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Performance Evaluation of Interval-Searching Conflict Resolution Algorithms

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Abstract

A single multiaccess channel is studied with the outcome of a transmission being either 'idle', 'success', or 'collision' (ternary channel). Packets involved in a collision must be retransmitted, and an efficient way to solve a collision is known in the literature as Gallager-Tsybakov-Mikhailov (GTM) algorithm, which falls into class of interval-searching contention resolution algorithms. Performance analysis of the algorithm was based on a numerical solution of some recurrence equations and on a numerical evaluation of some series. The obvious drawback of such an analysis is lack of insight into the behaviour of the algorithm. We shall present a new approach of looking at the algorithm and discuss some attempts of analyzing its performance. In particular, expected lengths of a resolved interval and a conflict resolution interval as well as throughput of the algorithm will be discussed using asymptotic approximation and "a small input rate" approximation. Finally, we generalize these results to cover a wider class of interval-searching algorithms.

1. INTRODUCTION

In a broadcast packet-switching network a number of users share a common communication channel. Since the channel is the only way of communications among the users, packet collisions are inevitable if a central coordination is not provided. The problem is to find an efficient algorithm for retransmitting conflicting packets. There are a number of algorithms, however, in recent years *conflict resolution algorithms* (CRA) [2], [3], [5], [11], [12] have become more and more popular. The basic idea of CRA is to solve each conflict by splitting it into smaller conflicts (divide-and-conquer algorithm). This is possible if each user observes the channel and learns whether in the past it was idle, success or collision transmissions. The partition of a conflict can be made on the basis of a random variable [3], [7], [9], [11] or on the basis of the time a user became active [2], [5], [12]. The former algorithm is known as Capetanakis-Tsybakov-Mikhailov algorithm (stack algorithm) while the latter as Gallager-Tsybakov-Mikhailov algorithm (interval-searching algorithm).

Performance analysis of CRA-algorithm is quite hard, since most quantities of interest are involved in quite sophisticated recurrence equations. In fact, previous analyses of CRA-algorithm were restricted to numerical evaluation of some recurrence equations. This was relaxed by Hofri [7], Fayolle et al [4] and Szpankowski [9], and lately by Kaplan [14] as well as by Mathys and Flajolet [15] for stack algorithms by solving some functional equations and applying asymptotic approximation technique. We use the same methodology to analyze interval-searching algorithms , however, since the problem is much more difficult than the previous one, we shall use some numerical analysis as well as an asymptotic approximation and so called "small input rate" approximation.

In the next section, we shortly describe the algorithm and formulate problems to solve. In Section 3, we reduce the problem to a simple one with a help of some simple numerical computations. Then a closed form solution for the modified problem will be given. Finally, we apply asymptotic and 'small input rate' approximation to obtain tractable formulas for quantities of interest, e.g. expected length of conflict resolution interval and throughput. In particular, we shall show that for Gallager-Tsybakov-Mikhailov algorithm [5],[12] the average time to solve a conflict of multiplicity n is $O(lgn)$ (compare with $O(n)$ for stack algorithm), and the average length of resolved interval is $O(n^{-1})$. However, to determine throughput of the algorithm we must estimate not only the average values of conflict resolution interval and resolved interval , but we need a tight approximation for generating functions for the above quantities. This is done through so called *small input rate approximation*. We shall show that the real throughput equal 0.48771 is approximated by 0.48819 in the small input rate approximation . Finally, in Section 5 we extend the above analysis in the sense that a general recurrence equation common for a class of interval-searching algorithms will be studied and solved. These considerations will be illustrated by performance evaluation of Berger's algorithm [2].

