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A QUEUEING MODEL FOR TOKEN-PASSING COMPUTER NETWORKS

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A QUEUEING MODEL FOR TOKEN-PASSING COMPUTER NETWORKS

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Abstract

Token-passing computer networks operating on ring and bus topologies can be modelled as multiqueueing systems. The free token behaves as a server (or key to channel access) that provides each station with a chance to use the channel for a finite time. The strictly cyclic service pattern and non-exhaustive service causes interference between queues, and consequently, dependencies between the different queueing processes. In the past, an assumption of independence between queues (station independence) has commonly been used for analyzing such systems. In this paper an exact method is presented, based on a Markov chain embedded in a certain semi-Markov process. The result yields a computational form for the steady-state distribution of the token's random cycle-time on th network. This result confirms the inadequacy of the station independence assumption at all but very extreme system loads. In addition to the exact res an assumption of packet independence in token-passing networks is introduced, in order to simplify computational effort. It is shown that unlike station independence, the packet independence assumption works remarkably well. Using asymmetric Poisson arrivals and otherwise general distributions, the cycle-time distribution of the token is computed exactly, as well as approximately under both kinds of independence assumption The method we introduce leads to a vacationing-server based queueing model for general, asymmetric token-passing systems. A few applications of cyle-times are included, for mean channel utilization, distribution of channel utilization, and busy and vacation periods of the token with respect to each station.

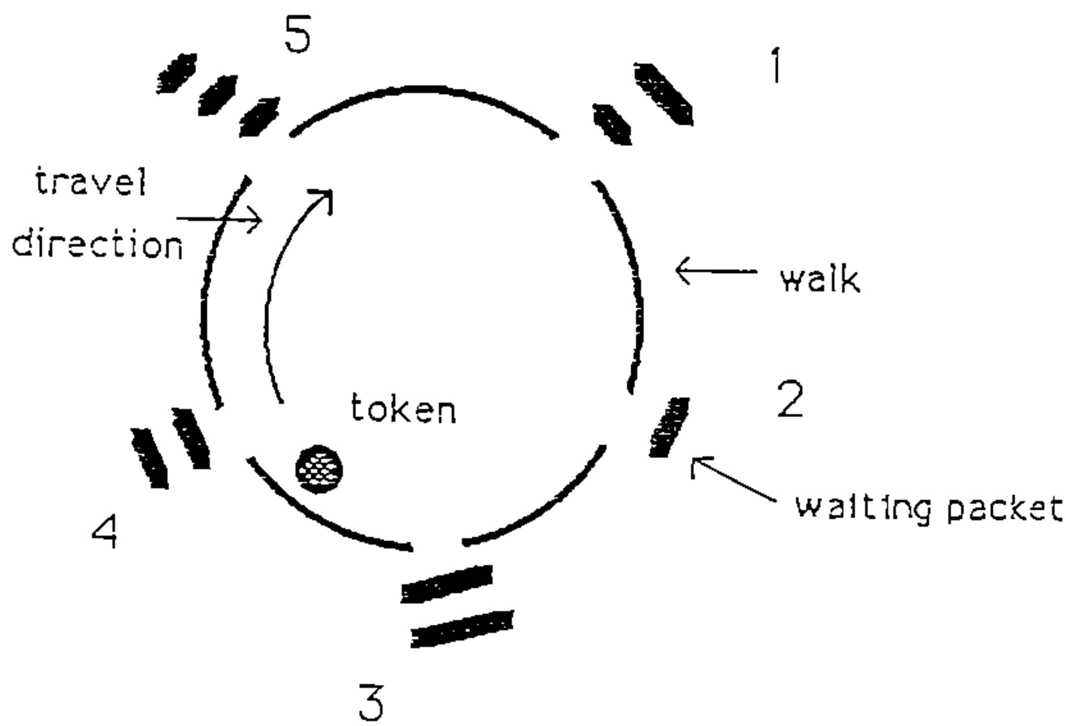

Fig. 1 Conceptual view of Token-passing on bus

1. INTRODUCTION

The token-passing (contention-free) protocols on ring and bus networks are two of the three access mechanisms presently being standardized by the IEEE standards committee [IEEE84a, IEEE84b]. In principle, the token-passing protocols are like the Newhall networks [FaNe69] in that they use a token to regulate channel access. A station that detects a free token is allowed to use the channel. If the station is ready to transmit and has a packet stored in its buffer, it immediately puts the packet onto the channel. Upon completing its transmission, a station passes the free token in an orderly fashion to the 'next' station on the channel. If a station has no packet to send or is not ready to transmit when it acquires a free token, it simply passes the token along to the next station. In either event, a *cyclic* token-passing sequence of stations is defined.

A token ring [IEEE84a, ECMA83a] is typically configured as a series of point-to-point cables between consecutive stations, with stations tapping onto the ring using active interfaces. The token is a unique signalling sequence of bits that circulates on the communication medium in one of two states, i.e., *free* and *busy*. A station that detects a free token may capture the token, change it to a busy token, and append to it a number of informative bits that go to make up a variable length *packet*. The resulting string of bits is read and forwarded bit by bit (since the topology is point to point) by consecutive stations on the ring. Only the destination station copies each bit of the packet as it passes. When its transmission is complete, the sending station performs certain tasks to ensure proper operation (including taking the packet off the ring) and then creates a new free token which it passes to the next station on the ring.

The token-passing bus is conceptually very similar to the token-passing ring [Buxw84] (see Fig. 1). A token bus [IEEE84b, ECMA83b] is configured as a passive medium, with stations tapping onto the medium via stubs in a multidrop fashion. The bus topology does not impose a sequential ordering of stations, as in the case of the ring. The token is made to circulate on a logical ring instead of a physical one, with a sequence of station addresses defining the token's path. Token bus protocols take advantage of broadcast mechanisms in executing the difficult tasks of establishing and maintaining the logical ring. Each station on the logical ring is required to know its predecessor and successor. In steady-state, the protocol is seen to alternate between packet broadcasts to destination stations, and token-passing broadcasts to successor stations.

Within the framework of the token-passing mechanism just described, there is room for flexibility in protocol design. For example, the *single token* rule [Buxw81] versus the *multiple token* rule [BCJK82]. A more important design issue is the maximum length of time that a station is allowed to retain control of the transmission medium, i.e., a station's *channel-retention* time. In this paper we are concerned with token-passing systems for which

(i) each station is assumed to have unlimited buffer space for queueing packets, and

(ii) each station may transmit at most one packet at each instance of free token acquisition.

The outline of the rest of the paper is as follows. A comparison between token-passing overhead on buses and rings is given in the following subsection. In section 2 the general asymmetric queueing model is introduced along with a brief review of past work, a motivation for this work, and a summary of our results. Due to page limitation we focus our attention mainly on one aspect of the model, i.e., the token's cycle-time distribution. An interesting application of this distribution is in obtaining the distribution of channel utilization on token-passing schemes. In section 3 we derive the token's cycle-time distribution using three different methods, two of them approximate (since they use independence assumptions) and an one method exact (i.e., with no assumptions besides mild conditions on input distributions). The system analysed is general, in that we assume except for Poisson arrivals, the other input distributions may be general for each station (i.e., an asymmetric system). In section 4 is presented a comparison of the three different approaches along with a numerical example. It turns out that one approximate method (using station independence) fails at all but very extreme loads. An approximation that we introduce (using packet independence) works exceedingly well due to its similarity to the exact method. All three methods require a computational effort that is dependent on the number of stations. We are not especially concerned with the computational requirements at this stage because this is the very first exact result on the token's cycle-time distribution, and this enables us to assess the quality of some oft-used independence assumptions. Techniques for improving computational effort is a next step. Some simple applications of the cycle-time approach are given in section 5.

1.2 Token-Passing Delays

It is instructive to examine token-passing delays on ring and bus networks before introducing a queueing model. On a ring, there are two kinds of delays. The first kind is the time required to pass the token from station j to station $(j+1)$. Assuming an asymmetric scenario, this time is clearly a function of the distance between the two particular stations involved, averaging to approximately five microseconds per km of cable. Denote this signal propagation time by $R_1(j, j+1)$. The second kind of delay, say $R_2(j+1)$, is due to station functions, including repeater delay, token alteration time, pattern matching time for token detection etc. Thus, for each pair of neighboring stations $(j, j+1)$, token-passing delay on a ring is simply $R_1(j, j+1) + R_2(j+1)$ microseconds. Note that $R_2(j+1)$ is typically in the order of one bit time.

Token-passing delays on a bus are considerably different from those on a ring. Since a bus operates in broadcast mode, the act of token-passing requires the transmission of a 152 bit long explicit token frame [IEEE84b]. Denote this transmission time for station i to its successor j by $B_1(i, j)$ microseconds. A second delay is the signal propagation delay between the two stations, say $B_2(i, j)$ microseconds. To ensure that all stations hear the token being passed from i to j, $B_2(i, j)$ is not less than the end-to-end bus propagation delay. The third delay is due to station

