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An important question in stochastic modeling is whether a denumerable state Markov chain describing a system is stable or not. Adopting a general definition of stability we present some conditions for instability. In particular, we slightly extend the results of Kaplan [1979], Sennott et al [1983] and Szpankowski [1985] giving a condition for non-ergodicity, and we present criteria for moments of a chain to be infinite. Finally, we apply these instability conditions to study some multidimensional Markov chains describing multiaccess protocols in a broadcast environment.

The stability and instability of many important systems is usually determined by the classification of a related Markov chain. The question of ergodicity, recurrence and transience of a Markov chain is of great importance; however, in practice a more general definition of stability is necessary. It is said that a system is stable if it possesses *required properties* in the presence of some *disturbances*. For stochastic models disturbances are usually associated with the input process, and stability or instability depend upon the definition of the required property. For example, assuming existence of steady-state distribution we deal with the ergodicity of a Markov chain; by considering finiteness of some moments we investigate a new type of stability. This paper discusses the instability property of a Markov chain, and we focus our attention on nonergodicity and infiniteness of some moments of the process.

Let N^t $(t = 0,1, \ldots,)$ denote an irreducible aperiodic Markov chain with state space C over the nonnegative integers. Let $P = (p_{ij})_{i,j \in C}$ be its transition matrix and π_i, $i \in C$ the steady-state probabilities of the chain. It is well known that for ergodic Markov chains the steady state probabilities are positive for all $i \in C$ and they satisfy the following system of linear equations

$$\pi_j = \sum_{i \in C} \pi_i \, p_{ij}, \quad j \in C. \tag{1}$$

A simple criteria for ergodicity are available through the so called Lyapunov (or test) function approach (see for example Foster [1953], Tweedie [1976]). Let $V(k)$, $k \in C$ be a nonnegative real-valued function called a Lyapunov function, and we denote by $AV(k)$, $k \in C$ an operator for a Markov chain with respect to that function if it exists. That is,

$$AV(k) = E\{V(N^{t+1}) - V(N^t) | N^t = k\}, \quad k \in C. \tag{2}$$

The operator $AV(k)$ is often called a generalized drift, since for $V(k) = k$ it becomes the average drift $d(k) = E\{N^{t+1} - N^t \,|N^t = k\}$. If the generalized drift is negative for all but finitely many states, then the Markov chain is ergodic (Foster [1953], Tweedie [1976]). Kaplan [1979], Sennott et al [1983] and Szpankowski [1985] proved that, under some simple condition on the test function, $AV(k) \ge 0$ for all but finitely many states implies nonergodicity , if an additional condition known as the (generalized) Kaplan condition is satisfied. However, in practice it is sometimes difficult to verify the (generalized) Kaplan condition (e.g. for multidimensional Markov chains).

In the next section, we show that under the same condition as in Sennott et al [1983] and Szpankowski [1985], the criterion $AV(k) \ge 0$ for infinitely many states (i.e., the condition may be violated for an infinite number of states) together with Kaplan condition is sufficient for nonergodicity. Moreover, if we drop the Kaplan condition in the last statement, then we shall prove that such a modified criterion implies $\lim_{t \to \infty} EV(N^t) = \infty$. Under some additional restrictions we show that for any $t > 0$ the inequality $EV(N^t) \ge EV(N^o) + \varepsilon t$, $\varepsilon > 0$ follows from $AV(k) \ge \varepsilon > 0$. These criteria find useful applications in instability analysis of some multidimensional Markov chains. In particular, we apply them to study instability problems of multiaccess protocols, e.g., exponential back-off algorithms (Rosenkrantz [1984], Kelly [1985]) and a decentralized dynamic control algorithm (Hajek and van Loon [1982]).

1. Main results

We begin our discussion with an extension of Proposition 4 and Theorem 4 of Sennott et al [1983]. We prove first

LEMMA 1. Let N^t be an ergodic, irreducible Markov chain, and $V(k)$, $k \in C$, a nonnegative Lyapunov function. If H is a subset of C such that

$$\inf_{k \in C-H} V(k) > \sup_{k \in H} V(k), \tag{3}$$

then

$$\sum_{i \in H} AV(i)\pi_i > 0. \tag{4}$$

Proof. It follows directly from the proof of Proposition 1 in Sennott et al [1983] with some minor changes. However, note that in our Lemma we do not assume that H is finite. For the sake of completeness we present here a sketch of the proof. Note first that from global balance equations one finds

$$\sum_{i \in H} \pi_i \sum_{j \in C-H} p_{ij} = \sum_{i \in C-H} \pi_i \sum_{j \in H} p_{ij}.$$

Then by the above and (1), after some algebra as in Sennott et al [1983], we obtain

$$\sum_{i \in H} \pi_i \, AV(i) \geq \left[\inf_{j \in C-H} V(j) - \sup_{j \in H} V(j) \right] \sum_{i \in H} \pi_i \sum_{j \in C-H} p_{ij}.$$

But, by (3) and irreducibility, the RHS of the above is positive, thus proving (4).

□

Criteria for nonergodicity involves the following function

$$\psi_k^V(z) = (z^{V(k)} - \sum_{j \in C} p_{kj} \, z^{V(j)})/(1-z), \quad z \in [0,1), \quad k \in C. \tag{5}$$

Note that for $|AV(k)| < \infty$ by l'Hospital rule $\lim_{z \to 1} \psi_k^V(z) = AV(k)$. Then, the following holds:

THEOREM 1. Suppose that except for the ergodicity of N' the hypotheses of Lemma 1 hold, and let $H_1 \subset C$ be a finite subset of C. If $|AV(k)| < \infty, k \in C$ and

$$AV(k) \geq 0 \quad k \in C - H \tag{6}$$

$$\psi_k^V(z) \geq -B \quad k \in C - H_1, \tag{7}$$

where $B \geq 0$ is a constant, then the chain is not ergodic.

