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Stabilizing Switched Linear Systems under Adversarial Switching*

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Abstract

The problem of stabilizing discrete-time switched linear control systems using continuous input by the user and against adversarial switching by an adversary is studied. It is assumed that the adversary has the advantage in that at each time it knows the user's decision on the continuous control input but not vice versa. Stabilizability conditions and bounds on the fastest stabilizing rates are derived. Examples are given to illustrate the results.

1 INTRODUCTION

Switched control systems as a family of hybrid control systems are controlled by two input signals: the (continuous) control input and the (discrete) switching signal (or mode sequence in the discrete-time case). Stabilization of switched control systems is the problem of designing control laws for the control input and possibly the switching signal so that the closed-loop systems are stable.

Stabilizability of switched control systems, especially switched linear control systems, has been a well studied problem [1, 2, 3, 4, 5, 6, 7]. Existing approaches can be roughly grouped into two categories. In the first category, both the continuous control input and the switching signal are utilized to stabilize the systems. Under this assumption, even if none of the subsystems is stabilizable by itself, properly designed continuous controllers and switching laws could still render the switched systems stable. Work in this category includes, for example, [1, 8, 5, 6, 7]. In the second category, only the continuous input is under control, while the switching signal is unknown or a disturbance subject to constraints on, e.g., switching frequency, dwell time, time delayed observability, etc. Typically, it is assumed that the continuous controller is aware of the current mode and can thus be of the form of a collection of mode-dependent state feedback controllers. Examples of prior work in the second category include [3, 9, 4, 10, 11].

The stabilization problem studied in this paper assumes that the user designs the continuous control to stabilize a discrete-time switched linear control system, while an adversary counters the user's effort with the most destabilizing switching sequence. This formulation differs from existing work in the second category above in that it has a different information structure: at any time the

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continuous input is first decided by the user without knowledge of the mode to be deployed; the mode is then chosen by the adversary with full knowledge of the user's decision. This information structure gives advantage to the adversary and makes the stabilization task much more difficult compared to the existing formulations. For example, even if each subsystem can be stabilized from any initial state to zero in one step, it is still possible that the switched system is not stabilizable under adversarial switching (see Example 1). A family of application examples of the problem studied in this paper can be found in the stabilization (or consensus, rendezvous) of networked control systems where the network is attacked by an informed saboteur (see Example 2).

This paper is organized as follows. The σ -resilient stabilization problem is formulated in Section 2. The concepts of irreducible and nondefective systems are introduced in Section 3. In Section 4, lower and upper bounds on the σ -resilient stabilizing rate are obtained. The notions of generating functions are introduced in Sections 5. Finally, some concluding remarks are given in Section 6.

2 Resilient Stabilization

Consider a switched linear controlled system (SLCS)

$$x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad t \in \mathbb{N} = \{0, 1, \dots\}, \quad (1)$$

where $x(\cdot) \in \mathbb{R}^n$ is the state, $u(\cdot) \in \mathbb{R}^p$ (or simply u) is the (continuous) control input, and $\sigma(\cdot) \in \mathcal{M} = \{1, \dots, m\}$ (or simply σ) is the (discrete) switching sequence. In the following, the SLCS will be denoted by $\{(A_i, B_i)\}_{i \in \mathcal{M}}$ with subsystem dynamics (A_i, B_i) for brevity. Denote by $x(\cdot; \sigma, u, z)$ the SLCS solution starting from the initial state $x(0) = z$ under u and σ .

In this paper, we assume that a user specifies the control input u for the purpose of stabilizing the system, while an adversary specifies the switching sequence σ to counter the user's effort. We further assume the following information structure.

Assumption 1 (*Information Structure*). Denote by $\mathcal{F}_t := (x_{0:t}, u_{0:t-1}, \sigma_{0:t-1})$ the causal information available at time $t \in \mathbb{N}$, where $x_{0:t}$ denotes $x(0), \dots, x(t)$, $u_{0:t-1}$ denotes $u(0), \dots, u(t-1)$, and similarly for $\sigma_{0:t-1}$, with the understanding that $\mathcal{F}_0 = (x(0))$. At each time $t \in \mathbb{N}$, assume the user and the adversary determine the control input $u(t)$ and the mode $\sigma(t)$ according to the functional forms $u(t) = u_t(\mathcal{F}_t)$ and $\sigma(t) = \sigma_t(\mathcal{F}_t, u(t))$, respectively.

In other words, the user and the adversary both have access to all the past information including the opponent's decisions when making their decisions at time t ; and the adversary has the additional advantage of knowing the user's decision at time t as well. Both decisions are causal as no future information is utilized. Although the policies $u_t(\mathcal{F}_t)$ and $\sigma_t(\mathcal{F}_t, u(t))$ are in general of the feedback type, the subscript t in both of them allows for open-loop policies, i.e., dependence on t only. The set of all user control policies u and adversary switching policies σ compatible with the assumed information structure are denoted by $\mathcal{U}$ and $\mathcal{S}$, respectively.

Definition 1. The SLCS is called σ -resiliently stabilizable if there exists a user control policy $u \in \mathcal{U}$ such that $x(t; \sigma, u, x(0)) \rightarrow 0$ as $t \rightarrow \infty$ for all $x(0) \in \mathbb{R}^n$ and all $\sigma \in \mathcal{S}$. It is called σ -resiliently exponentially stabilizable if we can find finite constants $K \geq 0$, $\rho \in [0, 1)$, and a user control policy $u \in \mathcal{U}$ so that

$$\|x(t; \sigma, u, x(0))\| \leq K\rho^t \|x(0)\|, \quad \forall t \in \mathbb{N}, \quad \forall x(0), \quad \forall \sigma \in \mathcal{S}. \quad (2)$$

The σ -resilient (exponential) stabilizing rate ρ^* is the infimum of all ρ for which (2) holds.

The σ -resilient stabilizing rate is the slowest exponential growth rate of the state solution achievable by the user's control input against adversarial switching, uniformly in all initial states. It provides a quantitative metric of the σ -resilient (exponential) stabilizability of the SLCS. The SLCS being σ -resiliently exponentially stabilizable is equivalent to $\rho^* < 1$.

Note that in (2), $\|\cdot\|$ can be any norm of $\mathbb{R}^n$. Since all such norms are equivalent, the notions of σ -resilient stabilizability and stabilizing rate do not depend on the choice of the norm. In subsequent analyses and examples, different $\|\cdot\|$ may be chosen depending on the occasion.

The following result follows immediately from the homogeneity of the SLCS.

Lemma 1. *For any $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$, the scaled SLCS $\{(\alpha A_i, \beta B_i)\}_{i \in \mathcal{M}}$ has the σ -resilient stabilizing rate $|\alpha| \cdot \rho^*$.*

Proof. The conclusion is trivial if $\alpha = 0$. Assume in the following $\alpha \neq 0$. First note that the SLCS $\{(\tilde{A}_i = \alpha A_i, \tilde{B}_i = \alpha B_i)\}_{i \in \mathcal{M}}$ has the σ -resilient stabilizing rate $|\alpha| \cdot \rho^*$ since its solutions $\tilde{x}(t; \sigma, \tilde{u}, z) = \alpha^t \cdot x(t; \sigma, u, z)$ when $\tilde{u}(t) = \alpha^t u(t)$, $\forall t$. Second, for any $\beta \neq 0$, the SLCS $\{(\hat{A}_i = A_i, \hat{B}_i = \beta B_i)\}_{i \in \mathcal{M}}$ has the σ -resilient stabilizing rate ρ^* since its solutions are given by $\hat{x}(t; \sigma, \beta^{-1}u, z) = x(t; \sigma, u, z)$, $\forall t$. Combining the above two results yields the desired conclusion. $\square$

Furthermore, the SLCS $\{(A_i, B_i = 0)\}_{i \in \mathcal{M}}$ with all B_i 's set to zero becomes an autonomous SLS whose solutions depend only on σ . In this case, the σ -resilient stabilizability is reduced to the stability under arbitrary switching (i.e., uniform stability) of the autonomous SLS; and the σ -resilient stabilizing rate becomes the joint spectral radius (JSR) [12] of the matrix set $\{A_i\}_{i \in \mathcal{M}}$.

For studying the σ -resilient stabilization problem, the information structure in Assumption 1 can be simplified without loss as follows. First, the set $\mathcal{S}$ of admissible adversary's switching policies can be replaced with the smaller set $\mathcal{M}^\infty$ consisting of all *open-loop* switching policies, namely, switching sequences $\sigma = (\sigma_0, \sigma_1, \dots)$ with $\sigma_t \in \mathcal{M}$ for $t \in \mathbb{N}$. Second, the functional form $u_t(\mathcal{F}_t)$ of admissible user's control policies can be simplified to $u(t) = u_t(x(t))$ since the stabilizability property is entirely based on the behavior (i.e., convergence) of the future state solution, which depends on the past u , σ , and x only through the current state¹. Third, the homogeneity of the SLCS dynamics implies that $u_t(x(t))$ can be assumed to be homogeneous: $u_t(\alpha x(t)) = \alpha u_t(x(t))$, $\forall \alpha \in \mathbb{R}$. Indeed, by taking the restriction of any stabilizing control policy on the unit sphere (denote by $\mathbb{S}^{n-1}$) and extending it to the whole $\mathbb{R}^n$ via homogeneity, we obtain a homogeneous stabilizing control policy.

Theorem 1. *A SLCS is σ -resiliently stabilizable if and only if it is σ -resiliently exponentially stabilizable.*

Proof. We only prove one direction as the other is trivial. Assume the SLCS (1) is σ -resiliently stabilized by a control policy $u \in \mathcal{U}$. Let $x(0) = z \in \mathbb{S}^{n-1}$ be arbitrary. Then, $x(t; \sigma, u, z) \rightarrow 0$ as $t \rightarrow \infty$ for any $\sigma \in \mathcal{S}$.

Claim: there exists $N_z < \infty$ such that, for any $\sigma \in \mathcal{S}$, $\|x(t; \sigma, u, z)\| < \frac{1}{2}$ for some $t \leq N_z$. (3)

Suppose otherwise. Then a sequence of switching sequences $\sigma^{(1)}, \sigma^{(2)}, \dots$ and an increasing sequence of times $N_1 < N_2 < \dots$ exist such that $\|x(t; \sigma^{(k)}, u, z)\| \geq \frac{1}{2}$, $\forall t = 0, \dots, N_k$, for $k = 1, 2, \dots$. At

¹The only exceptions in this paper are the modified control policies in the proofs of Theorem 1 and Theorem 2.

each fixed time t , since the sequence $\sigma^{(k)}(t)$, $k = 1, 2, \dots$, takes values in the finite set $\mathcal{M}$, at least one value, denoted by $\sigma^{(\infty)}(t) \in \mathcal{M}$, must appear infinitely often. Assemble $\sigma^{(\infty)}(t)$ for all t into a switching sequence and denote it by $\sigma^{(\infty)}$. By taking progressively finer subsequences of $(\sigma^{(k)})_{k=0,1,\dots}$ and induction on the time t , it is easy to prove that $x(t; \sigma^{(\infty)}, u, z) \geq \frac{1}{2}$ at all t . This contradicts with the assumption that u is stabilizing, thus proving the claim in (3).

We now modify the control policy u in a small open neighborhood U_z of z so that for any $z' \in U_z$ and any $\sigma \in \mathcal{S}$ the same control input sequence up to time $N_z - 1$ is applied as if the initial state was z . This can be achieved by the user running a system simulator with the simulated initial state z at the moment the state first enters U_z . The modified policy, denoted by $\tilde{u}$, is clearly admissible. Moreover, by shrinking U_z if necessary, we have $\|x(t; \sigma, \tilde{u}, z')\| < \frac{1}{2}$ for some $t \leq N_z$, $\forall z' \in U_z$, for all $\sigma \in \mathcal{S}$. Note that U_z will still be an open neighborhood of z since there are only a finite number of possible switching sequences σ up to time N_z . As $\mathbb{S}^{n-1}$ is compact, the above procedure can be carried out at a finite number of states $z^{(1)}, \dots, z^{(p)}$ for their corresponding neighborhoods $U_{z^{(i)}}$ to cover the entire $\mathbb{S}^{n-1}$. Denote by $\hat{u} \in \mathcal{U}$ the final modified control policy, which can be assumed without loss of generality to be homogeneous, and let $N_{\max} = \max_i N_{z^{(i)}}$. By the above construction, for any $x(0) \in \mathbb{R}^n$ and any $\sigma \in \mathcal{S}$, $\|x(t; \sigma, \hat{u}, x(0))\| < \frac{1}{2}\|x(0)\|$ for some $t \leq N_{\max}$. Further modifying $\hat{u}$ so that the policy restarts itself whenever the state norm is first reduced by at least a factor of two, we obtain an admissible control policy that exponentially stabilizes the SCLS regardless of $\sigma \in \mathcal{S}$. $\square$

A simple SLCS will now be studied to demonstrate the results in this section.

