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ABSTRACT

We study a recurrence equation of type $l_n(2^{n+s}-2)=a_n+\sum_{k=1}^{n-1}\binom{n}{k}l_k,\ n\geq N$ where a_n is any sequence and s, N are integers . This recurrence arises in many applications, e.g. investigating the average number of bit inspection for a random unsuccessful search by Patricia tree, analyzing the average conflict resolution interval for contention resolution algorithms in a broadcast communication environment, etc. We present a closed-form solution of the recurrence and then we establish an asymptotic approximation for it. In addition, we offer an approximation of a generating function, $l(z)$, of l_n for small values of z.

Key words: linear recurrence, Bernoulli numbers, Bernoulli polynomials, Bernoulli inverse relations, asymptotic approximation, Mellin transform.

1. INTRODUCTION

Let the infinite sequence $l_0, l_1, l_2, \ldots$ satisfy the following linear recurrence

$$f_n l_n = a_n + \gamma \sum_{k=0}^{n-1} p_{nk} l_k\ , \quad \sum_{k=0}^{n} p_{nk} = 1 \tag{1.1}$$

where f_n and a_n are given sequences, while γ is a constant. Such a recurrence arises quite often in practice [2],[4],[7],[8],[10],[12],[18]. There are some techniques which might be used to solve it. In particular, Greene and Knuth [8] describe a method *by repertoire*. If we know that y_n also satisfies (1.1) with the additive term b_n, then by linearity an equation with additive term $A\ a_n+B\ b_n$ will have the solution Al_n+By_n. The idea is to choose l_n first so as to make the sum tractable, then to fit additive term a_n which gives l_n. Once we build up a repertoire of enough additive terms, the original a_n can be constructed by linear combination.

This is a quite general method, but it is an uphill work to construct an appropriate a_n. However, a solution of (1.1) is easier to find if we restrict a class of distribution p_{nk}. From the practical point of view Bernoulli distribution of p_{nk} is very important. Assume now that

$$p_{nk} = \binom{n}{k} \alpha^k (1-\alpha)^{n-k} \quad 0 < \alpha < 1 \tag{1.2}$$

Then solution of (1.1) depends critically on sequence f_n. In particular, for $f_n = 1 - \gamma p_{nn}$ a closed form solution of (1.1) may be found [12],[15]. In fact, in that case a generalization of (1.1) was solved, namely in [15] p_{nk} was assumed to be multinominal Bernoulli distribution and l_k on the RHS of (1.1) was substituted by $l_{k_1} + l_{k_2} + \cdots + l_{k_\nu}$, $k_1 + k_2 + \cdots + k_\nu = n$ for some $\nu > 0$.

In this paper we study (1.1) when $f_n = 1 - \gamma p_{nn} - \gamma p_{n0}$. This is almost identical to the one described above, but appearance of γp_{n0} is enough to change the entire character of the recurrence, and the methods used before are wiped out. Because of that we are forced to further restriction on the class of distribution (1.2). We assume throughout the paper $\alpha = 0.5$, however, we present the solution for any sequence of a_n. Even with such restrictions the equation is very important in practice, especially for the performance evaluation of conflict resolution algorithms in a broadcast communication [2],[7],[16],[18].

Under the above assumption we present a closed form solution of (1.1) and asymptotic approximation of it. In addition, an approximation of the exponential generating function of l_n is given. To the author's knowledge such a solution was available only for a few specific values of a_n, namely: Knuth considered $a_n = 1 - 2^{1-n}$, $\gamma = 1$ [12, p.409] while Szpankowski [16] assumed either $a_n = 1$, $\gamma = 1$ or $a_n = 2^{-n-1}$, $\gamma = 0.5$. This paper generalizes these works.

2. PROBLEM FORMULATION

We shall study (1.1) under the following assumptions:

a) p_{nk} is given by (1.2) with $\alpha=0.5$,

b) $f_n = 1-\gamma p_{n0}-\gamma p_{nk} = 1-\gamma 2^{-n}-\gamma 2^{-n}$,

c) *for simplicity* we assume also that $\gamma=2^{-s}$ where s is an integer (later we shall point out that this assumption is irrelevant for the proposed method).