2. PROBLEM STATEMENT

Let us start with a short description of Gallager-Tsybakov-Mikhailov algorithm with ternary feedback [5],[12]. Assume a channel is slotted and a slot duration is equal to a packet transmission time. The algorithm defined below allows the transmission of the packets on the basis of their generation times. Assume packets are generated according to a Poisson point process with rate λ. Access to the channel is controlled by a window based on the current age of packets. This window will be referred to as the *enabled interval* (EI). Let s_i denote the starting point for the i-th EI, and t_i is corresponding starting point for the *conflict resolution interval* (CRI), where CRI represents the number of slots needed to resolve a collision . Initially, the enabled interval is set to be $[s_i, \min\{s_i+\tau, t_i\})$, where τ is a constant which will be further optimized. At each step of the algorithm we compute the endpoints of the EI based on the outcomes of the channel. If at most one packet falls in the initial EI, then the conflict resolution interval ends immediately, and $s_{i+1}=s_i+\min\{\tau, t_i-s_i\}$. Otherwise, the EI is split into two halves, and three cases must be considered:

(i) all users whose current age of packets fall into the first (left) half are allowed to transmit packets. If it causes next *collision*, all knowledge about the second half is erased, and the first half is immediately split into two halves,

(ii) if enabling the first half causes on *idle* slot, the second half is immediately split into two halves,

(iii) if the first half gives a *success*, the entire second half is enabled

A CRI that begins with a collision continues until enabling the second half of some pairs gives a success. The rate of successful transmission is called *throughput*. This algorithm is known in the literature as Gallager [5], Tsybakov- Mikhailov [12] algorithm (GTM algorithm). To analyze it we introduce below some notations.

Assume an initial collision of a CRI is of multiplicity n. Then all packets whose generation time fall into an interval $[s_i, s_{i+1})$ are successfully sent in the i-th CRI. The interval $[s_i, s_{i+1})$ is called the i-th *resolved interval* (RI). Let T_n and τW_n denote the expected value for CRI and RI, respectively, assuming you start with n-conflict. In [12], [2] it is proved that they satisfy the following recurrence equations:

$$T_0 = T_1 = 1$$
$$T_n = (2^n - 2)^{-1}[2^n + nT_{n-1} + \sum_{k=1}^{n-1} \binom{n}{k} T_k], \quad n \geq 2 \tag{1}$$

$$W_0 = W_1 = 1$$
$$W_n = (2^{n+1} - 1)^{-1}[1 + nW_{n-1} + \sum_{k=1}^{n-1} \binom{n}{k} W_k], \quad n \geq 2 \tag{2}$$

Moreover, let $T(x)$ and $W(x)$ denote exponential generating functions for T_n and W_n, respectively, that is,

$$W(x) = \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} \quad ; \quad T(x) = \sum_{n=0}^{\infty} T_n \frac{x^n}{n!} \tag{3}$$

In [12] it was also proved that the algorithm is *stable* if and only if the input rate $\lambda < \lambda_{max}$ where

$$\lambda_{max} = \sup_x \frac{xW(x)}{T(x)} \tag{4}$$

In the next section we analyze (1) - (4), and give easy computable formulas for the quantities of interests.

3. ANALYSIS OF GALLAGER-TSYBAKOV-MIKHAILOV ALGORITHM

To find a closed form solution and asymptotic approximation for T_n, W_n, we first reduce the problem of solving (1) and (2) to a simple one using some numerical analysis. Then an analytical solution of the modified problem will be given and finally we shall present some approximations.

Reduction to a simple problem

Solution of (1) and (2) are not known, but we are able to present a rigorous solution of the following non-trivial recurrence equations:

$$t_0 = t_1 = 1$$
$$t_n = (2^n - 2)^{-1}[2^n + \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix} t_k], \quad n \geq 2 \tag{5}$$

$$w_0 = w_1$$
$$w_n = (2^{n+1} - 1)^{-1}[1 + \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix} w_k] \quad n \geq 2 \tag{6}$$

Naturally, $t_n \leq T_n$ and $w_n \leq W_n$. Before we deal with (5) and (6) let us find a relationship between T_n, W_n and t_n, w_n. We have computed the differences $T_n - t_n$ and $(n+1)$ $[W_n - w_n]$ for $n \geq 2$. Numerical results reveal that T_n and W_n might be tightly estimated by the following approximate formulas:

$$T_n \approx t_n + a \quad n > 3 \tag{7}$$

$$W_n \approx w_n + \frac{b}{n+1} \quad n > 3 \tag{8}$$

where $a = 2.644$ and $b = 0.9122$. Hence, by (7) and (8) we reduce (1), (2) to solution of the recurrence equations (5) and (6).

Closed form solution

Let $t(x)$ and $w(x)$ be exponential generating function for t_n and w_n. We find explicit formulas for $t(x)$ and $w(x)$.

