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A p-adic spectral triple

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To my parents, whose absence I will always feel.

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ABSTRACT

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We construct a spectral triple for the C*-algebra of continuous functions on the space of p-adic integers. On the technical level we utilize a weighted rooted tree obtained from a coarse grained approximation of the space combined with the forward derivative D on the tree. Our spectral triple satisfies the properties of a compact spectral metric space and the metric on the space of p-adic integers induced by the spectral triple is equivalent to the usual p-adic metric. Furthermore, we show that the spectrum of the operator D^*D is closely related to the roots of a certain q-hypergeometric function and discuss the analytic continuation of the zeta function associated with D^*D.

1. Introduction

1.1 Background

The content of this thesis is part of an ongoing project to understand the parallelisms between noncommutative Riemann surfaces and the noncommutative theory of algebraic number fields. The classical version of this analogy was done by Dedekind and Weber in their groundbreaking paper in 1882 "Theory of algebraic functions of one variable" [9], which revolutionized the subject of algebraic geometry by introducing methods of algebraic number theory into it.

If R is a compact Riemann surface the set of meromorphic functions defined on R forms a field, which in fact is a finite extension of $\mathbb{C}(z)$, the field of rational functions in the indeterminate z. Two Riemann surfaces are isomorphic as Riemann surfaces if and only if their corresponding fields of meromorphic functions are $\mathbb{C}-$isomorphic. Thus, compact Riemann surfaces are completely characterized by their fields of meromorphic functions. Dedekind and Weber explored the analogies between algebraic number theory and algebraic geometry, via the analogies between algebraic number fields and algebraic function fields, which allowed subsequent mathematicians like Arakelov, Weil and Iwasawa to further develop these similarities between the two areas of mathematics.

The noncommutative analogue of this story is still at a very early stage. The objective of our project is to study local fields, i.e. finite algebraic extensions of the field of p-adic numbers, from the point of view of noncommutative geometry. As a first step, in this thesis, we give the construction of a spectral triple for the C*-algebra of continuous functions on the space of p-adic integers. The celebrated Gelfand-Naimark Theorem states that commutative unital C*-algebras are precisely the algebras of continuous functions on compact topological spaces. Consequently, unital C*-algebras (not necessarily commutative) are called compact quantum (noncommutative) spaces.

We also analyze additional structures on C*-algebras corresponding to geometrical notions on topological spaces. The seed of the concept of a quantum metric space was first planted by A. Connes in his paper [8]. He observed that for a compact spin Riemannian manifold one can recover its smooth structure, the Riemannian metric and many other properties from the standard Dirac operator; thus Dirac operators carry the metric information in noncommutative geometry. This observation led him to the concept of a spectral triple $(A, \mathcal{H}, \mathcal{D})$ where A is a $*$-algebra represented on $\mathcal{H}$ by bounded operators and $\mathcal{D}$ is a self-adjoint operator on $\mathcal{H}$ satisfying certain properties (precise definition will be given later). Spectral triples with a faithful representation of A, called unbounded Fredholm modules, have been used initially by Connes [6], for the cyclic cohomology of the noncommutative space defined by A. Additionally, he showed that the seminorm $L_{\mathcal{D}}$ arising from the spectral triple given by $L_{\mathcal{D}}(a) = \|[\mathcal{D}, a]\|$ for $a \in A$ induces a metric on the state space of A via the formula

$$d_{\mathcal{D}}(\mu, \nu) = \sup_{a}\{|\mu(a) - \nu(a)| \ : \ L_{\mathcal{D}}(a) \leq 1\}. \tag{1.1}$$

Inspired by Connes' work, in a series of papers ([19], [20], [21]), M. Rieffel proposed the concept of a compact quantum metric space based on the following observations.

If (X, ρ) is an ordinary compact metric space and $\phi \in C(X)$, the space of continuous functions on X, let $L_\rho : C(X) \to [0, \infty]$ be the Lipschitz seminorm defined by,

$$L_\rho(\phi) = \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{\rho(x, y)}.$$

Then (X, ρ) can be recovered from the data $(C(X), L_\rho)$ via Gelfand-Naimark Theorem and the distance formula

$$\rho(x, y) = \sup_{\phi}\{|\phi(x) - \phi(y)| \ : \ L_\rho(\phi) \leq 1\},$$

thereby reformulating the notion of a metric in terms of the commutative C*-algebra $C(X)$ and a seminorm. He then observed that it is natural to try reformulating metric data for a noncommutative unital C*-algebra A, by using a seminorm on A which plays an analogous role to L_ρ. For the precise definition of a compact quantum metric space see [21]. There are many Lipschitz-like seminorms on C*-algebras, but of particular interest to us are those seminorms coming from spectral triples [6].

Starting with a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for a C*-algebra A, we follow a more recent paper [3], in which the authors propose the term compact spectral metric space $(A, L_{\mathcal{D}})$ (see Definition 3). From the work of M. Rieffel it follows that $(A, L_{\mathcal{D}})$ is a spectral metric space if and only if the Connes metric defined on the state space of A via formula (1.1) is well defined and the topology induced by $d_{\mathcal{D}}$ is equivalent to the weak*-topology. In the commutative case this extension of a metric from the compact space X to its set of probability measures, which is the space of states for $C(X)$, had been defined and studied before by Kantorovich and Rubinstein [15].

Introducing and studying noncommutative notions like spectral triples perhaps allows to view the Cantor set of p-adic integers as more than a mere metric space, namely as some sort of a differentiable space. Moreover there is a number theoretic angle to considering operators related to p-adic numbers, as they may lead to interesting spectral functions.

1.2 Statement of results

In this thesis we present a graph theoretic construction of a spectral triple for the C*-algebra of continuous functions on the space of p-adic integers. Several constructions of similar spectral triples on Cantor sets can be found in the literature, including the original proposal in Connes' book [6] (see also [5], [18], [4]). We start with a coarse grained approximation of the space of p-adic integers $\mathbb{Z}_p$, which is essentially a weighted rooted tree, called the p-adic tree, whose vertices are balls in $\mathbb{Z}_p$. We then introduce a "forward" derivative D on this tree which leads to a more elaborate

Dirac-type operator $\mathcal{D}$. We investigate the distributional properties of D and more importantly show that D has a bounded inverse in the Hilbert space H consisting of weighted ℓ^2 functions on the vertices of the p-adic tree. The key tool at our disposal is the p-adic Fourier transform which is currently not available for general Cantor sets discussed in [18] and [4].

Using H and D introduced above we now construct the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where

$$\mathcal{H} = H \oplus H, \;\; \mathcal{D} = \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}$$

and $\mathcal{A}$ is the dense $*$-subalgebra of $C(\mathbb{Z}_p)$ consisting of Lipschitz functions on $\mathbb{Z}_p$. Then we prove one of our main results in this thesis:

Theorem *The triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even spectral triple.*

We also study the metric on p-adic integers induced by the spectral seminorm $L_{\mathcal{D}}$ and show that this metric is equivalent to the usual p-adic metric. Moreover, we verify that:

Theorem *The pair $(C(\mathbb{Z}_p), L_{\mathcal{D}})$ is a compact spectral metric space.*

In the second half of the thesis, we compute the spectrum of the operator D^*D introduced above. The operator D is invertible with compact inverse implying that D^*D has compact resolvent. Therefore the spectrum of D^*D is discrete with only possible accumulation point at infinity. A reparametrization of the p-adic tree allows us to decompose the Hilbert space into invariant subspaces, and hence the operator D^*D decomposes into the direct sum

$$D^*D = \bigoplus_{g \in \mathcal{G}_p} p^{2m} D_0^* D_0$$

where D_0 is the operator on $\ell^2(\mathbb{Z}_{\geq 0})$ given by $D_0 f_n = p^n(f_n - f_{n+1})$ and $\mathcal{G}_p = \{g = \frac{r}{p^m} \mid 0 \leq r < p^m, p \nmid r\}$ is the Prüfer p-group.

Then we prove that:

Theorem *The spectrum of the operator $D_0^* D_0$ consists of simple eigenvalues $\{\lambda_n\}_{n \geq 0}$ which are the roots of the q-hypergeometric function $\lambda \mapsto {}_1\phi_1\left({0 \atop q}; q, \lambda\right)$, where $q = p^{-2}$.*

In fact,

$$ {}_1\phi_1\left({0 \atop q}; q, \lambda\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k \lambda^k q^{\frac{k(k-1)}{2}}}{(q;q)_k^2} $$

is known as the third Jackson q-Bessel function. Analytic bounds for the roots of this function and their asymptotic behavior has already been studied in the literature [1]. Therefore, we have a good understanding of the spectrum of D^*D. In particular, using the results of [1], we also present several results on the analytic structure and analytic continuation of the zeta function of D^*D.

Part of the motivation for studying the spectrum of D^*D is that it might be relevant to the development of p-adic quantum mechanics. The operator D^*D, a natural analog of the Laplacian, can be taken as an alternative starting point for the theory of p-adic Schrödinger operators, see [22].

2. p-adic Fourier Analysis

2.1 The ring of p-adic integers

Throughout the rest of this thesis, we work with a fixed prime number p. Given any $x \in \mathbb{Q}$ we can uniquely write $x = p^{\gamma(x)} \cdot \frac{m}{n}$ where $m, n \in \mathbb{Z}$, $n \neq 0$ are not divisible by p and $\gamma(x) \in \mathbb{Z}$. The number $\gamma(x)$ is called the valuation of x.

The function $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ defined by

$$|x|_p = p^{-\gamma(x)}, \quad |0|_p = 0,$$

is a norm, called the p-adic norm of x. We can easily show that the p-adic norm satisfies $|xy|_p = |x|_p|y|_p$ and the strong triangle inequality $|x+y|_p \leq \max(|x|_p, |y|_p)$, where the equality holds if $|x|_p \neq |y|_p$. Thus the p-adic norm is called a non-Archimedean norm.

From the definition it is clear that the range of the p-adic norm consists of the countable set of values $p^\gamma, \gamma \in \mathbb{Z}$ and zero. This norm induces the p-adic metric ρ_p on $\mathbb{Q}$ given by $\rho_p(x, y) = |x - y|_p$. It follows that the p-adic metric satisfies the strong triangle inequality

$$\rho_p(x, y) \leq \max\{\rho_p(x, z), \rho_p(z, y)\}$$

for any $x, y, z \in \mathbb{Q}$. Thus, it is called an ultra metric.

Definition 1 *The field of p-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic ultra metric.*

Every p-adic number $x \neq 0$ has the canonical representation

$$x = x_\gamma p^\gamma + x_{\gamma+1} p^{\gamma+1} + \ldots + x_0 + x_1 p + \ldots \tag{2.1}$$

where $\gamma = \gamma(x)$ is the valuation of x, and the integral coefficients satisfy $0 \leq x_j \leq p-1$, $x_\gamma \neq 0$. Since the p-adic norm of the N^{th} term of the series , $|x_N p^N|_p = p^{-N} \to 0$ as

$N \to \infty$ the series (2.1) converges to x with respect to the p-adic norm. With this representation, arithmetic in $\mathbb{Q}_p$ can be performed using a system of carries.

The ultra metric ρ_p gives rise to a rather surprising topology on $\mathbb{Q}_p$. For example, among many other properties, one can show that with respect to this topology

1. balls (and circles) in $\mathbb{Q}_p$ are clopen,

2. every point inside a ball is its center,

3. any two balls are either disjoint or one is contained in the other, and

4. every open set in $\mathbb{Q}_p$ is a union of at most a countable set of disjoint balls.

Definition 2 *The space $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \ : |x|_p \leq 1\}$ is called the ring of p-adic integers.*

Equivalently, $\mathbb{Z}_p$ is the completion of the set of integers $\mathbb{Z}$, with respect to the p-adic metric ρ_p. In fact, the canonical representation (2.1) of a p-adic integer consists only of nonnegative powers of p. Moreover, $\mathbb{Z}_p$ is compact and totally disconnected while $\mathbb{Q}_p$ is locally compact and totally disconnected with respect to the p-adic topology.

Using the representation (2.1) of a p-adic number we can define the fractional part $\{x\}_p$ of $x \in \mathbb{Q}_p$ as

$$\{x\}_p = \begin{cases} 0 & ; \text{if } \gamma(x) \geq 0 \text{ or } x = 0 \\ p^{\gamma}(x_{\gamma} + x_{\gamma+1}p + \ldots + x_{-1}p^{-\gamma-1}) & ; \text{if } \ \gamma(x) < 0. \end{cases}$$

This p-adic fractional part can be thought of as a p-adic analogue of the usual fractional part of a real number.

Proposition 2.1.1 *For any $x, y \in \mathbb{Q}_p$ the following are true.*

1. $\{x\}_p \in [0, 1)$

2. $x - \{x\}_p \in \mathbb{Z}_p$

3. $\{x\}_p = 0$ *if and only* $x \in \mathbb{Z}_p$

4. $\{x\}_p + \{y\}_p - \{x + y\}_p \in \mathbb{Z}$

2.2 Character groups of $\mathbb{Q}_p$ and $\mathbb{Z}_p$

Recall that for a locally compact abelian group G, its dual group $\widehat{G}$ is defined as the set of continuous group homomorphisms, called characters, $\chi : G \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$, equipped with pointwise multiplication.

For a given $a \in \mathbb{Q}_p$, consider the map $\chi_a : \mathbb{Q}_p \to \mathbb{C}$ defined by

$$\chi_a(x) = e^{2\pi i\{ax\}}.$$

The character group $\widehat{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ is described by the following theorem.

Theorem 2.2.1 *For all $a, b, x, y \in \mathbb{Q}_p$ and χ_a defined as above we have,*

1. $\chi_a(x + y) = \chi_a(x)\chi_a(y), \ |\chi_a(x)| = 1$
2. $\chi_a(x)\chi_b(x) = \chi_{a+b}(x)$
3. $\chi_a = \chi_b$ *if and only if* $a = b$
4. *If ϕ is a character of the group $(\mathbb{Q}_p, +)$ then there is $a \in \mathbb{Q}_p$ s.t. $\phi = \chi_a$.*
5. *If $\chi_a(x) = 1$ for every $a \in \mathbb{Q}_p$ then $x = 0$.*

Consequently, $\widehat{\mathbb{Q}}_p \cong \mathbb{Q}_p$.

Any character on $\mathbb{Q}_p$ is also a character on $\mathbb{Z}_p$. The following theorem describes the character group $\widehat{\mathbb{Z}}_p$ of $\mathbb{Z}_p$.

Theorem 2.2.2 *If $\chi_a : \mathbb{Z}_p \to \mathbb{C}$ is defined as above then for all $a, b \in \mathbb{Q}_p$ and $x, y \in \mathbb{Z}_p$ we have*

1. $\chi_a(x + y) = \chi_a(x)\chi_a(y), \ |\chi_a(x)| = 1$ *for all* $a \in \mathbb{Q}_p$
2. $\chi_a(x)\chi_b(x) = \chi_{a+b}(x)$ *for* $a, b \in \mathbb{Q}_p$
3. $\chi_a = \chi_b$ *if and only if* $a - b \in \mathbb{Z}_p$
4. *If ϕ is a character of the group $(\mathbb{Z}_p, +)$ then there is $a \in \mathbb{Q}_p$ s.t. $\phi = \chi_a$*

5. *If* $\chi_a(x) = 1$ *for every* $a \in \mathbb{Q}_p$ *then* $x = 0$.

Consequently, $\widehat{\mathbb{Z}}_p \cong \mathbb{Q}_p/\mathbb{Z}_p$.

For proofs of Theorems 2.2.1 and 2.2.2, see [22].

The discrete group of $(\mathbb{Z}_p, +)$ is called the Prüfer group and has many presentations. On one hand it can be considered as an inductive limit of the groups $\mathbb{Z}/p^n\mathbb{Z}$. On the other hand, it can be identified with a group of roots of unity, namely,

$$\left\{z \in \mathbb{Z} : \exists\, n \in \mathbb{Z}_{\geq 0} \text{ with } z^{p^n} = 1\right\}.$$

All such roots of unity can be written uniquely as $z = e^{\frac{2\pi i k}{p^n}}$ with $p \nmid k$. So, the corresponding character on $\mathbb{Z}_p$ is

$$\chi_{n,k}(x) = e^{2\pi i \{\frac{kx}{p^n}\}}. \tag{2.2}$$

Consequently we have the presentation,

$$\widehat{\mathbb{Z}}_p \cong \left\{e^{2\pi i\{\frac{kx}{p^n}\}} : n \in \{0, 1, 2, \ldots\}, p \nmid k \in \mathbb{Z}, x \in \mathbb{Z}_p\right\}. \tag{2.3}$$

2.3 The space of test functions and distributions

Definition 2.3.1 *A function* $\phi : \mathbb{Z}_p \to \mathbb{C}$ *is called locally constant if for every* $x \in \mathbb{Z}_p$ *there exists a neighborhood* $U_x \ni x$ *such that* ϕ *is constant on* U_x.

