a *two step collocation* method as follows:

Step 1: Determine $v \in S_{2\Delta}$ such that is satisfies

$$[Lv - f]_{x=\tau_i} = 0, \quad i = 1 \text{ to } N, \tag{3.7a}$$

and

$$Bv \,|_{x = x_0, x_N} = 0. \tag{3.7b}$$

Step 2: (i) For $i = 2$ to $N-1$ estimate the higher derivatives of u at the data points τ_i by

$$u^{(4)}(\tau_i) \approx \Lambda\, v''(\tau_i), \tag{3.8a}$$

and

$$u^{(3)}(\tau_i) \approx \Lambda v'(\tau_i) \tag{3.8b}$$

and substitute these values into (3.1) to obtain more accurate right side terms:

$$\begin{aligned}
\bar{f}_1 &= f_1 - h^2[2\,\Lambda\, v_2'' - \Lambda\, v_3'' - p_1(2\,\Lambda\, v_2' - \Lambda\, v_3')]/24 \\
\bar{f}_i &= f_i - h^2[\Lambda\, v_i'' - p_i\,\Lambda\, v_i']/24 \qquad \text{for } i = 2, \ldots, N-1 \\
\bar{f}_N &= f_i - h^2[2\,\Lambda\, v_{N-1}'' - \Lambda\, v_{N-2}'' - p_N(2\,\Lambda\, v_{N-1}' - \Lambda\, v_{N-2}')]/24 \\
\bar{g}_0 &= h^2\,\alpha_{0,1}[5\,\Lambda\, v_2' - 3\,\Lambda\, v_3']/24 \\
\bar{g}_{N+1} &= h^2\,\alpha_{1,1}[5\,\Lambda\, v_{N-1}' - 3\,\Lambda\, v_{N-2}']/24.
\end{aligned}$$

(ii) Use these right sides to determine $u_\Delta \in S_{2\Delta}$ such that it satisfies the equations:

$$[Lu_\Delta - \bar{f}]_{x=\tau_i} = 0, \quad 1 \le i \le N, \tag{3.9a}$$

and

$$[Bu_\Delta - \bar{g}]_{x = x_0, x_N} = 0. \tag{3.9b}$$

In the general case ($p, q \ne 0$), the existence and convergence of this method is discussed in Section 4. For the case $p(x) = q(x) \equiv 0$ denote by Q the coefficient matrix of $\{u_\Delta''(\tau_i)\}_1^N$ in the system (3.5). In this case the solvability follows from the diagonal dominance of the corresponding system. Specifically we have the following:

Lemma 3.2. *If $p(x) \equiv q(x) \equiv 0$, then the system (3.5) is solvable,* $||Q^{-1}|| \le 1.5$ and $||z_\Delta''||_\infty \le 1.5||f||_\infty$.

Further we can show that the coefficient matrices of equations (3.7) and (3.9) are diagonally dominant in certain cases.

Lemma 3.3. *If $\alpha_{0,0} \cdot \alpha_{0,1} \le 0$, $\alpha_{1,0} \cdot \alpha_{1,1} \ge 0$ and $q(x) \le 0$ at the τ_i's, then the coefficient matrices of equations (3.7) and (3.9) are diagonally dominant for sufficient small h.*

Proof: After the substitution of $u_\Delta = \sum_{i=0}^{N+1} u_i\, B_i(x)$ in the boundary equation at $x = a$ we obtain the diagonal dominance condition

$$\left|\frac{2}{3}\,\alpha_{0,0} - \frac{4}{3h}\,\alpha_{0,1}\right| - \left|\frac{2}{3}\,\alpha_{0,0} + \frac{4}{3h}\,\alpha_{0,1}\right| \ge 0,$$

which is satisfied for sufficiently small h, provided $\alpha_{0,0} \cdot \alpha_{0,1} \le 0$. Similarly, the collocation equation obtained by the second boundary condition is diagonal dominant if $\alpha_{1,0} \cdot \alpha_{1,1} \ge 0$. In the case of interior collocation equations the diagonal dominance condition becomes

$$|-\frac{8}{3h^2}+q(\tau_i)| - \left\{ |\frac{4}{3h^2}-\frac{2}{3h}p(\tau_i)+\frac{1}{6}q(\tau_i)| + |\frac{4}{3h^2}+\frac{2}{3h}p(\tau_i)+\frac{1}{6}q(\tau_i)| \right\} \geq 0.$$

It is easy to see that this condition is satisfied when $q(\tau_i) \leq 0$ for $i = 1$ to N and h is sufficiently small.