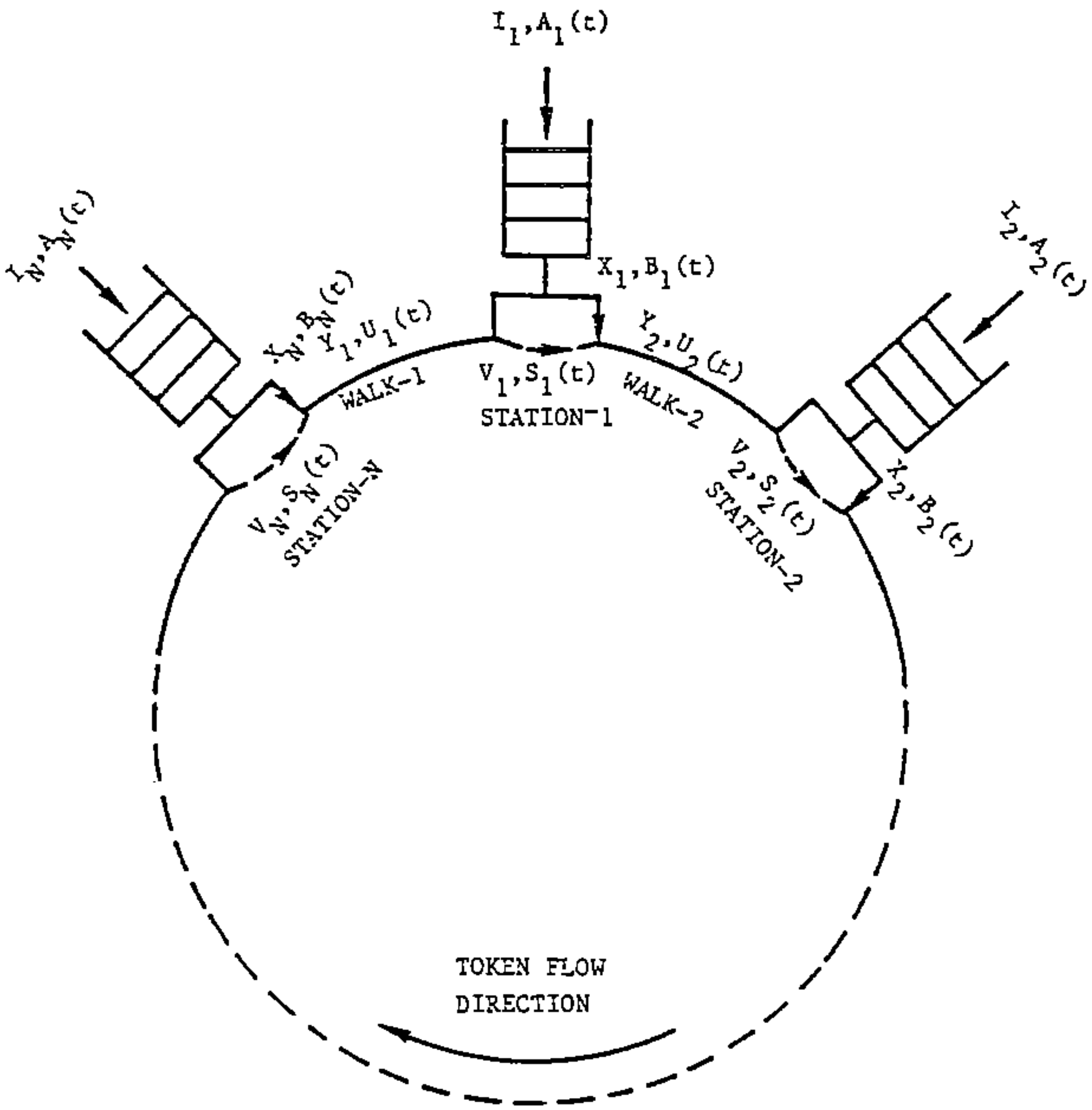

Figure 2. The token-passing protocol is modelled as a multiqueue and cyclic server (MQCS) queueing system.

functions, i.e., the time it takes a station to react from the instant it receives the token to the instant it creates either a token or a data packet for transmission. If we denote this delay by $B_3(j)$, then the token-passing delay on a bus is given by $B_1(i,j) + B_2(i,j) + B_3(j)$.

Because of the conceptual similarities in the token-passing operation on buses and rings, both schemes can be subjected to a uniform queueing analysis. However, it is clear that parameters in each model will differ. From our knowledge of delays we see that

$$\bar{R}_1(j,j+1)+\bar{R}_2(j+1)<\bar{B}_1(j,k)+\bar{B}_2(j,k)+\bar{B}_3(k) \tag{1}$$

in general, with $\bar{X}(\cdot)$ and $\bar{X}(\cdot,\cdot)$ used to denote averages over all stations and pairs of stations, respectively. In other words, the mean token-passing delay between any station and its successor, for comparable ring and bus topologies, is larger for a token bus than a token ring. It is reasonable to conclude that for comparably configured rings and buses (i.e. when the model parameters are made comparable in both meaning and value) the performance of the ring scheme is generally superior to that of the token bus. Note that this only deals with channel utilization and queueing delays and is a consequence of the relatively high token-passing overhead on buses. We make no case for comparisons regarding system reliability, fault tolerance, stability, algorithm and implementation complexity, etc.

2. A QUEUEING MODEL

The model described in this section is applicable to any system of asymmetric multiqueues where a single server provides strictly cyclic service to the system and at most one customer is served at each (infinite capacity) queue. A detailed list of all symbols used is given in the appendix. Consider a system of N independent buffers, chained together to form a ring by sections of varying cable lengths, as shown in Fig. 2. Packet arrivals at station j are generated by some process with interarrival distribution given by $A_j(t)=Pr(I_j \le t)$, where I_j is the interarrival time random variable at station j, $j \in S = \{1,2,...,N\}$.

Denote the *walk* between station $(j-1)$ and station j with label w_j, $j \in S$. Here $(j-1)$ indicates station j's predecessor and $(j+1)$ indicates station j's successor on the path of the token. If the circulating token finds a waiting packet at the buffer of station j, a transmission of random length X_j ensues, with probability distribution function $B_j(t) = Pr\,(X_j \le t)$. If not, it *switches* (i.e., a *small* random time to bypass the empty station) from walk w_j to walk w_{j+1}, taking a random time V_j, with distribution function $S_j(t) = Pr(V_j \le t)$. In most token-passing models it is usually assumed that the switching time is small in comparison to the token-passing and packet-transmission times and is consequently ignored. In any event, after leaving station j, the token spends a random time Y_{j+1} in walk $w_{j+1} \in W = \{w_1, w_2, ..., w_n\}$. The random vartiable Y_j has a distribution function given by $U_j(t) = Pr(Y_j \le t)$, $j \in S$.

We use the term distribution to denote the cumulative distribution function, while the term density is reserved for the probability density function for continuous random variables. For analytic convenience, we assume that all distributions possess finite first and second moments.

2.1 Review, Motivation and Results

Though there has been much work on multiqueueing systems in the past, models for which service is nonexhaustive are very scarce [Buxw84]. To the best knowledge of the authors, no exact model for one-packet service asymmetric token-passing models existed prior to this research effort. A detailed review of closely related nonexhaustive service models and exhaustive service models in the context of token-passing can be found in [Rego85].

The approximate multiqueueing methods found in the literature that are applied to analyze token-passing schemes can loosely be categorized as follows. One class of models are the *independence assuming* models, e.g., [Kueh79, HaOh72, Heym83], and the other class of models involve applications of "resembling" models, i.e., models whose behavior resemble the behavior of the system of interest. In actual implementations (e.g., Ringnet) token-passing schemes allow each station a fixed time for channel usage. Generally, this is translated to mean only one-packet-at-a-time. However, it is not unusual to find approximate models in the literature that use exhaustive service systems [KoMe74, Swar77] to model-token passing [Buxw81, ChLL82, LiHG82]. Another kind of resemblance model is the finite buffer model (i.e., buffering for at most one packet) [MaMW57, Kaye72] applied as in [Buxw81, WuCh75] etc. A recent exact result is one that analyses asymmetric systems but applies *gated* service (i.e., only those customers recorded by the server at the server's scan instant at each queue are served) [FeAm85]. With asymmetry, these systems are generally not as fair as the one-packet-at-a-time service systems. Another recent model involves a decomposition rule and has been applied in the case of infinite N [FuCo85] and symmetric queues [Fuhr84].

The contribution of this effort can briefly be summarized as follows. Following [Kueh79, HaOh72], we first make the assumption that the events "station j is empty" and "station k is empty" are independent for each pair of stations $j, k \in S, j \neq k$. Using this *assumption of station independence*, abbreviated as SIN, we present the token's cycle-time distribution. The details of this derivation can be found in [Rego85]. Next, we derive the *exact cycle-time distribution under no independence assumptions* (abbreviated NIN), and demonstrate the method for a two-station system. Following this, we make the observation that computation can be considerably simplified if we assume that, for a given station, two packets transmitted in two different cycles can be interchanged without affecting the analysis. This is an *assumption of packet independence*, or PIN, almost identical to the assumption used by Kleinrock in the Arpanet models [Klei76]. Indeed, using a numerical example we show that PIN performs exceedingly well for our example. The example demonstrated utilizes station loads chosen with some care, i.e., *moderate loads* that cause SIN to fail, while PIN and NIN compare very well. Finally, we present some simple applications of cycle-times such as the token's busy-period distribution and the exact distribution of channel utilization.

3. CYCLE-TIME DISTRIBUTIONS

The *cycle-time* of the token can be defined as the random time between two consecutive *appearances* of the token at an arbitrary reference station j, $j \in S$. For convenience the reference point is taken to be the point at which the token (or server) *enters* station j to scan for a packet (or customer). Thus, the cycle-time random variable C is the time between two consecutive scan instants at station j. Since the server actually *visits* each station in each cycle, we may equivalently take the reference point to be the point at which the server *exits* from station j.

The existence of a stationary distribution $F_C(\cdot)$ for the cycle-time random variable can be proved in a straightforward manner. We must assume that given an N-station system, all queue-length distributions are stationary. First assume that all queues are *stable*. (i.e., all mean queue lengths are finite). The condition for stability is presented in section 5. For a stable system, the time points corresponding to *all* empty queues are regeneration epochs, meaning that at these points in time the system regenerates itself, with a future that is independent of the past. By hypothesis, the first two moments of all input distributions are finite, i.e., the *mean* and *variance* of the cycle-time random variable are finite. With these conditions, it follows that the mean time between regeneration epochs is finite. As a consequence (see theorems 10-4,10-5 of [HeSo82]), the regenerative process and the cycle-time distribution are asymptotically stationary. Removing the assumption of stability for any queue, some combination of queues, or even all queues does not change things. That is, we can still identify regeneration points and show that the mean time between regeneration points is always finite. Thus, the existence of $F_C(\cdot)$ is proved.

3.1 An Approximation Under Station Independence

In this section is presented the distribution of cycle-time under SIN. For each station j in S, let us assume that $A_j(\cdot)$, $B_j(\cdot)$, $S_j(\cdot)$ and $U_j(\cdot)$ are negative exponential distributions, with parameters $\frac{1}{\lambda_j}$, $\frac{1}{\mu_{j0}}$, $\frac{1}{\mu_{j1}}$ and $\frac{1}{\alpha_j}$, respectively. Let L_i be a Bernoulli random variable reflecting the status of queue i at its steady-state scan instants, for all $i \in \mathrm{S}$. That is, $L_i = 0$ if the server finds queue i empty and $L_i = 1$ otherwise, for all $i \in \mathrm{S}$. The approximation we obtain is a direct result of the following assumption.