Let us start with $t(x)$. Multiplying both sides of (5) by $x^n/n!$ and using boundary conditions ($t_0 = t_1 = 1$) one finds that

$$t(x) - t(\frac{x}{2})(e^{x/2} - 1) = e^x - e^{x/2} - x - 1 \tag{9}$$

Define now a new function $H(x)$ as

$$H(x) = \frac{x}{e^x - 1} t(x) \tag{10}$$

then multiplying (9) by $x/(e^x-1)$ we obtain

$$H(x)=2H(\frac{x}{2})+\frac{x}{e^x-1}(e^x-e^{x/2}-x-1) \tag{11}$$

Using similar arguments for w_n we obtain

$$2w(x)-w(\frac{x}{2})(e^{x/2}+1)=\frac{x}{2} \tag{12}$$

and defining

$$h(x)=\frac{x}{e^x-1}w(x)$$

we transform (12) into

$$h(x)=h(\frac{x}{2})+\frac{x}{4}\cdot\frac{x}{e^x-1} \tag{13}$$

Note that (11) and (13) are of the same type. The common pattern is:

$$f(x)=a(x)f(px)+b(x) \tag{14}$$

where $0<p<1$ (in (11) and (13) $p=½$). Formally iterating (14) n times and assuming

$$(i) \quad \lim_{n\to\infty} f(p^{n+1}x)\prod_{j=0}^{n} a(p^j x) \overset{def}{=} f^*(x) \quad \textit{exists} \tag{15a}$$

$$(ii) \quad \sum_{k=0}^{\infty} b(p^k x)\prod_{j=0}^{k-1} a(p^j x) \quad \textit{is convergent} \tag{15b}$$

we find a general solution of (14) as

$$f(x)=f^*(x)+\sum_{k=0}^{\infty} b(p^k x)\prod_{j=0}^{k-1} a(p^j x) \tag{16}$$

To apply (16) successfully we have to compute the products in the right-hand side of (16). For Eq. (12) we have $a(x)=2$, so $\prod_{j=0}^{k-1} a(\frac{x}{2})=2^k$, and after some algebra we show that (9) has the following solution

$$t(x)=e^x+(e^x-1)\sum_{k=0}^{\infty}\left[1-\frac{x2^{-k}}{e^{x2^{-k}}-1}\right] \tag{17}$$

The same arguments may be applied to find $w(x)$. Note that in that case the product in (16) is equal to one, so (13) and (16) directly imply that

$$w(x)=\frac{e^x-1}{x}+\frac{1}{4}(e^x-1)\sum_{k=0}^{\infty}\frac{x2^{-2k}}{e^{x2^{-k}}-1} \tag{18}$$

is a solution of (12).

We use now (17) and (18) to derive explicit formula for t_n and w_n. Expanding the expression under the sum of (17) one obtains

$$t(x)=2e^x-1-x+(e^x-1)\sum_{k=1}^{\infty}\frac{B_k x^k}{k!(2^k-1)} \tag{19}$$

where B_n are Bernoulli numbers [1], [6], [8] defined as

$$\frac{z}{e^z-1}=\sum_{k=0}^{\infty}B_k\frac{z^k}{k!} \qquad |z|<2\pi \tag{19a}$$

Applying the rule of multiplication for series to the last component of (19) and comparing coefficients on both sides of (19) we find

$$t_n=2-\delta_{n0}-\delta_{n1}+\sum_{k=1}^{n-1}\binom{n}{k}\frac{B_k}{2^k-1} \tag{20}$$

where δ_{nk} is Kronecker delta. Using the same arguments to (18) we transform it into

$$w(x)=\frac{e^x-1}{x}+\frac{1}{4}x+\frac{1}{4}(e^x-1)\sum_{k=0}^{\infty}\frac{B_k x^k}{k!(2^{k+1}-1)} \tag{21}$$

and then one gets immediately

$$w_n=\frac{1}{n+1}+\frac{1}{4}\delta_{n1}+\frac{1}{4}\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_k}{2^{k+1}-1} \tag{22}$$

The values of T_n and W_n are obtained through the approximate formulas (7) and (8).