In addition, instead of a_n we consider $2^n a_n$ for the reasons which will be clear later. Then, the problem may be formulated as follows:

$$\begin{aligned} &\textbf{given:} \quad l_0, l_1, \ldots, l_N \\ &\textbf{solve:} \quad (2^{n+s}-2)l_n = 2^n a_n + \sum_{k=1}^{n-1} \binom{n}{k} l_k \quad n > N \end{aligned} \tag{2.1}$$

where s, N are integers such that $N>-s$, and a_n is any sequence.

Example 1a

Let us study Patricia tree. It is proved that the average number of bit inspections for a random unsuccessful search satisfies the following recurrence [12]

$$\begin{aligned} &c_0 = c_1 = 0 \\ &(2^n-2)c_n = 2^n - 2 + \sum_{k=1}^{n-1} \binom{n}{k} c_k \end{aligned} \tag{2.2}$$

This is equivalent to (2.1) with $s=0$, $N=1$, $a_n = 1-2^{1-n}$.

Example 2a

Assume an infinite number of users sharing a common communication channel. Since the channel is the only way of communications among the users packet collision is inevitable, if a central coordination is not provided. The problem is to find an efficient algorithm for retransmitting conflicting packets. It turns out that so called *conflict resolution algorithms* [2],[4],[7] are the most efficient, and among them Gallager-Tsybakov-Mikhailov (GTM) algorithm [7],[18] achieves the highest throughput. The idea of the algorithm is to partition a conflict of the multiplicity n into smaller conflicts by observing the channel and learning whether in the past it was idle, success or collision transmission (ternary feedback). The performance of the algorithm depends on two quantities T_n and W_n, where n is multiplicity of a conflict: T_n represents the average length of a *conflict resolution interval*, while W_n is the average length of so called *enabled interval* (for details see [7],[18]). It is proved that T_n and W_n satisfy recurrences [18]

$$\begin{gathered} T_0 = T_1 = 1 \\ (2^n - 2)T_n = 2^n + n\, T_{n-1} + \sum_{k=1}^{n-1} \binom{n}{k} T_k \end{gathered} \tag{2.3}$$

and

$$\begin{gathered} W_0 = W_1 = 1 \\ (2^{n+1} - 2)W_n = 1 + n\, W_{n-1} + \sum_{k=1}^{n-1} \binom{n}{k} W_k \end{gathered} \tag{2.4}$$

These recurrences are *not* of type (2.1), but in [16] we have proved that both recurrences might be solved if one finds a solution of the following recurrences

$$\begin{gathered} t_0 = t_1 = 1 \\ (2^n - 2)t_n = 2^n + \sum_{k=1}^{n-1} \binom{n}{k} t_k \end{gathered} \tag{2.5}$$

and

$$\begin{gathered} w_0 = w_1 = 1 \\ (2^{n+1} - 2)w_n = 1 + \sum_{k=1}^{n-1} \binom{n}{k} w_k \end{gathered} \tag{2.6}$$

which fall into class of (2.1). Moreover, it is shown that for Poisson message arrival process the maximum throughput λ_{max} of the algorithm is equal to

$$\lambda_{max} = \max_z \frac{zW(z)}{T(z)} \tag{2.7}$$

where

$$W(z) = \sum_{n=0}^{\infty} W_n \frac{z^n}{n!}\,; \quad T(z) = \sum_{n=0}^{\infty} T_n \frac{z^n}{n!}$$

Note that $W(z)$ and $T(z)$ are exponential generating functions of W_n and T_n, respectively.