Assumption of Station Independence (SIN)

The random variables L_i and L_j are independent, for all $i, j \in \mathrm{S}$, $i \neq j$.

The above assumption appears to have been used (in similar form) in multiqueues with unrestricted buffers in the very early sixties [Lieb61], and since then in similar applications [HaOh72, Kueh79, Rego85]. With the aid of this assumption and some manipulation, the approximate asymmetric and symmetric forms of the cycle time distribution can be obtained. Interested readers are referred to [Rego85] for details of the derivation. With SIN, C becomes a sum of independent random variables, i.e.,

$$C = \sum_{k \in \mathrm{S}} X_k^* + \sum_{k \in \mathrm{W}} Y_k \tag{2}$$

where X_k^* is distributed as the mixture $p_k B_k(\cdot) + q_k S_k(\cdot)$ representing time spent by the server at station k (in either serving or switching) and Y_k is distributed as $U_k(\cdot)$, representing the time taken by the server to walk from station $(k-1)$ to station k, for each k. Here, p_k is the probability that station k is found nonempty by the server, and $q_k = 1 - p_k$, for $k \in \mathrm{S}$. The distribution of C in the asymmetric case is given by

$$f_C(t) = \sum_{j \in \mathrm{S}} \sum_{\mathrm{k} \in \Theta} \prod_{i \in \mathrm{S}} a_{ik_i} \beta_j \left\{ \frac{e^{-\alpha_j t}}{\prod_{m \in S} (\mu_{mk_m} - \alpha_j)} + \sum_{n \in S} \frac{e^{-\mu_{nk_n} t}}{\prod_{\substack{m \in S \\ m \neq n}} (\mu_{mk_m} - \mu_{nk_n})(\alpha_j - \mu_{nk_n})} \right\} \tag{3}$$

where Θ is the set of all N-bit binary vectors $\mathrm{k} = (k_1, \ldots, k_N)$, $a_{j0} = p_j \mu_{j0}$, $a_{j1} = q_j \mu_{j1}$, $p_j + q_j = 1$, with $p_j = \lambda_j E(C)$, for all $j \in \mathrm{S}$. The coefficient β_j is a generalized Erlang coefficient, given by

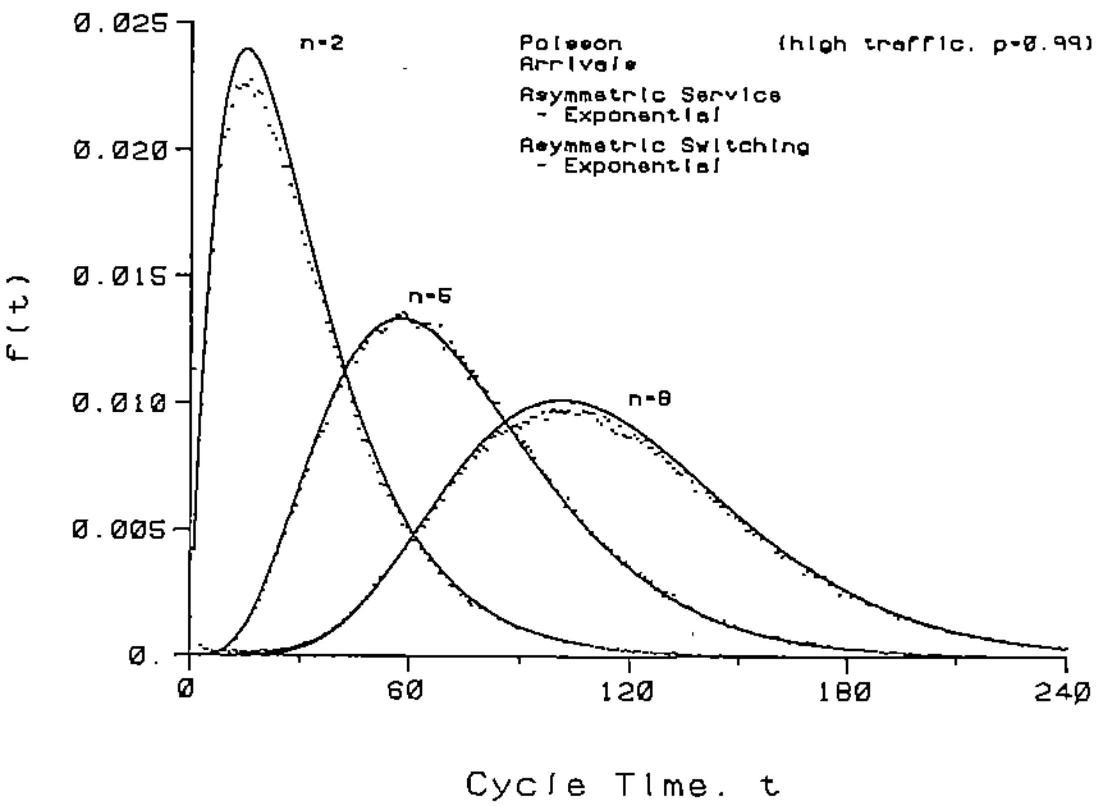

Fig. 3 Simulated versus approximate densities for high loads

$$\beta_j = [\prod_{\substack{i \in S \\ i \neq j}} \frac{\alpha_i}{(\alpha_i - \alpha_j)}]\,\alpha_j\,, \qquad \text{for all } j \in S. \tag{4}$$

The mean cycle time $E(C)$ is obtained as

$$E(C) = \frac{\sum_{j \in S} [E(Y_j) + E(V_j)]}{[1 - \sum_{i \in S} \lambda_i E(X_i) + \sum_{i \in S} \lambda_i E(V_i)]} \tag{5}$$

Mean Channel Utilization

As an immediate application, an approximate mean channel utilization measure is obtained as follows. For each $\mathbf{k} \in \Theta$, let $\pi_{\mathbf{k}}$ be the (joint) probability that the *particular configuration of stations* which transmit during a steady-state cycle are given by the nonzero bits of the vector $\mathbf{k}$. That is, $k_i = 1$ if station i is included in the set of stations that transmit in the same cycle, and 0 otherwise. Under the assumption of all stationary queue length distributions, the limiting joint distribution $\{\pi_{\mathbf{k}}\}$ will exist. As an example, in a two-station system, SIN will yield the distribution $\pi_{00}=q_1 q_2$, $\pi_{01}=q_1 p_2$, $\pi_{10}=p_1 q_2$, and $\pi_{11}=p_1 p_2$. Note that the mean cycle-time $E(C)$ can be got by using this joint distribution. That is,

$$E(C) \;=\; \sum_{\mathbf{k} \in \Theta} \pi_{\mathbf{k}} \sum_{j \in S} [k_j E(X_j) \;+\; (1-k_j) E(V_j)] \;+\; \sum_{j \in W} E(Y_j) \tag{6}$$

Let Y denote the total overhead in walking, i.e., $Y = Y_1\, Y_2 + ... + Y_N$. The mean channel utilization U^* of the entire system of stations is the sum of each individual station's mean channel utilization. Hence U^* is given (with the SIN assumption) by

$$U^* \;=\; \sum_{\mathbf{k} \in \Theta} \pi_{\mathbf{k}} \sum_{j \in S} \frac{[k_j E(X_j)]}{[E(Y) \;+\; (1-k_j) E(V_j) \;+\; k_j E(X_j)]} \tag{7}$$

A comparison of simulated and analytic densities (obtained under SIN) for $N = 2$, $N = 5$ and $N = 8$ station systems is shown in Fig. 3. From our observations, it appears that the approximation performs very well (when measuring cycle-times) under conditions of extremely high or extremely low loads. However, for other load situations, as we shall see in the next section, the SIN assumption fails.

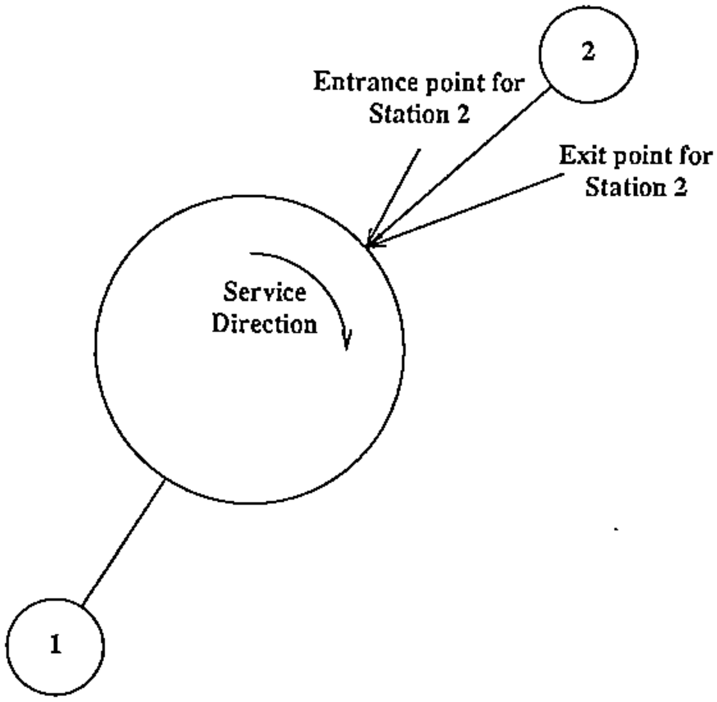
2
Entrance point for
Station 2
Exit point for
Station 2
Service
Direction
1