Example 1. Consider a 1D SLCS with two subsystems: $A_1 = a_1$, $B_1 = b_1$, $A_2 = a_2$, $B_2 = b_2$, with $b_1^2 + b_2^2 \neq 0$. Thus, at least one subsystem (a_i, b_i) has $b_i \neq 0$ and is controllable hence stabilizable. To characterize σ -resilient stabilizability, suppose at any time $t \in \mathbb{N}$ we have $x(t) = z$. Applying a control $u(t) = v$ leads to two possible outcomes of $x(t+1)$: $\{a_1 z + b_1 v, a_2 z + b_2 v\}$. To achieve the slowest state growth rate for $\sigma \in \mathcal{S}$, $u(t)$ should be chosen to minimize $\max\{|a_1 z + b_1 v|, |a_2 z + b_2 v|\}$.

$$\textbf{Claim: } \min_v \max\{|a_1 z + b_1 v|, |a_2 z + b_2 v|\} = |a_1 b_2 - a_2 b_1| / (|b_1| + |b_2|) \cdot |z|. \quad (4)$$

The above claim can be proved by differentiating the following three cases:

- (i) Suppose $b_1 b_2 < 0$. Then the minimizing v satisfies $a_1 z + b_1 v = a_2 z + b_2 v$, i.e., $v^* = -[(a_1 - a_2)/(b_1 - b_2)]z$. By choosing such v^* , $x(t+1) = [(a_2 b_1 - a_1 b_2)/(b_1 - b_2)]z$ regardless of $\sigma(t)$.
- (ii) Suppose $b_1 b_2 > 0$. Then the minimizing v satisfies $a_1 z + b_1 v = -(a_2 z + b_2 v)$, i.e., $v^* = -[(a_1 + a_2)/(b_1 + b_2)]z$. This results in $x(t+1) = \pm[(a_1 b_2 - a_2 b_1)/(b_1 + b_2)]z$, with the sign depending on $\sigma(t) \in \{1, 2\}$.
- (iii) Suppose $b_1 = 0$. In this case any v between $(a_1 - a_2)z/b_2$ and $-(a_1 + a_2)z/b_2$ is a minimizer of $\max\{|a_1 z|, |a_2 z + b_2 v|\}$, with the minimum being $|a_1 z|$.

The claim (4) implies that the σ -resilient stabilizing rate of the 1D SLCS is given by

$$\rho^* = |a_1 b_2 - a_2 b_1| / (|b_1| + |b_2|), \quad (5)$$

which satisfies the scaling property predicted by Lemma 1. We remark that if $a_1 a_2 b_1 b_2 \leq 0$, then $\rho^* = \nu|a_1| + (1-\nu)|a_2|$ with $\nu = |b_2|/(|b_1| + |b_2|)$ is between $|a_1|$ and $|a_2|$, namely, the stabilizing rates of the two individual autonomous subsystems. However, if $a_1 a_2 b_1 b_2 > 0$, then ρ^* can be smaller than

both $|a_1|$ and $|a_2|$. For example, suppose $a_1/b_1 = a_2/b_2$, i.e., the two subsystems are scaled versions of each other. Then $\rho^* = 0$. Indeed, from any $x(0) = z$, the control $u^*(0) = -(a_1/b_1)z = -(a_2/b_2)z$ ensures that $x(1) = 0$ regardless of $\sigma(0)$.

Finally, if $b_1 = b_2 = 0$, then the SLCS becomes an autonomous SLS, and ρ^* is given by the JSR of $\{a_1, a_2\}$, namely, $\max\{|a_1|, |a_2|\}$. This is exactly the limit superior of the expression in (5) as both $b_1, b_2 \rightarrow 0$. $\square$

Remark 1. *An observation from the above example is that the adversary will not gain any advantage if the user adopts the optimal state-feedback policy $u^*(\cdot)$ for all of its future control inputs and reveals it to the adversary at time 0. On the other hand, if the user adopts an open-loop control policy by implementing a fixed control input sequence, then the adversary by knowing such a sequence in advance will have a much greater advantage. In fact, it would be impossible to stabilize the system in Example 1 in the second setting. This observation remains valid for general SLCS's.*

Example 2. A family of problems is given by the distributed stabilization of networked systems. In such systems, a number of linear subsystems are interconnected, can exchange information, and have dynamics couplings, via network links. Under a distributed control that uses only local (network neighbors') information, the overall system dynamics can be written as $x(t+1) = A_{G(t)}x(t) + B_{G(t)}u(t)$, where $x(t)$ and $u(t)$ are the concatenation of subsystems' state and control and $G(t)$ is the network topology at time t . Assume the network is hacked by an adversary, which may disable, e.g., up to a certain number of network connections. Then, the problem of stabilizing the networked systems, or its variant such as consensus/rendezvous [13] can be formulated as a σ -resilient stabilization problem.

3 Reducibility and Defectiveness

Recall that the σ -resilient stabilizing rate ρ^* is defined to be the infimum of all ρ satisfying (2). We now study the class of SLCSs for which the infimum can be exactly achieved. We first introduce some relevant concepts.

Definition 2. A subset $\mathcal{V} \subset \mathbb{R}^n$ is called a *control σ -invariant set* of the SLCS if for any $z \in \mathcal{V}$ there exists a control $u \in \mathbb{R}^p$ such that $A_i z + B_i u \in \mathcal{V}$ for all $i \in \mathcal{M}$. If $\mathcal{V}$ is further a subspace of $\mathbb{R}^n$, then it is called a *control σ -invariant subspace*.

Two trivial control σ -invariant subspaces are given by $\{0\}$ and $\mathbb{R}^n$.

Definition 3. *The SLCS (1) is called*

- *irreducible if it does not have any control σ -invariant subspaces other than $\{0\}$ and $\mathbb{R}^n$. Otherwise, it is called reducible.*
- *nondefective if there exists a finite $K \geq 0$ and a control policy $u \in \mathcal{U}$ such that, for any $z \in \mathbb{R}^n$ and $\sigma \in \mathcal{S}$, $\|x(t; \sigma, u, z)\| \leq K(\rho^*)^t \|z\|$, $\forall t \in \mathbb{N}$. Otherwise, it is called defective.*

Note that defectiveness is independent of $\|\cdot\|$ due to the equivalence of all such norms.

Nondefective SLCSs are those systems for which the infimum ρ^* of the exponential growth rate in (2) can be exactly achieved. In particular, if the SLCS has its σ -resilient stabilizing rate at the

stability boundary, i.e., $\rho^* = 1$, then the SLCS has all bounded state solutions (we call such systems σ -resiliently marginally stable) if and only if it is nondefective.

If there exists a common coordinate change so that all the subsystem matrices have the same block structure $A_i = \begin{bmatrix} \star & \star \\ 0 & \star \end{bmatrix}$ and $B_i = \begin{bmatrix} \star \\ \star \end{bmatrix}$, then the SLCS is reducible with $\left\{ \begin{bmatrix} \star \\ 0 \end{bmatrix} \right\}$ being a nontrivial control σ -invariant subspace. See Example 3 in the next section for a reducible SLCS with a less straightforward control σ -invariant subspace. For an example of defective SLCSs, consider the one with a single subsystem (A, B) , where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. As B is zero, $\rho^* = 1$ is the spectral radius of A . However, $x(t) = A^t x(0) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0)$ is unbounded for some $x(0)$. For

another example, consider the SLCS with a single subsystem (A, B) with $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. As the LTI system is controllable to the origin in two steps, we have $\rho^* = 0$. The system is not controllable to the origin in one time step starting from some z ; hence it is defective.

A SLCS with $\rho^* = 0$ is nondefective if and only if it is resiliently controllable to the origin in one time step, i.e., for any $z \in \mathbb{R}^n$, there exists $v \in \mathbb{R}^p$ such that $A_i z + B_i v = 0$ for all $i \in \mathcal{M}$. Another equivalent condition is that the range space of $[A_1^T \ \cdots \ A_m^T]^T$ is contained in that of $[B_1^T \ \cdots \ B_m^T]^T$. For such a SLCS, any subspace of $\mathbb{R}^n$ will be control σ -invariant; thus the system is reducible if its state dimension is greater than one.

Assume $\rho^* \neq 0$ and let $\|\cdot\|$ be an arbitrary norm of $\mathbb{R}^n$. Define the extended-value function

$$\zeta(z) := \inf_{u \in \mathcal{U}} \sup_{\sigma \in \mathcal{S}} \sup_{t \in \mathbb{N}} \|x(t; \sigma, u, z)\| / (\rho^*)^t \in \mathbb{R}_+ \cup \{+\infty\}, \quad \forall z \in \mathbb{R}^n. \quad (6)$$

Obviously, $\zeta(z)$ is positively homogeneous of degree one: $\zeta(\alpha z) = \alpha \zeta(z)$, $\forall \alpha \geq 0$. Noting that $\|x(t; \sigma, u, z)\|$ is jointly convex in u and z for fixed t and σ and $\mathcal{U}$ is a vector space hence convex, by applying a result in [14, pp. 87], we deduce that ζ is convex. Thus, the set

$$\mathcal{W} := \{z \mid \zeta(z) < \infty\} \quad (7)$$

must be a subspace of $\mathbb{R}^n$. We claim that $\mathcal{W}$ is a control σ -invariant subspace. Indeed, for any $z \in \mathcal{W}$, $\zeta(z) < \infty$ implies that there exists a policy $u = (u_0, u_1, \dots) \in \mathcal{U}$ and a finite K such that $\|x(t; \sigma, u, z)\| \leq K(\rho^*)^t$ for all t and all $\sigma = (\sigma_0, \sigma_1, \dots) \in \mathcal{S}$. Let $v = u_0(z)$ be the control at time $t = 0$ specified by the policy u and let $\sigma_0 = i$ be arbitrary. Then the solution starting from $x(1) = A_i z + B_i v$ under the control policy $u_+ := (u_1, u_2, \dots)$ satisfies $\|x(t; \sigma_+, u_+, x(1))\| = \|x(t+1; \sigma, u, z)\| \leq K(\rho^*)^{t+1}$ for all $t \in \mathbb{N}$ and all $\sigma_+ := (\sigma_1, \sigma_2, \dots) \in \mathcal{S}$. As a result, $\zeta(x(1)) \leq K\rho^* < \infty$ and thus $x(1) \in \mathcal{W}$, proving that $\mathcal{W}$ is control σ -invariant.