4. CONVERGENCE ANALYSIS AND ERROR BOUNDS

In order to analyze the two quadratic spline collocation methods, we introduce an integral representation of equations (3.5), (3.7), (3.9) and the differential equation (1.1). For this purpose, we assume that the boundary value problem $u'' = 0, Bu = 0$ has a unique solution. This implies that there is a Green's function $G(x,t)$ for this problem. If we denote by $w_\Delta \equiv z_\Delta''$, $r \equiv u''$, $s \equiv v''$ and $r_\Delta \equiv u_\Delta''$ and assume that z_Δ, u, v and u_Δ satisfy the homogeneous boundary conditions (1.2), then z_Δ, u, v and u_Δ can be obtained via the Green's function. That is, we have

$$\begin{aligned}
z_\Delta(x) &= \int_a^b G(x,t)\, w_\Delta(t)dt, & z_\Delta' &= \int_a^b G_x(x,t)\, w_\Delta(t)dt,\\
u(x) &= \int_a^b G(x,t)\, r(t)dt, & u'(x) &= \int_a^b G_x(x,t)\, r(t)dt,\\
v(x) &= \int_a^b G(x,t)\, s(t)dt, & v'(x) &= \int_a^b G_x(x,t)\, s(t)dt,\\
u_\Delta(x) &= \int_a^b G(x,t)\, r_\Delta(t)dt, & u_\Delta'(x) &= \int_a^b G_x(x,t)\, r_\Delta(t)dt.
\end{aligned}$$

We introduce the operator K that maps $L_2(I)$ to $C(I)$ by

$$\mathrm{K}r(x) = p(x)\int_a^b G_x(x,t)\, r(t)dt + q(x)\int_a^b G(x,t)\, r(t)dt, \tag{4.1}$$

and the linear projection P_Δ which maps continuous functions to $S_{0,\Delta} \equiv P_{0,\Delta} \cap C^{-1}(I)$ via piecewise interpolation at the middle points $\{\tau_i\}_1^N$.

4.1 Convergence analysis of the two step method

Based on the notation introduced we can rewrite equations (3.7), (3.9) in the following form:

$$s + P_\Delta \mathrm{K}s = P_\Delta f \tag{4.2a}$$

$$r_\Delta + P_\Delta \mathrm{K}r_\Delta = P_\Delta \bar{f} \tag{4.2b}$$

respectively, since $P_\Delta\, s = s$ and $P_\Delta\, r_\Delta = r_\Delta$. Equation (1.1) can be written as

$$r + \mathrm{K}\, r = f. \tag{4.3}$$

According to the definition of P_Δ we can assume that $P_\Delta\, g = g(\tau_i)$ for $x \in (x_{i-1}, x_i)$ and $P_\Delta g = (g(\tau_{i+1}) + g(\tau_i))/2$ at interior nodal points. Thus $||P_\Delta g - g||_\infty$ tends to zero as h converges to zero for any continuous function g. This in turn implies that the sequence of operators P_Δ converges strongly to the identity operator $I: C \to L_2$. Further, according to Russell and Shampine [13], the operator K from $L_2([a,b])$ into C is completely continuous, $(I + P_\Delta \mathrm{K})^{-1}$ exists and is uniformly bounded for sufficiently small h. Following similar arguments as in [3], first we show convergence in step 1, that is, of the collocation approximation v.

Theorem 4.1. *If we assume that*

(a1) the coefficients p, q *and* f *are in* $C(I)$,

(a2) *the boundary value problem* $Lu = f$, $Bu = 0$ *has a unique solution in* $C^4(I)$,

and

(a3) the problem $u'' = 0, Bu = 0$ *has a unique solution,*

then

(r1) *the collocation approximation* $v \in S_{2\Delta}$ *defined by equations (3.7) exists,*

(r2) *we have the global error estimates*

$$||u - v||_\infty \le Ch^2, \ \ ||u' - v'||_\infty \le Ch^2 \ \ and \ \ ||u'' - v''||_\infty \le Ch, \tag{4.4}$$

and the local error estimates

(r3) $$|(u - v)^{(k)}(\tau_i)| \le Ch^2 \ \ for \ \ k = 0,1,2 \ \ and \ \ i = 1, \dots, N, \tag{4.5}$$

where C *is a generic constant independent of* h.

Proof: The solvability of equations (4.2) follows from the existence and uniform boundedness of $(I + P_\Delta K)^{-1}$. To establish (4.4), (4.5), consider the problem $s_\Delta \equiv S''$, $BS = O(h^2)$. Notice that there is a linear function w such that $BS = Bw = O(h^2)$ because of assumption (a3). Further more $||w||_\infty = O(h^2)$ and $||w'||_\infty = O(h^2)$. From (3.1a) and the solvability of $(S - w)'' = s_\Delta$, $B(S - w) = 0$ we conclude that

$$(I + P_\Delta K)(S'' - w'') = P_\Delta f + O(h^2). \tag{4.6}$$

Subtracting (4.2a) and (4.6) we obtain

$$(I + P_\Delta K)(S'' - w'' - v'') = O(h^2).$$

The uniform boundedness of $(I + P_\Delta K)^{-1}$ yields

$$||S'' - w'' - v''||_\infty = O(h^2). \tag{4.7}$$

Since $(S - w - v)'' = z, B(S - w - v) = 0$ is uniquely solvable (a3), we have

$$(S - w - v)^{(k)}(x) = \int_a^b D^k G(x,t)(S'' - w'' - v'')(t)dt.$$

This implies that

$$||S - w - v||_\infty = O(h^2), \ \ ||S' - w' - v'|| = O(h^2). \tag{4.8}$$

The error bounds (4.4) and (4.5) now follow from Theorem 2.1, the definition of w and relations (4.7) and (4.8). This concludes the proof.