Fig. 3a

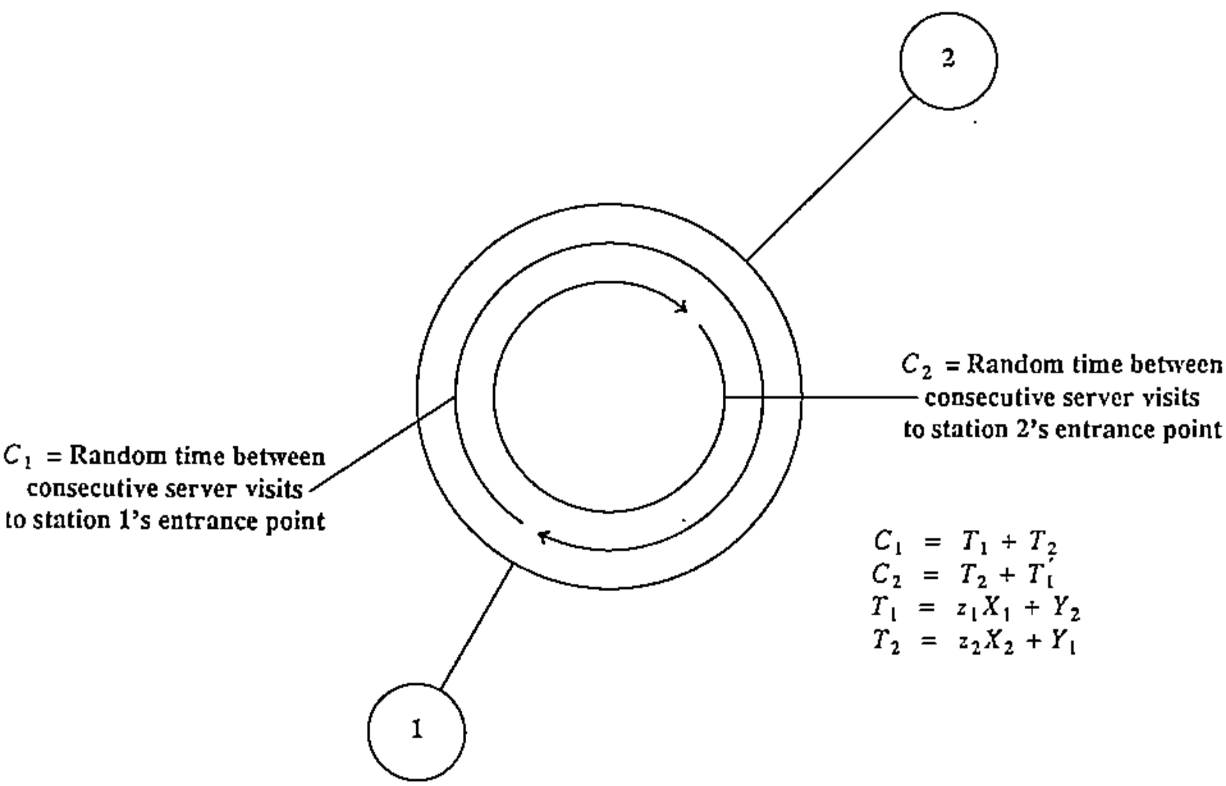
2
C_2 = Random time between
consecutive server visits
to station 2's entrance point
C_1 = Random time between
consecutive server visits
to station 1's entrance point
$C_1 = T_1 + T_2$
$C_2 = T_2 + T_1'$
$T_1 = z_1 X_1 + Y_2$
$T_2 = z_2 X_2 + Y_1$
1

Fig. 3b

3.2 An Exact Result for Cycle-Times

In this section we derive the cycle-time distribution of the token without any assumptions of independence (i.e., NIN). Besides Poisson arrivals, we allow for arbitrary input distributions and only require that the first two moments of each distribution be finite. In order to show how the distribution under NIN is derived without obscuring the technique, an example is demonstrated for a two-station token-passing system.

We focus our attention on the queueing analogue of the two-station system. For convenience, assume that both stations have zero switching times (i.e., $V_1 = V_2 = 0$). The server walks around the multiqueueing system in a cyclic fashion, making an *entrance* to and an *exit* from each station, as shown in Fig. 3a. On entering a station, at most one customer is served at the station before the server makes an exit from the station and begins to walk to the next station. Let the cycle-time random variable C denote the time between two consecutive server exits either from station 1 or station 2. Since C is the server's interappearance time regardless of the point at which these times are measured, C is independent of station index (see for example, [Kueh79]).

Consider an *observer* who stands at the *point of exit* from station 2 (shown in Fig. 3b) and measures instances of the random length C with respect to this exit point. In order to do this efficiently, the observer works out an agreement with the server to do the following. On every cycle made with respect to the observer's position, the server constructs a two-bit binary vector $z = \langle z_1, z_2 \rangle$, where $z_i = 1$ if a customer was served at station i, and $z_i = 0$ otherwise, for $i = 1, 2$. Each time the server passes by the observer, the server hands the observer the most recent service-vector z (abbreviated SV z) constructed in this way. Clearly, in the two-station case the observer receives a vector z from the set $\Theta_2 = \{00, 01, 10, 11\}$ each time he greets the server.

Define the *current state* of the observer to be his *most recently acquired* SV. The *next state* of the observer is given by the *next* SV given to him by the server. If SV $Z_n = \langle z_1, z_2 \rangle$ is given to the observer at time t_n, and SV $Z_{n+1} = \langle z_1', z_2' \rangle$ is given to the observer at time t_{n+1}, then we say that the observer makes a transition from *current state* Z_n to *next state* Z_{n+1} at time instant t_{n+1}. Thus, at steady-state operation, the observer can be viewed as a randomly moving particle in the finite set Θ_2, described in discrete transition steps by the process $\{Z_n, n \geq 0\}$. The following result shows that the observer behaves in a Markov fashion over the space Θ_2. In a forthcoming paper, we present the general counterpart of this result for Θ_N, i.e., in the N station case. Without any loss of generality, the two-station case is presented.

THEOREM *(On a Markov Property of the Observer)*

Let A_k be an exponential distribution function, and let B_k and U_k be general input distributions, all with finite first and second moments, $k = 1, 2$. Assume that an observer is placed at the exit point of station 2 and that that the queue-length distributions at both stations are stationary. At steady-state, let $Z_0, Z_1, Z_2, \cdots, Z_n, \cdots$ be a sequence of service-vectors transferred from server to observer at a strictly increasing sequence of times $t_0, t_1, t_2, \cdots, t_n$, respectively, $n \geq 0$, and let $Z(t)$ represent the observer's current state (i.e., last SV the observer received) at time t. Then

(a) $\{Z(t), t \geq 0\}$ is a semi-Markov chain.

(b) $\{Z_n, n \geq 0\}$ is the Markov chain embedded in $\{Z(t), t \geq 0\}$.

PROOF: The proof that follows is by construction. This will enable us to use the method in a numerical example. If we can prove either (a) or (b) then we are done because the other easily follows. We will prove (b). That is, we compute $P(Z_{n+1} \mid Z_n)$ and show that this probability is equal to $P(Z_{n+1} \mid Z_n, Z_{n-1}, \cdots, Z_0)$. Note that a semi-Markov chain (see [Cinl75]) is a semi-Markov process on a finite or countable state space. A semi-Markov process is a process in which the successive states visited forms a Markov chain, and the sojourn time of the process has a distribution which depends on the state being visited (i.e., current state) as well as the next state to be entered. So, the sojourn time of the process in any state is generally not an exponentially distributed random variable.

In the sequel, we compute the probability that the server goes from state Z_n to state Z_{n+1} by using information given by *only by these two states*. Define q_k as the *conditional* probability that, at steady-state, the server finds exactly one customer queued at station k, *conditioned* on this station being nonempty, $k = 1, 2$. By the assumption of stationarity for queue-length distributions, these steady-state conditional probabilities are constant in time and can be computed explicitly as will be shown.

For each transition $<z_1,z_2> \rightarrow <z_1',z_2'>$ made by the observer, define $T_i = Y_{i+1} + z_i X_i$ to be disjoint components of the random sojourn time (i.e., cycle-time) of the observer in the current state $<z_1,z_2>$, before he makes the transition to the next state $<z_1',z_2'>$, and similarly, define $T_i' = Y_{i+1} + z_i' X_i$ to be disjoint components of the observer's sojourn time (i.e., cycle-time) in the next state. Note that subscripts are incremented modulo 2 (i.e., Y_{i+1} is used to denote $Y_{(i \bmod 2)+1}, i = 1, 2$). In other words, during the server-cycle that generates SV $<z_1, z_2>$, T_1 is the time spent by the server in visiting with station 1 and then walking to station 2, and T_2 is described similarly for station 2. During the next server-cycle, which generates SV $<z_1', z_2'>$, T_1' is the time spent by the server in visiting with station 1 and then walking to station 2, and T_2' is described in similar fashion for station 2. Let C_k be the random time between two consecutive server exits from station k, $k = 1, 2$. We can write the system of cycle-times as

$$\begin{aligned} C_1 &= T_1 + T_2 \\ C_2 &= T_2 + T_1' \end{aligned} \tag{15}$$

where it must be noted that for each k, given the SVs $<z_1,z_2>, <z_1',z_2'>$, the random variables T_k and T_k' are *independent*. Given a current state and next state pair, for each $k, k = 1, 2$, T_k is the random time spent by the server at station k plus the time it takes him to walk to the next station in the current state of the observer. When the observer moves into the next state, the time spent by the server at station k plus the walk time to the next station will be *independent* of the server's corresponding walking time in the current state of the observer (since we are given the current and next state). Observe that a packet transmitted by station k (if any) while the observer is in the next state will have a random length that is *independent* of the length of a station k packet (if any) transmitted while the observer is in the current state. Consequently, the independence of T_k and T_k' follows, for $k = 1, 2$.

An observer transition of the general form $<z_1, z_2> \to <z_1', z_2'>$ can occur *only* if station 1 makes the transition $z_1 \to z_1'$ in the random time C_1, and station 2 makes the transition $z_2 \to z_2'$ in the random time C_2. From (15), we see that C_1 and C_2 are *dependent* random variables due to the overlap caused by the random variable T_1' appearing in both C_1 and C_2. Thus, in order to compute the probabilities of joint events at both stations, we require the joint distribution of C_1 and C_2. For now, let us denote the joint density function as $f_{C_1,C_2}(\cdot, \cdot)$, and let $p_k(n, C_k)$ denote the probability of n customer arrivals at station k in the time C_k, $k = 1, 2$. Since only three random variables are involved in the joint distribution for C_1 and C_2, and each random is defined by one bit in the current-state and next state pair, eight possible joint distributions can arise, *all of the same form* but each with different parameters.