Theorem 2. *An irreducible SLCS with $\rho^* \neq 0$ is nondefective.*

Proof. Suppose the SLCS is irreducible. Then the subspace $\mathcal{W}$ defined in (7) being control σ -invariant must be either $\{0\}$ or $\mathbb{R}^n$. We will show by contradiction that the former is impossible. Suppose $\mathcal{W} = \{0\}$. Then, for any $z \in \mathbb{S}^{n-1}$ and any $u \in \mathcal{U}$, there exist some $\sigma \in \mathcal{S}$ and $s \in \mathbb{N}$ such that $\|x(s; \sigma, u, z)\| > 2(\rho^*)^s$. Fix an arbitrary $u \in \mathcal{U}$. We claim that the time s can be chosen to be uniformly bounded (w.r.t. z). Suppose otherwise. Then there exist a sequence $\{z^{(k)}\}_{k=1,2,\dots}$ in $\mathbb{S}^{n-1}$ and an increasing sequence of times $s^{(1)} < s^{(2)} < \dots$ such that, for any $\sigma \in \mathcal{S}$, $\|x(t; \sigma, u, z^{(k)})\| \leq 2(\rho^*)^t$, $t = 0, \dots, s^{(k)}$, for $k = 1, 2, \dots$. By taking subsequences if necessary, we

can assume that $s^{(k)}$ converges to $s^* \in \mathbb{S}^{n-1}$. Then $\|x(t; \sigma, u, z^*)\| \leq 2(\rho^*)^t$, $\forall t \in \mathbb{N}$, for all $\sigma \in \mathcal{S}$. Note that for this conclusion to hold, it may be necessary to modify the policy u at z^* so that $u_t(z^*)$ at each time t is the limit of a converging subsequence of the bounded sequence $u_t(x(t; \sigma, u, z^{(k)}))$ as $k \rightarrow \infty$. The modified policy is clearly admissible. As a result, $\zeta(z^*) \leq 2$ hence $z^* \in \mathcal{W}$, contradicting our assumption $\mathcal{W} = \{0\}$. Therefore, $\mathcal{W} = \mathbb{R}^n$, i.e., $\zeta(\cdot)$ is finite on $\mathbb{R}^n$. As has been shown, $\zeta(\cdot)$ is convex and positively homogeneous; thus, it is a norm of $\mathbb{R}^n$. A constant $K < \infty$ then exists so that $\xi(\cdot) \leq K\|\cdot\|$, which is exactly the desired nondefectiveness conclusion. $\square$

For SLCSs with $\rho^* = 0$, the conclusion in Theorem 2 does not hold. A counter example is given by any defective SLCS with state space dimension one.

Remark 2. *The concepts of reducibility and defectiveness have been proposed in the study of the JSR and the stability of autonomous SLCSs [15, 16]. Results obtained here are extensions of them to SLCSs. In particular, the proof of Theorem 2 is an extension of the proof of [17, Theorem 2.1].*

4 Bounds of σ -Resilient Stabilizing Rate

In this section, we will derive lower and upper bounds of the σ -resilient stabilizing rate ρ^* via seminorms and norms.

4.1 A Motivating Example

We first study an example SLCS to demonstrate the techniques to be formalized in Section 4.2

Example 3. Consider the SLCS given by

$$A_1 = \begin{bmatrix} a_1 & 0 \\ 0 & f_1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} b_1 \\ g_1 \end{bmatrix}, \quad \text{and} \quad A_2 = \begin{bmatrix} a_2 & 0 \\ 0 & f_2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_2 \\ g_2 \end{bmatrix},$$

whose σ -resilient stabilizing rate is denoted by ρ^* . This SLCS is obtained from two 1D SLCS's, $\{(a_i, b_i)\}_{i=1,2}$ and $\{(f_i, g_i)\}_{i=1,2}$, with a shared control input u and switching signal σ . Denote by ρ_1^* and ρ_2^* the σ -resilient stabilizing rates of the two 1D SLCS's, respectively. Obviously, $\rho^* \geq \max\{\rho_1^*, \rho_2^*\}$. We will next derive more refined bounds of ρ^* .

Assume in the following that $A_1 \neq A_2$ and that B_1 and B_2 are not collinear, i.e., $b_1 g_2 \neq b_2 g_1$. This assumption implies that the constants

$$\alpha := (a_1 - f_1)g_2 - (a_2 - f_2)g_1, \quad \beta := (a_1 - f_1)b_2 - (a_2 - f_2)b_1$$

satisfy $\alpha^2 + \beta^2 \neq 0$. Define two nonnegative functions $V, W : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$V(z) := |\alpha z_1 - \beta z_2|, \quad W(z) := |\beta z_1 + \alpha z_2|, \quad \forall z = [z_1, z_2]^T \in \mathbb{R}^2.$$

Their null sets $\mathcal{N}_V := \{z \mid V(z) = 0\}$ and $\mathcal{N}_W := \{z \mid W(z) = 0\}$ are 1D subspaces orthogonal to each other. At any time t and for any $x(t) = [z_1 \ z_2]^T \in \mathbb{R}^2$, we have

$$\begin{aligned} \min_{u(t)} \max_{i=1,2} V(A_i x(t) + B_i u(t)) &= \min_{u(t)} \max_{i=1,2} |(\alpha b_i - \beta g_i)u(t) + (\alpha a_i z_1 - \beta f_i z_2)| \\ &= \frac{|(\alpha b_1 - \beta g_1)(\alpha a_2 z_1 - \beta f_2 z_2) - (\alpha b_2 - \beta g_2)(\alpha a_1 z_1 - \beta f_1 z_2)|}{\sum_{i=1}^2 |\alpha b_i - \beta g_i|} \\ &= \frac{|a_1 f_2 - a_2 f_1|}{|a_1 - f_1| + |a_2 - f_2|} V(x(t)) := \rho_0 \cdot V(x(t)). \end{aligned} \tag{8}$$

Here, (4) is used in deriving the second equality; and the $u(t)$ achieving the minimum is given by

$$u^*(t) = -\frac{(\alpha a_1 z_1 - \beta f_1 z_2) \pm (\alpha a_2 z_1 - \beta f_2 z_2)}{(\alpha b_1 - \beta g_1) \pm (\alpha b_2 - \beta g_2)}, \quad (9)$$

with the sign " $\pm$ " being "+" if $(a_1 - a_2)(f_1 - f_2) \geq 0$ and "-" if otherwise.

The result in (8) has several implications. First, $\mathcal{N}_V$ is a control σ -invariant subspace: for $x(t) \in \mathcal{N}_V$, we have $A_i x(t) + B_i u^*(t) \in \mathcal{N}_V$, for $i = 1, 2$. Second, if at each time t the adversary chooses $\sigma(t) = \arg \max_i V(A_i x(t) + B_i u(t))$, then $V(x(t+1)) \geq \rho_0 V(x(t))$ regardless of the user's choice of $u(t)$. As $V(x(t))$ is positively homogeneous of degree one in $x(t)$, we conclude that $x(t)$ cannot decay at a faster exponential rate than ρ_0 , i.e.,

$$\rho^* \geq \rho_0 = \frac{|a_1 f_2 - a_2 f_1|}{|a_1 - f_1| + |a_2 - f_2|}. \quad (10)$$

As a third consequence of (8), suppose the user adopts the feedback control strategy in (9). Then

$$V(x(t+1)) = V(A_{\sigma(t)} x(t) + B_{\sigma(t)} u^*(t)) \leq \rho_0 \cdot V(x(t)), \quad \forall \sigma(t) \in \{1, 2\}, \forall x(t). \quad (11)$$

If $\rho_0 < 1$, then $V(x(t)) \rightarrow 0$, i.e., $x(t) \rightarrow \mathcal{N}_V$, as $t \rightarrow \infty$ for any $\sigma \in \mathcal{S}$. To ensure that $x(t) \rightarrow 0$, one simply needs in addition that $x(t)$ will not diverge along $\mathcal{N}_V$. Pick any $x(t) \in \mathcal{N}_V$, i.e., $x(t) = [z_1 \ z_2]^T$ with $\alpha z_1 = \beta z_2$. Then it can be verified that, with the sign in (9) being either "+" or "-", we always have $W(A_i x(t) + B_i u^*(t)) = \rho_i \cdot W(x(t))$ where

$$\rho_i := \frac{|g_i(a_1 b_2 - a_2 b_1) - b_i(f_1 g_2 - f_2 g_1)|}{|b_1 g_2 - b_2 g_1|}, \quad \forall i = 1, 2. \quad (12)$$

Thus, $W(x(t+1)) \leq \max\{\rho_1, \rho_2\} \cdot W(x(t))$ regardless of $\sigma(t)$. This together with (11) implies that, if $\max\{\rho_0, \rho_1, \rho_2\} < 1$, the system is σ -resiliently stabilized by u^* . In other words, $\rho^* < 1$ if $\max\{\rho_0, \rho_1, \rho_2\} < 1$. Noting that $\max\{\rho_0, \rho_1, \rho_2\}$ has the exact same scaling properties as ρ^* in Lemma 1, we obtain via a scaling argument that

$$\rho^* \leq \max\{\rho_0, \rho_1, \rho_2\} = \max \left\{ \frac{|a_1 f_2 - a_2 f_1|}{|a_1 - f_1| + |a_2 - f_2|}, \max_{i=1,2} \left[\frac{|g_i(a_1 b_2 - a_2 b_1) - b_i(f_1 g_2 - f_2 g_1)|}{|b_1 g_2 - b_2 g_1|} \right] \right\}. \quad (13)$$

In particular, if $\rho_0 \geq \max\{\rho_1, \rho_2\}$, then $\rho^* = \rho_0$ by (10) and (13). For instance, if $a_1 b_2 = a_2 b_1$ and $f_1 g_2 = f_2 g_1$, then $\rho_1^* = \rho_2^* = 0$ by Example 1, while $\rho^* = \rho_0 > 0$ as long as $a_1 f_2 - a_2 f_1 \neq 0$. $\square$

4.2 Bounds via seminorms

We now formalize the bounding technique employed in Example 3. A seminorm of $\mathbb{R}^n$ is a mapping $\xi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ with the following properties: it is convex (hence continuous) and positively homogeneous (of degree one): $\xi(\alpha z) = |\alpha| \cdot \xi(z)$ for all $\alpha \in \mathbb{R}$ and $z \in \mathbb{R}^n$. It becomes a norm if it is positive definite: $\xi(z) > 0$ whenever $z \neq 0$.

Lemma 2. *For an arbitrary seminorm ξ on $\mathbb{R}^n$, define a mapping $\mathcal{T} : \xi \mapsto \xi_\#$ where*

$$\xi_\#(z) = \mathcal{T}[\xi](z) := \inf_{v \in \mathbb{R}^p} \max_{i \in \mathcal{M}} \xi(A_i z + B_i v), \quad \forall z \in \mathbb{R}^n. \quad (14)$$

Then, $\xi_\#$ is also a seminorm of $\mathbb{R}^n$. In other words, $\mathcal{T}$ is a self map of seminorms of $\mathbb{R}^n$.

Proof. Obviously $\xi_{\#}$ is finite on $\mathbb{R}^n$ as $\mathcal{M}$ is finite. Noting that $\max_i \xi(A_i z + B_i v)$ is convex in (z, v) , by [14, pp. 88], $\xi_{\#}$ is convex in z . For homogeneity, let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, be arbitrary. Then

$$\xi_{\#}(\alpha z) = \inf_{v \in \mathbb{R}^p} \max_{i \in \mathcal{M}} \xi(\alpha A_i z + B_i v) = \inf_{v' \in \mathbb{R}^p} \max_{i \in \mathcal{M}} |\alpha| \cdot \xi(A_i z + B_i v') = |\alpha| \cdot \xi_{\#}(z),$$

where $v' := v/\alpha$ above. When $\alpha = 0$, it is obvious from (14) that $\xi_{\#}(0) = 0$. $\square$

In particular, if $\xi(\cdot) = \|\cdot\|$ is a norm of $\mathbb{R}^n$, then $\xi_{\#}(\cdot)$, which we denote as $\|\cdot\|_{\#}$, is a seminorm of $\mathbb{R}^n$. Note that $\|\cdot\|_{\#}$ may not be a norm. For instance, if the two 1D subsystem dynamics in Example 1 are scaled version of each other, $a_1/b_1 = a_2/b_2$, then $|z|_{\#} = \inf_v \max\{|a_1 z + b_1 v|, |a_2 z + b_2 v|\} = 0$ for all z if we set $v = -(a_1/b_1)z = -(a_2/b_2)z$. Thus, $\|\cdot\|_{\#} \equiv 0$, which is not a norm of $\mathbb{R}$.

Remark 3. $\|\cdot\|_{\#}$ scales with matrices (A_i, B_i) in the same way as ρ^* does (see Lemma 1).

For the following lemma, we extend the definition (14) to include those extended-valued seminorms, i.e., mappings $\xi : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ that are convex and positively homogeneous.

Lemma 3. The mapping $\mathcal{T} : \xi \mapsto \xi_{\#}$ defined in (14) has the following properties.

- **(Monotonicity):** For two extended-valued seminorms ξ and ξ' with $\xi \leq \xi'$, $\mathcal{T}(\xi) \leq \mathcal{T}(\xi')$.
- **(Monotone Continuity):** Let $\{\xi_k\}$ be a monotone sequence of seminorms and denote $\xi_{\infty} = \lim_{k \rightarrow \infty} \xi_k$ its extended-valued limit. Then $\lim_{k \rightarrow \infty} \mathcal{T}(\xi_k) = \mathcal{T}(\xi_{\infty})$.