Example 3a

Consider now a conflict resolution algorithm as in Example 2 with a binary feedback, that is, a user distinguishes only two states of a channel: nothing or something. Then Berger [2] described an algorithm for which the average length of conflict resolution interval T_n and the average length of the enabled interval W_n satisfy recurrences similar to (2.3) and (2.4) except the first term which is either $2^{n+1}+n-1+n\,T_{n-1}$ or $1+n\,W_{n-1}$. Neglecting terms $n\,T_{n-1}$ and $n\,W_{n-1}$ (by the same reasons as above) we have to solve the following equations

$$\begin{aligned} &t_0=0 \quad t_1=1 \\ &(2^n-2)t_n=2^{n+1}+n-1+\sum_{k=1}^{n-1}\binom{n}{k}t_k \end{aligned} \tag{2.8}$$

$$\begin{aligned} &w_0=w_1=1 \\ &(2^{n+1}-2)w_n=1+\sum_{k=1}^{n-1}\binom{n}{k}w_k \end{aligned} \tag{2.9}$$

Moreover, the maximum throughput λ_{max} satisfies (2.7) with $T(z)$ and $W(z)$ defined as in (2.8) and (2.9), respectively.

□

These motivating examples suggest that from the practical point of view a closed form solution of (2.1) and generating function of the solution are interesting for us. In particular, for (2.7) much more important is to derive an approximation of $W(z)$ and $T(z)$ for small values of z, than exact closed form solution of (2.5) and (2.6).

3. SOLUTION OF THE RECURRENCE

Let $l(z)$ be exponential generating function for the sequence l_n, $n=0,1,\ldots,$ defined in (2.1). Let us also introduce a new sequence L_n, $n=0,1,\ldots,$ as follows

$$\begin{aligned} &L_n=l_n-l_0 \\ &L(z)=l(z)-l_0e^z \end{aligned} \tag{3.1}$$

where $L(z)$ is an exponential generating function for L_n. Note that $L_0=0$, and recurrence (2.1) is transformed into

$$L_0=0,\ L_1,\ldots,L_N \text{ given}$$
$$(2^{n+s}-2)L_n=2^n a_n+l_0(2^n-3)+\sum_{k=1}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix} L_k, \quad n>N \tag{3.2}$$

To solve (3.2) we use generating function method. Multiplying (3.2) for $n>N$ by $z^n/n!$ and taking into considerations initial conditions one shows that

$$2^s L(2z)-L(z)(e^z+1)=a(2z)-a_0-l_0(2^s-1)(e^{2z}-1)+$$
$$\sum_{k=1}^{N}\frac{z^k}{k!}\{L_k(2^{k+s}-1)+l_0(2^s-1)2^k-a_k 2^k-\sum_{i=0}^{k}\begin{bmatrix} k \\ i \end{bmatrix} L_i\} \tag{3.3}$$

where $a(z)$ is exponential generating function for a_n. Substituting now in (3.3) z by $z/2$ and using (3.1) we obtain

$$L(z)=2^{-s}L(z/2)(e^{z/2}+1)+b(z) \tag{3.4}$$

where

$$b(z)=2^{-s}[a(z)-a_0]-l_0(1-2^{-s})(e^z-1)+\sum_{k=1}^{N}\frac{z^k}{k!}g_k \tag{3.5a}$$

$$g_k=l_k(1-2^{-k-s})-a_k 2^{-s}-2^{-k-s}\sum_{i=1}^{k}\begin{bmatrix} k \\ i \end{bmatrix} l_i \quad k=1,2,\ldots,N \tag{3.5b}$$

To find a closed form solution for (3.2) we must solve functional equation (3.4). The easiest way is to introduce a new function $H(z)$ as follows

$$H(z)=L(z)\frac{z}{e^z-1} \tag{3.6}$$

Then, (3.4) becomes

$$H(z)=2^{1-s}H(z/2)+\frac{b(z)z}{e^z-1} \tag{3.7}$$

We prove that

Lemma 3.1. A general solution of functional equation (3.7) is given by

$$H(z)=H^*(z)+\sum_{k=0}^{\infty}2^{(1-s)k}\frac{b(z2^{-k})z2^{-k}}{e^{z2^{-k}}-1} \tag{3.8}$$

where $H^*(z)=\lim_{k\to\infty}2^{k(1-s)}H(z2^{-k})$, provided $H^*(z)$ exists and the series in (3.8) is convergent.