Given the current state $z = <z_1, z_2>$ and the next state $z' = <z_1', z_2'>$ of the observer, let $\xi_k(z_k' \mid <z, z'>)$ denote the marginal conditional probability that each station k makes the transition $z_k \to z_k'$ in the time C_k that the server stayed away from this station during this particular observer transition. We only make use of the current and next state information in describing this transition probability. This marginal conditional probability is computed for each station k as

$$\xi_k(z_k' \mid <z, z'>) = \begin{cases} p_k(0, C_k) & z_k = 0,\ z_k' = 0 \\ 1 - p_k(0, C_k) & z_k = 0,\ z_k' = 1 \\ p_k(0, C_k)\, q_k & z_k = 1,\ z_k' = 0 \\ 1 - p_k(0, C_k)\, q_k & z_k = 1,\ z_k' = 1 \end{cases} \qquad (16)$$

for $k = 1, 2$. Recall that q_k is the conditional probability that station k has exactly one customer queued given that it is nonempty, and by our assumption of stationary queues, it exists and is constant in time, for each k. In order to compute the entire vector transition probability for the observer's transition (which is a *joint* transition probability involving both stations), we must use the joint distribution $f_{C_1, C_2}(\cdot, \cdot)$ using the random variables described in (15). From (16), note that we only need to compute the probability $p_k(0, C_k)$ of zero arrivals in C_k, $k = 1, 2$, for each current state and next state pair of observer states. Thus, we compute any vector transition $z = <z_1, z_2> \rightarrow z' = <z_1', z_2'>$ seen by our observer as having the corresponding joint probability

$$P(Z_{n+1} = z' \mid Z_n = z) = \int_0^\infty \int_0^\infty \xi_1(z_1' \mid z, z')\xi_2(z_2' \mid z, z') f_{C_1, C_2}(c_1, c_2)\, dc_1\, dc_2 \qquad (17)$$

Since (16) defines the transition probability using only the random times C_1, C_2 (which depend only on the current and next state of the observer) and the steady-state conditional probabilities q_1, q_2, we have proved (b).

The proof for (a) follows from the fact that $\{Z_n\}$ is the embedded Markov chain (see [Cinl75]) for the semi-Markov chain $\{Z(t), t \geq 0\}$. The chain $\{Z(t), t \geq 0\}$ is semi-Markov because the observer makes transitions in a manner that depends on the current state, next state, *and* the (nonexponentially distributed) time spent in the current state.

□

3.2.1 Computation of the Probability Transition Matrix (under NIN)

Consider the following situation for a two-station system. Packet arrivals at station j are (independently) Poisson with parameter λ_j , $j = 1, 2$. Station j transmits packets with random length X_j which is exponentially distributed with parameter μ_j, $j = 1, 2$. The token-passing time from station $j-1$ to station j is given by the random variable Y_j which is exponentially distributed, with parameter α_j, $j = 1, 2$. For explanatory convenience, we assume that both stations have negligible (i.e., zero) switching times.

The theorem is now applied to compute the probability transition matrix corresponding to the embedded Markov chain of observer transitions. The first step is the computation of the joint cycle-time density $f_{C_1, C_2}(\cdot, \cdot)$ corresponding to each type of transition. This can be obtained by standard methods for the system in (15). Assuming that we have this density, consider a few example transitions. Let Z_n = <00> and Z_{n+1} = <10>. Using (16) we compute

$$\begin{aligned} \xi_1(1 \mid 00, 10) &= 1 - p_1(0, C_1) \\ \xi_2(0 \mid 00, 10) &= p_1(0, C_2) \end{aligned} \tag{18}$$

and from (17) we obtain

$$\begin{aligned} P(10 \mid 00) &= \int_0^\infty \int_0^\infty [1 - p_1(0, c_1)] p_1(0, c_2) f_{C_1, C_2}(c_1, c_2)\, dc_1\, dc_2 \\ &= \int_0^\infty \int_0^\infty (1 - e^{-\lambda_1 c_1})(e^{-\lambda_2 c_2}) f_{C_1, C_2}(c_1, c_2)\, dc_1\, dc_2 \end{aligned} \tag{19}$$

which, after substitution of the joint density of C_1 and C_2, yields the desired probability. As another example,

$$P(01 \mid 11) = \int_0^\infty \int_0^\infty (q_1 e^{-\lambda_1 c_1})(1 - q_2 e^{-\lambda_2 c_2}) f_{C_1, C_2}(c_1, c_2)\, dc_1\, dc_2 \tag{20}$$

where q_1, q_2 behave as constants (by the assumption of stationary queues).

3.3 An Approximation Under Packet Independence

Consider the system of equations in (15) involving the transition components z_i, z_i', and the random variables T_i and T_i', $i = 1, 2$. We were forced to compute the joint density $f_{C_1, C_2}(\cdot, \cdot)$ only because of *overlap*, i.e., the same random variable T_2 appeared in the cycle-time C_1 seen by station 1 as well as the cycle-time C_2 seen by station 2. If we assume that the random cycle-times C_1 and C_2 are independent, we basically assume that the random time T_2 in C_1 is *independent* of the random time T_2 in C_2. In other words, we remove the overlap by assuming that during any particular cycle, station 1 and station 2 each view a transmission from station k as having an *independently* random length from the *same* distribution $B_k(\cdot)$, $k=1,2$. This is another form of the independence assumption that was used extensively in the Arpanet models [Klei76]. Note that the walk times do not contribute much to the overlap in the sense that the random time (Y_1+Y_2) appears in every cycle-time regardless of the events (transmission or no transmission) at the stations.

Under PIN, C_1 and C_2 become indepent random variables. As a consequence, $f_{C_1,C_2}(c_1,c_2)=f_{C_1}(c_1)f_{C_2}(c_2)$, and we can rewrite (17) as

$$P^*(Z_{n+1}=z' \mid Z_n=z)=[\int_0^\infty \xi_1(z_1' \mid z,z')f_{C_1}(c_1)\,dc_1][\int_0^\infty \xi_2(z_2' \mid z,z')f_{C_2}(c_2)\,dc_2] \quad (21)$$

which is in a *product-form*. We use P^* to denote the transition matrix obtained under PIN, to differentiate it from the probability measure P of section 3.2 computed under NIN.

4. A Numerical Example

In this section we use the two-station example introducted in section 3.2.1 with numerical values for the parameters. The probability transition matrices P (under NIN) and P^* (under PIN) are computed to demonstrate our methods. Finally, we obtain the limiting distributions for service vectors seen by the observer at station 2 using the usual assumption of station independence (SIN) and our two new methods, to demonstrate the effect of queue interference (dependence between queues) at moderate loads.

We begin by computing the matrix P^* under PIN. Assume that the system is operating at steady-state. There are four possible service vectors that the observer (at the exit of station 2) can receive from the server, i.e., (00), (01), (10), and (11). The i^{th} entry of each vector tells the observer if station i did or did not transmit a packet in the corresponding cycle, for $i=1,2$. For example, <10> means that station 1 transmitted a packet while station 2 did not.

For a given SV $<i\ j>$, let x_{ij} be the probability that no customers arrive at station 1 during a service cycle represented by vector $<i\,j>$. The corresponding probability for station 2 is denoted by y_{ij}. The 4×4 transition matrix P^* is given by

$$P^* = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \left[\begin{array}{cccc} x_{00}y_{00} & x_{00}(1-y_{00}) & y_{01}(1-x_{00}) & (1-x_{00})(1-y_{01}) \\ x_{01}y_{10}q_2 & x_{01}(1-y_{10}q_2) & q_2y_{11}(1-x_{01}) & (1-q_2y_{11})(1-x_{01}) \\ x_{10}q_1y_{00} & x_{10}q_1(1-y_{00}) & y_{01}(1-x_{10}q_1) & (1-y_{01})(1-x_{10}q_1) \\ x_{11}y_{10}q_1q_2 & x_{11}q_1(1-y_{10}q_2) & y_{11}q_2(1-x_{11}q_1) & (1-y_{11}q_2)(1-x_{11}q_1) \end{array}\right] \end{array}$$

where q_1, q_2 are the conditional probabilities defined earlier. At this stage we have six unknowns, and these are the limiting probabilities for the observer's states, i.e., π_{00}, π_{01}, π_{10} and π_{11}, plus the conditional steady-state probabilities q_1 and q_2. There are several ways in which we can proceed to solve for these. As an example, we demonstrate one such method. Using M/G/1 theory we know that the probability that station i is empty when the server gets there is $p_{0i} = 1 - \lambda_i E(C)$, for $i = 1, 2$. Additionally, we know that

$$\pi_{00} + \pi_{01} = p_{01}$$
$$\pi_{00} + \pi_{10} = p_{02}$$

thus reducing our unknowns to only four. Thus, the problem is reduced to solving a system of four independent equations in four unknowns. On solving this system, we obtain the conditional probabilities q_1 and q_2. We briefly mention that this works in the N station case too. Details on the general result are not within the scope of the current paper. Consider an example for the (moderate load) values $\lambda_1 = 0.0032$, $\lambda_2 = 0.003492$, $\frac{1}{\mu_1} = 198$, $\frac{1}{\mu_2} = 100$, $\frac{1}{\alpha_1} = 1$, $\frac{1}{\alpha_2} = 2$, and $E(V_1) = E(V_2) = 0$. We obtain $E(C) = 174.4186$, $p_{01} = 0.44186$, $p_{02} = 0.39093$, $q_1 = 0.0870366$, and $q_2 = 0.0805674$. On utilizing the q_i, the final form of the matrix is

$$P^* = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \left[\begin{array}{cccc} .9801 & .0103 & .0056 & .0040 \\ .0491 & .7012 & .0097 & .2400 \\ .0478 & .0005 & .5568 & .3949 \\ .0024 & .0342 & .0373 & .9261 \end{array}\right] \end{array}$$

yielding the limiting SV probabilities as $\pi_{00} = 0.3604$, $\pi_{01} = 0.0717$, $\pi_{10} = 0.0497$ and $\pi_{11} = 0.5181$. Note that (6) may be used to ascertain that these probabilities yield the correct value for E(C) (i.e., as given by (5)). If $F_{00}(\cdot)$, $F_{01}(\cdot)$, $F_{10}(\cdot)$ and $F_{11}(\cdot)$ represent the distributions of the random variables corresponding to the service cycles (00), (01), (10), (11) as seen by the observer (i.e., in this case each is a generalized Erlangian of a simple form), we obtain the cycle-time distribution (under PIN) as

$$F_C(c) = \pi_{00} F_{00}(c) + \pi_{01} F_{01}(c) + \pi_{10} F_{10}(c) + \pi_{11} F_{11}(c)$$

In applying NIN, the transition probability matrix P is computed differently (i.e., using the joint distribution of C_1 and C_2). For example, observe the difference between (19) and the entry $P^*(10 \mid 00)$ in matrix P^*. On computing $f_{C_1,C_2}(\cdot,\cdot)$ and applying it in (17), we obtain the transition matrix for SV transitions *under no independence assumptions* as

$$P = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} .9802 & .0094 & .0063 & .0040 \\ .0473 & .6862 & .0080 & .2585 \\ .0512 & .0005 & .5748 & .3735 \\ .0025 & .0359 & .0354 & .9262 \end{bmatrix} \end{array}$$

The limiting SV probabilities under NIN are $\pi_{00} = 0.3614$, $\pi_{01} = 0.0702$, $\pi_{10} = 0.0499$ and $\pi_{11} = 0.5185$. Again, (6) may be used to ascertain that these probabilities yield the correct value for E(C). Observe that the probabilities obtained with PIN are very close to exact probabilities. In this case, the values obtained for the conditional steady-state probabilities are $q_1 = 0.08917018$ and $q_2 = 0.08081105$. As a by-product of our method, we obtain the steady-state probability q_k^* that station k has exactly one customer queued, $k = 1, 2$. That is, $q_1^* = q_1(1-p_{01}) = 0.0498$, and $q_2^* = q_2(1-p_{02}) = 0.0492$.