Approximations

Formulas (17), (18) and (20), (22) are one step forward to have better insight into the behavior of the system. However, from the qualitative point of view they are still too complex to judge of the real nature of the algorithm. Therefore, we apply asymptotic approximation to discover the nature of t_n and w_n, and we use so called "small input rate" approximation to find properties of $W(x)$ and $T(x)$.

Let us consider first $w(x)$ given by (18). Note that the function under the sum in (18) is of type $1/(e^u-1)$ where $u=x2^{-k}$. Using Mellin transform [6] we prove that [10] (see also [8])

$$\frac{1}{e^u-1}=\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\zeta(z)\Gamma(z)u^{-z}dz \qquad ;\ u-real$$

where $\zeta(z)$ and $\Gamma(z)$ are zeta and gamma functions [1], [6]. Hence, the sum in (18) becomes for $\mathrm{Re}(z) < 2$

$$\sum_{k=0}^{\infty} 2^{-2k} \frac{x}{e^{x2^{-k}}-1} = \frac{1}{e^x - 1} + \frac{x}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} \frac{\zeta(z)\Gamma(z)x^{-z}}{2^{2-z}-1}\, dz \tag{23}$$

The evaluation of the counter integral is routine, and it is equal minus the sum of residues of the function under the integral, right to the line of integration [6], [8]. The poles of the function under the integral are roots of the equation $2^{2-z} - 1 = 0$, that is, $z_k = 2 - 2\pi ik/ln\,2$, $k = 0, \pm 1, \pm 2, \ldots$ Noting that $\zeta(2) = \pi^2/6$ [1], [13], hence after some algebra we finally obtain

$$w(x) = \frac{e^x - 1}{x} + \frac{x}{4} + \frac{\pi^2}{24\ln 2}\frac{e^x - 1}{x} + \frac{e^x - 1}{4x} P_2(x) \tag{24}$$

where

$$P_2(x) = \frac{1}{\ln 2} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \zeta(2 - 2\pi ik/ln\,2)\Gamma(2 - 2\pi ik/\ln 2)\exp[2\pi ik\,lg x]$$

and $lg x = \log_2 x$. Formula (24) might be used twofold. At first, to compute $w(x)$ for sufficiently large real values of x, and secondly to find asymptotic approximation for w_n. To take advantage of (24) we must somehow evaluate $P_2(x)$. In fact we show that $P(x) = O(1)$. This is a consequence of the following well known facts:

(i) $|\exp(iy)| \le 1$, y - real

(ii) $\zeta(s + iy) = O(1)$ for s, y-real and $s > 1$ [13],

(iii) for any nonnegative integer s, and real-valued y [13], [8].

$$|\Gamma(s + iy)|^2 = \frac{\pi}{y \sinh y} \prod_{j=0}^{s-1} (j^2 + y^2)$$

Thus, the series is uniformly bounded, i.e. the bound does not depend on x. Then (24) may be rewritten as

$$w(x) = \frac{e^x - 1}{x} + \frac{x}{4} + \frac{\pi^2}{24\ln 2}\cdot\frac{e^x - 1}{x} + \frac{e^x - 1}{4x} O(1) \tag{25}$$

Comparing coefficients of the power of x on the both sides of (25) we find that

$$w_n = \frac{1}{n+1} + \frac{1}{4}\delta_{n_1} + \frac{\pi^2}{24\ln 2}\cdot\frac{1}{(n+1)} + O(n^{-1}) \tag{26}$$

The last approximation might be also derived from (22). In [10] we proved that

$$w_n = \frac{1}{n+1} + \frac{1}{4}\delta_{n_1} + \frac{\pi^2}{24\ln 2}\cdot\frac{1}{n} + \frac{1}{4n} f_2(n) + O(n^{-2}) \tag{27a}$$

where

$$f_2(n) = \frac{1}{\ln 2}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \zeta(2-2\pi ik/\ln 2)\Gamma(2-2\pi ik/\ln 2)\exp(2\pi ik lgn) \tag{27b}$$

Noting that $f_2(n)$ is bounded (the proof repeats the same arguments used above for $P(x)$) we conclude that (26) and (27) are of the same nature.