Proof. The monotonicity property is trivial. For continuity, suppose $\xi_k \uparrow \xi_{\infty}$ as $k \rightarrow \infty$. Then, $\{\mathcal{T}(\xi_k)\}$ is a nondecreasing sequence of seminorms upper bounded by $\mathcal{T}(\xi_{\infty})$. Thus its limit exists and has the same upper bound: $\lim_{k \rightarrow \infty} \mathcal{T}(\xi_k) \leq \mathcal{T}(\xi_{\infty})$. For each fixed $z \in \mathbb{R}^n$, let v^* be such that $\mathcal{T}(\xi_{\infty})(z) = \max_i \xi_{\infty}(A_i z + B_i v^*)$ (which may be of infinite value). Note that $\mathcal{T}(\xi_k)(z) \geq \max_i \xi_k(A_i z + B_i v^*)$, $\forall k$. By letting $k \rightarrow \infty$, we obtain $\lim_{k \rightarrow \infty} \mathcal{T}(\xi_k)(z) \geq \mathcal{T}(\xi_{\infty})(z)$, $\forall z$, i.e., $\lim_{k \rightarrow \infty} \mathcal{T}(\xi_k) \geq \mathcal{T}(\xi_{\infty})$. The case when $\xi_k \downarrow \xi_{\infty}$ can be proved similarly, using Dini's theorem to show that the convergence is uniform. The detailed proof is omitted. $\square$

Proposition 1. Let ξ be a non-zero seminorm of $\mathbb{R}^n$ and let $\alpha \in \mathbb{R}$ be a constant such that

$$\xi_{\#}(z) \geq \alpha \xi(z), \quad \forall z \in \mathbb{R}^n.$$

Then, the σ -resilient exponential stabilizing rate of the SLS (1) satisfies $\rho^* \geq \alpha$.

Proof. Assume the adversary adopts the switching strategy $\sigma(t) = \arg \max_i \xi(A_i x(t) + B_i u(t))$, $\forall t \in \mathbb{N}$, for the SLS (1) and assume $x(0) = z$ is such that $\xi(z) > 0$. Then,

$$\xi(x(t+1)) = \xi(A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t)) = \max_i \xi(A_i x(t) + B_i u(t)) \geq \xi_{\#}(x(t)) \geq \alpha \xi(x(t)),$$

for all $t \in \mathbb{N}$ and all admissible user input strategies $u \in \mathcal{U}$. This implies that the exponential growth rate of $\xi(x(t))$, hence that of $\|x(t)\|$, is at least α . Therefore, $\rho^* \geq \alpha$. $\square$

Proposition 1 has been applied in Example 3 with $\xi(\cdot) = V(\cdot)$ and $\alpha = \rho_0$ in equation (8).

Proposition 2. Let $\|\cdot\|$ be a norm of $\mathbb{R}^n$ such that $\|\cdot\|_{\#} \leq \alpha \|\cdot\|$ for some constant $\alpha \in \mathbb{R}$. Then, $\alpha_S^* \leq \alpha$.

Proof. Suppose the user adopts the input policy $u^*(t) = \arg \min_v \max_{i \in \mathcal{M}} \|A_i x(t) + B_i v\|$ for $t \in \mathbb{N}$, which is admissible. Note that $u^*(t)$ thus defined exists due to the convexity and nonnegativity of $\|\cdot\|$, though it may not be unique (in which case any choice suffices). Then, for any adversary's switching strategy $\sigma \in \mathcal{S}$ and any $t \in \mathbb{N}$,

$$\|x(t+1)\| = \|A_{\sigma(t)}x(t) + B_{\sigma(t)}u^*(t)\| \leq \max_{i \in \mathcal{M}} \|A_i x(t) + B_i u^*(t)\| = \|x(t)\|_{\#} \leq \alpha \|x(t)\|.$$

This implies that $\|x(t)\| \leq \alpha^t \|x(0)\|$, $\forall t \in \mathbb{N}$; hence $\rho^* \leq \alpha$. $\square$

The following result follows immediately from Proposition 1 and Proposition 2.

Corollary 1. *If $\alpha_1 \|\cdot\| \leq \|\cdot\|_{\#} \leq \alpha_2 \|\cdot\|$ for some norm $\|\cdot\|$ on $\mathbb{R}^n$, then $\alpha_1 \leq \rho^* \leq \alpha_2$.*

4.3 Extremal Norms

Owing to Corollary 1, associated with each norm $\|\cdot\|$ are the following two bounds of ρ^* :

$$\alpha_{\ell} := \sup \{ \alpha_1 \mid \alpha_1 \|\cdot\| \leq \|\cdot\|_{\#} \}, \quad \alpha_u := \inf \{ \alpha_2 \mid \|\cdot\|_{\#} \leq \alpha_2 \|\cdot\| \},$$

where ρ^* is guaranteed to lie in the interval $[\alpha_{\ell}, \alpha_u]$. A natural question arises: can such bounds be tight?

Definition 4. *A norm $\|\cdot\|$ on $\mathbb{R}^n$ is called an (upper) extremal norm if $\|\cdot\|_{\#} \leq \rho^* \|\cdot\|$, i.e., if the upper bound α_u it generates is precisely ρ^* .*

Suppose an extremal norm $\|\cdot\|$ exists. Then the property $\|\cdot\|_{\#} \leq \rho^* \|\cdot\|$ implies that, for any $z \in \mathbb{R}^n$, the user can find a control $v \in \mathbb{R}^p$ such that $\|A_i z + B_i v\| \leq \rho^* \|z\|$ for all $i \in \mathcal{M}$. This essentially specifies a state feedback control policy $u \in \mathcal{U}$ under which $\|x(t; \sigma, u, z)\| \leq (\rho^*)^t \|z\|$ for arbitrary $\sigma \in \mathcal{S}$. In particular, this implies that the SLCS must be nondefective. The following theorem says that the reverse is also true.

Theorem 3. *Extreme norms exist if and only if the SLCS is nondefective.*

Proof. The "only if" part has been proved above. We will prove the "if" part in the following.

Suppose the SLCS is nondefective with $\rho^* \neq 0$. It has been shown in Section 3 that $\zeta(z)$ defined in (6) is a norm of $\mathbb{R}^n$. For each $z \in \mathbb{R}^n$, we can rewrite

$$\zeta(z) = \inf_{u(0)} \inf_{u_+} \sup_{\sigma(0)} \sup_{\sigma_+} \sup_{t \in \mathbb{N}} \|x(t; \sigma, u, z)\| / (\rho^*)^t = \inf_{u(0)} \sup_{\sigma(0)} \inf_{u_+} \sup_{\sigma_+} \sup_{t \in \mathbb{N}} \|x(t; \sigma, u, z)\| / (\rho^*)^t,$$

where $(u(0), u_+)$ is a decomposition of $u \in \mathcal{U}$ and $(\sigma(0), \sigma_+)$ is a decomposition of $\sigma \in \mathcal{S}$. The reason that $\sup_{\sigma(0)}$ and $\inf_{u_+}$ can switch order is due to the observation in Remark 1: for the objective of maximizing $\sup_{t \in \mathbb{N}} \|x(t)\| / (\rho^*)^t$, knowing the optimal state feedback control policy u_+ at time 0 gives no extra advantage to the adversary's decision on $\sigma(0)$.

By denoting $u(0) = v$ and $\sigma(0) = i$, and noting $x(t+1; \sigma, u, z) = x(t; \sigma_+, u_+, x(1))$, we further write

$$\begin{aligned} \zeta(z) &= \inf_v \sup_i \inf_{u_+} \sup_{\sigma_+} \max \left(\|z\|, \sup_{t \in \mathbb{N}} \|x(t; \sigma_+, u_+, A_i z + B_i v)\| / (\rho^*)^{t+1} \right) \\ &= \inf_v \sup_i \max \left(\|z\|, \zeta(A_i z + B_i v) / \rho^* \right) \\ &= \max \left(\|z\|, \zeta_{\#}(z) / \rho^* \right), \end{aligned}$$

where in the last two steps we have used the fact that $\inf_x \max(c, f(x)) = \max(c, \inf_x f(x))$ and $\sup_x \max(c, f(x)) = \max(c, \sup_x f(x))$ for arbitrary function $f(x)$ and constant c . It follows then that $\zeta(\cdot) \geq \zeta_{\#}(\cdot)/\rho^*$, i.e., $\zeta_{\#}(\cdot) \leq \rho^* \cdot \zeta(\cdot)$, making $\zeta(\cdot)$ an extremal norm of the SLCS.

Finally, suppose the SLCS is nondefective with $\rho^* = 0$. Then, for any $z \in \mathbb{R}^n$, there exists $v \in \mathbb{R}^p$ such that $A_i z + B_i v = 0$ for all $i \in \mathcal{M}$. Pick any norm $\|\cdot\|$ on $\mathbb{R}^n$. It is easily seen that $\|\cdot\|_{\#} \equiv 0$, i.e., $\|\cdot\|$ is an extremal norm. $\square$

Definition 5. A nonzero seminorm $\xi(\cdot)$ on $\mathbb{R}^n$ is called a lower extremal seminorm if $\xi_{\#}(\cdot) \geq \rho^* \cdot \xi(\cdot)$.

Theorem 4. Lower extremal seminorms exist if the SLCS is nondefective.

Proof. Suppose the SLCS is nondefective and $\rho^* \neq 0$. Let $\|\cdot\|$ be an arbitrary norm of $\mathbb{R}^n$ and define a sequence of seminorms on $\mathbb{R}^n$ as $\xi^{(0)}(\cdot) := \|\cdot\|$,

$$\xi^{(t)}(\cdot) := \underbrace{\mathcal{T} \circ \cdots \circ \mathcal{T}}_{t \text{ times}}(\|\cdot\|), \quad \forall t = 1, 2, \dots, \quad (15)$$

where $\mathcal{T}$ is the $\#$ -operator defined in (14). Alternatively, $\xi^{(t)}$ is defined recursively by $\xi^{(t+1)} = \xi_{\#}^{(t)}$.

A useful fact that can be proved via induction is that

$$\xi^{(t)}(z) = \inf_{u(0)} \max_{\sigma(0)} \cdots \inf_{u(t-1)} \max_{\sigma(t-1)} \|x(t; \sigma, u, z)\| = \inf_{u \in \mathcal{U}} \sup_{\sigma \in \mathcal{S}} \|x(t; \sigma, u, z)\|, \quad \forall z \in \mathbb{R}^n, t \geq 1. \quad (16)$$

From this it is easy to see that $\xi^{(t)}/(\rho^*)^t \leq \zeta$ for all t . Thus, the following defined function

$$\eta := \limsup_{t \rightarrow \infty} \xi^{(t)}/(\rho^*)^t = \inf_{s \in \mathbb{N}} \sup_{t \geq s} \xi^{(t)}/(\rho^*)^t \quad (17)$$

also satisfies $\eta \leq \zeta$. The nondefective assumption implies that ζ , hence η , is finite on $\mathbb{R}^n$. Being the limit of a monotone decreasing sequence of seminorms $\sup_{t \geq s} \xi^{(t)}/(\rho^*)^t$ as $s \rightarrow \infty$, η is a seminorm as well. We next shown that $\eta \neq 0$. Suppose otherwise. Then for any $z \in \mathbb{S}^{n-1}$ we have $\lim_{t \rightarrow \infty} \xi^{(t)}(z)/(\rho^*)^t \rightarrow 0$. By (16), we can find time $N_z \in \mathbb{N}$ and control policy $u_z \in \mathcal{U}$ such that $\|x(N_z; \sigma, u_z, z)\| \leq \frac{1}{2}(\rho^*)^t$ for all $\sigma \in \mathcal{S}$. In the same way as in the proof of Theorem 1, we can modify u_z in a neighbor U_z of z to obtain a policy $\tilde{u}_z$ so that, for any $z' \in U_z$, $\|x(N_z; \sigma, \tilde{u}_z, z')\| \leq \frac{1}{2}(\rho^*)^t$ for all $\sigma \in \mathcal{S}$; obtain a finite number of such U_z to cover $\mathbb{S}^{n-1}$; patch the their $\tilde{u}_z$ together to form an overall policy $\tilde{u} \in \mathcal{U}$ under which, $\forall z \in \mathbb{S}^{n-1}$, $\|x(t; \sigma, \tilde{u}, z)\| \leq \frac{1}{2}(\rho^*)^t$ at some time t bounded by $N < \infty$ regardless of σ . This shows that the σ -resilient exponential stabilizing rate should be less than ρ^* , a contradiction. Therefore, $\eta(z) > 0$ for some $z \neq 0$ and hence η is a nonzero seminorm.