Proof. Assume the contrary that N' is ergodic and let π_k, $k \in C$ be its steady state distribution. Then, by (5) we find that $\sum_{k \in C} AV(k)\, \psi_k^V(z) = 0$. But, by (7) and Fatou's Lemma

$$0 = \lim_{z \to 1} \sum_{k \in C} \pi_k\, \psi_k^V(z) \geq \sum_{k \in C} AV(k)\pi_k = \sum_{k \in H} AV(k)\pi_k + \sum_{k \in C-H} AV(k)\pi_k > 0.$$

The last inequality follows from (6) and Lemma 1. This is the desired contradiction.

□

Remarks:

(i) If in (3) we replace strict inequality by a weak one, then for validity of Theorem 1 we must have strict inequality in (6) for some $k \in C - H$.

(ii) Another extension of Theorem 1 is possible if one notes that introducing a parametric Lyapunov function $V_k(z) = z^{V(k)}$, $k \in C$, $z \in [0,1)$ and a function $g(z) = 1 - z$, we may represent (5) as $\psi_k^V(z) = -AV_k(z)/g(x)$, and (6) is equivalent to $-AV'_k(1)/g'(1) \geq 0$ where $AV'_k(1)$ is the operator of the derivative of $V(z)$ at $z_0 = 1$. This approach was adopted in Szpankowski [1985], where a precise definition of a generalized parametric Lyapunov function $V_k(z)$ and additional function $g(z)$ were introduced , and conditions (3), (6) and (7) are expressed in terms of $AV_k(z)$ as above. These functions must satisfy some conditions, a few of which are that $V_k(z)$ is uniformly bounded for all $k \in C$, $z \in [a,b]$, and there exists such $z_0 \in [a,b]$ that $V_k(z_0)$ = const, and $g(z_0) = 0$ (for details see Szpankowski [1985]). A particular application of this is as follows. Let us define a new function

$L_k(z) = \sum_{m=0}^{n} n_{[n-m]} V_k^{(m)}(z)(1-z)^m$, $g(z) = (1-z)^{n+1}$ where n is an integer, $n_{[n-m]} \stackrel{\text{def}}{=} n(n-1) \cdots (n-m+1)$, and $V_k^{(m)}(z)$ is the m-th derivative of $V_k(z)$ with respect to z. Then, $-AL'_k(z) = -AV_k^{(n+1)}(z)(1-z)^n$, $g'(z) = (n+1)(1-z)^n$ and applying Proposition 1 of Szpankowski [1985] with $z_0 = 1$ one finds that a Markov chain N^t is not ergodic if $-AV_k^{(n+1)}(1) \geq 0$ for $k \in C - H$ and $-AL_k(z) \geq -B(1-z)^{n+1}$, $k \in C - H_1$, $B \geq 0$ where the subsets H and H_1 are defined above in Theorem 1. In particular, if $V_k(z) = z^k$ then the first condition is equivalent to $\sum_{j \in C} p_{kj}\{j(j-1) \cdots (j-n) - k(k-1) \cdots (k-n)\} \geq 0$ for $k \geq M$, where M is a constant, and it is proved (Kaplan [1983], Sennott [1985]) that the Kaplan condition is reduced to $k^n \psi_k(z) \geq -B$ where $\psi_k(z)$ is given by (5) with $V(k) = k$.

□

The applicability of Theorem 1 depends on verification of conditions (3), (6) and (7). In most cases (3) and (6) are easy to check, but (7) needs some additional work. However, it is proved (Kaplan [1979], Sennott et al [1983], Szpankowski [1985]) that for downward uniformly bounded Markov chains ($p_{kj} = 0$ for $j < k - m$, m is given) condition (7) is automatically satisfied. Nevertheless, in the general case a question arises: what are properties of the Markov chain that follow from conditions (3) and (6) alone?

To solve the problem we need the following:

LEMMA 2. Let τ be a Markov moment such that $Pr\{\tau < \infty\} = 1$. Then for a Markov chain N^t and a Lyapunov function $V(k)$, $k \in C$ such that $AV(k) < \infty$, $k \in C$ the following holds

$$EV(N^\tau) = EV(N^o) + E\sum_{j=0}^{\tau-1} AV(N^j). \tag{8}$$

Proof. We prove that $Z^t = V(N^t) - \sum_{j=0}^{t-1} AV(N^j)$ is a martingale. Let $F(t)$ be the history or the past of the process N^s up until time t, $s<t$. Then by Markov chain properties

$$E\{Z^{t+1}|F(t)\} = E\{V(N^{t+1})|N^t\} - E\{V(N^{t+1}) - V(N^t)|N^t\} - \sum_{j=0}^{t-1} AV(N^j) = Z^t$$

hence Z^t is a martingale and by optional sampling theorem (Karlin and Taylor [1975]) we find that $EZ^\tau = EZ^o = EV(N^o)$, which implies (8).

□

Defining $E_k V(N^t) = E\{V(N^t)|N^o = k\}$ Eq.(8) implies that

$$E_k V(N^\tau) = V(k) + E \sum_{j=0}^{\tau-1} AV(N^j). \tag{9}$$

Then we prove

THEOREM 2. Let N^t be a Markov chain and $H = \{k_o\}$ such that $V(k_o) \le \inf_{k \in C-H} V(k)$. If $\varepsilon > 0$ and

$$AV(k) \ge \varepsilon > 0 \quad k \in C - \{k_o\}, \tag{10}$$

then

$$E_k V(N^t) \ge V(k) + \varepsilon\, t. \tag{11}$$

Proof. Define a new Markov chain $\tilde{N}^t$ such that $\tilde{p}_{kj} = p_{kj}$ for $k \in C - H$ and $\tilde{p}_{k_o k_o} = 1$. Then $\tilde{A}V(k) = AV(k)$ for $k \in C - H$ and $\tilde{A}V(k_o) = 0$. Note also that by definition of k_o $V(N^t) \underset{st}{\ge} V(\tilde{N}^t)$, and then by (10) and (9) we find

$$E_k(V(N^t) \ge E_k V(\tilde{N}^t) \ge V(k) + \varepsilon\, t,$$

which is the desired inequality (11).