The functions $\chi_a(x)$ and the characteristic function of $\mathbb{Z}_p$ defined by

$$1_{\mathbb{Z}_p}(x) = \begin{cases} 1 & ;\ \text{if } |x|_p \leq 1 \\ 0 & ;\ \text{if } |x|_p > 1 \end{cases}$$

are locally constant functions. In fact, all locally constant functions are continuous and constant on every ball of sufficiently small radius. Let $n(\phi)$ be the smallest positive integer n such that ϕ is constant on every ball of radius p^{-n}.

Definition 2.3.2 *The space of locally constant functions on* $\mathbb{Z}_p$ *is called the space of test functions and is denoted by* $\mathcal{E}(\mathbb{Z}_p)$.

The convergence in $\mathcal{E}(\mathbb{Z}_p)$ is defined in the following way.

Definition 2.3.3 *If $\phi_k \in \mathcal{E}(\mathbb{Z}_p)$ is a sequence of test functions then $\phi_k \to 0$ as $k \to \infty$ if,*

1. *the functions ϕ_k are uniformly constant, i.e. the sequence of numbers $\{n(\phi_k)\}$ is bounded,*

2. *$\phi_k(x) \to 0$ as $k \to \infty$ for every $x \in \mathbb{Z}_p$.*

The space of linear functionals on $\mathcal{E}(\mathbb{Z}_p)$ is called the space of distributions (generalized functions), denoted by $\mathcal{E}^*(\mathbb{Z}_p)$, and is equipped with the weak*-topology (i.e. weak convergence of functionals).

Theorem 2.3.1 *Every distribution $T \in \mathcal{E}^*(\mathbb{Z}_p)$ is automatically continuous.*

For a proof of this theorem see [22].

2.4 The p-adic Fourier transform

Since $(\mathbb{Q}_p, +)$ is a locally compact additive abelian group there is a Haar measure on $\mathbb{Q}_p$ denoted by d_px which is normalized so that $\int_{\mathbb{Z}_p} d_px = 1$. Moreover, the Haar measure satisfies $d_p(xa) = |a|_p d_px$ for all $0 \neq a \in \mathbb{Q}_p$.

Definition 2.4.1 *The Fourier transform of a test function ϕ on $\mathbb{Z}_p$ is the function $\widehat{\phi}$ on the classes $\mathbb{Q}_p/\mathbb{Z}_p$ given by*

$$\widehat{\phi}([a]) = \int_{\mathbb{Z}_p} \phi(x)\overline{\chi_a(x)}\, d_px$$

where $\overline{\chi_a(x)}$ denotes the complex conjugation.

Proposition 2.4.1 *Suppose that $a \in \mathbb{Q}_p$ and $n \geq 0$. Then,*

$$\int_{|x|_p \leq p^{-n}} \chi_a(x) d_px = \begin{cases} p^{-n} & ;\ \text{if } \ |a|_p \leq p^n \\ 0 & ;\ \text{otherwise} \end{cases}$$

Proof Notice that

$$\int_{|x|_p \le p^{-n}} \chi_a(x) d_p x = p^{-n} \int_{|y|_p \le 1} \chi_a(p^n y) d_p y.$$

If $|a|_p \le p^n$ then $|ap^n y|_p \le |y|_p$ and hence $\chi_a(p^n y) = 1$ whenever $|y|_p \le 1$. Therefore

$$\int_{|x|_p \le p^{-n}} \chi_a(x) d_p x = p^{-n}.$$

If $|a|_p > p^n$ then $|p^n a|_p > 1$ and hence $\chi_a(p^n) \neq 1$. Moreover,

$$\begin{aligned}\int_{|x|_p \le p^{-n}} \chi_a(x) d_p x = \int_{|x-p^n|_p \le p^{-n}} \chi_a(x) d_p x &= \int_{|y|_p \le p^{-n}} \chi_a(y + p^n) d_p y \\ = \chi_a(p^n) \int_{|x|_p \le p^{-n}} \chi_a(x) d_p x.\end{aligned}$$

Therefore,

$$\int_{|x|_p \le p^{-n}} \chi_a(x) d_p x (1 - \chi_a(p^n)) = 0$$

from which it follows that $\int_{|x|_p \le p^{-n}} \chi_a(x) d_p x = 0$. ■

As a consequence of Proposition 2.4.1 only a finite number of Fourier coefficients of a locally constant function are nonzero. If we denote by $\mathcal{E}(\widehat{\mathbb{Z}}_p)$ the space of compactly supported functions on $\widehat{\mathbb{Z}}_p$ (i.e. space of functions which are zero almost everywhere), then the Fourier transform defines an isomorphism between $\mathcal{E}(\mathbb{Z}_p)$ and $\mathcal{E}(\widehat{\mathbb{Z}}_p)$ and the inverse Fourier transform is given by the formula:

$$\phi(x) = \sum_{[a] \in \widehat{\mathbb{Z}}_p} \widehat{\phi}([a]) \chi_a(x), \quad x \in \mathbb{Z}_p.$$

Definition 2.4.2 *The Fourier transform of a distribution $T \in \mathcal{E}^*(\mathbb{Z}_p)$ is the function $\widehat{T}$ on $\widehat{\mathbb{Z}}_p$ given by*

$$\widehat{T}([a]) = T\left(\overline{\chi_a(x)}\right).$$

The inverse Fourier transform of a distribution is given by:

$$T(\chi_a(x)) = \sum_{[a] \in \widehat{\mathbb{Z}}_p} \widehat{T}([a]) \chi_a(x).$$

Remark 1 *Since the test functions on $\widehat{\mathbb{Z}}_p$ are nonzero only at a finite number of points, the formal sum in the above definition makes distributional sense.*

If $\mathcal{E}^*(\widehat{\mathbb{Z}}_p)$ denotes the space of all functions on $\widehat{\mathbb{Z}}_p$ then the Fourier transform is an isomorphism between $\mathcal{E}^*(\mathbb{Z}_p)$ and $\mathcal{E}^*(\widehat{\mathbb{Z}}_p)$. The distributional Fourier transform $T \mapsto \widehat{T}$ defines a Hilbert space isomorphism

$$L^2(\mathbb{Z}_p, d_p x) \cong \ell^2(\widehat{\mathbb{Z}}_p).$$

3. The p-adic Tree

For our analysis we use a coarse grained approximation of the space $\mathbb{Z}_p$, called the p-adic tree, which in fact is a weighted rooted Cantorian tree associated via Michon's correspondence. In this section we explain some terminology and the construction of the p-adic tree.

3.1 Michon's correspondence between ultrametrics and Cantorian trees

We start with the following definition of a Cantor set C and a regular metric d on it.

Definition 3.1.1 *A cantor set C is a nonempty compact topological space which is perfect, metrizable and totally disconnected. A metric on a Cantor set C is called a regular metric if C is a Cantor set with respect to the topology induced by d.*

For the remainder of this section, we will assume that (C, d) is a Cantor set equipped with a regular ultrametric d.

Definition 3.1.2 *For a given $\epsilon > 0$ and $x, y \in C$, a sequence of points $x_0, x_1, \cdots, x_n$ in C with $x_0 = x, x_n = y$ and $d(x_j, x_{j+1}) < \epsilon$ for all $j \geq 0$ is called an $\epsilon - chain$.*

Let $\sim_\epsilon$ be the relation on C defined by $x \sim_\epsilon y$ if and only if there is an ϵ-chain between x and y. It is easily seen that $\sim_\epsilon$ is an equivalence relation and we denote the equivalence classes by $[x]_\epsilon$.

Definition 3.1.3 *The separation of x and y is $\delta(x, y) = \inf\{\epsilon > 0 \mid x \sim_\epsilon y\}$.*

Proposition 3.1.1 *If (C, d) is a Cantor set equipped with a regular ultrametric then $\delta = d$.*

For a proof of this proposition see [4].

Definition 3.1.4 *Let $\{R_\epsilon \mid \epsilon > 0\}$ be an increasing family of equivalence relations on C satisfying,*

1. *$\forall \epsilon > 0$, R_ϵ is open in $C \times C$ and $\exists \epsilon > 0$ for which $R_\epsilon = C \times C$*

2. *$\bigcup_{\epsilon' < \epsilon} R_{\epsilon'} = R_\epsilon$*

3. *the diagonal Δ of $C \times C$ is given by, $\Delta = \bigcap_{\epsilon > 0} R_\epsilon$.*

Then $\{R_\epsilon \mid \epsilon > 0\}$ is called a profinite structure on C.

Given a regular ultrametric d on a Cantor set C one can show that d gives rise to a unique profinite structure via the equivalence relation $\sim_\epsilon$. Conversely, given a profinite structure $\{R_\epsilon \mid \epsilon > 0\}$, it defines a unique regular ultrametric d via the formula

$$d(x, y) = \inf\{\epsilon \mid x \sim_\epsilon y\}.$$

Thus, we have the following proposition:

Proposition 3.1.2 *There is a one-to-one correspondence between regular ultrametrics on a Cantor set C and profinite structures on C.*

Proof For a proof of this proposition see [4]. ■

Before we explain how to associate a tree to a Cantor set we introduce some necessary terminology.

Recall that a graph $G = \{V, E\}$ consists of the set of vertices V and the set of edges E which are unordered pairs of elements of V. If $E \ni e = (v, w)$ for some $v, w \in V$, then we say that v and w are connected by an edge. A path in a graph is a finite or infinite sequence of edges which connect a sequence of vertices which are all distinct from one another. A tree is an undirected graph in which any two vertices are connected by exactly one path. A rooted tree is a tree with a distinguished vertex called the root. Thus, there is a unique path connecting any vertex to the root.

The set of vertices of a rooted tree can be totally ordered with respect to which the root is the minimal element. We will say that $u \leq v$ if and only if the path connecting the root and u passes through v. If $u \leq v$ then u is called a descendant of v and v is called an ancestor of v. In the special case that u and v are adjacent, we say that v is the father of u and u is a child of v. A vertex which has no children is called a dangling vertex. If all the vertices of a given tree have only a finite number of children, then the tree is called locally finite.

We can view the tree as a topological space with the discrete topology. In our discussion we are mainly dealing with trees with an infinite number of vertices, hence with respect to this topology the tree is not compact. For the rest of the discussion T will always denote an infinite rooted tree.

Definition 3.1.5 *In a rooted tree T, the set of infinite paths starting at the root is the called the boundary of the tree, denoted ∂T.*

The boundary of a rooted tree serves as the compactification of T. If a path connecting the vertices v_1 and v_n contains the set of vertices $v_1, v_2, \cdots v_n$ where $v_i \leq v_j$ for $i \geq j$ then we denote the path by $v_1 v_2 \cdots v_n$. Hence ∂T consists of infinite sequences of the form $v_1 v_2 v_3 \cdots$.

Definition 3.1.6 *If v is a vertex of the rooted tree T, then $[v] \subset \partial T$ consists of all infinite paths that start at the root and pass through v.*

In fact, the collection of sets $\{[v] \, : \, v \in V\}$ form a basis of open sets for a topology on the boundary ∂T.

Definition 3.1.7 *A rooted, locally finite tree with no dangling vertices and the property that each vertex has a descendant with at least 2 children is called a Cantorian tree. Equivalently, a tree T is Cantorian if and only if ∂T is a Cantor set.*

For the sake of the reader who wishes to follow the proof of Theorem 3.1.1 (see below) we discuss some boundary preserving operations on a tree.

Edge contraction is the operation in which if γ is a path connecting v to u where $u \leq v$ then the edges in γ are removed and the vertices are collapsed to form a single edge between v and u provided each vertex on the path, other than v, u, have only one child. *Edge splitting* is the reverse process of edge contraction. i.e., if e is an edge connecting v to u then we can replace e by a finite path connecting v to u such that all the vertices of the path have only one child. If a vertex u has more than two children $u_1, u_2, \cdots u_n$, then one can introduce a new child v of u such that $u_2, u_3, \cdots u_n$ are children of v. This operation on the tree is called *vertex splitting*. One can then verify that theses operations yield homeomorphisms between the boundary of the original tree and the boundary of the operated tree.

Definition 3.1.8 *If every vertex of a tree T having only one child is operated using edge contraction so that each vertex has at least two children, then the resulting tree is called a reduced tree.*

Definition 3.1.9 *A weighted rooted tree is an infinite rooted tree T with no dangling vertex, and a function $\omega : V \to \mathbb{R}_{\geq 0}$ satisfying:*

1. *If $u \leq v$ with $u \neq v$ then $\omega(u) < \omega(v)$*

2. *If $u_1 u_2 u_3 \cdots$ is an infinite path then $\lim_{n\to\infty} \omega(u_n) = 0$.*

A function satisfying the above two properties is called a weight function.

Now we can give the following one-to-one correspondence:

Theorem 3.1.1 (Michon's Correspondence) *[4], [17]*
There is a one-to-one correspondence between regular ultrametrics on a Cantor set C and reduced, weighted, rooted Cantorian trees such that the boundary of the Cantorian tree corresponding to an ultrametric d is isometric to the metric space (C, d). Moreover, the weight function $\omega : V \to \mathbb{R}_{\geq 0}$ satisfies $\omega(v) = \mathrm{diam}_d([v])$ where for $v \in V$, $\mathrm{diam}_d([v])$ denotes the diameter of the set $[v]$ with respect to d.

Proof For a proof of this theorem see [4], [17]. ■

3.2 The p-adic tree

Using Theorem (3.1.1) we can now associate a weighted rooted tree (V, E) to the metric space $(\mathbb{Z}_p, \rho_p)$. The tree is constructed in the following way.

1. The root is the set $\mathbb{Z}_p$, the unique ball of radius one.

2. Vertices of the tree are balls in $(\mathbb{Z}_p, \rho_p)$ of radius p^{-n} for $n \geq 1$. In fact, the set of vertices V can be written as $V = \bigcup_{n=0}^{\infty} V_n$ where $V_n = \{\text{balls of diameter } p^{-n}\}$.

3. The set of edges $E = \bigcup_{n=1}^{\infty} E_n$ where if $e \in E_n$ then $e = (v, v')$ with $v \in V_n$ and $v' \in V_{n+1}$, and $v' \subset v$. In this case we say that there is an edge (undirected) between vertices v and v'.

4. The weight function $\omega : V \to \mathbb{R}^+$ is given by $\omega(v) = p^{-n}$ for $v \in V_n$.

This tree will be called the p-adic tree. Since $\mathbb{Z}_p$ is a compact set, for each n the number of balls with diameter p^{-n}, i.e. the cardinality of V_n, and the degree of each vertex $v \in V$ are finite.

For our analysis, it is important to parametrize the vertices of the p-adic tree in a convenient way. To this end, we make the following observation which asserts that we can find a unique integer, which serves as the center, inside every ball of radius p^{-n}.

Proposition 3.2.1 *Every ball of radius p^{-n} contains exactly one integer k such that $0 \leq k < p^n$.*

Proof Given a ball B_n of radius p^{-n}, let $x = \sum_{i=0}^{\infty} x_i p^i$ be any element in it. Take $k = \sum_{i=0}^{n-1} x_i p^i$. Then clearly $k < p^n$ and moreover $k \in B_n$ because,

$$|x - k|_p = \left| \sum_{i=n}^{\infty} x_i p^i \right|_p \leq p^{-n}.$$

To prove uniqueness, suppose $0 \leq k_1, k_2 < p^n$ both belong to the ball B_n. Thus, $|k_1 - k_2|_p \leq p^{-n}$, which implies $p^n | (k_1 - k_2)$ and this is impossible due to the inequality $0 \leq k_1, k_2 < p^n$ unless $k_1 = k_2$. ∎

The above proposition implies that there is a one-to-one correspondence between the set of balls of radius p^{-n}, i.e. elements of V_n, and the set of nonnegative integers less than p^n. Thus we can parametrize the set of vertices V as follows:

$$V = \{(n, k)|\ n = 0, 1, 2, \dots,\ 0 \le k < p^n\}\,. \tag{3.1}$$

Then the ball $v' = (n+1, k')$ is contained in the ball $v = (n, k)$ (equivalently there is an edge between vertices v and v') if and only if the difference $k' - k$ is divisible by p^n . Therefore, given any vertex (n, k) there are exactly p edges connecting (n, k) to $(n + 1, k + ip^n)$ for $i = 0, 1, \dots, p - 1$. That is, the degree of each vertex is equal to $p + 1$. Furthermore, given an edge $e = (v, v')$ where $v \in V_n$ and $v' \in V_{n+1}$ then we see that the vertex v' uniquely determines e. Hence by assigning the coordinates of v' to e, the set of edges E can also be parametrized . Consequently, there is a natural one-to-one correspondence between the set of vertices minus the root, and the set of edges.