Using the monotone continuity property of the $\mathcal{T}$ operator in Lemma 3, we obtain

$$\eta_{\#} = \inf_{s \in \mathbb{N}} \mathcal{T} \left[\sup_{t \geq s} \xi^{(t)}/(\rho^*)^t \right] \geq \inf_{s \in \mathbb{N}} \sup_{t \geq s} \xi_{\#}^{(t)}/(\rho^*)^t = \rho^* \cdot \inf_{s \in \mathbb{N}} \sup_{t \geq s+1} \xi^{(t)}/(\rho^*)^t = \rho^* \cdot \eta.$$

This shows that η is a lower extremal seminorm of the SLCS.

Suppose the SLCS is nondefective and $\rho^* = 0$. Then, it is easily verified that any seminorm ξ satisfies $\xi_{\#} = 0$ and will be a lower extremal seminorm. $\square$

Remark 4. Theorem 4 remains valid if we replace the nondefective condition with the following (possibly strictly) weaker condition: $\sup_{t \in \mathbb{N}} \xi^{(t)}/(\rho^*)^t$ is finite on $\mathbb{R}^n$.

The converse of Theorem 4 is not true. For example, we have shown that the SLCS with a single subsystem (A, B) with $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has $\rho^* = 0$. Then, any seminorm will be a lower extremal seminorm. However, as shown in Section 2 the system is defective.

A norm that is both upper and lower extremal is called a Barabanov norm.

Definition 6. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is called a Barabanov norm if $\|\cdot\|_{\#} = \rho^* \|\cdot\|$.

Theorem 5. Barabanov norms exist if the SLCS is irreducible.

Proof. Suppose the SLCS is irreducible with $\rho^* \neq 0$. Define

$$\chi(z) := \inf_{u \in \mathcal{U}} \sup_{\sigma \in \mathcal{S}} \limsup_{t \rightarrow \infty} \|x(t; \sigma, u, z)\| / (\rho^*)^t, \quad \forall z \in \mathbb{R}^n.$$

It is easy to see that $\chi(\cdot)$ is convex and positively homogeneous on $\mathbb{R}^n$. Note that

$$\chi(z) \leq \inf_{u \in \mathcal{U}} \sup_{\sigma \in \mathcal{S}} \sup_{t \in \mathbb{N}} \|x(t; \sigma, u, z)\| / (\rho^*)^t = \zeta(z).$$

By Theorem 2, the SLCS is nondefective; thus $\zeta(\cdot)$ is bounded. The above inequality implies that $\chi(\cdot)$ is also bounded and thus is a seminorm of $\mathbb{R}^n$. Define the set $\mathcal{V} := \{z \mid \chi(z) = 0\}$ which is a subspace of $\mathbb{R}^n$. For each $z \in \mathcal{V}$, we can find a policy $u = (u_0, u_1, \dots) \in \mathcal{U}$ such that $\lim_{t \rightarrow \infty} \|x(t; \sigma, u, z)\| / (\rho^*)^t = 0$ for all $\sigma = (\sigma_0, \sigma_1, \dots) \in \mathcal{S}$. Let $v = u_0(z)$ be the control at time $t = 0$ specified by the policy u and let $\sigma_0 = i$ be arbitrary. Then the solution starting from $x(1) = A_i z + B_i v$ under the control policy $u_+ := (u_1, u_2, \dots)$ satisfies $\lim_{t \rightarrow \infty} \|x(t; \sigma_+, u_+, x(1))\| / (\rho^*)^t = \lim_{t \rightarrow \infty} \|x(t+1; \sigma, u, z)\| / (\rho^*)^t = 0$ for all $\sigma_+ := (\sigma_1, \sigma_2, \dots) \in \mathcal{S}$. In other words, $x(1) \in \mathcal{V}$. This shows that $\mathcal{V}$ is a control σ -invariant subspace. By the irreducibility assumption, $\mathcal{V}$ is either $\{0\}$ or $\mathbb{R}^n$. The latter is impossible since, otherwise, the scaled SLCS $\{(A_i/\rho^*, B_i/\rho^*)\}_{i \in \mathcal{M}}$ with its solutions $\tilde{x}(t; \sigma, u, z) = x(t; \sigma, u, z) / (\rho^*)^t$ is σ -resiliently stabilizable hence exponentially stabilizable, which implies that its σ -resilient stabilizing rate $\tilde{\rho}^* < 1$. By Lemma 1, the σ -resilient stabilizing rate of the original SLCS is $\rho^* \cdot \tilde{\rho}^*$, less than the assumed ρ^* , a contradiction. This shows that $\mathcal{V} = \{0\}$. Consequently, χ is a norm of $\mathbb{R}^n$.

We next show that $\chi_{\#} = \rho^* \cdot \chi$. Let $z \in \mathbb{R}^n$ be arbitrary and decompose $u \in \mathcal{U}$ and $\sigma \in \mathcal{S}$ as $u = (u_0, u_+)$ and $\sigma = (\sigma_0, \sigma_+)$ as above. Then

$$\begin{aligned} \chi(z) &= \inf_{u_0} \sup_{\sigma_0} \inf_{u_+} \sup_{\sigma_+} \limsup_{t \rightarrow \infty} \|x(t; \sigma, u, z)\| / (\rho^*)^t \\ &= \inf_{u_0} \sup_{\sigma_0} \inf_{u_+} \sup_{\sigma_+} \limsup_{t \rightarrow \infty} \|x(t; \sigma_+, u_+, A_{\sigma_0} z + B_{\sigma_0} u_0)\| / (\rho^*)^{t+1} \\ &= \inf_{u_0} \sup_{\sigma_0} \chi(A_{\sigma_0} z + B_{\sigma_0} u_0) / \rho^* \\ &= \chi_{\#}(z) / \rho^*. \end{aligned}$$

Again, in deriving the first equality we have used the observation in Remark 1 to exchange the order of $\inf_{u_+}$ and $\sup_{\sigma_0}$. As a result, $\chi_{\#} = \rho^* \cdot \chi$, proving that χ is a Barabanov norm.

For an irreducible SLCS with $\rho^* = 0$, any norm $\|\cdot\|$ satisfies $\|\cdot\|_{\#} = 0$ and is a Barabanov norm. $\square$

As a simple example, in Example 1 with $a_1/b_1 = a_2/b_2$, the 1D SLCS has a Barabanov norm $|\cdot|$.

To sum up, Theorems 3 and 4 imply that, when the SLCS is nondefective, suitable norms (resp. seminorms) exist that provide tight upper (resp. lower) bounds for ρ^* . If furthermore the SLCS is

irreducible, then by Theorem 5 it is possible to find a single norm that gives simultaneously tight lower and upper bounds of ρ^* . Although the proofs of these theorems are constructive, the theoretically constructed (semi)norms are difficult to compute numerically. Starting from Section 4.4, we will focus on special families of norms from which bounds on ρ^* can be computed numerically.

Remark 5. *The notions of extremal norms and Barabanov norms are originally proposed for the study of joint spectral radius and the stability of autonomous switched linear systems [16, 18, 19]. We extend them to the context of resilient stabilization of SLCSs. The proofs of Theorem 3 and Theorem 5 are inspired by those of [20, Theorem 3] and [16], respectively. See also [17, Theorem 2.1]. The concept of lower extremal seminorms and Theorem 4, on the other hand, do not have their counterparts in existing literature.*

4.4 Polytopic seminorms

For a given matrix $C = [c_1 \ \cdots \ c_\ell] \in \mathbb{R}^{n \times \ell}$ where $c_1, \dots, c_\ell \in \mathbb{R}^n$, define

$$\xi(z) := \max_{j=1, \dots, \ell} |c_j^T z|, \quad \forall z \in \mathbb{R}^n.$$

Obviously, ξ is a seminorm, with the set $\{z \mid \xi(z) \leq 1\}$ being a (possibly unbounded) polytope. We call ξ the polytopic seminorm with parameter C . If C has full column rank, then ξ becomes a norm, whose unit ball is a bounded symmetric convex polyhedron $\cap_{j=1, \dots, \ell} \{z \mid |c_j^T z| \leq 1\}$.

The following result is straightforward from the above definition.

Lemma 4. *Let ξ and $\tilde{\xi}$ be two polytopic seminorms on $\mathbb{R}^n$ with parameters $C = [c_1 \ \cdots \ c_\ell] \in \mathbb{R}^{n \times \ell}$ and $\tilde{C} = [\tilde{c}_1 \ \cdots \ \tilde{c}_\ell] \in \mathbb{R}^{n \times \ell}$, respectively. Denote by Conv the convex hull (of sets). Then $\xi \leq \tilde{\xi}$ if and only if $c_j \in \text{Conv}\{\pm \tilde{c}_1, \dots, \pm \tilde{c}_\ell\}$ for all $j = 1, \dots, \ell$.*

Lemma 5. *Suppose ξ is a polytopic seminorm of $\mathbb{R}^n$. Then $\xi_\#$ is also a polytopic seminorm.*

Proof. Suppose ξ is a polytopic seminorm of $\mathbb{R}^n$ with the parameter $C = [c_1 \ \cdots \ c_\ell] \in \mathbb{R}^{n \times \ell}$. For each $z \in \mathbb{R}^n$, $\xi_\#(z)$ defined in (14) is the solution to the following linear programming

$$\min_{v \in \mathbb{R}^p, r \in \mathbb{R}} r \quad \text{subject to} \quad \pm c_j^T (A_i z + B_i v) \leq r, \quad \forall i \in \mathcal{M}, \forall j \in \{1, \dots, \ell\}. \quad (18)$$

By introducing the Lagrange multipliers $\theta_{ij}^+ \geq 0$ and $\theta_{ij}^- \geq 0$ for $i \in \mathcal{M}$ and $j \in \{1, \dots, \ell\}$, each corresponding to a constraint in (18), the dual problem of (18) can be easily verified to be

$$\begin{aligned} & \max_{\theta_{ij}^+, \theta_{ij}^-} \sum_{i,j} (\theta_{ij}^+ - \theta_{ij}^-) c_j^T A_i z \\ & \text{subject to} \quad \sum_{ij} (\theta_{ij}^+ - \theta_{ij}^-) c_j^T B_i = 0, \quad \sum_{ij} (\theta_{ij}^+ + \theta_{ij}^-) = 1, \quad \text{and} \quad \theta_{ij}^+ \geq 0, \theta_{ij}^- \geq 0, \quad \forall i, j. \end{aligned} \quad (19)$$

Since problem (18) is strongly feasible (as r can be made arbitrarily large), the dual problem (19) has the identical solution $\xi_\#(z)$. By inspecting problem (19), its solution can be alternatively written as $\max\{\tilde{c}^T z \mid \tilde{c} \in \Omega_C\}$, where Ω_C is the subset of $\mathbb{R}^n$ given by

$$\Omega_C := \left\{ \sum_{i,j} (\theta_{ij}^+ - \theta_{ij}^-) A_i^T c_j \mid \sum_{ij} (\theta_{ij}^+ - \theta_{ij}^-) c_j^T B_i = 0, \sum_{ij} (\theta_{ij}^+ + \theta_{ij}^-) = 1, \theta_{ij}^+ \geq 0, \theta_{ij}^- \geq 0 \right\}. \quad (20)$$

Clearly, Ω_C is a bounded convex polyhedron. It is also symmetric w.r.t. the origin as the constraints in (20) are all invariant to the linear transformation that exchanges the roles of ξ_{ij}^+ and ξ_{ij}^- for each i, j . Let $C_\# \in \mathbb{R}^{n \times \ell_\#}$ be such that its columns, $\tilde{c}_1, \dots, \tilde{c}_{\ell_\#}$, consist of exactly those vertices of the polyhedron Ω_C on one side of a generic half-plane passing through the origin. Then we have

$$\xi_\#(z) = \max_{j=1, \dots, \ell_\#} |\tilde{c}_j^T z|.$$

In other words, $\xi_\#$ is the polytopic seminorm with the parameter $C_\#$. $\square$

Using the two previous lemmas, the results in Proposition 1 and Proposition 2 when restricted on polytopic (semi-)norms are easily shown to be given as follows.