Consider now (17) and assume $u = x2^{-k}$. Using once again Mellin transform we prove that [10]

$$\frac{1}{e^u - 1} - \frac{1}{u} = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \zeta(z)\Gamma(z)u^{-z}\,dz\,, \quad u - \text{real}$$

Then for $\mathbf{Re}(z) < 1$.

$$\sum_{k=1}^{\infty}\left[1 - \frac{u}{e^u - 1}\right] = -\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\zeta(z)\Gamma(z)x^{1-z}}{2^{1-z}-1}\,dz \tag{28}$$

The same idea as before might be used to evaluate the integral. However, now in addition to the poles $z_k = 1 - 2\pi ik/\ln 2$ $k = 0, \pm 1, \pm 2, \ldots,$ (roots of the denominator) there is one simple pole at $z_0 = 1$ of zeta function which coincides with the pole of the denominator for $k=0$. This double pole is the most difficult to handle, however using series expansions of the functions under the integral we prove that the residue at $z_0 = 1$ is equal to $lgx - 0.5$ [10]. Then, $t(x)$ may be evaluated as

$$t(x) = 2e^x - x - 1 + (e^x - 1)lgx - \frac{1}{2}(e^x - 1) + (e^x - 1)P_1(x) \tag{29}$$

where

$$P_1(x)=\frac{1}{\ln 2}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\zeta(1-2\pi ik/ln\,2)\Gamma(1-2\pi ik/ln\,2)\exp(2\pi ik\,lgx)$$

Since $P_1(x)$ is uniformly bounded we might use (29) as an asymptotic approximation for $t(x)$ (for $x>7$ it is a good approximation). However, it is much more complicated to determine an asymptotic approximation for t_n, because it is not easy to find coefficients in an expansion of $(e^x-1)lgx$. But alternatively (20) provides such an approximation, and in [10] we prove that

$$t_n=1.5-\delta_{n0}-\delta_{n1}+lgn+f_1(n)+O(n^{-1}) \tag{30}$$

where

$$f_1(n)=\frac{1}{\ln 2}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\zeta(1-2\pi ik/\ln 2)\Gamma(1-2\pi ik/\ln 2)\exp(2\pi ik\,lgn)$$

But, by the same arguments as before $f_1(n)$ is bounded and numerical analysis reveals that the value of the function is very small in comparison with the other terms of (30). Thus, we may safely ignore $f_1(n)$ for practical purposes.

Note, however, that to determine the maximum throughput λ_{max} given by (4) we must evaluate $W(x)$ and $T(x)$ (or $w(x)$ and $t(x)$) for small values of x. We deal now with such an approximation which is called a "small input rate" approximation. In fact we may use previously derived formulas. Considering in (21) only first five terms of the series we obtain for small values of x

$$w(x)=\frac{e^x-1}{x}+\frac{x}{4}+\frac{e^x-1}{4}\tilde{w}(x)+O(x^6) \tag{31a}$$

where

$$\tilde{w}(x)=1-\frac{x}{6}+\frac{x^2}{84}-\frac{x^4}{180\cdot 31} \tag{31b}$$

On the other hand, considering the first five terms of the series in (19) we might approximate $t(x)$ for small values of x by

$$t(x)=2e^x-1-x+(e^x-1)\tilde{t}(x)+O(x^6) \tag{32a}$$

where

$$\tilde{t}(x)=\frac{x}{2}-\frac{x^2}{36}+\frac{x^4}{24\cdot 450} \tag{32b}$$

Numerical analysis reveals that (31) and (32) very well approximate $t(x)$ and $w(x)$ for all $x \le 3$. As we shall see below the small value approximation is much more useful than asymptotic approximation to evaluate $W(x)$, $T(x)$ and λ_{max}.

4. NUMERICAL ANALYSIS AND DISCUSSIONS

In the previous section we have obtained explicit expressions for $t(x)$, $w(x)$, t_n, w_n, and we have found approximate formulas for W_n and T_n. Now, some numerical results will be presented, however, we restrict our consideration to analysis of the maximum throughput λ_{max} given by (4).