From the relations (4.7), (4.8) and (2.4) one can conclude that $\Lambda S_i^k = \Lambda v^k(\tau_i) + O(h^2)$ $k = 1,2$. Therefore the relations (3.2) can be written as

$$\begin{aligned} LS_i &= \bar{f}(\tau_i) + O(h^4), \quad && i = 1 \text{ to } N \text{ and} \\ BS_j &= \bar{g}(x_j) + O(h^4), \quad && j = 0, N+1. \end{aligned} \tag{4.9}$$

From the definition of u_Δ in equation (3.9) and (4.9) we obtain

$$L(S - u_\Delta)(\tau_i) = O(h^4), \quad 1 \le i \le N \tag{4.10a}$$

and

$$B(S - u_\Delta) = O(h^4). \tag{4.10b}$$

Notice that there is a linear function w such that $B(S - u_\Delta) = Bw = O(h^4)$ and $||w|| = O(h^4)$, $||w'|| = O(h^4)$. Equations (4.10a) can be written as

$$(I + P_\Delta K)(S'' - u_\Delta'' - w'') = O(h^4),$$

since $B(S - u_\Delta - w) = 0$ and (a3) holds. Applying the arguments used in proof of Theorem 4.1 and with $\lambda = (3 \pm \sqrt{(3)})/6$, we obtain the following optimal results.

Theorem 4.2. *Under the hypotheses of Theorem 4.1, and with $\lambda = (3 \pm \sqrt{3})/6$, we conclude that*

(i) u_Δ exists,

(ii) $||D^k(u - u_\Delta)||_\infty = O(h^{3-k}), k = 0,1,2,$

(iii) $|D(u(x_i + \lambda h) - u_\Delta(x_i + \lambda h))| = O(h^3),$

(iv) $|D^2(u(x_i + \frac{1}{2}h) - u_\Delta(x_i + \frac{1}{2}h))| = O(h^2),$

(v) $|u(x) - u_\Delta(x)| = O(h^4)$ for $x = x_i$ and τ_i.

4.2 Convergence analysis of the one step method

In order to represent equation (3.5) in integral form, we introduce the following notation: Let D_Δ be the vector value function $D_\Delta : C[a,b] \to R^N$ defined by $(D_\Delta g)_i = g(\tau_i)$ for $i = 1$ to N and E_p, E_q be the $N \times N$ diagonal matrices $E_p = \text{diag}(p(\tau_i))$, $E_q = \text{diag}(q(\tau_i))$. Consider the $N \times N$ tridiagonal matrices

$$T^{(1)} = \frac{1}{24}\, trid(-1, 26, -1), \quad T^{(2)} = \frac{1}{24}\, trid(1, 22, 1)$$

and define the $N \times N$ matrics Φ and Ψ such that

$$\Phi_{1,1} = \Phi_{N,N} = 22/24, \quad \Phi_{1,2} = \Phi_{N,N-1} = 5/24, \quad \Phi_{1,3} = \Phi_{N,N-2} = -4/24, \quad \Phi_{1,4} = \Phi_{N,N-3} = 1/24,$$

with the rest of the elements $\Phi_{i,j} = T_{i,j}^{(1)}$ and

$$\Psi_{1,1} = \Psi_{N,N} = 26/24, \quad \Psi_{1,2} = \Psi_{N,N-1} = -5/24, \quad \Psi_{1,3} = \Psi_{N,N-2} = 4/24, \quad \Psi_{1,4} = \Psi_{N,N-3} = -1/24,$$

with the rest of the elements $\Psi_{i,j} = T_{i,j}^{(2)}$. Then we can rewrite equation (3.5) as

$$w_\Delta + R_\Delta K\, w_\Delta = P_\Delta \Psi^{-1} D_\Delta f \tag{4.11}$$

where $R_\Delta K$ is the integral operator defined by

$$\begin{aligned} R_\Delta K\, g \equiv P_\Delta \Psi^{-1} E_p\, \Phi\, D_\Delta \int_a^b G_x(x,t)\, g(t)dt\, + \\ + P_\Delta \Psi^{-1} E_q\, D_\Delta \int_a^b G(x,t)\, g(t)dt. \end{aligned} \tag{4.12}$$

Lemma 4.1: *The sequence of operators $R_\Delta K$ defined in* (4.12) *converges strongly to the integral operator* K *in* L_2.