□

Note that by Theorem 2 condition (10) implies that $\lim_{t \to \infty} EV(N^t) = \infty$. We prove now that this property holds under much weaker assumptions.

THEOREM 3. Let N^t be an irreducible, ergodic Markov chain, and $V(k)$ a nonnegative Lyapunov function such that

$$\inf_{k \in C-H} V(k) > \sup_{k \in H} V(k), \tag{12}$$

for some $H \subset C$. If

$$AV(k) \geq 0 \quad k \in C-H, \tag{13}$$

then

$$\lim_{t \to \infty} EV(N^t) = \infty. \tag{14}$$

Proof. By ergodicity of N^t and Fatou's Lemma we find

$$\lim_{t \to \infty} EV(N^t) = \lim_{t \to \infty} \sum_{k \in C} V(k) Pr\{N^t = k\} \geq \sum_{k \in C} V(k)\pi_k. \tag{15}$$

We prove that RHS of (15) is equal to infinity. Let $\bar{V}$ denote the RHS of (15) and assume contrary that $\bar{V} < \infty$. By (12) and (13) Lemma 1 holds, and together with (13) we find that $\sum_{k \in C} AV(k)\pi_k > 0$. But (1), after some algebra, implies that

$$\bar{V} = \bar{V} + \sum_{k \in C} AV(k)\pi_k > \bar{V},$$

which is the desired contradiction.

□

We prove now that for nonergodic Markov chains (12) and a slightly modified (13) imply (14). Let us start with

LEMMA 3. Let N^t be an irreducible Markov chain satisfying (12) and for some $H \subset C$

$$AV(k) \geq \varepsilon > 0 \quad k \in C-H. \tag{16}$$

Then $V(k)$ is unbounded function in C.

Proof. By (12) $V(k)$ is not a constant function, hence (see Mitrinovic [1970], p.76)

$$\sum_{j \in C} p_{kj} V(j) < \sup_{j \in C} V(j).$$

But (2) and (16) imply that $\sum_{j \in C} p_{kj} V(j) \geq \varepsilon + V(k)$, for $k \in C - H$, hence by these facts and (12) we find

$$\sup_{j \in C - H} V(j) > \varepsilon + V(k) \quad k \in C - H. \tag{17}$$

Assume the contrary, that $V(k)$ is bounded and $\sup_{j \in C} V(j) = \sup_{j \in C - H} V(j) = A < \infty$. Then by (17) $A - V(k) > \varepsilon, k \in C - H$ and taking supremum of both sides of the latter we obtain

$$0 = A - \sup_{k \in C - H} V(k) \geq \varepsilon > 0,$$

which is the desired contradiction.

□

Then, we are able to prove

THEOREM 4. Let N^t be an irreducible nonergodic Markov chain satisfying (12), (16) and $\Delta V(k) = V(k) - V(k-1) \geq -B, k \in C$ for some $B \geq 0$. Then (14) holds.

Proof. By Lemma 3, $V(k)$ is unbounded. Noting that for any function $g(X)$ and a discrete random variable X, $Eg(X) = \sum_{k \in C} [g(k) - g(k-1)] Pr\{X \geq k\}$ we find

$$\begin{aligned}
\lim_{t \to \infty} E_k V(N^t) &= \lim_{t \to \infty} \sum_{j \in C} [V(j) - V(j-1)]\, Pr\{N^t \geq j \mid N^0 = k\}, \\
&\geq \sum_{j \in C} [V(j) - V(j-1)] \lim_{t \to \infty} Pr\{N^t \to j \mid N^0 = k\}, \\
&= \sum_{j \in C} [V(j) - V(j-1)] = \lim_{j \to \infty} V(j) = \infty,
\end{aligned}$$

where the inequality is a consequence of Fatou's Lemma and the following equality is derived from nonergodicity of the Markov chain. This implies (14).

□

Finally, Theorem 3 and 4 imply our main result

PROPOSITION. For an irreducible Markov chain satisfying (12) and (16) with $\Delta V(k)$ bounded below the property (14) holds, that is, $\lim_{t \to \infty} EV(N^t) = \infty$.

□

Remarks

(i) Conditions of the above Proposition are sufficient but not necessary. We can find a Markov chain with negative drift for all but one state such that the chain does not possess any moment. To show this , assume that $p_{nm} = 0$ for $m \neq 0,\ n+1$ and $p_{n,n+1} = p_n$, $p_{n,0} = q_n = 1 - p_n$ where $p_n < n/(n+1)$ for $n \neq 0$. Then $d(n) = (n+1)p_n - n < 0$ for $n \neq 0$, hence by Foster's criterion the chain is ergodic. Solving (1) we show that $\pi_n = 0.5\ \pi_0/[n(n-1)],\ n \neq 0, 1,\ \pi_0 > 0$. Then for any $r \geq 1$

$$EN^r \leq \frac{\pi_0}{2} \sum_{n=2}^{\infty} \frac{n^{r-1}}{n-1} = \infty,$$

hence the chain does not possess any moments EN^r, $r \geq 1$.

(ii) To prove Theorem 4, we might also use equation (9). Here is a sketch of the proof. Let $\tau_{k,H} = \min\{t : N^t \in H \mid N^0 = k\}$, and $\tau = \min\{\tau_{k,H}, t\}$. Then, by (9) and (16) we obtain

$$E_k\ V(N^\tau) = V(k) + E \sum_{j=0}^{\tau-1} AV(N^j) \geq V(k) + \varepsilon\, E\tau. \tag{18}$$

But nonergodicity of N^t and Fatou's Lemma imply that $\lim_{t \to \infty} E\tau \geq E \lim_{t \to \infty} \min\{\tau_{k,H}, t\} = E\ \tau_{k,H} = \infty$, hence $\lim_{t \to \infty} E_k V(N^\tau) = \infty$, and this implies (14) since $\tau \leq t$.