Proposition 3. *Let ξ be a polytopic seminorm of $\mathbb{R}^n$ with the parameter $C = [c_1 \ \dots \ c_\ell] \in \mathbb{R}^{n \times \ell}$, and let $\xi_\#$ be the polytopic seminorm with the parameters $C_\# = [\tilde{c}_1 \ \dots \ \tilde{c}_{\ell_\#}]$ derived in the proof of Lemma 5. Then, the σ -resilient stabilizing rate ρ^* of the SLCS satisfies*

$$\rho^* \geq \sup \{ \alpha \geq 0 \mid \alpha c_j \in \text{Conv}\{\pm \tilde{c}_1, \dots, \pm \tilde{c}_{\ell_\#}\}, \forall j = 1, \dots, \ell \}.$$

If furthermore C has full column rank (hence ξ is a norm), then

$$\rho^* \leq \inf \{ \alpha \geq 0 \mid \tilde{c}_j \in \text{Conv}\{\pm \alpha c_1, \dots, \pm \alpha c_\ell\}, \forall j = 1, \dots, \ell_\# \}.$$

In general, finding the parameter matrix $C_\#$ of $\xi_\#$ may not be an easy task. When the state dimension is low, we can approximate $\xi_\#$ in Lemma 4 by gridding half of the unit sphere and find $\xi_\#(z)$ for each grid point $z \in \mathbb{S}^{n-1}$ by solving the linear program in (18).

4.5 Bounds via Ellipsoidal Norms

Denote by $\mathbb{P}_{\succ 0}$ the set of all positive definite (p.d.) matrices, and by $\mathbb{P}_{\succeq 0}$ the sets of all positive semidefinite (p.s.d.) matrices. We write $P \succ 0$ if $P \in \mathbb{P}_{\succ 0}$ and $P \succeq 0$ if $P \in \mathbb{P}_{\succeq 0}$. For each $P \succeq 0$, $\|z\|_P := \sqrt{z^T P z}$, $\forall z \in \mathbb{R}^n$, defines a seminorm of $\mathbb{R}^n$. If further $P \succ 0$, then $\|\cdot\|_P$ is a norm, called an ellipsoidal norm as its unit ball is an ellipsoid. In this section, we will study the bounds on ρ^* derived from $\|\cdot\|_P$.

We first introduce some useful notations. Denote by

$$\Delta := \{ \theta \in \mathbb{R}^m \mid \theta_i \geq 0, \forall i \in \mathcal{M}, \sum_{i \in \mathcal{M}} \theta_i = 1 \}$$

the m -simplex. For each $\theta \in \Delta$ and $P \succeq 0$, define

$$\Gamma_\theta(P) := \sum_{i \in \mathcal{M}} \theta_i A_i^T P A_i - \left(\sum_{i \in \mathcal{M}} \theta_i A_i^T P B_i \right) \left(\sum_{i \in \mathcal{M}} \theta_i B_i^T P B_i \right)^\dagger \left(\sum_{i \in \mathcal{M}} \theta_i B_i^T P A_i \right),$$

where $\dagger$ denotes matrix pseudo inverse. Note that $\Gamma_\theta(P)$ is the (generalized) Schur complement [21, pp. 28] of the lower right block of the following p.s.d. matrix:

$$\Upsilon_\theta := \begin{bmatrix} \sum_{i \in \mathcal{M}} \theta_i A_i^T P A_i & \sum_{i \in \mathcal{M}} \theta_i A_i^T P B_i \\ \sum_{i \in \mathcal{M}} \theta_i B_i^T P A_i & \sum_{i \in \mathcal{M}} \theta_i B_i^T P B_i \end{bmatrix} = \sum_{i \in \mathcal{M}} \theta_i \begin{bmatrix} A_i & B_i \end{bmatrix}^T P \begin{bmatrix} A_i & B_i \end{bmatrix}.$$

From this observation, we conclude that: (i) $\Gamma_\theta(P) \succeq 0$; (ii) for a fixed P (resp. θ), $\Gamma_\theta(P)$ is a concave mapping of θ (resp. P) into $\mathbb{P}_{\succeq 0}$ equipped with the partial order $\preceq$. Define the set

$$\Gamma_\Delta(P) := \{ \Gamma_\theta(P) \mid \theta \in \Delta \} \subset \mathbb{P}_{\succeq 0}. \quad (21)$$

Lemma 6. For each $P \succeq 0$, denote $\|\cdot\|_{P\sharp} := \mathcal{T}(\|\cdot\|_P)$ where $\mathcal{T}$ is defined in (14). Then,

$$\|z\|_{P\sharp} = \sup_{\theta \in \Delta} \|z\|_{\Gamma_\theta(P)} = \sup_{Q \in \Gamma_\Delta(P)} \|z\|_Q, \quad \forall z \in \mathbb{R}^n. \quad (22)$$

Proof. By (14), $(\|z\|_{P\sharp})^2$ is the solution of the following optimization problem:

$$\begin{aligned} & \text{minimize } r \\ & \text{subject to } (A_i z + B_i v)^T P (A_i z + B_i v) \leq r, \quad \forall i \in \mathcal{M}, \end{aligned} \quad (23)$$

with the optimization variables being $v \in \mathbb{R}^p$ and $r \in \mathbb{R}$. By introducing the multipliers (dual variables) $\theta_i \geq 0$ for $i \in \mathcal{M}$, we can define the Lagrangian

$$\begin{aligned} L(v, r, \theta) &:= r + \sum_{i \in \mathcal{M}} \theta_i \cdot [(A_i z + B_i v)^T P (A_i z + B_i v) - r] \\ &= (1 - \sum_{i \in \mathcal{M}} \theta_i) r + [z^T \quad v^T] \Upsilon_\theta \begin{bmatrix} z \\ v \end{bmatrix}. \end{aligned}$$

The Lagrange dual function is easily verified to be

$$g(\theta) := \inf_{v, r} L(v, r, \theta) = \begin{cases} z^T \Gamma_\theta(P) z & \text{if } \sum_{i \in \mathcal{M}} \theta_i = 1 \\ -\infty & \text{if otherwise.} \end{cases}$$

Hence, the dual problem of (23) is

$$\text{maximize } z^T \Gamma_\theta(P) z \quad \text{subject to } \theta \in \Delta, \quad (24)$$

whose solution is exactly the square of the right hand side of (22). Since the original optimization problem (23) is both convex (indeed a second order cone programming) and strongly feasible (r can be made arbitrarily large), it has the same solution as that of (24). This proves the desired conclusion. $\square$

Remark 6. Let $\|\cdot\|$ be the Euclidean norm. By setting $v = 0$, Lemma 6 implies that

$$\sup_{\theta \in \Delta} z^T \Gamma_\theta(P) z = \inf_{v \in \mathbb{R}^p} \max_{i \in \mathcal{M}} \|A_i z + B_i v\|_P^2 \leq \max_{i \in \mathcal{M}} z^T (A_i^T P A_i) z \leq \max_{i \in \mathcal{M}} \|A_i^T P A_i\| \cdot \|z\|^2,$$

where the norm in $\|A_i^T P A_i\|$ is the L_2 -induced matrix norm. This implies that $\Gamma_\theta(P) \preceq \|A_i^T P A_i\| \cdot I$ for all $\theta \in \Delta$. In other words, although pseudo inverses are used in defining $\Gamma_\theta(P)$, the set $\Gamma_\Delta(P)$ is bounded. For example, if $B_i = 0$ for all i , then $\Gamma_\theta(P) = \sum_{i \in \mathcal{M}} \theta_i A_i^T P A_i$.

To apply Proposition 1 to the ellipsoidal norm associated with $P \succ 0$, we write the condition $\|z\|_{P\sharp} \geq \alpha \|z\|_P$, $\forall z$, equivalently as

$$\sup_{\theta \in \Delta} z^T \Gamma_\theta(P) z \geq \alpha z^T P z, \quad \forall z.$$

As noted before, with P given, $\Gamma_\theta(P)$ hence $z^T \Gamma_\theta(P) z$ is a concave function of θ for each z . Thus, a sufficient condition for the above is obtained by replacing Δ with the set of its vertices, i.e.,

$$\max_{i \in \mathcal{M}} z^T \left(A_i^T P A_i - (A_i^T P B_i)(B_i^T P B_i)^\dagger (B_i^T P A_i) \right) z \geq \alpha z^T P z, \quad \forall z.$$

The above holds if there exists $\theta \in \Delta$ such that

$$\sum_{i \in \mathcal{M}} \theta_i \left(A_i^T P A_i - (A_i^T P B_i)(B_i^T P B_i)^\dagger (B_i^T P A_i) \right) \succeq \alpha P.$$

From the above derivation and by Proposition 1, we have the following result.

Proposition 4. *For any $P \succ 0$, denote by α^* the solution to the following linear matrix inequality (LMI) problem:*

$$\begin{aligned} & \max_{\alpha \geq 0, \theta \in \Delta} \alpha \\ \text{subject to} & \quad \sum_{i \in \mathcal{M}} \theta_i \left(A_i^T P A_i - (A_i^T P B_i)(B_i^T P B_i)^\dagger (B_i^T P A_i) \right) \succeq \alpha P. \end{aligned} \tag{25}$$

Then, $\rho^* \geq \alpha^*$.

Example 4. *Consider the following SLCS:*

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Using Proposition 4 with 500 randomly generated $P \succeq 0$ and keeping the largest α^* , we find that $\rho^* \geq 0.3282$, which is achieved by $P = \begin{bmatrix} 0.8437 & 0.4611 \\ 0.4611 & 0.4658 \end{bmatrix}$.

We next apply Proposition 2 to ellipsoidal norms. For $P \succ 0$, the condition $\|\cdot\|_{P\#} \leq \alpha \|\cdot\|$ is equivalent to $\sup_{\theta \in \Delta} z^T \Gamma_\theta(P) z \leq \alpha z^T P z$, $\forall z$, or equivalently, $\Gamma_\theta(P) \preceq P$ for all $\theta \in \Delta$. This leads to the following result.

Proposition 5. *Let $P \succ 0$ be given, and let α^* be the solution to the following problem:*

$$\begin{aligned} & \min_{\alpha \geq 0} \alpha \\ \text{subject to} & \quad \sum_{i \in \mathcal{M}} \theta_i A_i^T P A_i - \left(\sum_{i \in \mathcal{M}} \theta_i A_i^T P B_i \right) \left(\sum_{i \in \mathcal{M}} \theta_i B_i^T P B_i \right)^\dagger \left(\sum_{i \in \mathcal{M}} \theta_i B_i^T P A_i \right) \preceq \alpha P, \quad \forall \theta \in \Delta. \end{aligned} \tag{26}$$

Then, $\rho^* \leq \alpha^*$.

The above problem is difficult to solve due to its infinite number of constraints. With the left hand side of the constraint inequality being a concave function of θ , it no longer suffices to have the inequality hold at only the vertices of Δ . Although solving the problem (26) is in general very difficult, the following example shows that it can yield useful bounds on ρ^* in some simple cases.