Proof.: First consider the convergence of $||R_\Delta Kg - P_\Delta D_\Delta Kg||_\infty$ for $g \in L_2$. According to the definition of $R_\Delta K$ and the use of the triangular inequality we obtain

$$||R_\Delta \mathrm{K}g - P_\Delta D_\Delta \mathrm{K}g||_\infty \le$$

$$\le ||P_\Delta \Psi^{-1} E_p \,\Phi D_\Delta \int_a^b G_x g dt - P_\Delta \Psi^{-1} \Psi E_p \, D_\Delta \int_a^b G_x g dt ||_\infty +$$

$$+ ||P_\Delta \Psi^{-1} E_q \, D_\Delta \int_a^b G g dt - P_\Delta \Psi^{-1} \Psi E_q \, D_\Delta \int_a^b G g dt ||_\infty$$

From the boundedness of $||P_\Delta||_\infty$ and $||\Psi^{-1}||_\infty$

$$||R_\Delta \mathrm{K}g - P_\Delta D_\Delta \mathrm{K}g||_\infty \le \Big[C ||E_p \,\Phi D_\Delta \int_a^b G_x g dt - \Psi E_p \, D_\Delta \int_a^b G_x g dt ||_\infty + ||E_q \, D_\Delta \int_a^b G g dt - \Psi E_q \, D_\Delta \int_a^b G g dt ||_\infty \Big]. \tag{4.13}$$

It is easy to observe that the relation $\Psi E_p = E_p \Psi + O(h)$ holds. This implies that the first norm in the right side of (4.13) can be bounded by

$$||E_p(\Phi - \Psi) D_\Delta \int_a^b G_x g dt ||_\infty + O(h).$$

If p is a least in L_g we conclude that $||E_p||_\infty$ is bounded. From the definition of Φ and Ψ we conclude that $(\Phi - \Psi) D_\Delta \int_a^b G_x g dt$ is bounded by the modulus of continuity ω of the continuous function $\int_a^b G_x g dt$ over a $3h$-interval.

The second term of the right side in (4.13) is $O(||(I - \Psi) D_\Delta g \int_a^b G g dt ||_\infty)$. It can be easily observed that this norm is bounded by $\omega(q \int_a^b G g dt, 3h)$. From the properties of P_Δ and the continuity of $\mathrm{K}g$ we conclude that $||P_\Delta D_\Delta \mathrm{K}g - \mathrm{K}g||_\infty$ converges to zero. This proves the assertion of the Lemma, since $||R_\Delta \mathrm{K}g - \mathrm{K}g||_\infty \le ||R_\Delta \mathrm{K}g - P_\Delta D_\Delta \mathrm{K}g||_\infty + ||P_\Delta D_\Delta \mathrm{K}g - \mathrm{K}g||_\infty$ holds.

Theorem 4.3. *Under the assumptions of Theorem 4.2 we have that z_Δ exists and for Dirichlet boundary conditions the following error bounds hold:*

$$\begin{aligned} &||D^k(u - z_\Delta)||_\infty = O(h^{3-k}), \;\; k = 0, 1, 2, \\ &|D(u(x_i + \lambda h) - z_\Delta(x_i + \lambda h))| = O(h^3), \\ &|D^2(u(x_i + \tfrac{1}{2} h) - z_\Delta(x_i + \tfrac{1}{2} h))| = O(h^2), \\ &|u(x) - z_\Delta(x)| = O(h^4) \;\; \text{for} \;\; x = x_i \;\; \text{and} \;\; \tau_i. \end{aligned} \tag{4.14}$$

Proof: From the existence of $(I + \mathrm{K})^{-1}$, Lemma 4.1 and the Neumann's Theorem, we conclude that $(I + R_\Delta \mathrm{K})^{-1}$ exists, is uniformly bounded for sufficiently small h and the system of collocation equations (4.11) has a unique solution. For homogeneous Dirichlet boundary conditions there is a linear function w such that $Bw = BS = O(h^4)$, $||w||_\infty = O(h^4)$ and $||w'|| = O(h^4)$. Following the same reasoning as in Theorem 4.2 we derive the error bounds (4.14).

It is worth noticing that in the case of mixed boundary conditions we obtain numerically error bounds similar to (4.14).

5. NUMERICAL RESULTS

In this section, we present a number of numerical results to demonstrate the convergence of the quadratic spline collocation method as implemented in the program P2C1COL. The second order method based on the first step is referred by P2C1COL (order = 2) and the fourth order one that corresponds to the two-step method is denoted by P2C1COL (order = 4). The program P2C1COL has an argument to select either second or fourth versions of the method. The choice is indicated here by the arguments order = 2 or order = 4. These results exhibit the various optimal error bounds obtained in Theorems 4.1 and 4.2. The examples were obtained from [3], [13] in order to allow a comparison with other collocation methods. All computations were carried out on a VAX 780 in double precision. For problem 2, we present some data for Galerkin method based on quadratic splines as implemented in the program (P2C1GAL). The data indicate complete agreement between the analytical and numerical behavior of the method.