Proof. Iterating (3.7) n times and taking limit for $n \to \infty$ we find (3.8) assuming the appropriate limits exist.

□

Let us now consider $H^*(z)$. We show that

Corollary 3.2. If $H(z)$ is differentiable $(1-s)^+$ times at $z=0$, where $a^+=\max\{0,a\}$, then

$$H^*(z)=z^{(1-s)^+}\frac{H^{(1-s)^+}(0)}{(1-s)^+!} \tag{3.9}$$

Proof: Assume first $s \geq 1$ and note that by (3.6) and (3.2) $H(0)=0$. Then

$$\lim_{k\to\infty} 2^{k(1-s)}H(z2^{-k})=0$$

Let now $s<1$ and $u=z2^{-k}$. Then applying l'Hospital rule $1-s$ times we find

$$\lim_{k\to\infty} 2^{k(1-s)}H(z2^{-k})=z^{1-s}\lim_{u\to 0}\frac{H(u)}{u^{1-s}}=z^{1-s}\frac{H^{(1-s)}(0)}{(1-s)!}$$

where $H^{(n)}(z_0)$ is the n-th derivative of $H(z)$ at z_0.

□

We present now sufficient and necessary conditions for convergence of the series in (3.8). Let b_n, $n=0,1,\ldots,$ be coefficients in Taylor expansions of $b(z)$ at $z=0$. By definition we assume also that $b_{-n}=0$, $n=0,1\ldots$. Then

Corollary 3.3. The series in (3.8) is convergent if and only if

$$b_0=b_1=\cdots=b_{1-s}=0 \tag{3.10}$$

provided $b(z)$ is $(1-s)^+$ - times differentiable at z=0.

Proof. Necessity. Let $z>0$ be a fixed real number and denote the series in (3.8) as $\sum_{k=0}^{\infty}\alpha_k$, which is assumed to be convergent. This implies that $\lim_{k\to\infty}\alpha_k=0$ [11], i.e., the following must be satisfied

$$\lim_{k\to\infty} 2^{(1-s)k}\frac{b(z2^{-k})z2^{-k}}{e^{z2^{-k}}-1} = z^{1-s}\lim_{u\to 0} u^{s-1}\frac{b(u)u}{e^{u}-1} = 0$$

where $u = z2^{-k}$. Assume first $s > 1$. Then

$$\lim_{u\to 0} u^{s-1}\frac{b(u)u}{e^{u}-1} = 0$$

for any values of b_k, $k = 0,1\ldots$ assuming $b(0)<\infty$ (in our case $b(0)=b_0=0$). Let now $s \le 1$ and apply l'Hospital rule $1-s$ times. Then

$$0 = \lim_{u\to 0} u^{s-1}\frac{b(u)u}{e^{u}-1} = \lim_{u\to 0}\frac{b(u)}{u^{1-s}} = \lim_{u\to 0}\frac{b^{(1)}(u)}{(1-s)u^{-s}} = \cdots = \lim_{u\to 0}\frac{b^{(1-s)}(u)}{(1-s)!}$$

hence (3.10) holds.

Sufficiency. Assume now that (3.10) is satisfied. By D'Alembert's criterion [11] the series is convergent if $\lim_{k\to\infty} \alpha_{k+1}/\alpha_k < 1$. Note that

$$\lim_{k\to\infty}\frac{\alpha_{k+1}}{\alpha_k} = \lim_{k\to\infty} 2^{-1}\frac{e^{z2^{-k}}-1}{e^{z2^{-k-1}}-1}\, 2^{1-s}\,\frac{b(z2^{-k-1})}{b(z2^{-k})} \tag{3.11}$$

But, by l'Hospital's rule

$$\lim_{k\to\infty} 2^{-1}\frac{e^{z2^{-k}}-1}{e^{z2^{-k-1}}-1} = \lim_{u\to 0} 2^{-1}\frac{e^{u}-1}{e^{u/2}-1} = 1 \tag{3.12}$$

hence by (3.11) and (3.12)

$$\lim_{k\to\infty}\frac{\alpha_{k+1}}{\alpha_k} = 2^{1-s}\lim_{u\to 0}\frac{b(u2^{-1})}{b(u)} = 2^{1-s}\lim_{u\to 0} 2^{-1}\frac{b'(u2^{-1})}{b'(u)} = \cdots = 2^{-1}\lim_{u\to 0}\frac{b^{(2-s)}(u2^{-1})}{b^{(2-s)}(u)} \le 2^{-1} < 1$$

hence the series is convergent.