Let us study (4). To determine maximum throughput we have to find maximum over $xW(x)/T(x)$ where x is a real-valued variable defined as $x=\lambda\tau$. Using numerical analysis we have shown above that for $n>3$ the relationship between T_n, W_n and t_n, w_n are given by (7) and (8). Since $T_0=T_1=W_0=W_1=t_0=t_1=w_0=w_1=1$ the only values of n which must be reconsidered are $n=2$ and $n=3$. Therefore, the generating function for T_n and W_n are given as:

$$T(x)=t(x)+\frac{x^2}{2}+\frac{5x^3}{12}+a(e^x-1-x-\frac{x^2}{2}-\frac{x^3}{6}) \tag{33}$$

and

$$W(x)=w(x)+\frac{x^2}{6}+\frac{x^3}{24}+\frac{b}{x}(e^x-1-x-\frac{x^2}{2}-\frac{x^3}{6}-\frac{x^4}{24}) \tag{34}$$

where $a=2.644$ and $b=0.9122$.

Note now that we have found above three different expressions for $t(x)$ and $w(x)$: exact formulas given by (17) and (18), asymptotic approximations (29), (25), and "small input rate approximation" given by (32) and (31). Asymptotic approximation is not useful for us since it is valid for $x>7$, while we are interested in $0<x<2$.

Numerical results reveal that $W(x)$ and $T(x)$ are very well estimated by the series and the small rate approximations. To compute λ_{max} given by (4) we use small input rate approximation. It is known that direct search over (4) using recurrences (2) and (3) provides $x_{op}=1.266$ and

$\lambda_{max}=0.48711$. Our approximation gives $x_{op}=1.277$ and $\lambda_{max}=0.48819$. In fact, in some situations we may simplify a little formulas (33), (34) noting that approximations (7) and (8) are good also for $n=3$. Then the terms with x^3 in (33) and (34) disappear , and numerical analysis shows that $x_{op}=1.254$ and $\lambda_{max}=0.48246$.

5. SOME GENERALIZATIONS AND CONCLUSIONS

In this section we generalize some of the results discussed before. Let us start with a motivating example. In [2] Berger described a class of CRA algorithm with a binary feedback. In particular, he introduced an algorithm called *something/nothing* with feedback that tells only whether or not the previous slot was empty. He has shown that conditional average conflict resolution interval, T_n, and conditional average resolved interval, W_n, satisfy the following recurrences (we present here a simple version of Berger's formula for symmetric algorithm):

$$\begin{aligned} &T_0=0, \quad T_1=1 \\ &T_n=(2^n-2)^{-1}[2^{n+1}+n-1+n\,T_{n-1}+\sum_{k=1}^{n-1}\binom{n}{k}T_k] \quad n\geq 2 \end{aligned} \tag{35a}$$

$$\begin{aligned} &W_0=W_1=1 \\ &W_n=(2^n-2)^{-1}[1+n\,W_{n-1}+\sum_{k=1}^{n-1}\binom{n}{k}W_k] \quad n\geq 2 \end{aligned} \tag{35b}$$

Moreover, maximum throughput λ_{max} is given by [2]

$$\lambda_{max}=\max_x \frac{xW(x)}{e^x+T(x)} \tag{36}$$

where $W(x)$ and $T(x)$ are exponential generating functions of W_n and T_n.

Recurrences (35) are of the same form as recurrences (1), (2) for GTM-algorithm, that is, they differ only by the first additive terms. This suggests that the previous approach might be applied to find approximate formula for T_n and W_n. However, instead of repeating once again the same derivations we present below a solution for a class of recurrences which is common for the problem we discuss here.

First of all, by numerical analysis we reduce recurrences (35) to a simple form, namely we define sequences t_n and w_n as in (5), (6) respectively, that is, terms $n\,T_{n-1}$ and $n\,W_{n-1}$ from (35a), (35b) are dropped in t_n and w_n. Then, numerical analysis reveals that for $n \geq 3$ the following approximation holds

$$T_n = t_n + \alpha\,, \qquad \alpha = 4.053 \tag{37}$$

and W_n is computed as in (8) (note that recurrences (2) and (35b) are the same). Concluding out, the problem is reduced to solution of the recurrence for t_n and w_n.