4.1 Exact versus Approximate Methods

Consider the joint probabilities $\pi_{00} = q_1 q_2$, $\pi_{01} = q_1 p_2$, $\pi_{10} = p_1 q_2$, and $\pi_{11} = p_1 p_2$, where $p_k = (1-p_{0k})$ and $q_k = (1-p_k)$, for $k = 1, 2$.. These probabilities constitute the joint distribution

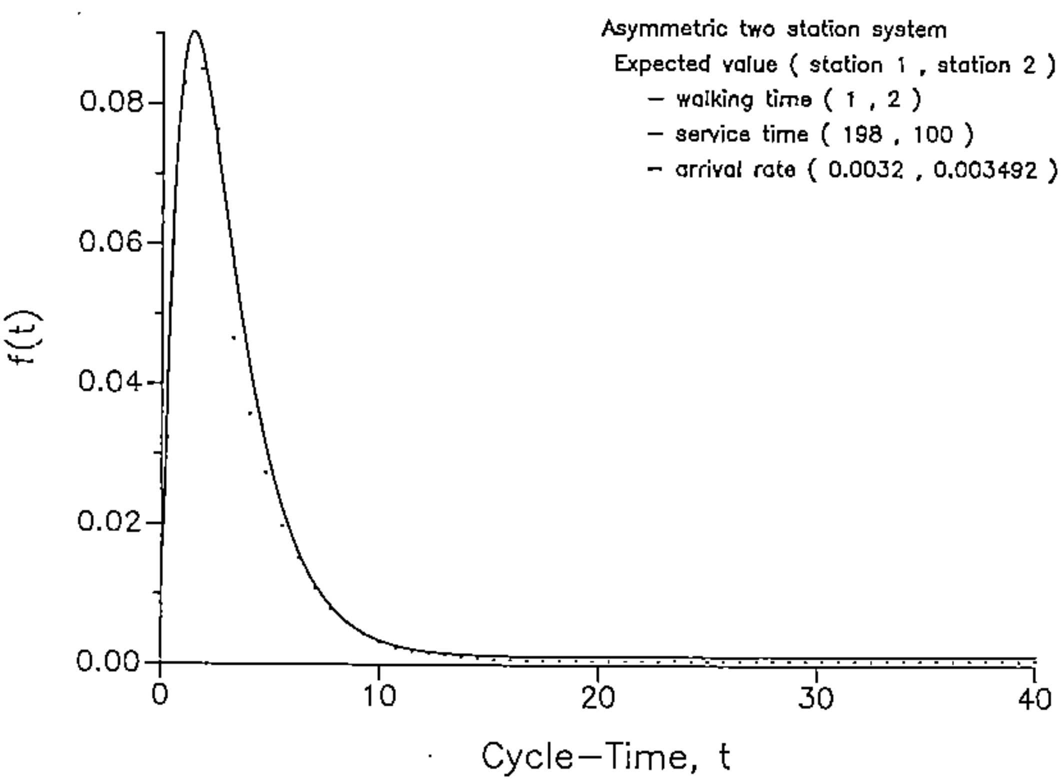

Fig.4 Enlarged view of peak in SV density

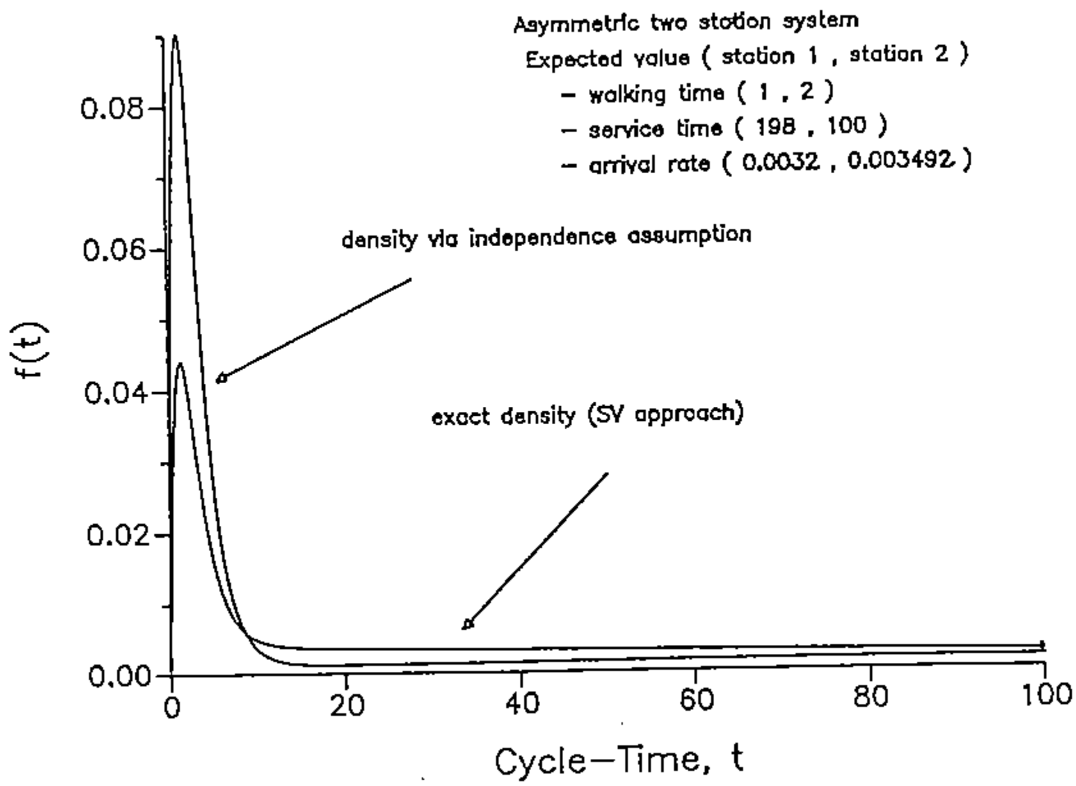

Fig. 5a Breakdown of independence assumption for moderate load

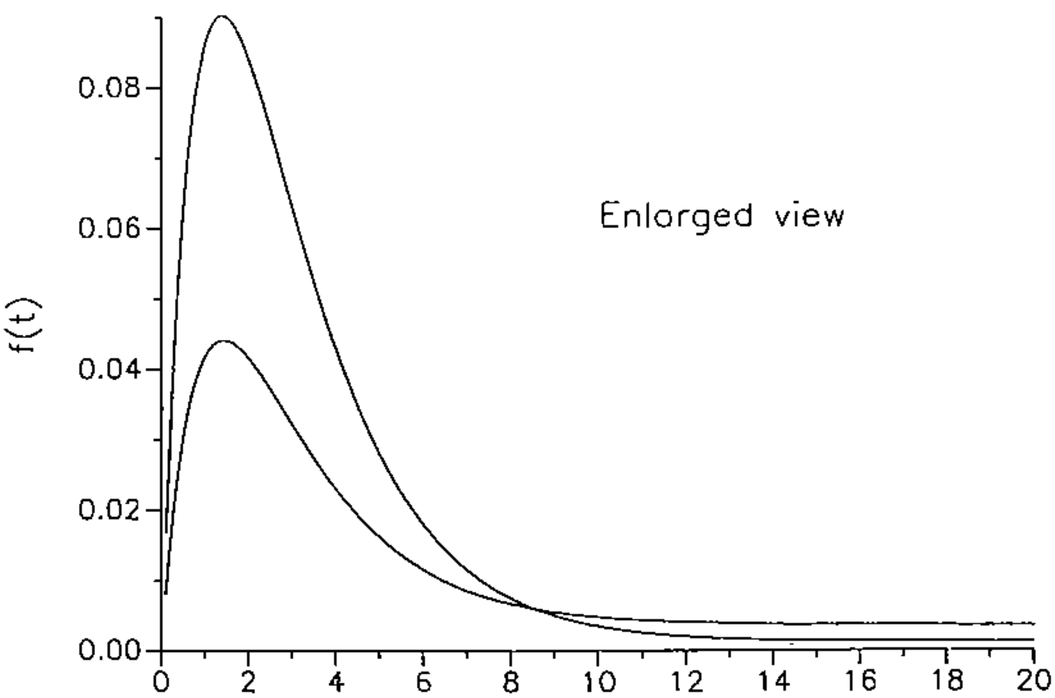

Fig. 5b Enlarged view of 5a

for SVs obtained with the SIN assumption. Though this joint distribution yields the exact mean cycle-time, the cycle-time distribution obtained by using these is incorrect. The reason for this is precisely the inappropriate application of the independence assumption for cycle-times.

One use of the cycle-time random variable is as a "service-time" in the M/G/1 approximation (applied to a single queue at a time) for token-passing queues. This approach has several drawbacks, some of which are discussed in section 5. The advantage of the service-vector approach is that it can be used to solve the queueing problem exactly via a semi-Markov approach [Rego85].