□

2. Applications to instability analysis of some multiaccess protocols

Assume that an infinite number of users sharing a common communication channel transmit fixed-length packets. The channel is slotted and a slot duration is equal to a packet transmission time. When two or more packets transmit at the same time, then there is a collision and no packet is transmitted. There is a variety of protocols for resolving these collisions, and they differ depending on how feedback information for the channel is used to resolve the collisions. To avoid subsequent collisions a probability of retransmitting a collided packet is introduced, which controls the number of retransmissions. This probability depends on the outcomes of the channel, the time, the number of users involved in a collision, etc. We use a letter f as a generic notation for this probability with an index to show a relationship between the probability and the state of the system. We describe below some examples of multiaccess protocols for which we investigate instability conditions.

EXAMPLE 1. *Finite-dimensional back-off (FDB) algorithm*
Kelly [1985], Hajek [1982]

We assume that each user has a counter which contains the number of times a packet was involved in a collision. Let f_k, $k = 0,1, \ldots, B$ be the probability of transmitting a packet where k is the number of times a packet has already been unsuccessfully transmitted. Hajek [1982a] assumed that B is finite and after B unsuccessful transmissions a packet is rejected and declared lost (so called *acknowledgement based retransmission control* algorithm). In this case, we assume that for the FDB algorithm, after B unsuccessful transmissions a packet is retransmitted with a constant probability f_B until success occurs. By this assumption, we can uniformly model the original ALOHA protocol for which $f_0 = 1$, $f_1 = r$, $B = 1$, and the ALOHA protocol without retransmission discrimination policy, where $f_0 = r$, $B = 0$ (Szpankowski [1984]). For the finite-dimensional exponential back-off algorithm we assume $f_k = 2^{-k}$, $0 \le k \le B < \infty$.

EXAMPLE 2. *Infinite-dimensional back-off algorithm* (IDB)
Rosenkrantz [1984], Kelly [1985], Goodman et al [1985]

This is exactly the same algorithm as in Example 1, except that $B = \infty$. Exponential back-off is characterized by $f_k = 2^{-k}$, $0 \le k \le \infty$, and for linear back-off we assume $f_0 = 1$, $f_k = 1/k$, $1 \le k < \infty$.

EXAMPLE 3. *Decentralized dynamic control algorithm*
Hajek and van Loon [1982], Kelly [1985]

Another strategy to select the probabilities f was proposed by Hajek and van Loon [1982]. Now each user contains a counter, S_t, $t = 0,1, \ldots,$ which is updated recursively at the end of each slot according to some rules common for all users. For example, Hajek and van Loon [1982] assumed $S_{t+1} = \max\{S_t, 1, a(Z_t)\}$ where Z_t is an outcome from the channel (idle, success or collision) and $a(\cdot)$ is a function of Z_t. On the other hand, Kelly [1985] proposed $S_{t+1} = \max\{1, S_t + a\ I[Z_t = 0] + B\ I[Z_t = 1] + c\ I[Z_t = \text{collision}]\}$, where a, b, c are constants and $I(\cdot)$ is an indicator function of an event. The probability of transmitting a packet is the same for all users at a time t and is equal to $f_t = 1/S_t$.

□

We present below some instability analyses for the above algorithms.

Finite-dimensional back-off algorithm

In this case the system is described by a B-dimensional Markov chain $\mathbf{N}^t = (N_1^t, N_2^t, \ldots, N_B^t)$ where N_k^t is the number of packets at the beginning of time t that have been unsuccessfully transmitted k times. Let $A(t)$ denote the number of new packets that arrive in a slot t. Assume $A(t)$, $t = 0,1, \ldots,$ is i.i.d. process and $c_i = Pr\{A(t)=i\}$, $\lambda = EA(t)$. Under this assumption $\mathbf{N}^t$ is Markov chain and the question of N^t's ergodicity arises.

Let us start with a special case $B = 1$. That is, we consider original ALOHA protocol with

$f_0 = 1, f_1 = r$. Then, the average drift, $d(n)$, is equal to

$$d(n) = E\{E^{t+1} - N^t \mid N^t = n\} = \lambda - c_1(1-r)^n - c_0\, nr(1-r)^{n-1}. \tag{19}$$

Noting that $\lim_{n \to \infty} d(n) = \lambda$ and applying Kaplan's theorem we immediately find that the system is not ergodic for all $\lambda \geq 0$.

For $B > 1$ the situation is much more sophisticated, however, we can prove that the system is also nonergodic for any $\lambda > 0$. Let $\mathbf{n} = (n_1, n_2, \ldots, n_B)$ and we introduce the Lyapunov function $V(\mathbf{n}) = \sum_{i=1}^{B} \alpha_i\, n_i$, α_i, $i = 1,2, \ldots, B$, are nonnegative constants. Note that $AV(\mathbf{n}) = \sum_{i=1}^{B} \alpha_i\, d_i(\mathbf{n})$ where $d_i(\mathbf{n}) = E\{N_i^{t+1} - N_i^t \mid \mathbf{N}^t = \mathbf{n}\}$, is the i-th component of the average drift-vector of $\mathbf{N}^t$. Denoting

$$f_\pi(\mathbf{n}) \overset{\text{def}}{=} \prod_{l=1}^{B} (1-f_l)^{n_l}, \tag{20}$$

we immediately obtain (see also Rosenkrantz [1984]),

$$d_1(\mathbf{n}) = \lambda - f_1\, n_1 - c_1\, f_0\, f_\pi(\mathbf{n}), \tag{21a}$$