Notice that if $v = (n, k) \in V_n$ then the map $(n, k) \mapsto k(\mod p^n)$ is a bijection between the sets V_n and $\mathbb{Z}/p^n\mathbb{Z}$, hence $V_n \cong \mathbb{Z}/p^n\mathbb{Z}$. Thus V_n can be given the structure of a finite additive group using addition modulo p^n.

4. Noncommutative spaces

In this chapter, we present some of the basic definitions and notions in noncommutative geometry which we will encounter in the subsequent chapters of this thesis.

4.1 C*-algebras and the Gelfand-Naimark Theorem

We start with the definition of a C*-algebra. In this thesis, we only consider unital C*-algebras.

Definition 4.1.1 *A C*-algebra $\mathcal{A}$ is a Banach algebra over the field of complex numbers with a map $* : \mathcal{A} \to \mathcal{A}$ $(a \mapsto a^*)$ satisfying*

1. $(a^*)^* = a$
2. $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$
3. $(ab)^* = b^* a^*$

and the C condition*

$$\|aa^*\| = \|a\|^2,$$

for all $a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$. A map $* : \mathcal{A} \to \mathcal{A}$ satisfying properties $(1) - (3)$ is called an involution.

If a nonzero C* algebra $\mathcal{A}$ has a unit 1, then necessarily $1^* = 1$ and $\|1\| = 1$. Below we will give three canonical examples of C*-algebras.

Example 1. If X is a compact space, $C(X)$ the space of all complex-valued continuous functions on X with the usual norm, $\|f\| = \sup_{x \in X} |f(x)|$, is a commutative unital C*-algebra. Multiplication in $C(X)$ is defined pointwise : $(f \cdot g)(x) = f(x) \cdot g(x)$

and the involution is the complex conjugation $f^*(x) = \overline{f(x)}$. The identity element is the function $i(x) = 1$ for all $x \in X$.

Example 2. If X is a locally compact space, $C_0(X)$ the space of all complex-valued continuous functions on X vanishing at infinity, equipped with the same operations and the norm as in Example 1, is a non-unital commutative C*-algebra.

Example 3. $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on a complex Hilbert space $\mathcal{H}$ where for any $A \in \mathcal{B}(\mathcal{H})$, A^* is given by the adjoint of A and the norm is the usual operator norm, is a unital C*-algebra.

Definition 4.1.2 *A representation π of a C*-algebra $\mathcal{A}$ is a map $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ satisfying*

1. $\pi(\lambda a + \mu b) = \lambda\pi(a) + \mu\pi(b)$
2. $\pi(ab) = \pi(a)\pi(b)$
3. $\pi(a^*) = \pi(a)^*$

(i.e. π is a $*$-homomorphism). A representation is said to be faithful if it is injective. If $\mathcal{A}$ has a unit then π is said to be non degenerate if $\pi(1) = id_{\mathcal{H}}$.

One of the most profound theorems in abstract C*-algebra theory was discovered by Gelfand and Naimark in their foundational 1943 paper.

Theorem 4.1.1 (Commutative Gelfand-Naimark Theorem)
Any commutative unital C-algebra is isometrically $*$-isomorphic to the C*-algebra of all continuous functions on a compact topological space.*

Thus, commutative C*-algebras were completely characterized as algebras of continuous functions on compact spaces and homomorphisms of commutative C*-algebras are related to continuous maps on the underlying compact spaces. Hence the category of unital commutative C*-algebras is dual to the category of compact topological spaces. This duality provides the foundation for the idea that the category of noncommutative C*-algebras may be regarded as the dual of the (otherwise undefined) category of noncommutative spaces.

In fact, any C*-algebra can be realized as a closed $*$-subalgebra of the algebra of bounded operators on a Hilbert space, a result which is due to Gelfand and Naimark:

Theorem 4.1.2 (Gelfand-Naimark Theorem) *Every unital C*-algebra is isometrically $*$-isomorphic to a C*-algebra of bounded operators on a Hilbert space.*

4.2 Spectral triples

Spectral triples provide a notion of Dirac operator in noncommutative geometry and a Riemannian type distance function for noncommutative spaces. As noted in the introduction, a motivating observation that led to the definition of a spectral triple is the following [12]: if M is a compact Riemannian spin manifold then its standard Dirac operator D acts on the Hilbert space $L^2(M,S)$ of L^2 spinors on M, as an unbounded selfadjoint operator. By letting $C^\infty(M)$ act on $L^2(M,S)$ by multiplication operators, one can show that for any $f \in C^\infty(M)$ the commutator $[D,f]$ extends to a bounded operator in $L^2(M,S)$. Moreover, the geodesic distance d on M can be recovered from the Connes distance formula

$$d(a,b) = \sup_f \{|f(a) - f(b)| \mid \|[D,f]\| \leq 1\},$$

for any $a, b \in M$. The triple $(C^\infty(M), L^2(M,S), D)$ is an example of a spectral triple.

In this thesis, we will use the following definition of a spectral triple:

Definition 4.2.1 *A spectral triple for a C*-algebra A is a triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ where $\mathcal{H}$ is a Hilbert space on which A is represented by bounded operators (i.e. there exists a $*$-homomorphism $\Pi : A \to \mathcal{B}(\mathcal{H})$), $\mathcal{A}$ is a dense $*-$subalgebra of A, and $\mathcal{D}$ is a self-adjoint operator (typically unbounded) on $\mathcal{H}$ satisfying:*

1. *for every $a \in \mathcal{A}$ the commutator $[\mathcal{D},\Pi(a)]$ is bounded,*

2. $(1+\mathcal{D}^2)^{-\frac{1}{2}}$ *is a compact operator.*

Additionally, we say that a spectral triple is even if there is a $\mathbb{Z}/2\mathbb{Z}$ grading on $\mathcal{H}$ with respect to which the representation Π is even, while the operator $\mathcal{D}$ is odd with respect to the grading.

The following is a prototypical example of a spectral triple.

Example. Let $A = C(S^1)$, the space of continuous functions on the unit circle, $\mathcal{H} = L^2(S^1)$ and $D = \frac{1}{i}\frac{d}{d\theta}$. There is a natural representation $\pi : A \to \mathcal{B}(\mathcal{H})$ given by $f \mapsto M_f$ where M_f is the multiplication operator. If we let $\mathcal{A}$ be the space of functions $f \in A$ whose first derivative is bounded then $\mathcal{A}$ is a dense subalgebra of A and the triple $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple.

In a more recent paper [3], the authors propose the following definition of a compact spectral metric space, which is the noncommutative analog of a compact complete metric space.

Definition 3 *Starting with a spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *for a* C^**-algebra* A *we define* $L_{\mathcal{D}}(a) := ||\ [\mathcal{D}, \Pi(a)]\ ||$. *If the conditions* $(1)-(3)$ *below are satisfied, then the pair* $(A, L_{\mathcal{D}})$ *is called a compact spectral metric space:*

1. *The representation of* A *in* $\mathcal{H}$ *is nondegenerate, i.e.* $A\mathcal{H} = \mathcal{H}$.

2. *The commutant* $A'_{\mathcal{D}} = \{a \in A : [\mathcal{D}, \Pi(a)] = 0\}$ *is trivial, i.e.* $A'_{\mathcal{D}} = \mathbb{C}I$.

3. *The image of the Lipschitz ball* $B_{\mathcal{D}} = \{a \in A : L_{\mathcal{D}}(a) \leq 1\}$ *is precompact in* $A/A'_{\mathcal{D}}$.

As mentioned in the introduction, it follows from the work of Rieffel that $(A, L_{\mathcal{D}})$ is a spectral metric space if and only if the Connes metric defined on the state space of A by formula (1.1) is well defined and the topology induced by this metric is equivalent to the weak*-topology.

5. The spectral triple

5.1 A forward derivative D on the tree.

An important component of the spectral triple is the Dirac operator acting on a Hilbert space associated with the space $\mathbb{Z}_p$. To this end, since there is no notion of differentiability for complex valued p-adic functions, we define a Dirac-type operator D which amounts to a forward derivative on the p-adic tree. In this section we give its definition and prove some properties of D which are essential for the construction of the self-adjoint operator $\mathcal{D}$ of the spectral triple.

First notice that due to the decomposition $V = \bigcup_{n=0}^{\infty} V_n$, a complex valued function f on the set V of vertices of the p-adic tree can be identified with a sequence of complex valued functions $\{f_n\}_{n\geq 0}$, where $f_n : V_n \to \mathbb{C}$. As mentioned in Section 3.2, the set V_n has the structure of the finite additive group $\mathbb{Z}/p^n\mathbb{Z}$. Hence, if f is a complex valued function on V, i.e., $f \in \mathcal{E}^*(V)$, then then the Fourier transform of f is the usual discrete Fourier transform on each V_n. This is given by the formula:

$$\widehat{f_n}(l) = \frac{1}{p^n} \sum_{k=0}^{p^n-1} f_n(k) e^{-2\pi i \frac{kl}{p^n}}, \ 0 \leq l < p^n. \tag{5.1}$$

To find a formula for the inverse Fourier transform, we first notice the following orthogonality condition for the characters.

Proposition 5.1.1 *For $k \in \mathbb{Z}$,*

$$\sum_{0\leq s<p^j} e^{\frac{-2\pi i k s}{p^j}} = \begin{cases} 0 & ;\ \textit{if}\ \ p^j \nmid k \\ p^j & ;\ \textit{if}\ \ p^j \mid k. \end{cases}$$

Proof If $p^j \nmid k$ then

$$\sum_{0\leq s<p^j} e^{\frac{-2\pi i k s}{p^j}} = \sum_{0\leq s<p^j} \left(e^{\frac{-2\pi i k}{p^j}}\right)^s = \frac{1-e^{-2\pi i k}}{1-e^{\frac{-2\pi i k}{p^j}}} = 0.$$

If $p^j \mid k$, let $k = p^j k'$. Then

$$\sum_{0 \le s < p^j} e^{\frac{-2\pi i k s}{p^j}} = \sum_{0 \le s < p^j} \left(e^{-2\pi i k'}\right)^s = p^j.$$

■

Thus we have the following Fourier inversion formula:

$$f_n(k) = \sum_{0 \le l < p^n} \widehat{f}_n(l) e^{\frac{2\pi i k l}{p^n}}. \tag{5.2}$$

Remark 2 *Since the group $\mathbb{Z}/p^n\mathbb{Z}$ is self-dual so is the set V_n and hence the p-adic tree is self-dual. That is, $V = \widehat{V}$ where $\widehat{V}$ is the "dual tree". Thus the Fourier transform is an isomorphism between $\mathcal{E}^*(V)$ and $\mathcal{E}^*(\widehat{V})$.*

Given two vertices $v, v' \in V$, we will write $v' \sqsubset v$ if $v' \subset v$ and v, v' are connected by an edge. Now define the forward derivative of a function $f \in \mathcal{E}^*(V)$ as,

$$Df(v) = \frac{1}{\omega(v)} \left(f(v) - \frac{1}{(\deg\, v - 1)} \sum_{\substack{v' \in V \\ v' \sqsubset\, v}} f(v') \right).$$

We are interested in the action of this operator on the Hilbert space H consisting of weighted ℓ^2 functions:

$$H = \ell^2(V, \omega) = \left\{ f : V \to \mathbb{C} \ : \ \sum_{v \in V} |f(v)|^2 \omega(v) < \infty \right\}.$$

The Hilbert space H has a natural decomposition, $H = \bigoplus_{n=0}^{\infty} \ell^2(V_n, p^{-n})$, induced by the decomposition of V. If we denote $\widehat{H} = \ell^2(\widehat{V})$, then the map $f \mapsto \widehat{f}$ induces a Hilbert space isomorphism $H \cong \widehat{H}$, between the weighted and the unweighted Hilbert spaces, via Parseval's identity,

$$\sum_{0 \le k < p^n} |f_n(k)|^2\, p^{-n} = \sum_{0 \le l < p^n} |\widehat{f}_n(l)|^2. \tag{5.3}$$

Formula (5.1) for the operator D can be rewritten as follows, using the labeling of formula (3.1):

$$Df_n(l) = p^n \left(f_n(l) - \frac{1}{p} \sum_{0 \le j < p} f_{n+1}(l + jp^n) \right). \tag{5.4}$$

Using the Fourier transfrom of f and Proposition (5.1.1) we obtain,

$$\begin{aligned} Df_n(l) &= p^n \left(\sum_{0\le k<p^n} \widehat{f}_n(k) e^{\frac{2\pi i k l}{p^n}} - \frac{1}{p} \sum_{0\le j<p} \sum_{0\le k<p^{n+1}} \widehat{f}_{n+1}(k) e^{\frac{2\pi i k(l+jp^n)}{p^{n+1}}} \right) \\ &= p^n \sum_{0\le k<p^n} \left(\widehat{f}_n(k) - \widehat{f}_{n+1}(pk) \right) e^{\frac{2\pi i k l}{p^n}}. \end{aligned} \tag{5.5}$$

Thus, in Fourier transform, the operator D becomes $\widehat{D}$ given by

$$\widehat{D}\widehat{f}_n(k) = p^n \left(\widehat{f}_n(k) - \widehat{f}_{n+1}(pk) \right). \tag{5.6}$$

The Fourier transform on the p-adic tree establishes a unitary equivalence between the operators D and $\widehat{D}$.

From here on we will consider the operators D and $\widehat{D}$ on their maximal Hilbert space domains,

$$\begin{aligned} &\mathcal{D}_{\max}(D) = \{ f \in H \mid Df \in H \} \quad \text{and} \\ &\mathcal{D}_{\max}(\widehat{D}) = \{ \widehat{f} \in \widehat{H} \mid \widehat{D}\widehat{f} \in \widehat{H} \}. \end{aligned}$$

Next we prove the following important proposition about the invertibility of D.

Proposition 5.1.2 *The kernel of the operator D defined on its maximal domain $\mathcal{D}_{\max}(D)$ in H is trivial.*

Proof Formula (5.5) shows that,

$$\mathrm{K}er D = \{ f = (f_n) \mid \widehat{f}_n(k) = \widehat{f}_{n+1}(pk) \}. \tag{5.7}$$

Suppose $f \in \mathcal{D}_{\max}(D)$ such that $Df = 0$. Then the Fourier coefficients of f satisfy $\widehat{f}_n(k) = \widehat{f}_{n+1}(pk)$. If we assume that $\widehat{f}_{n_0}(k_0) \neq 0$ for some n_0, k_0 then estimating the ℓ^2 norm of f we see that,

$$\begin{aligned} \|f\|^2 &= \sum_{n\ge 0} \sum_{0\le k<p^n} |\widehat{f}_n(k)|^2 \ge \sum_{n\ge n_0} \sum_{0\le k<p^n} |\widehat{f}_n(k)|^2 \\ &\ge \sum_{i\ge 0} |\widehat{f}_{n_0+i}(p^i k_0)|^2 = \sum_{i\ge 0} |\widehat{f}_{n_0}(k_0)|^2 = \infty \end{aligned}$$

since for $f \in \mathrm{Ker}D$, $\widehat{f}_{n_0}(k_0) = \widehat{f}_{n_0+i}(p^i k_0)$ for each i. This is a contradiction to the fact that $f \in \ell^2(V, \omega)$. Thus, all Fourier coefficients of f are zero and hence $f \equiv 0$. ■

Theorem 5.1.1 *The operator D defined on its maximal domain in H is invertible and has a bounded inverse.*

Proof Given $g \in H$, we can solve $Df_n(l) = g_n(l)$ for $f \in \mathcal{D}_{\max}(D)$ using Fourier transforms of f and g. Using formula (5.6) we obtain,

$$\widehat{g}_n(k) = \widehat{D}\widehat{f}_n(k) = p^n\Big(\widehat{f}_n(k) - \widehat{f}_{n+1}(pk)\Big). \tag{5.8}$$

One can easily check that the unique solution of the above equation in H is given by,

$$\widehat{f}_n(k) = \sum_{i=n}^{\infty} \frac{\widehat{g}_i(lp^{\alpha-n+i})}{p^i} =: \widehat{D}^{-1}\widehat{g}_n(k), \tag{5.9}$$

where $k = lp^\alpha$ with $p \nmid l$. The limits in the above sum are determined by the requirement $\widehat{f}_n(k) \to 0$ as $n \to \infty$. Estimating the pointwise norm of $\widehat{D}^{-1}$ using the Cauchy-Schwartz inequality we see that:

$$|\widehat{D}^{-1}\widehat{g}_n(k)| \le \left(\sum_{i=n}^{\infty} \widehat{g}_i(lp^{\alpha-n+i})^2\right)^{1/2} \cdot \left(\sum_{i=n}^{\infty} \frac{1}{p^{2i}}\right)^{1/2} = \frac{p^{-n}}{\sqrt{1-p^{-2}}} \cdot \|\widehat{g}\| \tag{5.10}$$

which shows that the above formula for $\widehat{f}_n(k)$ is well defined.