Example 5. Consider the following controlled double integrator system sampled at integer times:

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ b(t) \end{bmatrix} u(t),$$

where $b(t) \in \{b_1, b_2\}$ has two possible values chosen by the adversary to derail the stabilizing effort. The underlying SLCS is

$$A_1 = A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ b_1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ b_2 \end{bmatrix}.$$

Consider a generic (homogenized) $P = \begin{bmatrix} 1 & \beta \\ \beta & \gamma \end{bmatrix} \succ 0$. Then we must have $\gamma > 0$ and $\beta^2 < \gamma$. The constraint (27) is satisfied if and only if for all $\theta_1, \theta_2 \in [0, 1]$ with $\theta_1 + \theta_2 = 1$,

$$\begin{bmatrix} 1 & 1 + \beta \\ 1 + \beta & 1 + 2\beta + \gamma \end{bmatrix} - \underbrace{\frac{(\theta_1 b_1 + \theta_2 b_2)^2}{\gamma(\theta_1 b_1^2 + \theta_2 b_2^2)} \begin{bmatrix} \beta \\ \beta + \gamma \end{bmatrix} \begin{bmatrix} \beta \\ \beta + \gamma \end{bmatrix}^T}_{Q} \preceq \alpha \begin{bmatrix} 1 & \beta \\ \beta & \gamma \end{bmatrix}. \quad (28)$$

Observe that $Q \succeq 0$ and

$$\min_{\theta_1, \theta_2 \geq 0, \theta_1 + \theta_2 = 1} \frac{(\theta_1 b_1 + \theta_2 b_2)^2}{\theta_1 b_1^2 + \theta_2 b_2^2} = \begin{cases} \frac{4b_1 b_2}{(b_1 + b_2)^2} := M_b & \text{if } b_1 b_2 > 0 \\ 0 & \text{if } b_1 b_2 \leq 0. \end{cases}$$

Assume $b_1 b_2 \leq 0$. Then (28) is equivalent to

$$\begin{bmatrix} 1 & 1 + \beta \\ 1 + \beta & 1 + 2\beta + \gamma \end{bmatrix} \preceq \alpha \begin{bmatrix} 1 & \beta \\ \beta & \gamma \end{bmatrix}.$$

By letting $\alpha = 1$, it can be easily verified that the above constraint cannot be satisfied. Thus, $\alpha^* \geq 1$ and Proposition 5 fails to determine the σ -resilient stabilizability of the system.

Assume in the following $b_1 b_2 > 0$. Then the condition (28) is equivalent to

$$\gamma \begin{bmatrix} 1 - \alpha & 1 + \beta(1 - \alpha) \\ 1 + \beta(1 - \alpha) & 1 + 2\beta + \gamma(1 - \alpha) \end{bmatrix} \preceq M_b \begin{bmatrix} \beta \\ \beta + \gamma \end{bmatrix} \begin{bmatrix} \beta \\ \beta + \gamma \end{bmatrix}^T.$$

Since β and γ can be freely chosen (as long as $P \succ 0$), we set $\gamma = 2\beta^2$ where $\beta > 0$. Then, it can be verified that the above matrix inequality is equivalent to

$$M_b \geq \max \left\{ \frac{1 - \alpha}{2\beta^2}, \frac{2}{1 + 2\beta} + \frac{4\beta^2(1 - \alpha)}{(1 + 2\beta)^2}, \frac{2\beta^2(1 - \alpha)^2 - 2\alpha^2}{2\beta^2(1 - \alpha) - 2\alpha\beta - \alpha} \right\}.$$

Note that the right hand side converges to $\max\{0, 1 - \alpha\}$ as $\beta \rightarrow +\infty$. Thus, for any $\alpha > 1 - M_b$, we can find $\beta > 0$ large enough such that the above inequality, hence the constraint (27), is satisfied. By Proposition 5, this implies that, if $b_1 b_2 > 0$,

$$\rho^* \leq 1 - M_b = \frac{(b_1 - b_2)^2}{(b_1 + b_2)^2} < 1.$$

In conclusion, the SLCS is σ -resiliently stabilizable if $b_1, b_2 \neq 0$ are of the same sign. $\square$

5 σ -Resilient Generating Function

Another approach to characterize the σ -resilient stabilizing rate is via the generating functions. Define for each $\lambda \geq 0$ and $k \in \mathbb{N}$ the following function:

$$F_\lambda^k(z) := \inf_{u(0)} \sup_{\sigma(0)} \cdots \inf_{u(k-1)} \sup_{\sigma(k-1)} \left[\sum_{t=0}^k \lambda^t \|x(t; \sigma, u, z)\|^2 \right], \quad \forall z \in \mathbb{R}^n. \quad (29)$$

Obviously, $F_\lambda^k(\cdot)$ is finite on $\mathbb{R}^n$.

Proposition 6. *The functions $F_\lambda^k(\cdot)$, $k \in \mathbb{N}$, can be obtained iteratively as follows:*

$$\begin{aligned} F_\lambda^0(z) &= \|z\|^2 \\ F_\lambda^k(z) &= \|z\|^2 + \lambda \cdot \mathcal{T} \left[F_\lambda^{k-1} \right] (z), \quad \forall z, k = 1, 2, \dots \end{aligned} \quad (30)$$

For each $k \in \mathbb{N}$, $F_\lambda^k(\cdot)$ is nonnegative, convex, and homogeneous of degree two on $\mathbb{R}^n$. Moreover, the sequence of functions $F_\lambda^k(\cdot)$, $k \in \mathbb{N}$, is nondecreasing as k increases.

Proof. That $F_\lambda^0(z) = \|z\|^2$ is trivial. Denote $v = u(0)$, $u' = (u(1), \dots, u(k-1))$, $i = \sigma(0)$, and $\sigma' = (\sigma(1), \dots, \sigma(k-1))$. Then,

$$\begin{aligned} F_\lambda^k(z) &= \inf_v \sup_i \inf_{u(1)} \sup_{\sigma(1)} \cdots \inf_{u(k-1)} \sup_{\sigma(k-1)} \left[\|z\|^2 + \lambda \sum_{t=0}^{k-1} \|x(t; \sigma', u', A_i z + B_i v)\|^2 \right] \\ &= \|z\|^2 + \lambda \cdot \inf_v \sup_i \left[\inf_{u(1)} \sup_{\sigma(1)} \cdots \inf_{u(k-1)} \sup_{\sigma(k-1)} \sum_{t=0}^{k-1} \|x(t; \sigma', u', A_i z + B_i v)\|^2 \right] \\ &= \|z\|^2 + \lambda \cdot \inf_v \sup_i F_\lambda^{k-1}(A_i z + B_i v). \end{aligned}$$

For each $k \in \mathbb{N}$, the nonnegativeness of $F_\lambda^k(\cdot)$ is trivial. Its convexity can be proved by using the fact that $\mathcal{T}$ maps convex functions to convex functions (see Lemma 3). The homogeneity of $F_\lambda^k(\cdot)$ follows directly from the fact that $x(\cdot; \sigma, \alpha u, \alpha z) = \alpha \cdot x(\cdot; \sigma, u, z)$ for $\alpha \in \mathbb{R}$. Finally, the monotonicity of $F_\lambda^k(\cdot)$ in k is straightforward from the definition (29). $\square$

Definition 7. *The σ -resilient control generating function (σ -CGF) of the SLCS (1) is defined as*

$$F_\lambda(z) := \lim_{k \rightarrow \infty} F_\lambda^k(z), \quad \forall z \in \mathbb{R}^n, \quad \forall \lambda \geq 0. \quad (31)$$

The radius of convergence of $F_\lambda(\cdot)$, denoted λ_F^* , is defined as

$$\lambda_F^* := \sup \{ \lambda \geq 0 \mid F_\lambda(z) < \infty, \forall z \in \mathbb{R}^n \}.$$

The monotonicity of $F_\lambda^k(\cdot)$ in k implies that $F_\lambda(\cdot)$ is well defined, though possibly of infinite value. For $\lambda = 0$, we have $F_\lambda^k(z) = \|z\|^2$ for all k , hence $F_\lambda(z) = \|z\|^2$.

Remark 7. By setting $u(0), \dots, u(k-1)$ to zero in (29), we have the bound

$$\begin{aligned} F_\lambda^k(z) &\leq \sup_{\sigma(0), \dots, \sigma(k-1)} \sum_{t=0}^k \lambda^t \|x(t; \sigma, 0, z)\|^2 \\ &\leq \sup_{\sigma} \sum_{t=0}^{\infty} \lambda^t \|x(t; \sigma, 0, z)\|^2 := G_\lambda(z), \quad \forall k \in \mathbb{N}. \end{aligned}$$

Hence, $F_\lambda(z) \leq G_\lambda(z)$ where $G_\lambda(z)$ is the (strong) generating function of the autonomous SLS with subsystems $\{A_i\}_{i \in \mathcal{M}}$. The radius of convergence of $G_\lambda(z)$, $\lambda_G^* := \sup\{\lambda \geq 0 \mid G_\lambda(\cdot) < \infty\}$, thus provides a lower bound of λ_F^* : $\lambda_F^* \geq \lambda_G^*$. It is shown in [22] that λ_G^* characterizes the fastest exponential growth rate $\hat{\rho}^*$ of the state solutions of the autonomous SLS via $\hat{\rho}^* = (\lambda_G^*)^{-1/2}$. A similar relation between λ_F^* and ρ^* will be derived later on. Thus, we have $\rho^* \leq \hat{\rho}^*$.

The following result can be directly obtained from Proposition 6 and the definition (31).

Corollary 2. $F_\lambda(\cdot)$ is nonnegative, convex, and homogeneous of degree two on $\mathbb{R}^n$. Moreover,

$$F_\lambda(z) = \|z\|^2 + \lambda \cdot \mathcal{T}[F_\lambda](z), \quad \forall z \in \mathbb{R}^n. \quad (32)$$

Corollary 3. For any $\lambda \geq 0$, the set

$$\mathcal{N}_\lambda := \{z \in \mathbb{R}^n \mid F_\lambda(z) < \infty\}$$

is a control σ -invariant subspace of $\mathbb{R}^n$.

Proof. That $\mathcal{N}_\lambda$ is a subspace follows from the convexity and homogeneity of $F_\lambda(\cdot)$. The control-invariant property is a direct consequence of the Bellman equation (32). $\square$

Theorem 6. The SLCS (1) is σ -resiliently exponentially stabilizable if and only if the radius of convergence λ_F^* of its σ -CGF $F_\lambda(\cdot)$ satisfies $\lambda_F^* > 1$.

Proof. Suppose $\lambda_F^* > 1$. Then $F_\lambda(\cdot)$ with $\lambda = 1$, $F_1(\cdot)$, is finite on $\mathbb{R}^n$. As $F_1(\cdot)$ is also convex by Proposition 6, it must be continuous on $\mathbb{R}^n$. The continuity implies that $\alpha_2 := \sup_{\|z\|=1} F_1(z)$ is a finite constant. By homogeneity, we have $\|z\|^2 \leq F_1(z) \leq \alpha_2 \|z\|^2$, $\forall z$. Furthermore, by letting $\lambda = 1$ in (32), we have, for all $z \in \mathbb{R}^n$,

$$\mathcal{T}[F_1](z) = \inf_{v \in \mathbb{R}^p} \max_{i \in \mathcal{M}} F_1(A_i z + B_i v) = F_1(z) - \|z\|^2 \leq \frac{\alpha_2 - 1}{\alpha_2} F_1(z),$$

or equivalently, $\mathcal{T}[\sqrt{F_1}](\cdot) \leq \sqrt{(\alpha_2 - 1)/\alpha_2} \cdot \sqrt{F_1(\cdot)}$. Noting that $\sqrt{F_1(\cdot)}$ is a nonzero seminorm, by Proposition 2, we have $\rho^* \leq \sqrt{(\alpha_2 - 1)/\alpha_2} < 1$. Hence, the SLCS is σ -resiliently exponentially stabilizable.