Problem 1: This example is chosen to test convergence of P2C1COL (order = 4) for various smoothness assumptions on u.

$$u''(x) + \left[\frac{16x}{1+4x^2} \right] u'(x) + \frac{8}{1+4x^2} u(x) = f \quad \text{for } 0 < x < 1,$$

subject to boundary conditions

$$A_0 u(a) + B_0 u_x(a) = g_0 \quad \text{and} \quad A_1 u(b) + B_1 u_x(b) = g_1 .$$

The functions f, g_0 and g_1 are chosen so that $u(x) = x^{\alpha/2}$. Three values, 11, 9 and 7 of α, are used which put $u(x)$ in $C^{5.5}$, $C^{4.5}$ and $C^{3.5}$, respectively. We present tables of the norms of the observed errors for n=8 to 256 points in the partition Δ (see Tables 1, 3 and 5). From these we derive estimates of the orders of convergence which are shown in Tables 2, 4 and 6. The λ points are those of Theorem 4.2 with values $x_i + \lambda h$ where $\lambda = (3 \pm \sqrt{3})/6$. In all cases the estimated orders of convergence agree quite well with those predicted from Theorem 4.2.

n	$\|u - u_\Delta\|_{\tau_i, x_i}$	$\|u - u_\Delta\|_\infty$	$\|u' - u_\Delta'\|_{\lambda_i}$	$\|u' - u_\Delta'\|_\infty$	$\|u'' - u_\Delta''\|_{\tau_i}$	$\|u'' - u_\Delta''\|_\infty$
8	5.3-2 4.4-2	6.0-2	1.4-1	1.4-1	3.6-1	4.3+0
16	3.5-3 3.2-3	3.6-3	8.5-2	1.6-2	3.6-2	2.1+0
32	1.9-4 1.9-4	1.9-4	4.4-4	4.6-3	8.8-3	1.1+0
64	1.1-5 1.1-5	1.1-5	2.5-5	1.1-3	2.2-3	5.1-1
128	7.1-7 7.0-7	7.0-7	1.5-6	2.8-4	5.5-4	2.6-1
256	4.4-8 4.4-8	4.4-8	2.0-7	6.5-5	1.4-4	1.4-1

Table 1. Errors of P2C1COL (order = 4) for the case $\alpha = 11$ with $A_0 = B_0 = 1, A_1 = 0$ and $B_1 = 1$. The notation $x.y - k$ means $x.y * 10^{-k}$ and n is the number of subintervals in the partition Δ. The double entries in the first column are errors measured at the points τ_i and x_i respectively.

	u convergence		u' convergence		u'' convergence	
n	at τ_i, x_i	global	at λ points	global	at τ_i	global
8						
	3.94 3.79	4.05	4.03	3.15	3.29	1.02
16						
	4.17 4.11	4.22	4.28	1.82	2.05	0.90
32						
	4.06 4.04	4.08	4.12	2.08	2.00	1.18
64						
	4.02 4.01	4.02	4.06	1.97	2.00	0.94
128						
	4.01 4.00	4.01	2.93	2.09	2.00	0.92
256						

Table 2. Estimated orders of convergence of the P2C1COL (order = 4) for $\alpha = 11$ (Table 1).

n	$\|u-u_\Delta\|_{\tau_i,x_i}$	$\|u-u_\Delta\|_\infty$	$\|u'-u'_\Delta\|_{\lambda_i}$	$\|u'-u'_\Delta\|_\infty$	$\|u''-u''_\Delta\|_{\tau_i}$	$\|u''-u''_\Delta\|_\infty$
8	7.3-3 5.0-3	9.2-3	3.6-2	3.8-2	1.3-1	2.0+0
16	4.7-4 4.1-4	5.1-4	2.1-3	8.8-3	1.2-2	1.0+0
32	1.9-5 1.7-5	1.9-5	9.4-5	2.2-3	2.4-3	5.3-1
64	1.2-6 1.1-6	1.6-6	5.4-6	5.6-4	6.0-4	2.4-1
128	9.6-8 9.4-8	1.8-7	4.5-7	1.5-4	1.5-4	1.2-1
256	8.2-9 8.1-9	2.0-8	5.6-8	3.5-5	3.8-5	6.4-2

Table 3. Errors of P2C1COL (order = 4) for the case $\alpha = 9$ and the rest of parameters as in Table 1. The notation of Table 1 is used.

n	u convergence at τ_i, x_i	u convergence global	u' convergence at λ points	u' convergence global	u'' convergence at τ_i	u'' convergence global
8						
	3.94 3.62	4.17	4.09	2.11	3.38	0.97
16						
	4.65 4.55	4.74	4.49	2.01	2.36	0.94
32						
	3.98 3.92	3.63	4.11	1.97	2.00	1.13
64						
	3.63 3.60	3.14	3.60	1.91	2.00	0.97
128						
	3.55 3.54	3.13	2.99	2.09	2.00	0.95
256						