Thus, we now have the formula,

$$D^{-1}g_n(l) = f_n(l) = \sum_{0\le k<p^n} \widehat{f}_n(k)e^{\frac{2\pi ikl}{p^n}} = \sum_{0\le k<p^n} \sum_{i=n}^{\infty} \frac{\widehat{g}_i(lp^{\alpha-n+i})}{p^i}\, e^{\frac{2\pi ikl}{p^n}}.$$

Moreover, using formulas (5.9) and (5.6) with $k = lp^\alpha$ we see that,

$$\widehat{D}^{-1}\widehat{D}\widehat{f}_n(k) = \sum_{i=n}^{\infty} \frac{\widehat{D}\widehat{f}_i(lp^{\alpha-n+i})}{p^i} = \sum_{i=n}^{\infty} \widehat{f}_i(lp^{\alpha-n+i}) - \sum_{i=n}^{\infty} \widehat{f}_{i+1}(lp^{\alpha-n+i+1}) = \widehat{f}_n(k)$$

and

$$\begin{aligned}\widehat{D}\widehat{D}^{-1}\widehat{g}_n(k) &= p^n\big(\widehat{D}^{-1}\widehat{g}_n(k) - \widehat{D}^{-1}\widehat{g}_{n+1}(pk)\big)\\ &= p^n\left(\sum_{i=n}^{\infty} \frac{\widehat{g}_i(lp^{\alpha-n+i})}{p^i} - \sum_{i=n+1}^{\infty} \frac{\widehat{g}_{i+1}(lp^{\alpha-n+i})}{p^i}\right) = \widehat{g}_n(k).\end{aligned}$$

Hence $D^{-1}Df = f$ and $DD^{-1}g = g$.

Using formula (5.10) we can estimate the norm of D^{-1} as follows:

$$\begin{aligned}\|D^{-1}g\|^2 &= \sum_{n=0}^{\infty}\sum_{0\le k<p^n} |\widehat{D}^{-1}\widehat{g}_n(k)|^2 \le \sum_{n=0}^{\infty}\sum_{0\le k<p^n}\left(\frac{p^{-2n}}{1-p^{-2}}\right)\|\widehat{g}\|^2 \\ &= \frac{1}{(1-p^{-2})(1-p^{-1})}\|\widehat{g}\|^2.\end{aligned}$$

Hence D^{-1} is bounded as well.

■

Next we discuss the unique continuation property of the operator D on the space $\mathcal{E}^*(V)$. First, observe that if $\phi \in \mathcal{E}(\mathbb{Z}_p)$ then we can find a positive integer n_ϕ and a function $\tilde{\phi}$ such that,

1. $\tilde{\phi} \in \ell^2(V_n, \omega)$ for all $n \ge n_\phi$
2. $\tilde{\phi}(v) = \phi(x)$ for every $x \in v$.

Moreover, we have the formula,

$$\int \phi(x)\, d_p x = \sum_{v\in V_n} \tilde{\phi}(v)\, p^{-n}, \quad \text{for every } n \ge n_\phi.$$

Now we can define the concept of a limit of a function $f \in \mathcal{E}^*(V)$ at the boundary of the tree as follows.

Definition 5.1.1 *Let $f = \{f_n\}_{n=0}^{\infty} \in \mathcal{E}^*(V)$ be a function on V. Then f is said to converge weakly to a distribution $T \in \mathcal{E}^*(\mathbb{Z}_p)$ on the boundary of the tree $\{V, E\}$, if for every $\phi \in \mathcal{E}(\mathbb{Z}_p)$,*

$$\lim_{n\to\infty} \sum_{v\in V_n} f_n(v)\tilde{\phi}(v)\, p^{-n} := T(\phi)$$

exists. In this case, we say that f_n has a limit at the boundary, namely T.

The lemma below provides a necessary and sufficient condition for the weak convergence of a sequence of functions $f \in \mathcal{E}^*(V)$.

Lemma 1 *Let $f = \{f_n(k) \;:\; n \in \mathbb{N},\ 0 \le k < p^n\}$ be a complex-valued function on the vertices of the p-adic tree. Then f has a limit at the boundary if and only if $\lim_{n\to\infty} \widehat{f_n}(lp^{n-m})$ exists for every $p \nmid l$ and $m \le n$.*

Proof Suppose f_n has a limit at the boundary. Then formula (5.1) implies that

$$\widehat{f_n}(lp^{n-m}) = \frac{1}{p^n} \sum_{k=0}^{p^n-1} f_n(k) e^{-2\pi i \frac{kl}{p^m}}.$$

Notice that the character $\chi_{n,k}$ in formula (2.2) is also a test function and so the expression $e^{-2\pi i \frac{kl}{p^m}}$ appearing on the right hand side of the above equation is equal to $\tilde{\chi}_{m,l}(n,k)$. By our assumption, the limit of the right hand side exists as $n \to \infty$ for any $p \nmid l$ and $m \le n$. So the limit of the left hand side also exists as $n \to \infty$.

Now suppose that $\lim_{n\to\infty} \widehat{f_n}(lp^{n-m})$ exists for all $p \nmid l$ and $m \le n$. Every test function is a finite linear combination of characters because test functions have finite Fourier expansions. Consequently, by linearity, it is sufficient to check that the limit

$$\lim_{n\to\infty} \sum_{v \in V_n} f_n(v) \tilde{\chi}_{m,l}(v)\, p^{-n}$$

exists. But the existence of the above limit is guaranteed by the hypothesis since

$$\sum_{v \in V_n} f_n(v) \tilde{\chi}_{m,l}(v)\, p^{-n} = \frac{1}{p^n} \sum_{k=0}^{p^n-1} f_n(k) e^{-2\pi i \frac{kl}{p^m}} = \widehat{f_n}(lp^{n-m}).$$

■

Theorem 5.1.2 (The Unique Continuation Property of D)
Let $f \in \mathrm{Ker}D$. Then the following are true.

1. *f has a limit at the boundary.*

2. *If the limit of f at the boundary is equal to zero then f is identically zero.*

3. *If $T \in \mathcal{E}^*(\mathbb{Z}_p)$ then there exists $g \in \mathrm{Ker}D$ such that the limit of g at the boundary is equal to T.*

Proof To prove the first part of the theorem we observe from formula (5.5) that the operator D on $\mathcal{E}^*(V)$ has an infinite dimensional kernel. Using Lemma 1 it suffices to show that if $f \in \text{Ker } D$ then the limit $\lim\limits_{n\to\infty} \widehat{f}_n(lp^{n-m})$ exists for every $p \nmid l$ and m. If $p \nmid l$ and $n \geq m$ then

$$\begin{aligned}\widehat{f}_n(lp^{n-m}) &= \widehat{f}_{n-1}(lp^{n-m-1}) = \widehat{f}_{n-2}(lp^{n-m-2}) = \dots \\ &= \widehat{f}_m(l).\end{aligned}$$

So $\lim_{n\to\infty} \widehat{f}_n(lp^{n-m})$ exists and is equal to $\widehat{f}_m(l)$.

To prove the second part of the theorem, we first establish another formula for $f_n(k)$ in terms of its Fourier coefficients.

Starting with formula (5.2), we decompose the sum on the right hand side of (5.2) such that one sum is over all l divisible by p and the other over all l such that $p \nmid l$ to obtain

$$f_n(k) = \sum_{\substack{p\,\nmid\, l \\ 0\leq l<p^n}} \widehat{f}_n(l)e^{\frac{2\pi ikl}{p^n}} + \sum_{0\leq l<p^{n-1}} \widehat{f}_n(lp)e^{\frac{2\pi ikl}{p^{n-1}}}.$$

Iterating this we get,

$$\begin{aligned}f_n(k) &= \sum_{\substack{p\,\nmid\, l \\ 0\leq l<p^n}} \widehat{f}_n(l)e^{\frac{2\pi ikl}{p^n}} + \sum_{\substack{p\,\nmid\, l \\ 0\leq l<p^{n-1}}} \widehat{f}_n(lp)e^{\frac{2\pi ikl}{p^{n-1}}} + \dots + \sum_{0\leq l<p} \widehat{f}_n(lp^{n-1})e^{\frac{2\pi ikl}{p}} \\ &= \sum_{m=1}^{n} \sum_{\substack{(l,p)=1 \\ 0\leq l<p^m}} \widehat{f}_n(lp^{n-m})e^{\frac{2\pi ikl}{p^m}}.\end{aligned} \tag{5.11}$$

If the limit of f at the boundary is equal to zero then

$$\lim_{n\to\infty} \frac{1}{p^n} \sum_{0\leq k<p^n} f_n(k)e^{-\frac{2\pi ikl}{p^m}} = \lim_{n\to\infty} \widehat{f}_n(lp^{n-m}) = 0$$

which in turn implies from the first part of the theorem that $\widehat{f}_m(l) = 0$ for each m and $0 \leq l < p^m$ such that $p \nmid l$. Using formula (5.11) above we see that

$$f_n(k) = \sum_{m=1}^{n} \sum_{\substack{(l,p)=1 \\ 0\leq l<p^m}} \widehat{f}_n(lp^{n-m})e^{\frac{2\pi ikl}{p^m}} = \sum_{m=1}^{n} \sum_{\substack{(l,p)=1 \\ 0\leq l<p^m}} \widehat{f}_m(l)e^{\frac{2\pi ikl}{p^m}}.$$

Therefore $f_n(k) \equiv 0$ for each n.

Finally, we prove the third part of the statement. Any distribution $T \in \mathcal{E}^*(\mathbb{Z}_p)$ is completely determined by its Fourier coefficients $T_{n,k} := T(\overline{\chi_{n,k}})$ which, moreover, can be arbitrary numbers. For given $T_{m,l}$ we would like to find $g \in \text{Ker } D$ such that

$$\lim_{n\to\infty}\sum_{v\in V_n} g_n(v)\tilde{\chi}_{m,l}(v)\, p^{-n} = \lim_{n\to\infty} p^{-n}\sum_{0\le k<p^n} g_n(k)e^{\frac{-2\pi ikl}{p^m}} = \lim_{n\to\infty}\widehat{g}_n(lp^{n-m})$$
$$= T_{m,l}.$$

If $p \nmid l$ then from the first part of the theorem we see that the Fourier coefficients of g are given by $\widehat{g}_m(l) = T_{m,l}$. If $p \mid l$ let $l = p^r k$ with $\gcd(p,k) = 1$. Then:

$$\widehat{g}_n(lp^{n-m}) = \widehat{g}_{m-r}(k) = T_{m-r,l}.$$

Thus $g \in \mathrm{Ker}\ D$ is completely determined by the conditions,

$$\widehat{g}_m(l) = \begin{cases} T_{m,l} & \text{if } p \mid l \\ T_{m-r,l} & \text{if } l = p^r k \text{ with } \gcd(p,k) = 1. \end{cases}$$

■

5.2 The construction of the spectral triple

In this section we construct a spectral triple for the C*- algebra $A = C(\mathbb{Z}_p)$, the space of continuous functions on $\mathbb{Z}_p$. We use the Hilbert space H and Dirac operator D introduced in the previous section to construct a new Hilbert space $\mathcal{H} = H \bigoplus H$ and a selfadjoint operator $\mathcal{D}$ on $\mathcal{H}$ given by:

$$\mathcal{D} = \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}.$$

Notice that the Hilbert space $\mathcal{H}$ has a natural $\mathbb{Z}/2\mathbb{Z}$ grading with respect to which the operator $\mathcal{D}$ is odd (off diagonal). Moreover, there is a natural representation $\pi : A \to \mathcal{B}(H)$ given by

$$\pi(\phi)f_n(k) = \phi(k)f_n(k).$$

We choose $\mathcal{A}$ to be the algebra of Lipschitz functions on $\mathbb{Z}_p$ which is a dense $*$-subalgebra of A. This choice of $\mathcal{A}$ is natural in the sense that it is a maximal sub algebra of $C(\mathbb{Z}_p)$ which has a bounded commutator with $\mathcal{D}$, as we will see later in Corollary 1.

For the sake of the reader, we recall that a function $\phi : \mathbb{Z}_p \to \mathbb{C}$ is called Lipschitz (or Lipschitz continuous) if there is a constant $c > 0$ such that $|\phi(x)-\phi(y)| \leq c\,\rho_p(x,y)$ for all $x, y \in \mathbb{Z}_p$. The smallest such constant c is called the Lipschitz norm of ϕ and is denoted by $L(\phi)$:

$$L(\phi) = \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{\rho_p(x,y)}.$$

The algebra A can be represented on $\mathcal{H}$ via the representation $\Pi : A \to \mathcal{B}(\mathcal{H})$ given by $\Pi = \pi \oplus \pi$. i.e.,

$$\Pi(\phi) \begin{pmatrix} f_n(k) \\ g_n(k) \end{pmatrix} = \begin{pmatrix} \phi(k) f_n(k) \\ \phi(k) g_n(k) \end{pmatrix}.$$

Also notice that the representation Π is even (diagonal) with respect to the grading on $\mathcal{H}$ and has the following property:

Proposition 5.2.1 *The representation* $\Pi : A \to \mathcal{B}(\mathcal{H})$ *is faithful and nondegenerate.*

Proof Because $\pi(\phi)$ is a diagonal operator on H,

$$||\Pi(\phi)|| = ||\pi(\phi)|| = \sup_{k \in \mathbb{Z}_{\geq 0}} |\phi(k)|.$$

Since ϕ is continuous and the set of nonnegative integers is dense in $\mathbb{Z}_p$ we have that $||\Pi(\phi)|| = \sup_{x \in \mathbb{Z}_p} |\phi(x)|$ and so the representation of A is faithful. Notice that Π is unit preserving, $\Pi(1) = I$, and hence the representation is non-degenerate (i.e. $\Pi(A)\mathcal{H} = \mathcal{H}$). ■

Now we present one of our main theorems in this thesis.

Theorem 5.2.1 *The triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *defined above is an even spectral triple.*

Proof First we verify that the compactness of $(1 + \mathcal{D}^2)^{-1/2}$. For this, it is sufficient to show that D^{-1} is a Hilbert-Schmidt operator, for then the result follows by using functional calculus. So we estimate the Hilbert-Schmidt norm of D^{-1} as follows.

With the use of formula (5.9), D^{-1} can be written in Fourier transform as,

$$\widehat{D}^{-1}\widehat{g}_n(k) = \sum_{i=0}^{\infty} \sum_{0\le r<p^i} K(n,k,i,r)\widehat{g}_i(r)$$

where the integral kernel K is given by

$$K(n,k,i,r) = \begin{cases} \frac{1}{p^i} & ; \text{ if } i \ge n \text{ and } r = lp^{\alpha-n+i} \\ 0 & ; \text{ otherwise} \end{cases}$$

where α is the integer such that $k = lp^\alpha$ with $p \nmid l$. Hence the Hilbert-Schmidt norm of D^{-1} can be estimated as,

$$\begin{aligned} \|D^{-1}\|_{HS}^2 &= \sum_{i=0}^{\infty} \sum_{0\le r<p^i} \sum_{n=0}^{\infty} \sum_{0\le k<p^n} |K(n,k,i,r)|^2 = \sum_{n=0}^{\infty} \sum_{0\le k<p^n} \sum_{i=n}^{\infty} \left|K(n,k,i,lp^{\alpha-n+i})\right|^2 \\ &= \sum_{n=0}^{\infty} \sum_{0\le k<p^n} \sum_{i=n}^{\infty} \frac{1}{p^{2i}} = \frac{1}{(1-p^{-1})(1-p^{-2})} < \infty. \end{aligned}$$

Hence D^{-1} is a Hilbert-Schmidt operator.