For the other direction, assume the SLCS is σ -resiliently exponentially stabilizable with the parameters $K > 0$ and $\rho \in [0, 1)$ in (2). Then, for any $z \in \mathbb{R}^n$, there exists a user input u such that $\|x(t; \sigma, u, z)\| \leq K \rho^t \|z\|$, $\forall t, \forall \sigma \in \mathcal{S}$. By adopting the first k inputs $u(0), \dots, u(k-1)$ in (29) and noting that the $\sigma(0), \dots, \sigma(k-1)$ in (29) are the first k steps of an admissible switching sequence $\sigma \in \mathcal{S}$, we obtain

$$\begin{aligned} F_\lambda^k(z) &\leq \sum_{t=0}^k \lambda^t K^2 \rho^{2t} \|z\|^2 \leq \frac{K^2}{1 - \lambda \rho^2} \|z\|^2 \\ \Rightarrow F_\lambda(z) &\leq \frac{K^2}{1 - \lambda \rho^2} \|z\|^2 < \infty, \quad \forall z, \end{aligned}$$

for all $\lambda < 1/\rho^2$. This implies that $\lambda_F^* \geq 1/\rho^2 > 1$. $\square$

Corollary 4. *The σ -resilient stabilizing rate ρ^* of the SLCS (1) is characterized by the radius of convergence of its σ -CGF $F_\lambda(\cdot)$ by*

$$\rho^* = (\lambda_F^*)^{-1/2}.$$

Proof. Consider the scaled SLCS with subsystem dynamics $(\tilde{A}_i, \tilde{B}_i) = (\beta A_i, \beta B_0)$ for some $\beta > 0$. The function $\tilde{F}_\lambda^k(\cdot)$ defined in (29) but with $x(t; \sigma, u, z)$ replaced with $\tilde{x}(t; \sigma, u, z)$ satisfies $\tilde{F}_\lambda^k(z) = F_{\beta^2 \lambda}^k(z)$, $\forall z$, due to the fact that $\tilde{x}(t; \sigma, \tilde{u}, z) = \beta^t \cdot x(t; \sigma, u, z)$, where $\tilde{u}(t) := \beta^t u(t)$, $\forall t$. By letting $k \rightarrow \infty$, we have $\tilde{F}_\lambda(z) = F_{\beta^2 \lambda}(z)$, $\forall z$. As a result, the radius of convergence of the scaled SLCS satisfies $\lambda_{\tilde{F}}^* = \lambda_F^* / \beta^2$. On the other hand, by Lemma 1, the σ -resilient stabilizing rate $\tilde{\rho}^*$ of the scaled SLCS satisfies $\tilde{\rho}^* = \beta \cdot \rho^*$. The scaled SLCS is σ -resiliently exponentially stabilizable if and only if $\tilde{\rho}^* < 1$, i.e., $\beta < (\rho^*)^{-1}$. According to Theorem 6, another equivalent condition is given by $\lambda_{\tilde{F}}^* > 1$, i.e., $\beta < (\lambda_F^*)^{1/2}$. This proves the desired conclusion. $\square$

Example 6. *Consider the 1D SLCS in Example 1, $A_1 = a_1$, $B_1 = b_1$, $A_2 = a_2$, $B_2 = b_2$, with $b_1^2 + b_2^2 \neq 0$. By homogeneity, $F_\lambda^k(z) = f_k \cdot z^2$, $\forall z \in \mathbb{R}$, for some constant $f_k \geq 0$; thus we need only to compute f_k . At $k = 0$, $f_0 = 1$. For $k \geq 1$, by letting $z = 1$ in (30), we have*

$$f_k = 1 + \lambda f_{k-1} \cdot \inf_v \max\{(a_1 + b_1 v)^2, (a_2 + b_2 v)^2\}. \quad (33)$$

Note that $\max_{i=1,2}(a_i + b_i v)^2$ achieves minimum at v^ that solves $(a_1 + b_1 v)^2 = (a_2 + b_2 v)^2$. There are at most two such solutions $v_1 := -(a_1 - a_2)/(b_1 - b_2)$ and $v_2 := -(a_1 + a_2)/(b_1 + b_2)$ (only one solution if $b_1 = \pm b_2$), and v^* is the one with the smaller value for $(a_1 + b_1 v)^2 = (a_2 + b_2 v)^2$. This implies*

$$\begin{aligned} \inf_v \max_{i=1,2} (a_i + b_i v)^2 &= \begin{cases} \min_{i=1,2} (a_i + b_i v_i)^2 & \text{if } b_1 \neq \pm b_2 \\ (a_1 + b_1 v_1)^2 & \text{if } b_1 = -b_2 \\ (a_1 + b_1 v_2)^2 & \text{if } b_1 = b_2 \end{cases} \\ &= c, \end{aligned}$$

where $c := (a_1 b_2 - a_2 b_1)^2 / (|b_1| + |b_2|)^2$. Then the iteration (33) becomes $f_k = (\lambda c) f_{k-1} + 1$, which converges as $k \rightarrow \infty$ to $f_\infty = 1/(1 - \lambda c) < \infty$ if and only if $\lambda < 1/c$. In conclusion, the σ -CGF is

$$F_\lambda(z) = \begin{cases} \frac{z^2}{1 - \lambda c} & \text{if } \lambda \in [0, 1/c) \\ \infty & \text{if } \lambda \geq 1/c, \end{cases}$$

which has a radius of convergence $\lambda_F^ = 1/c = (|b_1| + |b_2|)^2 / (a_1 b_2 - a_2 b_1)^2$. As predicted by Corollary 4, λ_F^* is exactly $(\rho^*)^{-2}$ for the ρ^* derived in Section 2. In particular, if $a_1/b_1 = a_2/b_2$, then $c = 0$, $f_k \equiv 1$, $F_\lambda(z) = \|z\|^2 < \infty$ for all $\lambda > 0$, hence $\lambda^* = \infty$.*

For higher dimensional systems, the σ -CGF will have a more complicated form. An approach to computing $F_\lambda(\cdot)$ approximately is to find $F_\lambda^k(\cdot)$ for a large enough k , which in turn can be derived using the iteration (30). To elaborate, we first introduce some notions. For a subset $\mathcal{P}$ of p.s.d. matrices, denote by $\mathcal{P}^-$ the augmented subset:

$$\mathcal{P}^- := \{P \mid P \succeq 0, P \preceq P' \text{ for some } P' \in \mathcal{P}\},$$

which consists of all p.s.d. matrices dominated by matrices in $\mathcal{P}$. Then $\mathcal{P}^-$ defines the same pointwise maximum quadratic function as $\mathcal{P}$:

$$\sup_{P \in \mathcal{P}^-} z^T P z = \sup_{P \in \mathcal{P}} z^T P z.$$

Moreover, $\mathcal{P}^-$ has nonempty interior in the set of p.s.d. matrices if $\mathcal{P}$ has at least an element $P \succ 0$. In this case, $\mathcal{P}^-$ inherits the Lebesgue measure from the set of n -by- n symmetric matrices considered as an $n(n+1)/2$ -dimensional Euclidean space. Thus, a nonnegative measurable function $f(\cdot)$ defined on $\mathcal{P}^-$ can be integrated as $\int_{\mathcal{P}^-} f(P) dP$.

Proposition 7. *For each $k \in \mathbb{N}$,*

$$F_\lambda^k(z) = \sup_{P \in \mathcal{P}_k} z^T P z, \quad \forall z, \quad (34)$$

for some bounded subset $\mathcal{P}_k$ of p.s.d. matrices.

Proof. We prove by induction on k . At $k = 0$, the conclusion holds trivially since $F_\lambda^0(z) = \|z\|^2$ implies $\mathcal{P}_0 = \{I\}$. Suppose it holds at indices $0, \dots, k$. Then for $k+1$, we have, by (30),

$$F_\lambda^{k+1}(z) = \|z\|^2 + \lambda \cdot \inf_{v \in \mathbb{R}^p} \max_{i \in \mathcal{M}} F_\lambda^k(A_i z + B_i v). \quad (35)$$

Using the induction hypothesis, we have

$$J(z) := \inf_v \max_i F_\lambda^k(A_i z + B_i v) = \inf_v \max_i \max_{P \in \mathcal{P}_k^-} (A_i z + B_i v)^T P (A_i z + B_i v),$$

which is the solution to the following optimization problem

$$\text{minimize } r \text{ subject to } (A_i z + B_i v)^T P (A_i z + B_i v) \leq r, \quad \forall i \in \mathcal{M}, \forall P \in \mathcal{P}_k^-, \quad (36)$$

with the optimization variables being v and r . Define the multiplier function $\theta = (\theta_1, \dots, \theta_m)$ where, for each i , $\theta_i : \mathcal{P}_k^- \rightarrow \mathbb{R}$ is a nonnegative and integrable function; and define $Q_i \succeq 0$ as

$$Q_i := \int_{\mathcal{P}_k^-} \theta_i(P) P dP. \quad (37)$$

Then, the Lagrangian is given by

$$\begin{aligned} L(v, r, \theta) &:= r + \sum_i \int_{\mathcal{P}_k^-} \theta_i(P) [(A_i z + B_i v)^T P (A_i z + B_i v) - r] dP \\ &= \left(1 - \sum_i \int_{\mathcal{P}_k^-} \theta_i(P) dP \right) r + [z^T \quad v^T] \begin{bmatrix} \sum_i A_i^T Q_i A_i & \sum_i A_i^T Q_i B_i \\ \sum_i B_i^T Q_i A_i & \sum_i B_i^T Q_i B_i \end{bmatrix} \begin{bmatrix} z \\ v \end{bmatrix}. \end{aligned}$$

The Lagrange dual function $g(\theta) := \inf_{v, r} L(v, r, \theta)$ is

$$g(\theta) = \begin{cases} z^T \Upsilon_\theta z & \text{if } \sum_i \int_{\mathcal{P}_k^-} \theta_i(P) dP = 1 \\ -\infty & \text{if otherwise,} \end{cases}$$

where

$$\Upsilon_\theta := \sum_i A_i^T Q_i A_i - \sum_i A_i^T Q_i B_i \left(\sum_i B_i^T Q_i B_i \right)^\dagger \sum_i B_i^T Q_i A_i. \quad (38)$$

Since $\mathcal{P}_k$ hence $\mathcal{P}_k^-$ is bounded by the induction hypothesis, the problem (36) is strongly feasible. By strong duality, we have

$$J(z) = \sup_\theta g(\theta) = \sup_{\theta \in \Delta(\mathcal{P}_k^-)} z^T \Upsilon_\theta z,$$

where $\Delta(\mathcal{P}_k^-)$ denotes the set of all multiplier functions θ satisfying $\sum_i \int_{\mathcal{P}_k^-} \theta_i(P) dP = 1$. As a result, (35) implies

$$F_\lambda^{k+1}(z) = \sup_{\theta \in \Delta(\mathcal{P}_k^-)} z^T [I + \lambda \Upsilon_\theta] z = \sup_{P \in \mathcal{P}_{k+1}} z^T P z,$$

where $\mathcal{P}_{k+1}$ is the subset defined by

$$\mathcal{P}_{k+1} := \{I + \lambda \Upsilon_\theta \mid \theta \in \Delta(\mathcal{P}_k^-)\}. \quad (39)$$

Note that $\mathcal{P}_{k+1}$ is bounded since $F_\lambda^{k+1}(\cdot)$ is finite. This completes the proof of the induction step hence the proposition. $\square$

It is worth pointing out that the iterative procedure to obtain $\mathcal{P}_{k+1}$ from $\mathcal{P}_k$ has been described in the above proof, specifically, equations (37), (38), and (39). On the other hand, it would be impractical to exhaust all $\theta \in \Delta(\mathcal{P}_k^-)$ in the iteration (39). Instead, finite dimensional (e.g., polyhedral) outer approximations of $\mathcal{P}_k^-$ should be sought to compute over approximations of the σ -CGF. This will be explored in our future work.

The σ -resilient control generating function provides a way to construct an asymptotic approximation of a Barabanov norm. To see this, let $F_\lambda(\cdot)$ be the σ -CGF of the SLCS (1) and let $\lambda_F^* > 0$ be its radius of convergence. Note that $F_{\lambda_F^*}(z_0) = \infty$ at some $z_0 \in \mathbb{R}^n$. Assume the SLCS (1) is irreducible. Then Corollary 2 implies that $F_{\lambda_F^*}(z) = \infty$ for all $z \neq 0$. By choosing $\lambda = \lambda_F^* - \varepsilon$ for an arbitrarily small $\varepsilon > 0$, $F_\lambda(\cdot)$ is finite, convex, and homogeneous of degree two; thus

$$\|z\|_\lambda := \sqrt{F_\lambda(z)}, \quad \forall z \in \mathbb{R}^n,$$

defines a norm of $\mathbb{R}^n$. As the magnitude of $\|\cdot\|_\lambda$ is much larger than the Euclidean norm, in the Bellman equation (32), we have approximately

$$\|z\|_\lambda \simeq \sqrt{\lambda} \cdot \inf_{v \in \mathbb{R}^p} \max_{i \in \mathcal{M}} \|A_i z + B_i v\|_\lambda = \sqrt{\lambda} \cdot \|z\|_{\lambda, \#}.$$

In other words, as $\varepsilon \rightarrow 0$, $\|\cdot\|_\lambda$ becomes arbitrarily close to being a Barabanov norm of scale $1/\sqrt{\lambda} \simeq 1/\sqrt{\lambda_F^*}$, which is exactly ρ^* by Corollary 4.

6 CONCLUSIONS

The concept of switching-resilient stabilization of switched linear control systems is introduced. Stabilizability conditions and bounds on the σ -resilient stabilization rate are derived.

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