Table 4. Estimated orders of convergence of P2C1COL (order = 4) for $\alpha = 9$ (Table 3).

n	$\|u-u_\Delta\|_{\tau_i,x_i}$	$\|u-u_\Delta\|_\infty$	$\|u'-u'_\Delta\|_{\lambda_i}$	$\|u'-u'_\Delta\|_\infty$	$\|u''-u''_\Delta\|_{\tau_i}$	$\|u''-u''_\Delta\|_\infty$
8	2.0-3 2.0-3	2.1-3	3.7-3	1.5-2	4.6-2	7.0-1
16	2.0-4 1.9-4	2.0-4	4.7-4	3.7-3	4.9-4	3.6-1
32	3.7-5 3.6-5	3.8-5	1.0-4	8.6-4	1.8-3	1.8-1
64	6.0-6 5.9-6	6.0-6	1.9-5	2.1-4	6.0-4	9.1-1
128	1.0-6 1.0-6	9.9-7	3.3-6	5.8-5	2.1-4	4.6-2
256	1.8-7 1.7-7	1.7-7	5.8-7	1.4-5	7.1-5	2.3-2

Table 5. Errors of P2C1COL (order = 4) for the case $\alpha = 7$ with $A_0 = B_0 = 1$, $A_1 = 0$ and $B_1 = 1$. The notation of Table 1 is used.

n	u convergence at τ_i	at x_i	u convergence global	u' convergence at λ points	u' convergence global	u'' convergence at τ_i	u'' convergence global
8							
	3.34	3.37	3.39	3.00	2.02	3.23	0.98
16							
	2.40	2.42	2.42	2.21	2.11	1.48	0.98
32							
	2.64	2.61	2.64	2.44	2.01	1.55	1.00
64							
	2.57	2.55	2.60	2.50	1.87	1.54	0.98
128							
	2.52	2.52	2.52	2.50	2.09	1.53	1.03
256							

Table 6. Estimated orders of convergence of P2C1COL (order = 4) for $\alpha = 7$ (Table 5).

We now consider another version of Problem 1 where f, g_0 and g_1 are chosen to make $u(x) = 1/(1 + 4x^2)$. This should give the highest possible order of convergence and, as the results of Table 7 show, the observed errors are smaller. The estimated orders of convergence seen in Table 8 are as predicted by Theorem 4.2 and essentially the same as in Table 2.

n	$\|u - u_\Delta\|_{\tau_i,x_i}$		$\|u - u_\Delta\|_\infty$	$\|u' - u_\Delta'\|_{\lambda_i}$	$\|u' - u_\Delta'\|_\infty$	$\|u'' - u_\Delta''\|_{\tau_i}$	$\|u'' - u_\Delta''\|_\infty$
4							
	3.9-4	2.8-4	7.7-4	6.5-3	3.7-2	1.1-1	2.2+0
8							
	6.1-5	5.8-5	1.3-4	1.1-3	1.1-2	6.4-2	1.1+0
16							
	3.6-6	3.5-6	1.2-5	1.7-4	2.6-3	1.7-2	5.4-1
32							
	2.0-7	2.0-7	1.3-6	2.3-5	5.8-4	4.0-3	2.6-1
64							
	1.2-8	1.2-8	1.5-7	2.9-6	1.8-4	9.8-4	1.4-1
128							
	7.6-10	7.6-10	1.7-8	3.7-7	4.2-5	2.4-4	7.0-2
256							

Table 7. Errors of P2C1COL (order = 4) for Problem 1 with Dirichlet boundary conditions ($A_0 = A_1 = 1$, $B_0 = B_1 = 0$), and $u(x) = 1/(1 + 4x^2)$. The notation of Table 1 is used.

n	u convergence			u' convergence		u'' convergence	
	at τ_i, x_i		global	at λ points	global	at τ_i	global
8							
	2.68	2.28	2.62	2.53	1.68	0.74	0.98
16							
	4.06	4.05	3.35	2.73	2.11	1.94	1.01
32							
	4.16	4.12	3.21	2.89	2.20	2.06	1.05
64							
	4.06	4.05	3.13	2.97	1.70	2.02	0.88
128							
	4.02	4.01	3.19	2.99	2.01	2.01	1.03
256							

Table 8. Estimated orders of convergence of P2C1COL (order = 4) for the case of $u = 1/(1 + 4x^2)$ (Table 7).