Using SIN, Keuhn [Kueh79] computed the cycle-time variance, and (via simulation) concluded that the results underestimated true cycle-time variance. This can be attributed to dependence of cycle-times, and in particular, the tendency for neighboring cycles to be positively correlated. So SIN can be expected to perform well under very heavy and very light traffic conditions, respectively. For these two extreme conditions, the dependency or covariance between cycles grows small, and actually disappears for unstable and zero traffic conditions. However, for other traffic conditions, SIN can be shown to fail. For example, consider the probabilities obtained in the previous section using the exact and the approximate methods for a system load that is not extreme. SIN appears to neglect covariance information between stations altogether (as is to be expected). It is not clear at this stage whether PIN will do well in all situations. Our current knowledge about the behaviour of PIN is not sufficient to warrant any conjectures. However, we can expect that it will always perform better than SIN, and due to its computational similarity to NIN, we can expect that it will behave well. An investigation of the applicability of these approximations in various situations is presently under way.

In Fig. 4 is shown a comparison of the exact (i.e., obtained under NIN) and simulated cycle-time density for a moderate traffic, asymmetric two station system. In Figs. 5a and 5b the same analytic density is compared with the approximate density obtained from the expression in

Eq.(3) for asymmetric systems (i.e., via SIN). This clearly demonstrates a condition under which SIN fails. In fact, the station independence assumption can be show to degrade for all traffic condition that are not either extremely high or extremely low, especially as N increases. In contrast, PIN does well in comparison to NIN for this numerical example.

5. SOME APPLICATIONS OF CYCLE-TIMES

A (token-passing) system of N stations can be analyzed approximately by viewing each station in isolation as a GI/G/1 queueing system with $F_C(\cdot)$ as a "service-time" random variable. This is a standard approach and has already been used by Hashida and Ohara [Haoh72], and later by Kuehn [Kueh79] via an M/G/1 queueing analysis. Both use the cycle-time distribution obtained under the SIN assumption to obtain the variance of the cycle-time process. In fact, they obtain the variance by resorting to the Laplace-Stieltjes transform and do not actually derive the distribution. Kuehn recognized that this approach tended to underestimate the actual cycle-time variance (i.e., the variance that is obtained by considering a large number of consecutive cycles) and consequently also the mean waiting time. As an alternative, and a way to increase cycle-time variance, Kuehn considered cycle-times seen by station j to be of two types. One type involved customer service while the other type had no customer served at reference station j. The variance of the new cycle-time, created as a mixture of the two different cycle-times, was shown to be an improvement. However, it was observed that, as a queueing approximation, the SIN approach degraded as variance of the $E(X_i)$, $E(Y_i)$, or N, or some combination of these parameters was increased.

Let C_j^* be the random length of a cycle *conditioned* on the event that a station j customer is served during the cycle, $j \in S$. The queueing process at station j is known ([Loyn62]) to posses a steady-state distribution if the condition $\lambda_j < \dfrac{1}{E(C_j^*)}$ is satisfied. Using a result due to Tweedie ([Twee83]), it follows that the queue at station j is *stable* (i.e., has a finite mean) if this condition holds and the cycle-time variance is finite. The finiteness of the variance follows from our assumption that all random variables involved have finite first and second moments.

The approach using cycle-times as service-times for station j has two problems. One is caused by the serial dependency of consecutive cycles. Clearly, long cycles follow tend to follow long cycles, while short cycles tend to follow short cycles, thereby leading to a positive covariance between cycles close together in a sequence. Since service times are supposed to be i.i.d random variables in an M/G/1 system, cycle-time correlations reduce general M/G/1 analytic measures to approximations. The other problem crops up when a station j customer arrives to find an empty queue. Unlike the standard M/G/1 system, the server generally is not readily available. If there are many such critical arrivals, the M/G/1 analysis must surely yield an underestimate in waiting-times, since the forward recurrence time of corresponding cycle-times is neglected. In the next subsection we introduce the notion of ''server vacations'' in order to remedy the situation.

5.1 Distribution of Token Vacation Periods

On an N station system, we place an observer at the exit point of station j and pretend that this is a single server queueing station. Thus, we can embed the queueing process at this station in a server vacation model. The token begins a vacation (as far as station j customers are concerned) at the instant that the token scans an empty queue at station j. The term ''vacation'' is used because a station j customer arriving at the queue *after* this scan instant must wait for the next scan instant to meet the free token.

Define a station j token vacation period to be the random time $v = t_{n-1} - t_{e-1}$, where

(i) $n > e\,,\ e, n \in I^+$

(ii) $z_e \in \Theta_0, z_n \in \Theta_1, z_{e-1} \in \Theta_1$,

(iii) $z_m \in \Theta_0$, $e \le m < n$,

with the sets Θ_0, Θ_1 given by $\Theta_1 = \{z \in \Theta \mid z_j = 1\}$, $\Theta_0 = \Theta - \Theta_1$.

The first $m = 2^{N-1}$ rows and columns of the transition matrix P [see (17)] can be seen to correspond to transitions between elements of Θ_0. Let us label this submatrix as $\mathbf{P}_0$ and define $\mathbf{N}^0(i)$ to be an m-bit column vector $\mathbf{N}^0(i) = 1 - \sum_{k=1}^{m} \mathbf{P}_0(i,k)$. With the elements of Θ_0 labelled as $\{1, 2, ..., m\}$, consider the Markov chain defined on the set $\{1, 2, ..., m+1\}$ with probability transition matrix given by

$$\mathbf{N} = \begin{bmatrix} \mathbf{P}_0 & \mathbf{N}^0 \\ 0 & 1 \end{bmatrix} \tag{22}$$

describing transitions between service vectors for which station j remains empty. Let $F_v(\cdot)$ denote the distribution of the vacation-time random variable. Each vacation cycle-time is a sum of independent random variables [see (15)] and will have a distribution which can be given by a finite convolution. Since each vacation cycle-type vector $z \in \Theta_0$ occurs with a steady-state probability $\pi_z^{(j)}$ (where the index j indicates a limiting vector obtained by an observer standing at the exit point of station j), there is a natural way to formulate the distribution of a random vacation cycle-time as a finite mixture of finite convolutions. Since a token vacation period is comprised of a random number of such cycle-times, $F_v(\cdot)$ is a compound distribution given by repeated convolutions of the mixture. Because of the repeated convolutions, this form is not suited to computation. An alternative form is obtained via an application of phase-type distributions [Neut81], [Lato82].

Assume that each of the N stations utilizes exponentially distributed times for service, switching and walking, with parameters given by $\frac{1}{\beta_i}$, $\frac{1}{\gamma_i}$ and $\frac{1}{\alpha_i}$ respectively, for each $i \in S$. This assumption is made for analytic convenience, since generalized Erlangian distributions lend themselves easily to phase representations. For each vector $z \in \Theta_0$ let $d(z)$ be a unique label in the label set $\{1, 2, ..., m\}$, and let $a_{d(z)}$ be a 2N-bit vector with all entries as zero except for the very first entry, which is a 1. Define an indicator variable $\kappa_i(z)$ to be 1 if bit i in z is a 1, and 0 otherwise. The pair $(a_{d(z)}, T)$ is said to be a representation of the Erlangian distribution $F_{d(z)}(\cdot)$ with entry (i, k) of the order 2N square matrix T being given by

$$T = \begin{cases} -\alpha_i & i \ odd, \ i = k \\ \alpha_i & i \ odd, \ k = (i+1) \\ \kappa_i(z)\beta_i + (1-\kappa_i(z))\gamma_i & i \ even, \ i = k \\ -\kappa_i(z)\beta_i - (1-\kappa_i(z))\gamma_i & i \ even, \ k = (i+1) \end{cases} \tag{23}$$

The number of generalized Erlangian distributions required to completely specify the different vacation cycle-times is $m = 2^{N-1}$. Each vacation cycle-time is thus comprised of 2N phases of a generalized Erlangian distribution, N phases given by the service and switching times, and N phases given by the walk times. We now define a point process [Lato82] with events governed by epochs of transitions of the Markov chain N defined in Eq. (22). Note that each state of this order $(m+1)$ chain has a positive probability of being visited before absorption. If the chain has made a transition to the state i, $1 \le i \le m$, the next transition is to state k, with probability p_{ik}, and the time between transitions has a PH-distribution $F_i(\cdot)$ of order 2N.

Define the vector $\upsilon = (\upsilon_1, ..., \upsilon_m)$ to be the invariant vector corresponding to the ergodic chain given by N. Note that these are conditional probabilities in the sense that we are restricting transitions to be between elements of Θ_0. Let $S(t)$ and $\pi(t)$ denote the state of the Markov chain N at time t and the phase of the Markov chain $T_{S(t)}$ at time t, respectively. Assume that the last event occurred at time τ at which time the chain N made a transition to the state $S(\tau) = k$. Let t denote the current time. Recall that the initial vector chosen for the Markov chain T_k is of order 2N, given by $a_k = (1, 0, ..., 0)$ for all k, $1 \le k \le m$. In the interval $(\tau, t]$, the Markov chain T_k triggers through zero, one, or more than one transition, without entering its absorbing state. At time t , $S(t)=k$, and the chain T_k is in phase $\pi(t)$. We must assume that for $t>0$, the inter-event intervals are conditionally independent given the path of N, so as to make $\{S(t),\pi(t)\}$ a continuous time Markov process.

Given the generator of the process $\{S(t),\pi(t)\}$, the distribution $F_v(\cdot)$ can be obtained as a PH-distribution. Let e be a 2N-bit unit vector. Define an order $m * 2N$ square block-partitioned matrix $\mathbf{A}^*$ with block-entry (i,k) as $\mathbf{A}^*(i,k)=p_{ik}\,\mathbf{e}\,\mathbf{a}_k$, where the vector product denotes the product of a column vector by a row vector. Define also an order $m * 2N$ square block-diagonal matrix $\mathbf{T}^*$ with diagonal block entry i as $\mathbf{T}_i$. The infinitesimal generator of the Markov process $\{S(t),\pi(t)\}$ is given by $\mathbf{T}'=\mathbf{T}^*(\mathbf{I}-\mathbf{A}^*)$, where I is the order $m * 2N$ identity matrix. Thus, the token's vacation-period distribution is a PH-distribution with the representation $(\upsilon,\mathbf{T}')$. A similar procedure can be applied to obtain the token's busy-period distribution as a PH-distribution.

5.2 Distribution of Channel Utilization

As another application of cycle-times and the ideas introduced in the previous section, define U to be the random utilization of the channel by the N-station system at steady-state. If we can determine the distribution $F_U(\cdot)$ of the random variable U , we have a valuable steady-state measure of channel behavior. Observe that U must lie in the interval [0, 1] to be considered a utilization measure, with $U=\alpha$ taken to mean that the channel is busy $100\alpha\%$ of the time during steady-state operation.