Generalizing these two examples, we consider the following recurrence for l_n:

$$\begin{aligned} &\textbf{given}: \ l_0, l_1, \ldots, l_N \\ &\textbf{solve}: \quad (2^{n+s}-2)l_n = 2^n a_n + \sum_{k=1}^{n-1} \binom{n}{k} l_k \quad n > N \end{aligned} \tag{38}$$

where s and N are given integers, $N > -s$, and a_n is any given sequence, $n = 0,1,\ldots$. This type of recurrence was analyzed in [10]. Using the same arguments as in the previous section, we show that solution of (38) is reduced to determining a solution of the following functional equation

$$L(z) = 2^{-s} L(z/2)(e^{z/2}+1) + b(z) \tag{39}$$

where $L(z)$ is exponential generating function of $L_n = l_n - l_0$ and

$$b(z) = 2^{-s}\,[\,a(z) - a_0\,] - l_0(1-2^{-s})(e^z - 1) + \sum_{k=1}^{N} \frac{z^k}{k!}\, g_k \tag{40a}$$

$$g_k = l_k(1-2^{k-s}) - a_k 2^{-s} - 2^{-k-s} \sum_{i=1}^{k} \binom{k}{i} l_k, \quad k = 1,2,\ldots,N \tag{40b}$$

In (40a) $a(z)$ is exponential generating function for a_n.

To present a closed form solution for (38) and (39) we introduce a simple sequence transformation. Let us define for any sequence x_n a new sequence $\hat{x}_n$ by the following equation

$$\hat{x}_n = \sum_{k=0}^{n} \binom{n}{k} B_k x_{n-k} \tag{41a}$$

where B_k are Bernoulli numbers (see (19a)). Note also that exponential generating function for $\hat{x}_n$, $\hat{x}(z)$, is equal to $\hat{x}(z)=z\,x(z)/(e^z-1)$. Sequences x_n and $\hat{x}_n$ are known in the literature [6] as Bernoulli inverse relation. We prove that

Theorem 1. If $l(z)$ is $(1-s)^+$-times differentiable, than exponential generating function $l(z)$ of l_n is given by

$$l(z)=l_0e^z+z^{(1-s)^+}l_s^*(e^z-1)+b(z)+(e^z-1)\sum_{k=1}^{\infty}2^{-sk}\frac{b(z\,2^{-k})}{e^{z2^{-k}}-1} \tag{42}$$

or in another form

$$l(z)=l_0e^z+z^{(1-s)^+}l_s^*(e^z-1)+b(z)+\frac{e^z-1}{z}\sum_{k=(2-s)^-}^{\infty}\frac{\hat{b}_k z^k}{k!(2^{k+s-1}-1)} \tag{43}$$

where

$$l_s^*=\hat{l}_{(1+s)^+}-l_0\delta_{1,(1-s)^+}-l_0B_{(1-s)^+}$$

$a^-=\min\{0,a\}$ and $\hat{b}_k$, $k=0,1,\ldots$ are coefficients in the series expansion of $\hat{b}(z)$ (see (45a) and (45b)).

Proof. Eq.(42) follows from solution of functional equation (39) by the same arguments as we used to derive (16), (19) or (21). Eq.(43) is a consequence of (42). For details see [10].

□

Theorem 2. If hypothesis of Theorem 1 holds, then recurrence (38) possesses the following solution

$$l_n=l_0+(1-\delta_{n0})l_s^*\frac{n!}{(n+s)!}+b_n+\frac{1}{n+1}\sum_{k=(2-s)^-}^{n}\binom{n+1}{k}\frac{\hat{b}_k}{2^{k+s-1}-1} \tag{44}$$

where

$$b_n=2^{-s}a_n-l_0(1-2^{-s})+g_n\chi_{(n\le N)} \tag{45a}$$

$$\hat{b_k} = 2^{-s}(\hat{a}_k - a_0 B_k) - l_0(1-2^{-s})\delta_{k1} + (1-\delta_{k0}) \sum_{i=1}^{\min\{k,N\}} \left\{ {k \atop i} \right\} \tag{45b}$$

and χ_A is equal to one if A holds, otherwise it is zero.

Proof. Eq.(44) follows directly from (43) and is a consequence of multiplication formula for series.