Problem 2: This is a trivial second order problem used very often for verifying the convergence of various numerical methods, the equation is

$$u'' - 4u = 4\cosh(1)$$

subject to boundary conditions $u(0) = u(1) = 0$. It has the true solution

$$u(x) = \cosh(2x - 1) - \cosh(1).$$

The computational results show almost exact agreement with the orders of convergence predicted by Theorem 4.2.

n	$\|u - u_\Delta\|_{\tau_i, x_i}$		$\|u - u_\Delta\|_\infty$	$\|u' - u_\Delta'\|_{\lambda_i}$	$\|u' - u_\Delta'\|_\infty$	$\|u'' - u_\Delta''\|_{\tau_i}$	$\|u'' - u_\Delta''\|_\infty$
8	4.0-5	3.5-6	1.5-4	7.1-4	7.1-3	1.4-2	4.6-1
16	2.8-6	1.4-7	1.8-5	9.2-5	1.8-3	3.8-3	2.3-1
32	1.8-7	1.2-8	2.1-6	1.2-5	5.0-4	9.8-4	1.2-1
64	1.1-8	8.0-10	2.7-7	1.5-6	1.2-4	2.5-4	5.5-2
128	7.2-10	5.1-11	3.4-8	1.9-7	2.9-5	6.2-5	2.8-2
256	4.5-11	3.2-12	3.8-9	2.4-8	7.0-6	1.6-5	1.5-2

Table 9. Errors of P2C1COL (order = 4) for Problem 2.
The notation of Table 1 is used.

n	u convergence at τ_i, x_i		u convergence global	u' convergence at λ points	u' convergence global	u'' convergence at τ_i	u'' convergence global
8							
16	3.84	4.63	3.13	2.94	1.97	1.87	1.01
32	3.94	3.56	3.10	2.97	1.86	1.95	0.90
64	3.98	3.90	2.96	2.98	2.11	1.98	1.18
128	3.99	3.98	2.98	2.99	1.99	1.99	0.95
256	4.00	3.99	3.13	3.00	2.07	2.00	0.92

Table 10. Estimated orders of convergence of P2C1COL (order = 4) for Problem 2.

n	$\|u-u_\Delta\|_{\tau_i,x_i}$	$\|u-u_\Delta\|_\infty$	$\|u'-u'_\Delta\|_{\lambda_i}$	$\|u'-u'_\Delta\|_\infty$	$\|u''-u''_\Delta\|_{\tau_i}$	$\|u''-u''_\Delta\|_\infty$
8	9.6-4 1.0-3	1.0-4	4.1-3	9.0-3	3.8-3	4.5-1
16	2.5-4 2.5-4	2.5-4	1.1-3	2.5-3	9.8-4	2.3-1
32	6.2-5 6.2-5	6.2-5	2.8-4	5.7-4	2.5-4	1.2-1
64	1.5-5 1.6-5	1.6-5	7.3-5	1.5-4	6.2-5	5.5-1
128	3.9-6 3.9-6	3.9-6	1.8-5	4.0-5	1.6-5	2.8-2
256	9.7-7 9.7-7	9.7-7	4.6-6	1.0-5	3.9-6	1.5-2

Table 11. Errors of P2C1COL (order = 2) for Problem 2.
The notation of Table 1 is used.

n	u convergence at τ_i, x_i	u convergence global	u' convergence at λ points	u' convergence global	u'' convergence at τ_i	u'' convergence global
8						
	1.96 2.01	2.01	1.90	1.87	1.96	0.94
16						
	1.99 2.00	2.00	1.94	2.11	1.99	0.91
32						
	2.00 2.00	2.00	1.97	1.93	2.00	1.18
64						
	2.00 2.00	2.00	1.98	1.91	2.00	0.95
128						
	2.00 2.00	2.00	1.99	1.96	2.00	0.92
256						

Table 12. Estimated orders of convergence for P2C1COL (order = 2) for Problem 2.

We also solved this problem using the program P2C1GAL which implements the Galerkin method using quadratic spline basis functions. The results shown in Tables 13 and 14 indicate the rates of convergence expected for such a method. Comparing with Tables 9 and 10 for P2C1COL (order = 4), we see that the quadratic spline collocation method is slightly more accurate and they both exhibit the same order of convergence. The collocation method is more general as it does not require a self-adjoint operator.

n	$\|u - u_\Delta\|_{\tau_i, x_i}$		$\|u - u_\Delta\|_\infty$	$\|u' - u_\Delta'\|_{\lambda_i}$	$\|u' - u_\Delta'\|_\infty$	$\|u'' - u_\Delta''\|_{\tau_i}$	$\|u'' - u_\Delta''\|_\infty$
8	2.1-5	2.3-5	1.4-4	7.3-4	1.2-2	1.4-2	4.9-1
16	1.5-6	1.6-6	1.8-5	1.0-4	3.0-3	3.9-3	2.6-1
32	9.9-8	1.0-7	2.3-6	1.3-5	7.6-4	1.0-3	1.3-1
64	6.4-9	6.4-9	2.9-7	1.7-6	1.9-4	2.7-4	6.6-1
128	4.1-10	4.0-10	3.6-8	2.1-7	4.8-5	6.8-5	3.3-2

Table 13. Error estimates for quadratic spline galerkin (P2C1GAL) for Problem 2. The notation of Table 1 is used.

n	u convergence		u' convergence		u'' convergence	
	at τ_i, x_i	global	at λ points	global	at τ_i	global
8						
	3.81 3.91	2.98	2.87	1.96	1.85	0.95
60						
	3.90 3.96	2.99	2.93	1.98	1.91	0.97
32						
	3.95 3.98	3.00	2.96	1.99	1.95	0.99
64						
	3.97 3.99	3.00	2.98	2.00	1.98	0.99
128						

Table 14. Orders of convergence for P2C1GAL for Problem 2.