Next we would like to prove that the commutator $[\mathcal{D}, \Pi(\phi)]$ is bounded for any $\phi \in \mathcal{A}$. First, notice that,

$$[\mathcal{D}, \Pi(\phi)] \begin{pmatrix} f_n(k) \\ g_n(k) \end{pmatrix} = \begin{pmatrix} D\pi(\phi)g_n(k) - \pi(\phi)Dg_n(k) \\ D^*\pi(\phi)f_n(k) - \pi(\phi)D^*f_n(k) \end{pmatrix}. \tag{5.12}$$

Evaluating the first component of the above expression we get

$$\begin{aligned} D\pi(\phi)g_n(k) - \pi(\phi)Dg_n(k) &= p^n \left[\phi(k)g_n(k) - \frac{1}{p}\sum_{i=0}^{p-1} \phi(k+ip^n)g_{n+1}\big(k+ip^n\big)\right] \\ &\quad - p^n \left[\phi(k)g_n(k) - \frac{1}{p}\sum_{i=0}^{p-1} \phi(k)g_{n+1}\big(k+ip^n\big)\right] \\ &= p^{n-1} \sum_{i=0}^{p-1} \big[\phi(k) - \phi(k+ip^n)\big] g_{n+1}(k+ip^n). \end{aligned} \tag{5.13}$$

We can estimate the pointwise norm of the commutator using the Lipschitz condition:

$$\begin{aligned}\left|D\pi(\phi)g_n(k)-\pi(\phi)Dg_n(k)\right| &\le p^{n-1}\sum_{i=0}^{p-1}\left|\phi(k)-\phi(k+ip^n)\right||g_{n+1}(k+ip^n)| \\ &= \frac{L(\phi)}{p}\sum_{i=0}^{p-1}\left|g_{n+1}(k+ip^n)\right|.\end{aligned}$$

Hence,

$$\begin{aligned}\|D\pi(\phi)g-\pi(\phi)Dg\|^2 &= \sum_{n=0}^{\infty}\sum_{0\le k<p^n} p^{-n}\left|D\pi(\phi)g_n(k)-\pi(\phi)Dg_n(k)\right|^2 \\ &\le \sum_{n=0}^{\infty}\sum_{0\le k<p^n} p^{-n-2}L(\phi)^2\left(\sum_{i=0}^{p-1}|g_{n+1}(k+ip^n)|\right)^2.\end{aligned}$$

Using the fact that $\left(\sum_{k=1}^n a_k\right)^2 \le n\sum_{k=1}^n a_k^2$ for any positive $\{a_k\}_{k=1}^n$ we see that,

$$\begin{aligned}\|D\pi(\phi)g-\pi(\phi)Dg\|^2 &\le \sum_{n=0}^{\infty}\sum_{0\le k<p^n}\sum_{0\le i\le p-1} p^{-n-1}L(\phi)^2\,|g_{n+1}(k+ip^n)|^2 \\ &\le \sum_{n=0}^{\infty}\sum_{0\le l<p^{n+1}} p^{-n-1}L(\phi)^2\,|g_{n+1}(l)|^2 \\ &= L(\phi)^2\|g\|^2.\end{aligned}$$

Hence $\|D\pi(\phi)-\pi(\phi)D\|\le L(\phi)$. Thus, we also have the norm estimate $\|D^*\pi(\phi)-\pi(\phi)D^*\| = \|D\pi(\phi)-\pi(\phi)D\|\le L(\phi)$. Consequently the commutator $[\mathcal{D},\Pi(\phi)]$ is bounded for any $\phi\in\mathcal{A}$.

Thus we have proved that $(\mathcal{A},\mathcal{H},\mathcal{D})$ is an even spectral triple.

■

5.3 The distance

The goal of this section is to study metric properties of the above spectral triple, by comparing the two distance functions induced by the Lipschitz seminorm L and the spectral seminorm $L_{\mathcal{D}}$ through the Connes distance formula. We will consider the Lipschitz seminorm as a function on the whole C*-algebra $A=C(\mathbb{Z}_p)$ with possibly having infinite value, i.e. $L: A\to[0,\infty]$. For $\phi\in A$, the spectral seminorm $L_{\mathcal{D}}(\phi)$, is defined by $L_{\mathcal{D}}(\phi) = \|[\mathcal{D}, \Pi(\phi)]\|$, where unbounded operators are considered to have infinite norm. From formula (5.12) it follows that $L_{\mathcal{D}}(\phi) = \|[D, \pi(\phi)]\|$.

Via the Connes distance formula, seminorms L and $L_{\mathcal{D}}$ induce two metrics, dist and $\mathrm{dist}_{\mathcal{D}}$ respectively, on the space of p-adic integers $\mathbb{Z}_p$:

$$\mathrm{dist}(x,y) = \sup_{\phi\in A}\big\{|\phi(x)-\phi(y)| : L(\phi)\le 1\big\},\ x,y\in\mathbb{Z}_p, \qquad \text{and}$$
$$\mathrm{dist}_{\mathcal{D}}(x,y) = \sup_{\phi\in A}\big\{|\phi(x)-\phi(y)| : L_{\mathcal{D}}(\phi)\le 1\big\},\ x,y\in\mathbb{Z}_p.$$

As mentioned in the introduction, the distance induced by the Lipschitz seminorm is equal to the original metric. Below we verify this general result.

Proposition 5.3.1 *The metric,* dist(x, y)*, induced by the Lipschitz seminorm is equal to the usual p-adic metric.*

Proof The inequality $\mathrm{dist}(x, y) \le |x-y|_p$ is clear since for all $\phi \in A$ with $L(\phi) \le 1$ we have $|\phi(x) - \phi(y)| \le |x-y|_p$. Now consider the function given by $\phi_x(z) = |z-x|_p$, which is Lipschitz continuous with Lipschitz norm $L(\phi_x) = 1$. Moreover, $|\phi_x(x) - \phi_x(y)| = |x-y|_p$. Hence $\mathrm{dist}(x, y) = \rho_p(x, y)$. ∎

The following lemma describes an explicit formula for the spectral seminorm $L_{\mathcal{D}}$.

Lemma 2 *For $\phi \in \mathcal{A}$ we have:*

$$L_{\mathcal{D}}(\phi) = \|[D,\pi(\phi)]\| = \left(\sup_{\substack{n\\ 0\le k<p^n}} \frac{1}{p}\sum_{i=1}^{p-1}\frac{|\phi(k)-\phi(k+ip^n)|^2}{|k-(k+ip^n)|_p^2}\right)^{1/2}. \tag{5.14}$$

Proof Since ϕ is a Lipschitz function, the ratio $\sum_{i=1}^{p-1}\frac{|\phi(k)-\phi(k+ip^n)|^2}{|k-(k+ip^n)|_p^2}$ is uniformly bounded, hence the sup in formula (5.14) exists. Formula (5.13) implies that,

$$\begin{aligned}|D\pi(\phi)g_n(k) - \pi(\phi)Dg_n(k)| &= \left|p^{n-1}\sum_{i=0}^{p-1}\big[\phi(k)-\phi(k+ip^n)\big]g_{n+1}(k+ip^n)\right| \\ &\le \frac{1}{p}\sum_{i=0}^{p-1}\frac{|\phi(k)-\phi(k+ip^n)|}{p^{-n}}|g_{n+1}(k+ip^n)|.\end{aligned}$$

Therefore,

$$\|D\pi(\phi)g - \pi(\phi)Dg\|^2 \leq \sum_{n=0}^{\infty} \sum_{0\leq k<p^n} p^{-n} \frac{1}{p^2} \left(\sum_{i=0}^{p-1} \frac{|\phi(k) - \phi(k+ip^n)|}{p^{-n}} |g_{n+1}(k+ip^n)| \right)^2 .$$

The above norm can be estimated using Cauchy-Schwartz inequality as below.

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{0\leq k<p^n} p^{-n} \frac{1}{p^2} \left(\sum_{i=0}^{p-1} \frac{|\phi(k) - \phi(k+ip^n)|}{p^{-n}} |g_{n+1}(k+ip^n)| \right)^2 \\
&\leq \sum_{n=0}^{\infty} \sum_{0\leq k<p^n} \frac{p^{-n}}{p^2} \left(\sum_{i=0}^{p-1} \frac{|\phi(k) - \phi(k+ip^n)|^2}{p^{-2n}} \right) \left(\sum_{j=0}^{p-1} |g_{n+1}(k+jp^n)|^2 \right) \\
&\leq \sup_{n\geq 0} \sup_{0\leq k<p^n} \frac{1}{p} \sum_{i=0}^{p-1} \frac{|\phi(k) - \phi(k+ip^n)|^2}{p^{-2n}} \left(\sum_{n=0}^{\infty} \sum_{0\leq k<p^n} \sum_{j=0}^{p-1} p^{-n-1} |g_{n+1}(k+jp^n)|^2 \right) \\
&= \sup_{n\geq 0} \sup_{0\leq k<p^n} \frac{1}{p} \sum_{i=1}^{p-1} \frac{|\phi(k) - \phi(k+ip^n)|^2}{|k-(k+ip^n)|_p^2} \|g\|^2.
\end{aligned}$$

This establishes the inequality,

$$\|D\pi(\phi) - \pi(\phi)D\|^2 \leq \sup_{\substack{n \\ 0\leq k<p^n}} \frac{1}{p} \sum_{i=1}^{p-1} \frac{|\phi(k) - \phi(k+ip^n)|^2}{|k-(k+ip^n)|_p^2}.$$

To establish the equality we use a particular choice of a function g as described below. For any $\epsilon > 0$ there are numbers n_ϵ, k_ϵ such that,

$$0 \leq \sup_{n\geq 0} \sup_{0\leq k<p^n} \left(\frac{1}{p} \sum_{i=1}^{p-1} \frac{|\phi(k) - \phi(k+ip^n)|^2}{|k-(k+ip^n)|_p^2} \right) - \frac{1}{p} \sum_{i=1}^{p-1} \frac{|\phi(k_\epsilon) - \phi(k_\epsilon+ip^{n_\epsilon})|^2}{|k_\epsilon-(k_\epsilon+ip^{n_\epsilon})|_p^2} < \epsilon.$$

Thus, given n_ϵ, k_ϵ define the function g^ϵ by:

$$g^\epsilon_{n+1}(k+ip^n) = \begin{cases} \frac{\overline{\phi(k_\epsilon)} - \overline{\phi(k_\epsilon+ip^{n_\epsilon})}}{p^{-n_\epsilon}} & ; \text{ if } (n,k) = (n_\epsilon, k_\epsilon) \\ 0 & ; \text{ otherwise} \end{cases}$$

where $i = 0, 1, \ldots, p-1$. Then,

$$\begin{aligned}
\|D\pi(\phi)g^\epsilon - \pi(\phi)Dg^\epsilon\|^2 &= \sum_{n=0}^{\infty} \sum_{0\le k<p^n} p^{-n}\frac{1}{p^2}\left|\sum_{i=1}^{p-1}\frac{\big(\phi(k)-\phi(k+ip^n)\big)}{|k-(k+ip^n)\big)|_p}g^\epsilon_{n+1}(k+ip^n)\right|^2 \\
&= \frac{p^{-n_\epsilon}}{p^2}\left|\sum_{i=1}^{p-1}\frac{\big(\phi(k_\epsilon)-\phi(k_\epsilon+ip^{n_\epsilon})\big)}{p^{-n_\epsilon}}\cdot\frac{\big(\overline{\phi(k_\epsilon)}-\overline{\phi(k_\epsilon+ip^{n_\epsilon})}\big)}{p^{-n_\epsilon}}\right|^2 \\
&= \frac{p^{-n_\epsilon}}{p^2}\left(\sum_{i=1}^{p-1}\frac{|\phi(k_\epsilon)-\phi(k_\epsilon+ip^{n_\epsilon})|^2}{p^{-2n_\epsilon}}\right)^2 \\
&= \frac{p^{-n_\epsilon}}{p^2}\left(\sum_{i=1}^{p-1}\frac{|\phi(k_\epsilon)-\phi(k_\epsilon+ip^{n_\epsilon})|}{p^{-n_\epsilon}}\cdot|g^\epsilon_{n_\epsilon+1}(k_\epsilon+ip^{n_\epsilon})|\right)^2 \\
&= \frac{1}{p}\left(\sum_{i=1}^{p-1}\frac{|\phi(k_\epsilon)-\phi(k_\epsilon+ip^{n_\epsilon})|^2}{p^{-2n_\epsilon}}\right)\|g^\epsilon\|^2.
\end{aligned}$$

This computation shows that for every ϵ,

$$\begin{aligned}
\|[D,\pi(\phi)]\|^2 &\ge \frac{1}{p}\left(\sum_{i=1}^{p-1}\frac{|\phi(k_\epsilon)-\phi(k_\epsilon+ip^{n_\epsilon})|^2}{p^{-2n_\epsilon}}\right) \\
&\ge \sup_{n\ge 0}\sup_{0\le k<p^n}\frac{1}{p}\sum_{i=1}^{p-1}\left|\frac{\phi(k)-\phi(k+ip^n)}{p^{-n}}\right|^2-\frac{\epsilon}{p}.
\end{aligned}$$

Since this is true for every ϵ, by letting $\epsilon\to 0$ we obtain,

$$\|[D,\pi(\phi)]\|^2 \ge \sup_{n\ge 0}\sup_{0\le k<p^n}\frac{1}{p}\sum_{i=1}^{p-1}\left|\frac{\phi(k)-\phi(k+ip^n)}{p^{-n}}\right|^2.$$

Thus the equality follows and the Lemma is proved. ∎

Using Lemma 2 we can now show that the two seminorms $L_{\mathcal{D}}$ and L are equivalent.

Lemma 3 *For every $\phi\in\mathcal{A}$ we have:*

$$\frac{(p-1)}{2p\sqrt{p}}\,L(\phi)\le L_{\mathcal{D}}(\phi)\le\sqrt{\frac{p-1}{p}}\,L(\phi).$$

Proof Let $\phi\in\mathcal{A}$ be a Lipschitz function. Using the Lipschitz property of ϕ we observe that,

$$\frac{1}{p}\sum_{i=1}^{p-1}\frac{|\phi(k)-\phi(k+ip^n)|^2}{|k-(k+ip^n)|_p^2}\le\frac{1}{p}\sum_{i=1}^{p-1}L(\phi)^2=\frac{p-1}{p}L(\phi)^2.$$

By taking the supremum of both sides, Lemma 2 implies that $L_{\mathcal{D}}(\phi) \leq \sqrt{\frac{p-1}{p}} L(\phi)$.

To prove the reverse inequality, notice that for any $\phi \in \mathcal{A}$, all n and $0 \leq k < p^n$ we have:

$$\frac{1}{p}\sum_{i=1}^{p-1}\left|\frac{\phi(k)-\phi(k+ip^n)}{p^{-n}}\right|^2 \leq L_{\mathcal{D}}(\phi)^2.$$

Since the sum on the left hand side consists of positive terms each term in the sum can be estimated as,

$$\frac{|\phi(k)-\phi(k+ip^n)|}{p^{-n}} \leq \sqrt{p}\, L_{\mathcal{D}}(\phi). \tag{5.15}$$

Let $\phi \in \mathcal{A}$, and $x \neq y$ such that $|x-y|_p = p^{-n}$ for some $n \geq 0$. Then there is a nonnegative integer $k < p^n$ such that x, y have the following p-adic representations:

$$\begin{aligned} x &= k + a_0 p^n + a_1 p^{n+1} + \ldots + a_i p^{n+i} + \ldots \\ y &= k + b_0 p^n + b_1 p^{n+1} + \ldots + b_j p^{n+j} + \ldots, \end{aligned}$$

with $a_0 \neq b_0$. Now we estimate the following partial difference using the triangle inequality and inequality (5.15):

$$\begin{aligned} &\left|\phi(k + a_0p^n + a_1p^{n+1} + \ldots + a_Np^{n+N}) - \phi(k)\right| \\ &\leq |\phi(k) - \phi(k+a_0p^n)| + \left|\phi(k+a_0p^n) - \phi(k+a_0p^n+a_1p^{n+1})\right| + \ldots + \\ &\left|\phi(k+a_0p^n+\ldots+a_{N-1}p^{n+N-1}) - \phi(k+a_0p^n+\ldots+a_Np^{n+N})\right| \\ &\leq \sqrt{p}\, L_{\mathcal{D}}(\phi)(p^{-n} + p^{-(n+1)} + p^{-(n+2)} + \ldots + p^{-(n+N)}) \\ &= \sqrt{p}\, L_{\mathcal{D}}(\phi)\, \frac{p^{-n}(1-p^{-N-1})}{(1-p^{-1})}. \end{aligned}$$

Taking the limit as $N \to \infty$, and using the continuity of ϕ we obtain:

$$|\phi(x) - \phi(k)| \leq \sqrt{p}\, L_{\mathcal{D}}(\phi)\, \frac{p^{-n}}{(1-p^{-1})}.$$

A similar computation gives,

$$|\phi(k) - \phi(y)| \leq \sqrt{p}\, L_{\mathcal{D}}(\phi)\, \frac{p^{-n}}{(1-p^{-1})}.$$

Therefore, using the triangle inequality we obtain,

$$|\phi(x) - \phi(y)| \leq \sqrt{p}\, L_{\mathcal{D}}(\phi)\, \frac{2p^{-n}}{(1-p^{-1})} = \frac{2\sqrt{p}\, L_{\mathcal{D}}(\phi)}{(1-p^{-1})}|x-y|_p.$$

Since the above inequality is true for any x and y, we see that $L(\phi) \leq \frac{2\sqrt{p}\, L_{\mathcal{D}}(\phi)}{1-p^{-1}}$. Thus, $L_{\mathcal{D}}(\phi) \geq \frac{(p-1)}{2p\sqrt{p}}\, L(\phi)$ and the theorem is proved.