$$d_k(\mathbf{n}) = f_{k-1}\, n_{k-1} - f_k\, n_k - c_0\, n_{k-1}\, f_{k-1}(1-f_{k-1})^{-1} f_\pi(\mathbf{n}), \quad 2 \leq k \leq B-1 \tag{21b}$$

$$d_B(\mathbf{n}) = f_{B-1}\, n_{B-1} - c_0\, f_\pi(\mathbf{n})\, [\, n_{B-1}\, f_{B-1}(1-f_{B-1})^{-1} + n_B\, f_B(1-f_B)^{-1}\,], \tag{21c}$$

and

$$AV(\mathbf{n}) = \sum_{k=1}^{B-1} f_k\, n_k\, (\alpha_{k+1} - \alpha_k) + \alpha_1 \lambda - f_\pi(\mathbf{n}) \left[\alpha_1\, f_0\, c_1 + c_0 \sum_{k=1}^{B} \alpha_{k+1}\, n_k\, \frac{f_k}{1-f_k} \right], \tag{22}$$

where $\alpha_{B+1} \overset{\text{def}}{=} \alpha_B$. In particular, if $\alpha_1 = \alpha_2 = \cdots = \alpha_B = 1$ Eq. (22) reduces to a simple form

$$AV(\mathbf{n}) = \lambda - f_\pi(\mathbf{n}) \left[c_1\, f_0 + c_0 \sum_{k=1}^{B} n_k\, \frac{f_k}{1-f_k} \right], \tag{23}$$

which states that the average drift is equal to the input rate minus average throughput.

To study instability of the algorithm we must investigate properties of the following expression $f_\pi(\mathbf{n}) \sum_{k=1}^{B} n_k f_k/(1-f_k)$, with $f_\pi(\mathbf{n}) = \prod_{l=1}^{B} (1-f_l)^{n_l} = \exp\left[-\sum_{l=1}^{B} n_l \ln(1-f_l)^{-1}\right]$.

Let us introduce a function

$$g(\mathbf{x}) = \left[\sum_{l=1}^{B} a_i\, x_i\right] \exp\left[-\sum_{l=1}^{B} b_i\, x_i\right], \tag{24}$$

where $a_i, b_i > 0$, $i = 1, \ldots, B$ are nonnegative constants and $\mathbf{x} = (x_1, x_2, \ldots, x_B) \ge 0$. In our case, $a_i = f_i/(1-f_i)$, $b_i = \ln(1-f_i)^{-1}$ and $x_i = n_i$, $i = 1,2, \ldots, B$. In the Appendix A we prove that under the above constraints on **a**, **b**, and **x**

$$g(\mathbf{x}) \le e^{-1} \max\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_B}{b_B}\}, \tag{25}$$

and the maximum value of $g(\mathbf{x})$ is reached at one of the boundary points $(b_1^{-1}, 0, \ldots, 0)$ or $(0, b_2^{-1}, \ldots, 0)$ or ... or $(0, 0, \ldots, b_B^{-1})$. Let us also divide the Euclidean space $\mathbf{E} = \{\mathbf{x} : x_i \ge 0 \;\; i = 1,2, \ldots, B\}$ into two subspaces $\mathbf{E}_1 = \{\mathbf{x} \in E : b_1 x_1 + \cdots + b_B x_B \le 1\}$ and $\mathbf{E}_2 = \{\mathbf{x} \in E : b_1 x_1 + b_2 x_2 + \cdots + b_B x_B > 1\}$. Then, in the Appendix A we prove that $g(\mathbf{x})$ is 'decreasing' in E_2 in the sense that $g(\mathbf{x})$ defined on a line $L = \{\mathbf{x} \in E : x_1 = x_1,$ $x_2 = \beta_2 x_1, \ldots, x_B = \beta_B x_1, \beta_i \ge 0 \;\; i = 2,\ldots,B\}$ is decreasing for $x \in L \cap E_2$. In particular, defining $H_M = \{x \in E : x_1 + x_2 + \cdots + x_M < M\}$ where $M = \max_{1 \le i \le B} \{b_i^{-1}\}$ we prove that for any $\varepsilon > 0$ there exists such an $M > 0$ that

$$g(x) < \varepsilon \;\; \text{for} \;\; x \in E - H_M. \tag{26}$$

Now we are ready to prove

COROLLARY 1. For any finite B and $f_i > 0$ for all $i = 1,2, \ldots, B$ the system is not ergodic for $\lambda > 0$.

Proof. Note first that N^t is a uniformly downward bounded Markov chain. Hence by Theorem 1 and the above to prove the theorem we must only check (3) and (6), since (7) is automatically satisfied (see also Szpankowski [1985]). Let C be a state space consisting of B-tuples of nonnegative integers and let for some M $H_M = \{\mathbf{n} \in C : n_1 + n_2 + ... + n_B < M\}$. Then the function $V(\mathbf{n}) = \sum_{i=1}^{B} n_i$ satisfies condition (3), that is, $\inf_{\mathbf{n} \in C - H_M} V(\mathbf{n}) > \sup_{\mathbf{n} \in H_M} V(\mathbf{n})$. We show now that $AV(\mathbf{n}) > 0$ for any $\lambda > 0$, where $AV(\mathbf{n})$ is given by (23). Let $f^* = \min_{1 \le l \le B} f_l > 0$. Then for $\mathbf{n} \in C - H_M$

$$f_\pi(\mathbf{n}) \le (1 - f^*)^{\sum_{l=1}^{B} n_l} \le (1 - f^*)^M, \tag{27}$$

hence we may find such an M_1 that for $\mathbf{n} \in C - H_{M_1}$ $f_\pi(\mathbf{n}) < \delta$ where $\delta > 0$. Note now that the third term in (23) is our function $g(\mathbf{n})$ defined in (24) with $a_i = f_i/(1 - f_i)$ and $b_i = ln(1 - f_i)^{-1}$. Hence, by (26) we can find an $M_2 > \max_{i \le i \le B} \{ln^{-1}(1 - f_i)^{-1}\}$ such that for $\mathbf{n} \in C - H_{M_2}$