In an N-station system, U can be defined as follows. Each vector $z\in\Theta$ given to the observer at station j (by the server) defines a set of stations $\{k\in S \mid z_k=1 \ in \ z\}$ that contributed to channel usage during that particular cycle. The probability π_z corresponding to each $z\in\Theta$ is already known via the methods of the previous section. The random variable U can thus be defined as

$$U = \sum_{z \in \Theta} \frac{b_z \, [\sum_{k=1}^{N} z_k X_k]}{C_N(z)} \tag{24}$$

where b_z is Bernoulli with parameter π_z and $C_N(z)$ is the server's cycle-time, corresponding to the vector z, as seen by the observer at station N. As an example, we present the distribution of U for a two station system.

We assume that the times corresponding to packet interarrivals, packet transmissions and token-passing can be modelled as exponentially distributed random variables. Additionally, we assume that switching times are negligible. Denote these parameter sets for station 1 and station 2 as $(\frac{1}{\lambda_1}, \frac{1}{\beta_1}, \frac{1}{\alpha_1})$ and $(\frac{1}{\lambda_2}, \frac{1}{\beta_2}, \frac{1}{\alpha_2})$, respectively. There are four possible service vectors (as seen by an observer at station 1) and these are 00, 01, 10 and 11. Let the corresponding steady-state probabilities for cycles generating these service vectors be $\pi_{00}, \pi_{01}, \pi_{10}$ and π_{11}, respectively. For binary digits i and j , π_{ij} is the probability that a cycle of length $C_{ij} = Y_1 + Y_2 + i\,X_1 + j\,X_2$ has just occurred, and the actual channel utilization during this cycle is given by $u_{ij} = i\,X_1 + j\,X_2$. If b_{ij} is a Bernoulli random variable with parameter π_{ij} for $(ij) \in S$, then the random variable describing the system's steady-state channel utilization can be given by

$$U = \sum_{(ij) \in S} \left(\frac{u_{ij}}{C_{ij}}\right) b_{ij} \tag{25}$$

where $S = \{00, 01, 10, 11\}$.

The three nonzero ratio random variables defined in Eq. (25), i.e., $D_1 = \frac{X_1}{Y_1+Y_2+X_1}$, $D_2 = \frac{X_2}{Y_1+Y_2+X_2}$, and $D_3 = \frac{X_1+X_2}{Y_1+Y_2+X_1+X_2}$ can be regarded as Dirichlet components [Joko72] of three respective generalized Dirichlet distributions. In fact, since negative exponential random variables are being used, this becomes a special case. With some labour, it can be shown that for $0 \le d_1 < 1$,

$$f_{D_1}(d_1) = \alpha_1\alpha_2\beta_1\{\frac{1}{\alpha_2+d_1(\beta_1-\alpha_2)} - \frac{1}{\alpha_2+d_1(\beta_1-\alpha_2)+(\alpha_2-\alpha_1)(1-d_1)}\} \qquad (26)$$

and for $0 \le d_2 < 1$,

$$f_{D_2}(d_2) = \alpha_1\alpha_2\beta_2\{\frac{1}{\alpha_2+d_2(\beta_2-\alpha_2)} - \frac{1}{\alpha_2+d_2(\beta_2-\alpha_2)+(\alpha_2-\alpha_1)(1-d_2)}\} \qquad (27)$$

and, for $0 \le d_3 < 1$,

$$f_{D_3}(d_3) = A\ d_3 \quad [\frac{1}{(a-b)}\{\ln(\frac{2-d_3}{a+b-bd_3})+\ln(a)\} \qquad (28)$$

$$- \frac{1}{(p-q)}\{\ln(\frac{2-d_3}{p+q-q\ d_3})+\ln(p)\}],$$

where $A = \frac{2\alpha_1\alpha_2\beta_1\beta_2}{(\alpha_1-\alpha_2)}$, $a = \alpha_2+d_3(\beta_2-\alpha_2)$, $b = \alpha_2+d_3(\beta_1-\alpha_2)$, $p = \alpha_1+d_3(\beta_3-\alpha_2)$ and $q = \alpha_1+d_3(\beta_1-\alpha_1)$. Finally, the exact distribution of the channel utilization random variable U can be given as

$$F_U(t) = \pi_{10}\int_0^t f_{D_1}(x)\,dx + \pi_{01}\int_0^t f_{D_2}(x)\,dx + \pi_{11}\int_0^t f_{D_3}(x)\,dx \qquad (29)$$

6. SUMMARY AND CONCLUSIONS

The intent of this paper was to address a problem suggested by Paul Kuehn [Kueh79] to be an open problem, i.e., the cycle-time distribution of a single cyclic server in a multiqueueing system when the service discipline at each queue is strictly at-most-one customer at a time. A motivating factor was the applicability of ideas developed here in the performance of token-passing computer networks.

We introduced a queueing model comprised of a system of dependent queueing processes, and Poisson arrivals. Without specifying distribution particulars (other than desirable properties, such as existence of the first two moments) for the multiqueueing system, we determine that an exact solution for the token's cycle-time distribution can be obtained. The distribution is obtained as a finite mixture, where the mixing distribution is the limiting distribution of an ergodic Markov chain. In the general case, the final form of the cycle-time distribution will require the computation of the distributions of sums of independent random variables. There are a number of methods available for doing this (see for example [AlOb82], [MaSa73], [Math83]).

A numerical example is presented for a two-station system to demonstrate the general method. For completeness, we derive and compute approximate forms of the cycle-time distributions based on an independence assumption made by Kuehn [Kueh79] and previously Hashida and Ohara [HaOh72] (we call it SIN, or station independence). However, these authors worked with the Laplace-Stieltjes transform to obtain the first two moments and did not actually compute the distribution. By computing both the exact as well as the approximate distributions, we demonstrate a breakdown of the SIN assumption and confirm some empirical observations made by Kuehn. The exact and approximate results are also compared to results obtained through simulation.

In the course of the analysis, we determine that the computation is simplified if we assume that packets transmitted by stations in different cycles have independently random lengths (we call this PIN, or packet independence). Applying PIN to the numerical values showed that PIN did well in this case. How well the PIN assumption does in a variety of situations is a subject that is presently being studied.

The queueing process at a single station can be embedded in a single-server model, where the server is prone to taking vacations [Rego85]. Unlike the classical server-vacation models, the vacation periods of this system turn out to be dependent on service times. As an application of the service vector approach, we obtain the distribution of server vacation periods as a PH-distribution, a form suitable for computation. As another simple application of the exact methods, we define a random variable as a function of service vectors to represent the stochastic channel utilization of the steady-state system. For a two station system it is shown how the distribution of random channel utilization may be readily computed. This gives us the mean channel utilization and also all the moments.

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APPENDIX

SIN	assumption of station independence
PIN	assumption of packet independence
NIN	no independence (assumptions)
N	number of stations
S	set of stations $\{1, 2, \cdots, N\}$
W	set of "walks" $\{w_1, w_2, \cdots, w_N\}$
S^*	$S \cup W$
$A_j(\cdot)$	interarrival time *cdf* for station j arrivals, $j \in S$, ($r{\cdot}v$ is I_j)
$\frac{1}{\lambda_j}$	mean of $A_j(\cdot)$,
$B_j(\cdot)$	packet transmission length *cdf* for station j, $j \in S$, ($r{\cdot}v$ is X_j)
$\frac{1}{\mu_j}$	mean of $B_j(\cdot)$
$S_j(\cdot)$	token's switching time *cdf* for station j, $j \in S$, ($r{\cdot}v$ is V_j)
$U_j(\cdot)$	token's walk time (from station $j-1$ to station j) *cdf*, $j \in S$ ($r{\cdot}v$ is Y_j)
$\frac{1}{\alpha_j}$	mean of Y_j
C	cycle-time of the token (server)
L_i	Bernovll $r{\cdot}v$, set to 0 if queue i is found empty (at steady-state), and 1 otherwise
θ	set of all N-bit binary vectors $\mathbf{k} = (k_1, k_2, \cdots, k_N)$
$Q_{ij}(\cdot)$	kernel (matrix) of transition functions for the semi-Markov process introduced under SIN
$Q(+\infty)$	embedded Markov chain for this semi-Markov process
$\{\pi_j, j \in S^*\}$	limiting state probabilities obtained for embedded Markov Chain (under SIN)
p_*	standard normalization probability used for semi-Markov limiting probabilities
$\{\phi_j, j \in S^*\}$	limiting state probabilities obtained for semi-Markov Chain (under SIN)

U^*	approx. mean channel utilization (under SIN)
z_i	tells if station i transmitted in *current* state of observer
z_i'	tells if station i transmitted in *next* state of observer
SV	service vector
$<z_1, z_2>, <z_1', z_2'>$	current and next state SV_β
θ_2	$\{00, 01, 10, 11\}$
Z_n, Z_{n+1}	observe state at n^{th} and $(n+1)^{th}$ transition steps
$\{Z(t), t \geq 0\}$	semi-Markov Chain given by observer transitions (under NIN)
$\{Z_n, n \geq 0\}$	Markov chain of observer transitions (under NIN)
q_i	steady-state *conditional* probability that station i has exactly one customer queued, *conditioned* on the event that it is nonempty.
T_i	$Y_i + z_i X_i$, station i's component of cycle-time in current state of observer
T_i'	$Y_i + z_i' X_i$, station i's component of cycle-time in next state of observer
C_k	random time between consecutive server exits from station k, $k = 1, 2$
$p_k(0, C_k)$	probability of zero packet arrivals at station k during cycle C_k generated by corresponding pair $<z_1, z_2>, <z_1', z_2'>$, $k = 1, 2$
$\xi_k<z_k' \mid <z, z'>)$	probability that station k makes its part of the transition
$f_{C_1, C_2}(\cdot, \cdot)$	joint probability density for C_1 and C_2
P^*	probability transition matrix obtained under PIN
P	probability transition matrix obtained under NIN
C_j^*	random length of cycle conditioned on the event that a station j customer is served in the cycle
$\boldsymbol{\theta}_1$	$\{z \in \theta \mid j^{th}$ bit in Z is a $1\}$
$\boldsymbol{\theta}_0$	$\boldsymbol{\theta} - \boldsymbol{\theta}_1$
N	conditional Markov chain (see eq. 22) matrix

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