■

Corollary 1 *1. The algebra $C^1(A) := \{\phi \in A;\ \|[\mathcal{D}, \pi(\phi)]\| < \infty\}$ is the algebra of Lipschitz functions $\mathcal{A}$.*

2. The commutant $A'_{\mathcal{D}} = \{\phi \in A;\ [\mathcal{D}, \pi(\phi)] = 0\}$ is trivial. i.e., $A'_{\mathcal{D}} = \mathbb{C}I$.

Proof From Lemma 3 it follows that $L_{\mathcal{D}}(\phi) < \infty$ if and only if $L(\phi) < \infty$. Consequently, the operator $[\mathcal{D}, \pi(\phi)]$ is bounded if and only if $L(\phi) < \infty$, i.e., $\phi \in \mathcal{A}$. This proves the first part of the corollary.

To verify the second part of the Theorem, we use Lemma 3 again to observe that $L_{\mathcal{D}}(\phi) = 0$ if and only if $L(\phi) = 0$. But the only functions with zero Lipschitz norm are constant functions, hence the commutant is trivial. ■

The next theorem is one of the main result of this section, namely that the metric induced by the spectral seminorm is equivalent to the p-adic norm on $\mathbb{Z}_p$.

Theorem 5.3.1 *For any $x, y \in \mathbb{Z}_p$ we have*

$$\sqrt{\frac{p}{p-1}}|x-y|_p \leq \ \mathrm{dist}_{\mathcal{D}}(x,y) \leq \frac{2p\sqrt{p}}{(p-1)}|x-y|_p.$$

Proof Notice that we can rewrite the formula for $\mathrm{dist}_{\mathcal{D}}$ as

$$\mathrm{dist}_{\mathcal{D}}(x,y) = \sup_{\phi : L_{\mathcal{D}}(\phi) \neq 0} \frac{|\phi(x) - \phi(y)|}{L_{\mathcal{D}}(\phi)}.$$

Since $L_{\mathcal{D}}(\phi) = 0$ iff $L(\phi) = 0$ we write,

$$\frac{|\phi(x) - \phi(y)|}{L_{\mathcal{D}}(\phi)} = \frac{|\phi(x) - \phi(y)|}{L(\phi)} \frac{L(\phi)}{L_{\mathcal{D}}(\phi)}.$$

and now the inequalities of Lemma 3 give the desired result. ■

Finally, we verify that the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfies the conditions of a compact spectral metric space.

Theorem 5.3.2 *The pair $(A, L_{\mathcal{D}})$ defined above is a compact spectral metric space.*

Proof In the proof of Proposition 5.2.1 it was verified that the representation Π is nondegenerate and in the proof of Corollary 1 we established that the commutant $A'_{\mathcal{D}}$ is trivial.

To verify that the image of the Lipschitz ball $B_{\mathcal{D}} = \{\phi \in A : \; L_{\mathcal{D}}(\phi) \leq 1\}$ is precompact in $A/A'_{\mathcal{D}}$ we first observe that we have a natural identification of $A/A'_{\mathcal{D}} = C(\mathbb{Z}_p)/\mathbb{C}I$ with $A_{\{0\}} := \{\phi \in C(\mathbb{Z}_p) \; : \; \phi(0) = 0\}$. Let $\{\phi_n\}_{n=1}^{\infty}$, be a sequence of functions in $A_{\{0\}}$ such that $L_{\mathcal{D}}(\phi_n) \leq 1$. To prove that $\{\phi_n\}$ has a convergent subsequence we use Lemma 3 and the Ascoli-Arzela theorem.

It follows from Lemma 3 that:

$$L(\phi_n) \leq \frac{2p\sqrt{p}}{(p-1)}\, L_{\mathcal{D}}(\phi) \leq \frac{2p\sqrt{p}}{(p-1)}.$$

Consequently, for every n and for every $x, y \in \mathbb{Z}_p$, we have:

$$|\phi_n(x) - \phi_n(y)| \leq \frac{2p\sqrt{p}}{(p-1)}\, \rho_p(x, y),$$

which implies that the family $\{\phi_n\}$ is equicontinuous. It is also uniformly bounded because

$$|\phi(x)| = |\phi_n(x) - \phi_n(0)| \leq \frac{2p\sqrt{p}}{(p-1)}\, |x|_p \leq \frac{2p\sqrt{p}}{(p-1)}.$$

So, the Ascoli-Arzela theorem implies the existence of a convergent subsequence, and consequently the precompactness of the image of the Lipschitz ball $B_{\mathcal{D}}$ in $A/A'_{\mathcal{D}}$. This completes the proof of the theorem.

■

6. Spectral properties of the p-adic tree

In this chapter, we study spectral properties of the p-adic tree. Recall that, in Section 5.2 we verified that the operator D is invertible with compact inverse, implying that D^*D has compact resolvent. Consequently, the spectrum of D^*D is discrete with only possible accumulation point at infinity. Here we show that the eigenvalues of D^*D are the roots of the q-Bessel function ${}_1\phi_1\left({0 \atop q}; q, \lambda\right)$ and we also study the analytic continuation of the zeta function associated to D^*D.

6.1 Invariant Subspaces of H.

In order to compute the (discrete) spectrum of D^*D, we first decompose the Hilbert space $H = \ell^2(V, p^{-n})$ into invariant subspaces. To this end, we re-parametrize the p-adic tree which allows us to decompose D^*D into a direct sum of much simpler operators, hence making the computation of the spectrum easier.

In Section 5.1 we introduced the forward derivative D on $\mathcal{E}^*(V)$ as,

$$(Df)_n(k) = p^n \left(f_n(k) - \frac{1}{p} \sum_{0 \le j < p} f_{n+1}(k + jp^n) \right)$$

on the Hilbert space $H = \ell^2(V, p^{-n})$.

Since

$$\begin{aligned}
\langle Df, g \rangle &= \sum_{n=0}^{\infty} \sum_{0 \le k < p^n} \left(\overline{f_n(k)} - \frac{1}{p} \sum_{0 \le i \le p-1} \overline{f_{n+1}(k + ip^n)} \right) g_n(k) \\
&= \sum_{n=0}^{\infty} \sum_{0 \le k < p^n} \overline{f_n(k)} g_n(k) - \frac{1}{p} \sum_{n=1}^{\infty} \sum_{0 \le k < p^{n-1}} \sum_{0 \le i \le p-1} \overline{f_n(k + ip^{n-1})} g_{n-1}(k) \\
&= \sum_{n=0}^{\infty} \frac{1}{p^n} \sum_{0 \le k < p^n} \overline{f_n(k)} \left(g_n(k) - \frac{1}{p} g_{n-1}(k \mod p^{n-1}) \right) p^n,
\end{aligned}$$

the adjoint D^* of D is given by

$$D^* g_n(k) = p^n \left[g_n(k) - \frac{1}{p} g_{n-1}(k \mod p^{n-1}) \right] \tag{6.1}$$

assuming $g_{-1}(0) = 0$. We will also need the following formula for the adjoint $\widehat{D^*}$ of $\widehat{D}$:

$$\widehat{D^*}\widehat{g}_n(l) = \begin{cases} p^n \widehat{g}_n(l) & ; \text{if } \ p \nmid l \\ p^n \left(\widehat{g}_n(l) - \frac{1}{p}\widehat{g}_{n-1}\left(\frac{l}{p}\right)\right) & ; \text{otherwise.} \end{cases} \tag{6.2}$$

In Section 5.2 we proved that D (hence also D^*) is invertible with compact inverse. The key observation that allows us to find the spectrum of D^*D is that we can decompose the Hilbert space H into invariant subspaces by means of the following parametrization of the p-adic tree.

Recall that the original parametrization (3.1) of the set of vertices of p-adic tree was done by using the set

$$S := \left\{ (n, k) : n = 0, 1, 2, \ldots, 0 \leq k < p^n \right\}.$$

Given a pair (n, k) in S we can write $k = rp^l$ with $p \nmid r$ and $l \in \{0, 1, \ldots n-1\}$, by factoring out the highest power of p that divides k. Such a representation of k will be uniquely determined by r and l. If we associate n with $\frac{k}{p^n} = \frac{r}{p^{n-l}} = \frac{r}{p^m}$ where $m = n - l$, which is a unique representation of n in terms of r and m, then we have the correspondence $(n, k) \mapsto (\frac{r}{p^m}, l)$.

Conversely, given a pair $(\frac{r}{p^m}, l)$ where $0 \leq r < p^m$, $p \nmid r$ and $l \in \{0, 1, 2, \ldots\}$ we can make the unique association $(\frac{r}{p^m}, l) \mapsto (m + l, rp^l)$. Thus, if

$$S' := \left\{ \left(\frac{r}{p^m}, l \right) : 0 \leq r < p^m, \ p \nmid r, \ l \in \mathbb{Z}_{\geq 0} \right\}$$

then we have the one-to-one correspondence between the sets S and S' given by $(n, k) \leftrightarrow (\frac{r}{p^m}, l)$.

The set of numbers $\left\{ g = \frac{r}{p^m} : \ 0 \leq r < p^m, p \nmid r \right\}$ is isomorphic to the Prüfer group $\mathcal{G}_p$ defined in (2.3). Therefore, $V \cong \widehat{V} \cong \mathcal{G}_p \times \mathbb{Z}_{\geq 0}$. Consequently we have the following new decomposition of the Hilbert space $\widehat{H}$:

$$\widehat{H} = \ell^2(S) \cong \ell^2(S') = \bigoplus_{\frac{r}{p^m} \in \mathcal{G}_p} \ell^2(\mathbb{Z}_{\geq 0}) =: \bigoplus_{g \in \mathcal{G}_p} \widehat{H}_g$$

where $\widehat{H}_g = \ell^2(\mathbb{Z}_{\geq 0})$ for each g.

Using formulas (5.6) and (6.2) we can compute the operators $\widehat{D}$ and $\widehat{D^*}$ in the new coordinates as follows.

$$\widehat{D}\widehat{f}\left(\frac{r}{p^m}, l\right) = p^{m+l}\left(\widehat{f}\left(\frac{r}{p^m}, l\right) - \widehat{f}\left(\frac{r}{p^m}, l+1\right)\right)$$

and

$$\widehat{D^*}\widehat{f}\left(\frac{r}{p^m}, l\right) = \begin{cases} p^{m+l}\widehat{f}\left(\frac{r}{p^m}, 0\right) & ; \text{if } l = 0 \\ p^{m+l}\left(\widehat{f}\left(\frac{r}{p^m}, l\right) - \frac{1}{p}\widehat{f}\left(\frac{r}{p^m}, l-1\right)\right) & ; \text{otherwise .} \end{cases}$$

Assuming that $\widehat{f}\left(\frac{r}{p^m}, -1\right) = 0$ for any r, m we can rewrite the formula for $\widehat{D^*}$ as

$$\widehat{D^*}\widehat{f}\left(\frac{r}{p^m}, l\right) = p^{m+l}\left(\widehat{f}\left(\frac{r}{p^m}, l\right) - \frac{1}{p}\widehat{f}\left(\frac{r}{p^m}, l-1\right)\right). \tag{6.3}$$

Observe that, in the new coordinates, the operators $\widehat{D}$ and $\widehat{D^*}$ affect only the second coordinate l and consequently each H_g is an invariant subspace. By letting $\widehat{D}_g := \widehat{D}|_{H_g}$ and $\widehat{D^*_g} := \widehat{D^*}|_{H_g}$, we get the following decompositions of the operators $\widehat{D}$ and $\widehat{D^*}$:

$$\widehat{D} = \bigoplus_{g \in \mathcal{G}_p} \widehat{D_g} \quad \text{and} \quad \widehat{D^*} = \bigoplus_{g \in \mathcal{G}_p} \widehat{D^*_g}. \tag{6.4}$$

Introduce $\widehat{D_0}$ to be the operator on $\ell^2(\mathbb{Z}_{\geq 0})$ given by $\widehat{D_0}f(l) = p^l[f(l) - f(l+1)]$, written more conveniently using the subscript notation as,

$$(\widehat{D_0}f)_n = p^n\left(f_n - f_{n+1}\right). \tag{6.5}$$

The adjoint of $\widehat{D_0}$ is given by

$$(\widehat{D^*_0 g})_n = p^n\left(g_n - \frac{1}{p}g_{n-1}\right). \tag{6.6}$$

If $g = \frac{r}{p^m}$ then formula (6.3) implies that

$$\widehat{D_g} = p^m\widehat{D_0} \quad \text{and} \quad \widehat{D^*_g} = p^m\widehat{D^*_0}.$$

Consequently, $\widehat{D^*}\widehat{D}$ has the decomposition given by

$$\widehat{D^*}\widehat{D} = \bigoplus_{g\in\mathcal{G}_p} \widehat{D_g^*}\widehat{D_g} = \bigoplus_{g\in\mathcal{G}_p} p^{2m}(\widehat{D_0^*}\widehat{D_0}). \tag{6.7}$$

Thus, the key step in finding the spectrum of D^*D is to compute the spectrum of the operator $D_0^*D_0$ on $\ell^2(\mathbb{Z}_{\geq 0})$.

6.2 Spectrum of D^*D

As mentioned previously, because D^{-1} is compact, the operators D^*D and $D_0^*D_0$ have compact resolvent. Consequently, the spectrum of the unbounded operator $D_0^*D_0$ consists of eigenvalues diverging to infinity.

Using formulas (6.5) and (6.6) we obtain the following system of equations for $\widehat{D_0^*}\widehat{D_0}$.

$$\begin{aligned}
(\widehat{D_0^*}\widehat{D_0}f)_0 &= f_0 - f_1\\
(\widehat{D_0^*}\widehat{D_0}f)_n &= p^n\left((\widehat{D_0}f)_n - \frac{1}{p}(\widehat{D_0}f)_{n-1}\right)\\
&= p^{2n-2}[-p^2 f_{n+1} + (1+p^2)f_n - f_{n-1}] \text{ for any } n \geq 1.
\end{aligned}$$

Remark 3 *Equivalently, we could study the spectrum of $\widehat{D_0}\widehat{D_0^*}$; however the equations for the latter operator are not any simpler than the formulas for $\widehat{D_0^*}\widehat{D_0}$. Nevertheless, because the kernel of the two operators are trivial, the eigenvalue equations for both operators yield the same eigenvalues.*

Now we focus on solving the following eigenvalue equations for $\widehat{D_0^*}\widehat{D_0}$:

$$\begin{aligned}
p^{2n-2}\left[-p^2 f_{n+1} + (1+p^2)f_n - f_{n-1}\right] &= \lambda f_n; \text{ for } n \in \mathbb{Z}_{\geq 1}\\
f_0 - f_1 &= \lambda f_0,
\end{aligned} \tag{6.8}$$

with $f_n \in \ell^2(\mathbb{Z}_{\geq 0})$.

The key step in solving the system of equations (6.8) is the following theorem which asserts that all eigenvectors of $\widehat{D_0^*}\widehat{D_0}$ take the special exponential sum form

$f_n = \sum_{k=0}^{\infty} c(k)p^{-nk}$ with rapidly decaying coeffients $c(k)$. This result is a form of elliptic regularity of the operator $\widehat{D_0^*}\widehat{D_0}$.

Theorem 6.2.1 *Let $\{f_n(\lambda)\}$ be an eigenvector of $\widehat{D_0^*}\widehat{D_0}$ with eigenvalue λ. Then the following statements are true.*

1. *The sequence $\{f_n(\lambda)\}_{n=0}^{\infty}$ belongs to $\ell^1(\mathbb{Z}_{\geq 0})$.*

2. *The eigenvector $f_n(\lambda)$ can be uniquely expressed in the form*
$$f_n(\lambda) = \sum_{k=1}^{\infty} c(2k)p^{-2nk}, \tag{6.9}$$
where the coefficients $c(2k)$ decay exponentially in k.