Problem 3: This example was considered in [10]. The equation and boundary conditions are

$$u'' + xu'(x) - u(x) = xe^{x} - |x| \, (6 - 12x + 2x^2 - 3x^3)$$
$$u(-1) = e^{-1} + 2, \quad u(1) = e.$$

which has the unique solution

$$u(x) = \begin{cases} e^{x} - x^3 + x^4 & x \geq 0 \\ e^{x} + x^3 - x^4 & x \leq 0 \end{cases}$$

The derivatives degree three and four of $u(x)$ have jump discontinuities at the origin. All partitions used include the origin.

n	$\|u - u_\Delta\|_{\tau_i, x_i}$	$\|u - u_\Delta\|_\infty$	$\|u' - u_\Delta'\|_{\lambda_i}$	$\|u' - u_\Delta'\|_\infty$	$\|u'' - u_\Delta''\|_{\tau_i}$	$\|u'' - u_\Delta''\|_\infty$
8	9.4-3 1.2-2	1.2-2	3.0-2	9.8-2	1.2-1	3.1+0
16	2.5-3 2.9-3	2.9-3	7.3-3	2.4-2	4.5-2	1.6+0
32	6.5-4 7.0-4	6.9-4	1.8-3	6.3-3	1.9-2	8.2-1
64	1.6-4 1.7-4	1.7-4	4.4-4	1.6-3	8.7-3	3.7-1
128	4.1-5 4.2-5	4.1-5	1.1-4	4.1-4	4.1-3	1.9-1
256	1.0-5 1.1-5	1.0-5	2.7-5	9.7-5	2.0-3	9.9-2

Table 15. Errors of P2C1COL (order = 4) for Problem 3.
The notation of Table 1 is used.

	u convergence		u' convergence		u'' convergence	
n	at τ_i, x_i	global	at λ points	global	at τ_i	global
8						
	1.91 2.09	2.09	2.06	2.02	1.39	0.97
16						
	1.95 2.05	2.06	2.03	1.94	1.24	0.94
32						
	1.98 2.03	2.04	2.01	2.00	1.14	1.14
64						
	1.99 2.02	2.03	2.01	1.92	1.07	0.96
128						
	1.99 2.01	2.00	2.00	2.10	1.04	0.95
256						

Table 16. Estimated orders of convergence of P2C1COL (order = 4) for Problem 3.

REFERENCES

1. **Archer, D.A.,** Some collocation methods for differential equations, Rice University, Houston, TX, (1973), Ph.D. Thesis.
2. **Christara, C. C.,** Algorithms and architectures for the numerical solution of partial differential equations, Ph.D. thesis, in preparation.
3. **Daniel, J.W.** and **Swartz, B. K.,** Extrapolated collocation for two point boundary value problems using cubic splines, J. Inst. Math. Appl., **16** (1975), 161–174.
4. **De Boor, C.,** *A Practical Guide to Splines*, Springer-Verlag (1978).
5. **Fyfe, D.J.,** The use of cubic splines in the solution of two-point boundary value problems, Comput. J., 17 (1968), 188–192.
6. **Irodotou-Ellina, M.,** Optimal spline collocation methods for high degree two-point boundary value problems, University of Thessaloniki, Greece, Ph.D. Thesis, in preparation.
7. **Kammener, W. J., Reddien, G. W.** and **Varga, R. S.,** Quadratic interpolatory splines, Num. Math., 22, (1974), 241–259.
8. **Khalifa, A. K.** and **Eilbeck, J. C.,** Collocation with quadratic and cubic splines, IMA J. Num. Anal., 2, (1982), 111–121.
9. **Marsden, M. J.,** Quadratic spline interpolation, Bull. Amer. Math. Soc., **30**, (1974), 903–906.
10. **Sakai, M.** and **Usmani, R.,** Quadratic spline solutions and two-point boundary value problems, Publ. RIMS, Kyoto University, 19, (1983), 7–13.
11. **Sakai, M.** and **Usmani, R.,** A note on quadratic spline interpolation at midpoints, BIT, 22 (1982), 261–267.
12. **Sharma, A.** and **Tzimbalardo, J.,** Quadratic splines, J. of Approx. Th., **19**, (1977), 186–193.
13. **Russell, R.D.** and **Shampine, L. F.,** A collocation method for boundary value problems, Num. Math., **19**, (1972), 1–28.