$$f_\pi(\mathbf{n}) \sum_{l=1}^{B} \frac{f_l}{1 - f_l} n_l < \varepsilon, \tag{28}$$

where $\varepsilon > 0$. Finally, for $M = \max\{M_1, M_2\}$ and $\mathbf{n} \in C - H_M$ we obtain $AV(\mathbf{n}) \ge \lambda - f_0 c_1 \delta - c_0 \varepsilon$. Since $f_0 c_1 + c_0 < 1$ we find that for $\mathbf{n} \in C - H_M$ the drift $AV(\mathbf{n}) > 0$ whenever $\lambda > \delta + \varepsilon$. This implies the theorem as a consequence of arbitrary $\varepsilon > 0$ and $\delta > 0$.

□

In spite of the fact that the chain is not ergodic we can prove that $\lim_{t \to \infty} \sup EN_k^t \le \lambda f_k^{-1} < \infty$ for $k = 1,2, \ldots, B - 1$, and hence by the corollary, we must have $\lim_{t \to \infty} EN_B^t = \infty$. Indeed, by (21a) we find

$$E\{N_1^{t+1} \mid N_1^t\} - N_1^t = \lambda - f_1 N_1^t - c_1 f_0 f_\pi(\mathbf{N}^t) \le \lambda - f_1 N_1^t$$

and by recursive arguments we can show that $EN_1^t \le (1-f_1)^t EN_1^0 + \lambda f_1^{-1}$. This implies that $\limsup EN_1^t \le \lambda f_1^{-1} \stackrel{def}{=} M_1$ as $t \to \infty$. Let $M_1, M_2, \ldots, M_{k-1}$ denote upper bounds on EN_i^t for $i=1,\ldots,k-1$. Then by (21b) we find that for $k=2,3,\ldots,B-1$

$$EN_1^{t+1} \le (1-f_k) EN_k^t + f_{k-1} EN_{k-1}^t \le (1-f_k)^t EN_1^0 + \frac{f_{k-1}}{f_k} M_{k-1},$$

hence $M_k = \lim_{t \to \infty} \sup EN_k^t \le \frac{f_{k-1}}{f_k} M_{k-1}$. Naturally, recursion on M_k shows that $M_k = \lambda f_k^{-1}$, $k = 1,2, \ldots, B-1$. On the other hand, by (21c) $EN_B^{t+1} \le EN_B^0 + t f_{B-1} M_{B-1}$ and by Corollary 1 we conclude that $\lim_{t \to \infty} EN_B^t = \infty$.

Infinite-dimensional back-off algorithm

We assume now that $B = \infty$. This implies that N^t is an infinite dimensional Markov chain. Formulas (21a) and (21b) for the k-th component of the average drift are valid (however we do not need formula (21c)). In particular, (23) becomes

$$AV(\mathbf{n}) = \lambda - f_\pi(\mathbf{n}) \left[c_1 f_0 + c_0 \sum_{k=1}^{\infty} n_k \frac{f_k}{1-f_k} \right], \tag{29}$$

and $f_\pi(\mathbf{n}) = \prod_{l=1}^{\infty} (1-f_l)^{n_l}$. In this case the chain is obviously not downward uniformly bounded (it has infinite downward jumps), and therefore condition (7) in Theorem 1 should also be also verified. Moreover, the set H_M defined in the proof of Corollary 1 is infinite, and if $\lim_{k \to \infty} f_k = 0$, then M_2 defined in the proof of Corollary 1 is also infinite. To avoid these difficulties (in particular, to bypass verification of Kaplan's condition (7)) we apply Theorem 2 and Proposition. We prove that

COROLLARY 2. Let $f_{\max} = \max_{1 \le l \le \infty} f_l$, $f_0 = 1$ and $\gamma = -\frac{1-f_{\max}}{f_{\max}} \ln(1-f_{\max})$. If λ_0 is a solution of the following equation

$$\lambda - \frac{c_0}{\gamma} \exp [\ \gamma c_1/c_0) - 1\ ] = 0, \tag{30}$$

then for $\lambda > \lambda_0$ the average backlog grows linearly in time, that is, there exists such $\varepsilon>0$ that $E \sum_{i=1}^{\infty} N_i^t \geq \varepsilon t$, assuming $\mathbf{N}^0 = \mathbf{0}$.

Proof: We apply Theorem 2 with H being an empty set, and $V(\mathbf{n}) = \sum_{i=1}^{\infty} n_i$. Then $AV(\mathbf{n})$ is given by (23) with $B = \infty$. We find first on upper bound for $f_\pi(\mathbf{n}) = \exp \left[\sum_{l=1}^{\infty} n_l \ln(1 - f_l) \right]$. In Appendix B we prove that $\ln(1 - x) \leq -\gamma \frac{x}{1 - x}$ for $0 \leq x \leq f_{\max}$, where $\gamma = -\frac{1 - f_{\max}}{f_{\max}} \ln(1 - f_{\max})$. From this we immediately obtain

$$f_\pi(\mathbf{n}) \leq \exp \left[-\gamma \sum_{l=1}^{\infty} n_l \frac{f_l}{1 - f_l} \right]. \tag{31}$$

Let $s = \sum_{l=1}^{\infty} n_l \frac{f_l}{1 - f_l}$. Then by (31) and (23) we find

$$AV(\mathbf{n}) = \lambda - f_\pi(\mathbf{n}) \left[c_1 + c_0 \sum_{l=1}^{\infty} n_l \frac{f_l}{1 - f_l} \right] \geq \lambda - \max_{0 \leq s < \infty} \{ e^{-\gamma s}(c_1 + c_0 s) \}. \tag{32}$$

But the second term of (32) is equal to $\frac{c_0}{\gamma} \exp [\ \gamma c_1/c_0 - 1\ ]$ for $s^* = 1/\gamma - c_1/c_0$. Hence, equation (30) follows, and by Theorem 2 we prove the corollary.