3. *The coefficients $c(2k)$ satisfy the equations*
$$c(2) = \left(\frac{\lambda}{1-p^{-2}}\right)\sum_{k=0}^{\infty} f_k,$$
and, for $k \geq 2$,
$$c(2k) = \left(\frac{-\lambda}{1-p^{-2}}\right)^{k-1} \frac{c(2)p^{k(k-1)}(p^2-1)^{k-2}}{(p^4-1)^2(p^6-1)^2 \ldots (p^{2k-2}-1)^2(p^{2k}-1)}. \tag{6.10}$$

4. *If the remainder $r_n(2N)$ is defined by the formula:*
$$f_n(\lambda) = c(2)p^{-2n} + c(4)p^{-4n} + c(6)p^{-6n} + \ldots + c(2N-2)p^{-(2N-2)n} + r_n(2N),$$
then $\{r_n(2N)\}_{n=0}^{\infty} \to 0$ as $N \to \infty$ in the ℓ^1 norm.

Proof The main idea of the proof is to rewrite the equations (6.8) in an integral equation form and then use it iteratively to produce the solution. To this end we regroup the terms in the first equation of system (6.8) above to obtain:
$$(f_n - f_{n-1}) - p^2(f_{n+1} - f_n) = \lambda p^{2-2n} f_n.$$

Using the notation $\Delta f_n := f_{n+1} - f_n$, we can then rewrite the system of equations (6.8) as follows.
$$\begin{aligned} &\Delta f_n = p^{-2}\left(\Delta f_{n-1} - \lambda p^{2-2n} f_n\right) \quad \text{for } n \geq 1 \\ &\Delta f_0 = f_1 - f_0 = -\lambda f_0. \end{aligned} \tag{6.11}$$

Using equations (6.11) iteratively , we obtain the following formula for Δf_n:

$$\Delta f_n = -p^{-2n}\lambda(f_0 + f_1 + \ldots + f_n), \; n \geq 0. \tag{6.12}$$

Equation (6.12) is a one-step linear difference equation, so it has one-parameter family of solutions. Since we are looking for the solution in the Hilbert space $\ell^2(\mathbb{Z}_{\geq 0})$ we need to choose one that vanishes at infinity. This leads to the following formula for f_n:

$$f_n = \sum_{l=n}^{\infty} \lambda p^{-2l} \sum_{k=0}^{l} f_k.$$

Interchanging the summation indices in the above formula we obtain:

$$\begin{aligned} f_n &= \sum_{k=0}^{n}\sum_{l=n}^{\infty} \lambda p^{-2l} f_k + \sum_{k=n+1}^{\infty}\sum_{l=k}^{\infty} \lambda p^{-2l} f_k \\ &= \frac{\lambda}{(1-p^{-2})}\left[p^{-2n}\sum_{k=0}^{n} f_k + \sum_{k=n+1}^{\infty} p^{-2k} f_k\right]. \end{aligned} \tag{6.13}$$

Thus we can estimate:

$$\sum_{n=0}^{\infty} |f_n| \leq \frac{\lambda}{(1-p^{-2})}\left[\sum_{n=0}^{\infty} p^{-2n}\sum_{k=0}^{n} |f_k| + \sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty} p^{-2k}|f_k|\right].$$

By interchanging the summation indices in the first sum above and evaluating the sum over n we obtain:

$$\sum_{n=0}^{\infty} p^{-2n}\sum_{k=0}^{n} |f_k| = \sum_{k=0}^{\infty} |f_k| \left(\frac{p^{-2k}}{1-p^{-2}}\right).$$

Using Cauchy-Schwartz inequality and the fact that $f \in \ell^2(\mathbb{N})$ we conclude that this sum is bounded:

$$\sum_{k=0}^{\infty} |f_k| \left(\frac{p^{-2k}}{1-p^{-2}}\right) \leq \left(\frac{1}{1-p^{-2}}\right)\sqrt{\frac{1}{1-p^{-4}}}\, \|f\|_2 < \infty.$$

Notice that for the second sum we have

$$\sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty} p^{-2k}|f_k| = \sum_{n=0}^{\infty} p^{-2n}\sum_{l=1}^{\infty} p^{-2l}|f_{n+l}|.$$

Once again using Cauchy-Schwartz inequality we see that the second sum is finite:

$$\sum_{n=0}^{\infty} p^{-2n} \sum_{l=1}^{\infty} p^{-2l} |f_{n+l}| \leq \sqrt{\frac{p^{-4}}{1-p^{-4}}} \left(\frac{1}{1-p^{-2}} \right) \|f\|_2 < \infty.$$

This verifies that $\{f_n\} \in \ell^1(\mathbb{N})$.

To prove the second part of Theorem 6.2.1, we observe that equation (6.13) gives;

$$f_n = \frac{\lambda p^{-2n}}{(1-p^{-2})} \left[\sum_{k=0}^{\infty} f_k - \sum_{k=n+1}^{\infty} f_k + \sum_{l=1}^{\infty} p^{-2l} f_{n+l} \right].$$

Rearranging the terms on the right hand side of the equation to isolate the coefficient of p^{-2n} we get

$$f_n = \left(\frac{\lambda}{1-p^{-2}} \sum_{k=0}^{\infty} f_k \right) p^{-2n} - \lambda p^{-2n} \sum_{l=1}^{\infty} \left(\frac{1-p^{-2l}}{1-p^{-2}} \right) f_{n+l}, \tag{6.14}$$

from which we extract the coefficient

$$c(2) := \left(\frac{\lambda}{1-p^{-2}} \right) \sum_{k=0}^{\infty} f_k.$$

Notice that $c(2)$ is well defined due to part (1).

Recursively applying this formula for f_n on the right hand side of equation (6.14) we obtain:

$$f_n = c(2)p^{-2n} - \frac{\lambda}{p^{2n}} \sum_{l=1}^{\infty} \left(\frac{1-p^{-2l}}{1-p^{-2}} \right) \left(c(2)p^{-2n-2l} - \frac{\lambda}{p^{2n+2l}} \sum_{k=1}^{\infty} \left(\frac{1-p^{-2k}}{1-p^{-2}} \right) f_{n+l+k} \right).$$

Once again we rearrange the terms to extract the coefficient $c(4)$ of p^{-4n}.

$$\begin{aligned} f_n = c(2)p^{-2n} &+ \left(-\lambda c(2) \sum_{l=1}^{\infty} \left(\frac{p^{-2l} - p^{-4l}}{1-p^{-2}} \right) \right) p^{-4n} \\ &+ \frac{\lambda^2}{p^{4n}} \sum_{l=1}^{\infty} \left(\frac{p^{-2l} - p^{-4l}}{1-p^{-2}} \right) \sum_{k=1}^{\infty} \left(\frac{1-p^{-2k}}{1-p^{-2}} \right) f_{n+l+k}. \end{aligned}$$

This gives:

$$c(4) = \frac{-\lambda c(2)}{(1-p^{-2})} \cdot \frac{p^2}{(p^4-1)}.$$

By repeatedly applying this process we can obtain an expansion of f_n in powers of p^{-2n}, provided the remainder $r_n(2N)$ goes to zero as $N \to \infty$. We prove a stronger

ℓ^1 estimate on $r_n(2N)$ below, implying the pointwise convergence needed for the existence of the expansion of f_n.

Using induction we readily establish that the coefficients $c(2k)$ of this expansion are in general given by the formula:

$$c(2k) = \left(\frac{-\lambda}{1-p^{-2}}\right)^{k-1} \frac{c(2)p^{k(k-1)}(p^2-1)^{k-2}}{(p^4-1)^2(p^6-1)^2\dots(p^{2k-2}-1)^2(p^{2k}-1)}\ . \tag{6.15}$$

Next we estimate the growth of the coefficients $c(2k)$. Simplifying the formula for $c(2k)$ we obtain

$$\begin{aligned} c(2k) &= \frac{c(2)\lambda^{k-1}}{p^{(k-2)(k-1)}(1-\frac{1}{p^2})(1-\frac{1}{p^4})^2(1-\frac{1}{p^6})^2\dots(1-\frac{1}{p^{2k-2}})^2(1-\frac{1}{p^{2k}})} \\ &= \frac{\lambda^k(1-\frac{1}{p^{2k}})\sum_{m=0}^{\infty} f_m}{p^{(k-2)(k-1)}(1-\frac{1}{p^2})^2(1-\frac{1}{p^4})^2(1-\frac{1}{p^6})^2\dots(1-\frac{1}{p^{2k-2}})^2(1-\frac{1}{p^{2k}})^2}\ . \end{aligned}$$

Since $\prod_{i=1}^{k}(1-p^{-2i})^2 \geq \prod_{i=1}^{\infty}(1-p^{-2i})^2$ and the infinite product is a finite nonzero number, we obtain the following estimate for the coefficients $c(2k)$:

$$|c(2k)| \leq \frac{|\lambda|^k \|f\|_1}{p^{(k-2)(k-1)}\prod_{i=1}^{\infty}(1-p^{-2i})^2}\ ,$$

which verifies that the coefficients decay exponentially. This establishes both the second and the third parts of the theorem.

Finally we estimate the remainder term $r_n(2N)$. Using induction it is easily established that the remainder $r_n(2N)$ satisfies the following summation formula:

$$\begin{aligned} r_n(2N) =& \frac{(-\lambda)^N}{p^{2nN}} \sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\dots\sum_{l_N=1}^{\infty} \frac{(p^{-(2N-2)l_1}-p^{-2Nl_1})}{1-p^{-2}} \frac{(p^{-(2N-4)l_2}-p^{-(2N-2)l_2})}{1-p^{-2}} \\ & \dots \frac{(1-p^{-2l_N})}{1-p^{-2}} f_{n+l_1+l_2+\dots+l_N}. \end{aligned}$$

Estimating the ℓ^1 norm we see that

$$\begin{aligned} \sum_{n=1}^{\infty}|r_n(2N)| \leq & \sum_{l_1,l_2,\dots,l_N=1}^{\infty} \frac{|\lambda|^N}{p^{2nN}(1-p^{-2})^N}(p^{-(2N-2)l_1}-p^{-2Nl_1})(p^{-(2N-4)l_2}-p^{-(2N-2)l_2}) \\ & \dots(1-p^{-2l_N})|f_{n+l_1+l_2+\dots+l_N}|. \end{aligned} \tag{6.16}$$

Notice that the term $\sum_{l_N=1}^{\infty}(1-p^{-2l_N})|f_{n+l_1+l_2+\ldots+l_N}|$ can be estimated as follows:

$$\begin{aligned}\sum_{l_N=1}^{\infty}(1-p^{-2l_N})|f_{n+l_1+l_2+\ldots+l_N}| &\leq \sup_{l_N\geq 1}(1-p^{-2l_N})\sum_{l_N=1}^{\infty}|f_{n+l_1+l_2+\ldots+l_N}| \\ &\leq \|f\|_1\end{aligned}$$

where the last line can be justified by changing the summation index in the previous line appropriately. Moreover, we can explicitly calculate each sum that appears in formula (6.16). For example:

$$\begin{aligned}\sum_{n=1}^{\infty}p^{-2Nn} &= \frac{p^{-2N}}{1-p^{-2N}}, \\ \sum_{l_1=1}^{\infty}(p^{-(2N-2)l_1}-p^{-2Nl_1}) &= \frac{(1-p^{-2})p^{-(2N-2)}}{(1-p^{-(2N-2)})(1-p^{-2N})}, \\ \sum_{l_2=1}^{\infty}(p^{-(2N-4)l_2}-p^{-(2N-2)l_2}) &= \frac{(1-p^{-2})p^{-(2N-4)}}{(1-p^{-(2N-4)})(1-p^{-(2N-2)})},\end{aligned}$$

and so on. Substituting these values into the formula (6.16) we get the following estimate:

$$\begin{aligned}\sum_{n=1}^{\infty}|r_n(2N)| \leq & \frac{|\lambda|^N}{(1-p^{-2})^N}\cdot\frac{p^{-2N}}{(1-p^{-2N})}\cdot\frac{(1-p^{-2})p^{-(2N-2)}}{(1-p^{-(2N-2)})(1-p^{-2N})}\cdots \\ & \cdots\frac{(1-p^{-2})p^{-2}}{(1-p^{-4})(1-p^{-2})}.\end{aligned}$$

Simplifying this expression we obtain

$$\sum_{n=1}^{\infty}|r_n(2N)| \leq \frac{|\lambda|^N}{p^{N(N+1)}\prod_{k=1}^{N}(1-p^{-2k})^2} \leq \frac{|\lambda|^N}{p^{N(N+1)}\prod_{k=1}^{\infty}(1-p^{-2k})^2}.$$

Since $\prod_{k=1}^{N}(1-p^{-2k})^2<\infty$ we see that $\sum_{n=1}^{\infty}|r_n(2N)|\to 0$ as $N\to\infty$.

Finally we will prove uniqueness of the expansion for $f_n(\lambda)$. Consider the analytic function $f(z)=\sum_{k=1}^{\infty}c(2k)z^k$. From the above estimate of the coefficients $c(2k)$ we see that the radius of convergence R of the power series for f is given by:

$$\frac{1}{R}=\limsup_{k\to\infty}\sqrt[k]{|c(2k)|}\leq|\lambda|\ \limsup_{k\to\infty}\sqrt[k]{\frac{\|f\|_1}{\prod_{i=1}^{\infty}(1-p^{-2i})^2}\cdot\frac{1}{p^{k-3+2/k}}}.$$

Therefore, $R = \infty$ and the function $f(z)$ is entire. Therefore, in particular:

$$f(p^{-2n}) = \sum_{k=1}^{\infty} c(2k)p^{-2nk} = f_n(\lambda),$$

and so the coefficients $c(2k)$ are uniquely determined by $f_n(\lambda)$, because an analytic function is completely determined by its values on a convergent sequence of points [2].

■

Remark 4 *The collection of ℓ^2 functions with a power series representation of the form* (6.9) *is fairly restrictive which is clear from the fact that* $\lim_{n\to\infty} p^{2n} f(n) = c(2)$. *It can be easily shown that the set of ℓ^2 functions with this power series representation is dense in the space of all ℓ^2 functions.*

We can now state our main theorem.

Theorem 6.2.2 *The spectrum of the operator $\widehat{D_0^*}\widehat{D_0}$ consists of simple eigenvalues $\{\lambda_n\}$ which are the roots of the q-hypergeometric function $\lambda \mapsto {}_1\phi_1\left({0 \atop q}; q, \lambda\right)$, with $q = \frac{1}{p^2}$.*

Proof Substituting $f_n = \sum_{k=1}^{\infty} c(2k)p^{-2nk}$ and formula (6.15) into the initial condition of system (6.8) and dividing throughout by $c(2)$ we obtain the following:

$$\frac{1}{p^2} + \lambda - 1 + \sum_{k=2}^{\infty} \frac{\left(p^{-2k} + \lambda - 1\right) \lambda^{k-1} p^{2k-2}}{\prod_{j=2}^{k}(1 - p^{2j})\left(1 - p^{2-2j}\right)} = 0. \tag{6.17}$$

The infinite sum on the left hand side of the above equation, call it S_1, can be simplified by first breaking it up into two terms, extracting some of the terms and then recombining as follows:

$$\begin{aligned} S_1 &= \sum_{k=2}^{\infty} \frac{\left(p^{-2k} - 1\right) \lambda^{k-1} p^{2k-2}}{\prod_{j=2}^{k}(1 - p^{2j})\left(1 - p^{2-2j}\right)} + \sum_{k=3}^{\infty} \frac{\lambda^{k-1} p^{2k-4}}{\prod_{j=2}^{k-1}(1 - p^{2j})\left(1 - p^{2-2j}\right)} \\ &= \frac{\left(p^{-4} - 1\right) \lambda p^2}{\left(1 - p^4\right)\left(1 - p^{-2}\right)} + \sum_{k=3}^{\infty} \frac{p^{2k-4}\lambda^{k-1}\left[p^2(p^{-2k} - 1) + \left(1 - p^{2k}\right)\left(1 - p^{2-2k}\right)\right]}{\prod_{j=2}^{k}(1 - p^{2j})\left(1 - p^{2-2j}\right)} \; . \end{aligned}$$

Using the substitution $q = \frac{1}{p^2}$ equation (6.17) can be written as

$$(q-1) + \frac{\lambda}{(1-q)} + \sum_{k=3}^{\infty} \frac{q^{2-k}\lambda^{k-1}}{\prod_{j=2}^{k-1}(1-q^{-j})\prod_{j=2}^{k}(1-q^{j-1})} = 0.$$