□

In order to apply (43) and (44) we must compute explicit formula for $\hat{a}_n$ for a given sequence a_n. From the practical point of view, the most interesting is the case $a_n = \left[{n \atop r} \right] q^n$, where r is an integer and q is a constant. But then, using well known properties of Bernoulli numbers one finds [1],[6]

$$\hat{a}_n = \sum_{k=0}^{n} \left[{n \atop k} \right] B_k \left[{n-k \atop r} \right] q^{n-k} = \left[{n \atop r} \right] q^r \sum_{k=0}^{n-r} \left[{n-r \atop k} \right] B_k q^{n-r-k} = \left[{n \atop r} \right] q^r B_{n-r}(q) \tag{46}$$

where $B_n(x)$ is Bernoulli polynomial [1], [6].

Eq.(44) is used in [10] to derive asymptotic analysis for l_n. However, as long as we are interested in estimating maximum throughput for CRA algorithms we need only small value approximation for generating function $l(z)$ as it was shown in the previous section for GTM-algorithm. Then, assuming in (43) that $|z| < \beta$, β is a small real value, we find immediately the following approximation for $l(z)$:

$$l(z) = l_0 e^z + z^{(1-s)^+-1} l_s^*(e^z-1) + b(z) + (e^z-1) \sum_{k=(2-s)^-}^{M} \frac{\hat{b_k} z^{k-1}}{k!(2^{k+s-1}-1)} + O(z^{M+1}) \tag{47}$$

where $M > (2-s)^-$, and M is a rather small integer.

Now we are prepared to evaluate maximum throughput for CRA algorithms. The reader is asked to check that previously obtained results for GTM-algorithm follow immediately from (42)-(47). For Berger's algorithm we must compute $T(x)$ and $W(x)$. The latter was found before in (34). For $T(x)$ we must first compute $t(x)$ (see (37)). But t_n satisfies a recurrence of type (38), hence (47) is valid with $s=0$, $N=1$ $a_n=2+n2^{-n}-2^{-n}$. Then, by (40) $g_1=-2$ and $b_n=a_n-\delta_{n0}-\delta_{n1}$. To find $\hat{b_n}$ we use (45b) and (46). By (46) $\hat{a}_n=2(B_n+\delta_{n1})+0.5k$ $B_{k-1}(1/2)-B_k(1/2)$. But it is well known that [1] $B_n(1/2)=(2^{1-n}-1)$ B_n, hence $\hat{b_k}=B_k-1.5kB_{k-1}+2k(2^{-k}-2^{-1})B_{k-1}-(2^{1-k}-1)B_k+1.5\delta_{n1}$. Finally, by (47) we find

$$t(x)=3e^x-2-2x+e^{x/2}(0.5x-1)+(e^x-1)f(x) \tag{48a}$$

$$f(x)=\frac{9}{8}x-\frac{9}{18\cdot 8}x^2-\frac{15}{240\cdot 7\cdot 24}x^3+O(x^4) \tag{48b}$$

To compute $T(x)$ we use (37) (note that (37) is valid only for $n>3$). Evaluating T_i for $i=0,1,2,3$ by recurrence (35a), we find after some algebra

$$T(x)=\alpha(e^x-1-x-\frac{x^2}{2}-\frac{x^3}{6})+0.5x^2+3.75\frac{x^3}{6}+t(x) \tag{49}$$

where $\alpha=4.053$ and $t(x)$ is given by (48). Then maximum throughput λ_{max} (see (36)) might be found. Numerical analysis reveals that $\lambda_{max}=0.27677$ for $x=0.92$. Direct computation over (35) - (36) show that the exact value of λ_{max} is 0.27676, so the approximation is perfect.

Further generalizations of the above study are possible. In particular, we might be interested in asymptotic approximation of l_n (Eq.(44)). Such an analysis is presented in [10]. In particular, we proved that t_n for Berger's algorithm is asymptotically equal to $t_n=3+0.5\,lg\,(n^4/\pi)$ $+\,0.5\,\gamma/\ln 2+O(n^{-1})$, where γ is Euler constant [1],[6]. Moreover, further studies should deal with exact solution of recurrences for T_n, W_n instead of reducing the problem to solution of recurrences for t_n and w_n. Moreover, asymmetric GTM-algorithm and Berger's algorithm should be analyzed (by asymmetric we mean that enabled internal is not divided into two equal parts, but a fraction of the interval is used to solve a conflict.)

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