□

In particular, for Poisson input traffic with $c_0 = e^{-\lambda}$, $c_1 = \lambda e^{-\lambda}$ and $f_{\max} = 0.5$ (e.g. binary exponential back-off algorithm and linear back-off) equation (30) reduces to

$$\lambda \ln 2 - \exp [\lambda (ln 2 - 1) - 1] = 0 \tag{33}$$

and $\lambda_0 = 0.461$. For binary exponential back-off algorithm this result was also obtained by Sennot [1985a]. This can be compared with $\lambda_0 = 0.72$ for the same type of stability proved by Rosenkrantz [1984]. Note also that smaller the f_{max} is, smaller the λ_0 is. For example, for $f_l = 10^{-l}$ by equation (30) we find that $\lambda_0 = 0.381$. Finally, we can prove, using the same arguments as for finite-dimensional back-off algorithm, that $\lim_{t \to \infty} \sup EN_k^t \le \lambda f_k^{-1} < \infty$ (see Rosenbrantz [1984]).

A stronger result is available through the Proposition for some special choice of probabilities f_k. We prove that

COROLLARY 3. Let $\inf_{i \le i \le \infty} f_k = f^* > 0$. Then for $\lambda > 0$ the total backlog is unbounded, i.e.,

$$\lim_{t \to \infty} E \sum_{i=1}^{\infty} N_i^t = \infty. \tag{33}$$

Proof: We apply the Proposition and use the same line of proof as in Corollary 1. Let $V(\mathbf{n}) = \sum_{i=1}^{\infty} n_i$, $AV(\mathbf{n})$ is given by (23) with $B = \infty$ and define $H_{M_2} = \{\mathbf{n} : \sum_{i=1}^{\infty} n_i < M_2\}$ where $M_2 > \sup_{i \le i < \infty} \{\ln^{-1}(1 - f_i)^{-1}\} = \ln^{-1}(1 - f^*) < \infty$. Then one shows that (27) and (28) hold with $B = \infty$. Using the same arguments as in the proof of Corollary 1 we establish the result with the aid of the Proposition.

$\square$

Remarks.

(i) Kelly [1985] showed that any back-off algorithm with $f_0 = 1$ is not ergodic for $\lambda < 0.567$. (In his elegant and very simple proof he used only equilibrium arguments equating arrival and departure rates). In addition, he proved (see also Kelly and MacPhee [1986]) that for exponential back-off protocol the number of successful transmissions is infinite for

$\lambda < \log 2 = 0.693$, and finite for $\lambda \geq 0.693$.

(ii) *Conjecture.* Extension of Corollary 3 to the case $\inf_{1 \leq k \leq \infty} f_k = 0$ fails mainly because we cannot properly define the set H_{M_2} (M_2 is infinite in that case) such that condition (12) is satisfied. However, we know that properties (25) and (26) of the function (24) (which plays an essential role in our proofs) still hold for the set $E_2 = \{x \in E : \sum_{k=1}^{\infty} b_i\, x_i > 1\}$ where $b_i = \ln(1 - f_i)^{-1}$. Therefore, we conjecture that (33) holds also for $\inf_{1 \leq k < \infty} f_k = 0$.

(iii) In our analysis we assume an infinite model, however, this is not necessary. Indeed, consider a finite number of buffered users with total input rate λ. Let each packet in a buffer be transmitted with probability f_k, where k is the number of unsuccessful transmissions for that packet. Then, the above considerations hold for this case too. However, for the finite model a better approach is available. Instead of allowing transmission of two or more packets simultaneously from the same user (which leads to an obvious collision), we may assume that only packets from the head of a queue are sent, and all other packets from the same buffer are blocked until successful transmission of the first packet in the queue. Such a model with an exponential back-off algorithm was considered by Goodman et al [1985] and they prove that there exists a $\lambda_o > 0$ such that the system is ergodic for $\lambda < \lambda_o$. Intuitively it seems "obvious" since a "less stable" finite buffered model with the original ALOHA protocol is ergodic in that case (Szpankowski [1984]).

□

Decentralized dynamic control algorithm

As explained in Example 3, the system is described by a two dimensional Markov chain (N^t, S^t) where N^t is the backlog and S_t is a counter maintained by a user according to some recursive formula. The probability of transmitting a packet is the same for all blocked users and equal to $f_t = 1/S_t$, $S_t \geq 1$. The N-th component of the average drift is given by (see Kelly

[1985])

$$d(n,s) = E\{N^{t+1} - N^t \mid N^t = n \;\; S^t = s\} = \lambda - \frac{n}{s}\,(1 - \frac{1}{s})^{n-1}. \tag{34}$$

We prove that

COROLLARY 4. If $\lambda > e^{-1} + \varepsilon$, $\varepsilon > 0$, then for any recursive formula on S_t, $S_t \geq 1$, the average backlog is infinite, that is, $\lim_{t\to\infty} EN^t = \infty$.