Notice that at $k = 2$ the expression

$$\frac{q^{2-k}\lambda^{k-1}}{\prod_{j=2}^{k-1}(1-q^{-j})\prod_{j=2}^{k}(1-q^{j-1})}$$

yields the value $\frac{\lambda}{(1-q)}$. Thus the above equation is in fact equal to

$$(q-1) + \sum_{k=2}^{\infty} \frac{q^{2-k}\lambda^{k-1}}{\prod_{j=2}^{k-1}(1-q^{-j})\prod_{j=2}^{k}(1-q^{j-1})} = 0.$$

Thus,

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{q^{2-k}\lambda^{k-1}}{\prod_{j=2}^{k-1}(1-q^{-j})\prod_{j=2}^{k}(1-q^{j-1})} &= \sum_{k=2}^{\infty}(-1)^k\lambda^{k-1}\frac{(1-q)q^{\frac{(k-2)(k-1)}{2}}}{\prod_{j=1}^{k-1}(1-q^j)^2} \\
&= -\sum_{k=1}^{\infty}(-1)^k\lambda^k\frac{(1-q)q^{\frac{k(k-1)}{2}}}{\prod_{j=1}^{k}(1-q^j)^2} \\
&= 1 + \sum_{k=1}^{\infty}(-1)^k\lambda^k\frac{q^{\frac{k(k-1)}{2}}}{\prod_{j=1}^{k}(1-q^j)^2} - (q-1).
\end{aligned}$$

By using the notation

$$(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$$

and the above computation, we can rewrite the eigenvalue equation as

$$1 + \sum_{k=1}^{\infty}(-1)^k\lambda^k\frac{q^{\frac{k(k-1)}{2}}}{(q;q)_k^2} = 0.$$

The function ${}_1\phi_1$ of four variables a_0, b_1, q, z is defined as

$${}_1\phi_1\left(\begin{matrix} a_0 \\ b_1 \end{matrix}\ ;\ q^2,\ z\right) = \sum_{n=0}^{\infty}\frac{(a_0;q^2)_n}{(q^2;q^2)_n(b_1;q^2)_n}(-1)^n q^{2\binom{n}{2}} z^n.$$

Thus, if λ is an eigenvalue, we get:

$${}_1\phi_1\left({0 \atop q}; q, \lambda\right) = 1 + \sum_{k=1}^{\infty}(-1)^k\lambda^k\frac{q^{k\frac{(k-1)}{2}}}{(q;q)_k^2} = 0,$$

showing that the eigenvalues of the operator $\widehat{D_0^*}\widehat{D_0}$ are the roots of the above q - hypergeometric function. Conversely, the above calculation shows that given a root λ of $\lambda \mapsto {}_1\phi_1\left({0 \atop q}; q, \lambda\right)$ the formula (6.9) with arbitrary $c(2)$ and other coefficients $c(2k)$ given by (6.10) gives, up to a constant, the unique eigenvector of $\widehat{D_0^*}\widehat{D_0}$ corresponding to eigenvalue λ. By the analysis in [14] the whole spectrum of $\widehat{D_0^*}\widehat{D_0}$ consists of such eigenvalues. ■

Computation of the spectrum of D^*D is based on decomposition (6.7) and the analysis of the spectrum of $\widehat{D_0^*}\widehat{D_0}$ above.

Theorem 6.2.3 *Let $\{\lambda_n\}$ be the eigenvalues of the operator $\widehat{D_0^*}\widehat{D_0}$ and let $\widehat{D}_g^*\widehat{D}_g$ be as in formula* (6.4)*. Then,*

1. *the spectrum of $\widehat{D_g^*}\widehat{D_g}$ consists of simple eigenvalues $\{p^{2m}\lambda_n\}$ i.e., $\sigma(\widehat{D_g^*}\widehat{D_g}) = \bigcup_n\{p^{2m}\lambda_n\}$.*
2. *$\sigma(D^*D) = \sigma(\widehat{D^*}\widehat{D}) = \bigcup_{m,n}\{p^{2m}\lambda_n\}$. Moreover, each eigenvalue of $\widehat{D^*}\widehat{D}$ occurs with multiplicity $p^m(1-\frac{1}{p})$.*

Proof The above results follow directly from the decomposition (6.7). Since the number of different values of r less than p^m that are relatively prime to p is equal to $p^m - p^{m-1}$, each eigenvalue of $\widehat{D}^*\widehat{D}$ in H has multiplicity $p^m(1-\frac{1}{p})$. ■

Corollary 2 *The operator $(D^*D)^{-1}$ belongs to the s-th Schatten class for all $s \geq 1$.*

Proof From the decomposition (6.7) we see that:

$$(D^*D)^{-s} = \bigoplus_{\frac{r}{p^m} \in \mathcal{G}_p} p^{-2ms}\,(D_0^*D_0)^{-s},$$

from which we compute the following trace:

$$\begin{aligned} Tr(D^*D)^{-s} &= \sum_{\frac{r}{p^m} \in \mathcal{G}_p} p^{-2ms}\,Tr(D_0^*D_0)^{-s} \\ &= \sum_{m=0}^{\infty} \sum_{\substack{0 \leq r < p^m \\ p \nmid r}} p^{-2ms}\,Tr(D_0^*D_0)^{-s}. \end{aligned}$$

Since the number of nonnegative r's less than p^m and relatively prime to p is equal to the Euler number of p^m, we can compute the sum over m provided that $s > \frac{1}{2}$:

$$\begin{aligned} Tr(D^*D)^{-s} &= \sum_{m=0}^{\infty}(p^m - p^{m-1})p^{-2ms}\, Tr(D_0^*D_0)^{-s} \\ &= \left(1 - \frac{1}{p}\right)\left(\frac{1}{1-p^{1-2s}}\right) Tr(D_0^*D_0)^{-s}. \end{aligned} \tag{6.18}$$

From [1] we have that $\lambda_n \leq p^n$, so we can estimate the trace $Tr(D_0^*D_0)^{-s} = \sum_{n=0}^{\infty}(\lambda_n)^{-s}$ as follows, provided $s > 0$:

$$Tr(D_0^*D_0)^{-s} = \sum_{n=0}^{\infty}(\lambda_n)^{-s} \leq \sum_{n=0}^{\infty} p^{-ns} = \frac{1}{1-p^{-s}}.$$

Summing up this information we see that,

$$Tr(D^*D)^{-s} \leq \left(1 - \frac{1}{p}\right)\left(\frac{1}{1-p^{1-2s}}\right)\left(\frac{1}{1-p^{-s}}\right)$$

whenever $s > \frac{1}{2}$. Thus for any $s \geq 1$ the s-th Schatten norm of $(D^*D)^{-1}$ is finite. ■

6.3 Analytic continuation of the zeta functions

Using formula (6.18) we can express the zeta function associated with the operator D^*D, denoted $\zeta_D(s)$, in terms of $\zeta_{D_0}(s)$, the zeta function associated with the operator $D_0^*D_0$:

$$\zeta_D(s) = \left(1 - \frac{1}{p}\right)\left(\frac{1}{1-p^{1-2s}}\right)\zeta_{D_0}(s). \tag{6.19}$$

We now consider the analytic continuation of $\zeta_{D_0}(s)$.

Theorem 6.3.1 *$\zeta_{D_0}(s)$ is holomorphic for $\Re s > 0$ and can be analytically continued to a meromorphic function for $\Re s > -2$.*

Proof To show that $\zeta_{D_0}(s)$ is holomorphic in the region $\Re s > 0$ we estimate:

$$|\zeta_{D_0}(s)| \leq \sum_{n=1}^{\infty}\left|\frac{1}{\lambda_n^{\Re s + i\,\Im s}}\right| = \sum_{n=1}^{\infty}\frac{1}{\lambda_n^{\Re s}}\ ,$$

since $\lambda_n^{-i\Im s}$ is unimodular. From [1] we know that the eigenvalues λ_n of $D_0^*D_0$ satisfy the following upper and lower bounds:

$$p^n\left(1-\frac{p^{-2n}}{1-p^{-2n}}\right)<\lambda_n<p^n. \tag{6.20}$$

Thus we get

$$|\zeta_{D_0}(s)|\leq\sum_{n=1}^{\infty}\frac{1}{\left(p^n\left(1-\frac{p^{-2n}}{1-p^{-2n}}\right)\right)^{\Re s}}\quad\text{for }\Re s>0.$$

We have an elementary inequality

$$\frac{p^{-2n}}{1-p^{-2n}}=1-\frac{1}{p^{2n}-1}\geq\frac{p^2-2}{p^2-1},$$

which holds since the left-hand side is an increasing function of n, while the right-hand side is its value at $n=1$. Therefore, for $\Re s>0$ we get

$$|\zeta_{D_0}(s)|\leq\left(\frac{p^2-1}{p^2-2}\right)^{\Re s}\sum_{n=1}^{\infty}\frac{1}{p^{n\Re s}},$$

which is convergent for $\Re s>0$. Consequently, $\zeta_{D_0}(s)$ is holomorphic in $\Re s>0$.

We now show that $\zeta_{D_0}(s)$ can be meromorphically continued to $\Re s>-2$. Since λ_n behaves like p^n for large n, the analytic continuation of $\zeta_{D_0}(s)$ will be achieved by a perturbative argument from the meromorphic function obtained from the zeta function by replacing λ_n with p^n. First we write,

$$p^{-ns}-\lambda_n^{-s}=e^{-sn\ln p}-e^{-s\ln(\lambda_n)}=\int_{-\ln(\lambda_n)}^{-n\ln p}\frac{d}{dt}e^{ts}dt=s\int_{-\ln(\lambda_n)}^{-n\ln p}e^{ts}dt.$$

Thus, we obtain:

$$|p^{-ns}-\lambda_n^{-s}|\leq|s|\int_{-\ln(\lambda_n)}^{-n\ln p}|e^{ts}|dt=|s|\int_{-\ln(\lambda_n)}^{-n\ln p}e^{t\Re s}dt.$$

In this integral we can estimate the integrand by its maximum on the interval of integration $[-\ln(\lambda_n),-n\ln p]$ to arrive at the following estimate:

$$|p^{-ns} - \lambda_n^{-s}| \le \begin{cases} |s|(n\ln p - \ln(\lambda_n))e^{-n\ln p \Re s} & ; \text{if } \Re s \le 0 \\ |s|(n\ln p - \ln(\lambda_n))e^{-\ln(\lambda_n)\Re s} & ; \text{if } \Re s > 0. \end{cases}$$

Inequality (6.20) implies that

$$\ln\left(\frac{p^n}{\lambda_n}\right) < -\ln\left(1 - \frac{p^{-2n}}{1-p^{-2n}}\right) = \frac{p^{-2n}}{1-p^{-2n}} \sum_{k=0}^{\infty} \frac{1}{k+1}\left(\frac{1}{p^{2n}-1}\right)^k.$$

Since $1 - p^{-2n} > \frac{1}{2}$ for $n \ge 1$, we can estimate the above as:

$$\ln\left(\frac{p^n}{\lambda_n}\right) < 2p^{-2n} \sum_{k=0}^{\infty} \frac{1}{k+1}\left(\frac{1}{p^2-1}\right)^k = -2p^{-2n}\ln\left(1 - \frac{1}{p^2-1}\right).$$

Consequently, if $\Re s \le 0$, we have

$$|p^{-ns} - \lambda_n^{-s}| \le -2|s|\ln\left(1 - \frac{1}{p^2-1}\right)p^{-2n}p^{-n\Re s}.$$

This allows us to estimate the difference of the series as follows:

$$\left|\sum_{n=1}^{\infty}(p^{-ns} - \lambda_n^{-s})\right| \le -2\ln\left(1 - \frac{1}{p^2-1}\right)|s|\sum_{n=1}^{\infty} p^{-n(2+\Re s)}.$$

The series $\sum_{n=1}^{\infty} p^{-n(2+\Re s)}$ is convergent for $\Re s > -2$ hence, by the Weierstrass M test, the series $\sum_{n=1}^{\infty}(p^{-ns} - \lambda_n^{-s})$ converges uniformly for $\Re s > -2$ and hence it is analytic for $\Re s > -2$.

Moreover, since

$$\sum_{n=1}^{\infty} p^{-ns} = \frac{p^{-s}}{1-p^{-s}}$$

is meromorphic in the complex plane with poles at $s = \frac{2\pi i k}{\ln p}$, $k \in \mathbb{Z}$, we obtain that the zeta function $\zeta_{D_0}(s)$ extends to a meromorphic function for $\Re s > -2$ with the above mentioned poles. ■

Corollary 3 *$\zeta_D(s)$ is meromorphic for $\Re s > -2$ with poles at $s = \frac{2\pi i k}{\ln p}$, and $s = \frac{1}{2}\left(1 - \frac{2\pi i k}{\ln p}\right)$, where $k \in \mathbb{Z}$.*

Proof The proof of this corollary follows from Theorem 6.3.1 above and equation (6.19). ■

APPENDIX

A. Appendix

In this appendix we recall some basic properties and identities satisfied by the q - hypergeometric function ${}_1\phi_1$ encountered in the previous chapter. More on q - hypergeometric functions can be found in [10]. We start with the general definition of these type of functions:

$$ {}_{r+1}\phi_s \left(\begin{array}{c} a_0, a_1, \ldots, a_r \\ \\ b_1, b_2, \ldots, b_s \end{array} ; \, q, \, z \right) = \sum_{n=0}^{\infty} \frac{(a_0;q)_n (a_1;q)_n \ldots (a_r;q)_n}{(q;q)_n (b_1;q)_n \ldots (b_s;q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{s-r} z^n $$

where $b_j \neq q^{-n}$ for any j, n.

Here we used the notation $(a;q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1})$. We remark that $(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$. When $s > r$ the above series converges for all z while it converges for $|z| < 1$ when $s = r$.

In this thesis, we are interested in the special case where $r = 0, s = 1$ and $a = 0, b = q$, which leads to the formula

$$ {}_1\phi_1 \left(\begin{array}{c} 0 \\ \\ q \end{array} ; \, q, \, z \right) = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n (q;q)_n} (-1)^n q^{\binom{n}{2}} z^n. $$

The function ${}_1\phi_1$ satisfies the Cauchy's sum:

$$ {}_1\phi_1 \left(\begin{array}{c} a \\ \\ b \end{array} ; \, q, \, b/a \right) = \frac{(b/a;q)_\infty}{(b;q)_\infty}. $$

In [1] the authors investigated the roots of the third Jackson q-Bessel function:

$$ J_\nu^{(3)}(z;q) := z^\nu \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \, {}_1\phi_1 \left(\begin{array}{c} 0 \\ \\ q^{\nu+1} \end{array} ; \, q, \, qz^2 \right), $$

where $0 < q < 1$ and ν is a complex parameter. This function has infinitely many simple zeros all of which are real. When $\nu = 0$ we see that the function (A) is equal to the function ${}_1\phi_1$ which we used in this thesis.

We record the following transformation property of ${}_1\phi_1$:

$$ {}_1\phi_1\left(\begin{array}{c} 0 \\ b \end{array} ;\, q,\, z\right) = \frac{(z;q)_\infty}{(b;q)_\infty}\, {}_1\phi_1\left(\begin{array}{c} 0 \\ z \end{array} ;\, q,\, b\right). $$

Starting with this transformation the authors in [1] deduce that if $q < (1-q)^2$ then the positive roots $\omega_k(q)$, $k \in \mathbb{Z}_{\geq 1}$, of $J_0^{(3)}(z;q)$, arranged in the increasing order satisfy the following:

$$ q^{-k/2+\alpha_k(q)} < \omega_k(q) < q^{-k/2}, $$

where

$$ \alpha_k(q) = \frac{\log\left(1 - \frac{q^k}{1-q^k}\right)}{\log q}. $$

In particular, this gives the asymptotic behavior $\omega_k \sim q^{-k/2}$ as $k \to \infty$. Additionally, those results give an upper and lower bound (6.20) for the roots of the specific ${}_1\phi_1$ function needed in this thesis.

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VITA

VITA

Sumedha Rathnayake is an avid Sherlock Holmes fan. Below is a quote from "A study in scarlet".

"In solving a problem of this sort, the grand thing is to be able to reason backwards. That is a very useful accomplishment, and a very easy one, but people do not practise it much. In the every-day affairs of life it is more useful to reason forwards, and so the other comes to be neglected. There are fifty who can reason synthetically for one who can reason analytically...Let me see if I can make it clearer. Most people, if you describe a train of events to them, will tell you what the result would be. They can put those events together in their minds, and argue from them that something will come to pass. There are few people, however, who, if you told them a result, would be able to evolve from their own inner consciousness what the steps were which led up to that result. This power is what I mean when I talk of reasoning backwards, or analytically."

Sherlock Holmes-A Study in Scarlet