Proof: We apply the Proposition for $V(n,s) = n$. Let us define for any number $M > 0$ an (infinite) set $H_M = \{n, s): \; n < M\}$. Then condition (12) is satisfied and for the corollary we must verify only condition (16). But $AV(n,s) = d(n,s)$ given by (34). Using the inequality $1 - x \leq e^{-x}$ we find that

$$\frac{n}{s}\,(1 - \frac{1}{s})^{n-1} \leq \frac{n}{s} \exp\,[\,(n-1)/s\,] = \bar{f}_n(s). \tag{35}$$

But, by simple algebra we obtain

$$\max \bar{f}_n(s) = e^{-1} + \frac{1}{n-1}\,e^{-1} \;\text{ for }\; s \geq 1. \tag{36}$$

For $\delta < \varepsilon$ and for $M > 1 + \frac{1}{\delta e}$, we bound the second term in (36) above by δ. Then for $\lambda > e^{-1} + \varepsilon$, we obtain, for $(n,s) \in C - H_M$ and $M > 1 + \frac{1}{\delta e}$, that $AV(n,s) \geq \lambda - e^{-1} - \delta \geq \varepsilon - \delta > 0$. Hence condition (16) is satisfied, and therefore $\lim_{t\to\infty} EV(N^t, S^t) = \lim_{t\to\infty} EN^t = \infty$.

□

Hajek [1982b] proved that for a special multiplicative recursion on S_t (see also Example 3 and Kelly [1985]), the Markov chain (N^t, S^t) is geometrically ergodic for $\lambda < e^{-1}$, and all moments of N^t in the steady state exist and are finite. We have shown above that for $\lambda > e^{-1} + \varepsilon$

the average backlog is infinite for any recursive formula on S^t.

APPENDIX A

Let us consider a function

$$g(\mathbf{x}) = \left[\sum_{i=1}^{B} a_i \, x_i \right] \exp \left[- \sum_{i=1}^{B} b_i \, x_i \right], \tag{A1}$$

for $\mathbf{x} = (x_1, x_2, \ldots, x_M) \in E = \{\mathbf{x} : x_i \geq 0, i = 1,2, \ldots, B\}$ and $a_i, b_i > 0, i = 1,2, \ldots, B$. We cover the space E by a *pencil of rays* defined as: $x_i = \alpha_i \, x_1, \; \alpha_i > 0, \; i = 2,3, \ldots, B$. Then, the function (A1) defined on a line $L_\alpha = \{x \in E : x_i = \alpha_i \, x_1, \; \alpha_i \geq 0, \; i = 1,2, \ldots, B\}$ $\alpha = (1, \alpha_2, \alpha_3, \ldots, \alpha_B)$, is a function of one variable x_1 and it possesses $B - 1$ parameters $\alpha_i \geq 0 \; i = 2, \ldots, B$. Let

$$g_\alpha(x_1) = x_1 \left(\sum_{i=1}^{B} \alpha_i \, a_i \right) \exp \left\{ - x_1 \left(\sum_{i=1}^{B} \alpha_j b_j \right) \right\}. \tag{A2}$$

This function for a given α has one maximum with respect to x_1 for

$$x_i{}^* = \left[\sum_{i=1}^{B} \alpha_i \, b_i \right]^{-1}, \tag{A3}$$

and

$$\max_{x_1} g_\alpha(x_1) = g_\alpha(x_1{}^*) = e^{-1} \frac{\sum_{i=1}^{B} \alpha_i \, a_i}{\sum_{i=1}^{B} \alpha_i \, b_i}. \tag{A4}$$

Finding now the maximum of (A4) over α we finally obtain

$$\max_{\mathbf{x} \in E} g(\mathbf{x}) = \max_{\alpha} g(x_1{}^*) = e^{-1} \max \left\{ \frac{a_1}{b_1}, \frac{a_2}{b_1}, \cdots, \frac{a_B}{b_B} \right\}, \tag{A5}$$

hence for $\mathbf{x} \in E$

$$g(\mathrm{x}) \le e^{-1} \max_{1 \le i \le B} \left\{ \frac{a_i}{b_i} \right\} \tag{A6}$$

Fix α and consider (A2) defined on the line L_α. The above analysis implies that $g_\alpha(x_1)$ increases for $x_1 \le x_1^*$ and then decreases to zero for $x > x_1^*$. But for a given α and x_1^* defined as in (A3) the corresponding values of $x_2^*, x_3^*, \ldots, x_B^*$ are equal to $x_i^* = \alpha_i x_1^*$, $i = 2, \ldots, B$. Multiplying each x_i^* by b_i and taking into account (A3), we find that the maxima of $g(\mathrm{x})$ defined on the lines L_α, $\alpha \ge 0$ lie on the following hyperplane

$$b_1 x_1 + b_2 x_2 + \cdots + b_B x_B = 1. \tag{A7}$$

Let $E_1 = \{\mathrm{x} \in E : \sum_{i=1}^{B} b_i x_1 \le 1\}$ and $E_2 = E - E_1$. In particular, we have shown that for $\mathrm{x} \in L_\alpha \cap E_2$ the function $g(\mathrm{x})$ decreases to zero as x_1 tends to infinity. This also implies that for a sufficiently large $M > \max_{1 \le i \le B} \{b_i^{-1}\}$ and given $\varepsilon > 0$ $g(\mathrm{x}) < \varepsilon$ for $x \in E - H_M$ where $H_M = \{\mathrm{x} \in E : x_1 + x_2 + \cdots + x_B < M\}$ (see condition (26)). Fig. 1A illustrates the above analysis for $B = 2$ (note that in this case we need only α_2).

Appendix B.

We prove that

$$\ln(1 - x) \le -\gamma \frac{x}{1 - x}, \tag{B1}$$

for $0 \le x < f_{\max}$, where $\gamma = -\frac{1 - f_{\max}}{f_{\max}}$. Let us define a function

$$f(x) = \ln(1 - x) + \frac{ax}{1 - x} \tag{B2}$$

for $0 \le x < 1$ and $0 \le a \le 1$ for $0 < x < f_{\max}$. Note that $f(x)$ reaches minimum value $f_{\min} = \ln a + 1 - a < 0$ for $x^* = 1 - a$. Since $f(0) = 0$ and $f(x)$ is continuous, hence $f(x) \le 0$ for all $0 < x < x_0$ where $f(x_0) = 0$. Now we choose a in (B2) such that $f(f_{\max}) = 0$.

This implies that $a = \gamma$, hence we prove (B1).

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