□

In order to find an explicit formula for $l(z)$ let us introduce Bernoulli inverse relation. For a given sequence A_n, $n = 0,1$, we introduce a sequence $\hat{A}_n$, $n = 0,1,\ldots,$ as [17]

$$\hat{A}_n = \sum_{k=0}^{n} \binom{n}{k} B_k A_{n-k} \tag{3.13}$$

where B_k, $k=0,1,\ldots,$ are Bernoulli numbers defined by [1]

$$\frac{z}{e^z-1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \qquad |z| < 2\pi \tag{3.14}$$

(In Appendix A we list some properties of Bernoulli numbers and Bernoulli polynomials which are used extensively in this paper.) Note also that by (3.13) and (3.14) exponential generating function $\hat{A}(z)$ of $\hat{A}_n$ is given by [1],[17]

$$\hat{A}(z) = A(z)\frac{z}{e^z-1} \tag{3.15}$$

and

$$A_n = \sum_{k=0}^{n} \binom{n}{k} (k+1)^{-1} \hat{A}_{n-k}$$

(by the above equation and (3.13) A_n and $\hat{A}_n$ is a pair of inverse relations). Then, we prove our first main result of this section

Theorem 3.4. If (3.10) and hypothesis of Corollary 3.3 hold, then exponential generating function of l_n is given by

$$l(z) = l_0 e^z + z^{(1-s)^+-1} l_s^* (e^z-1) + b(z) + (e^z-1)\sum_{k=1}^{\infty} 2^{-sk} \frac{b(z2^{-k})}{e^{z2^{-k}}-1} \tag{3.16}$$

and

$$l(z) = l_0 e^z + z^{(1-s)^+-1} l_s^* (e^z-1) + b(z) + \frac{(e^z-1)}{z} \sum_{k=(2-s)^-}^{\infty} \frac{\hat{b}_k z^k}{k!(2^{k+s-1}-1)} \tag{3.17}$$

where

$$l_s^* = \hat{l}_{(1-s)^+} - l_0 \delta_{1,(1-s)^+} - l_0 B_{(1-s)^+} \tag{3.18}$$

and $a^- = \min\{a, 0\}$, δ_{nk} is Kronecker delta, while $b(z)$ is defined in (3.5a).

Proof. Eq. (3.16) follows directly from (3.1), (3.6), (3.8) and (3.9). We must only derive (3.18). But, by (3.6) and (3.15) $H(z) = \hat{l}(z) - l_0 z - l_0 z/(e^z-1)$. Note now that $H^*(z)$ given by (3.9) is a coefficient of Taylor expansion of $H(z)$ at $z^{(1-s)^+}$.

To prove (3.17) consider the series in (3.16), Corollary 3.3 and (3.15). By (3.13) condition (3.10) is equivalent to $\hat{b}_0=\hat{b}_1=\cdots=\hat{b}_{1-s}=0$. Then, the series in (3.16) is equal to

$$\sum_{k=1}^{\infty} 2^{(1-s)k}\frac{b(z2^{-k})z2^{-k}}{e^{z2^{-k}}-1}=\sum_{k=0}^{\infty} 2^{(1-s)k}\hat{b}(z2^{-k})=\sum_{k=1}^{\infty} 2^{(1-s)k}\sum_{i=(2-s)^-}^{\infty}\frac{\hat{b}_i}{i!}z^i 2^{-ki}=$$

$$\sum_{k=(2-s)^-}^{\infty}\frac{\hat{b}_k z^k}{k!}\sum_{i=1}^{\infty}2^{-i(k+s-1)}=\sum_{k=(2-s)^-}^{\infty}\frac{\hat{b}_k z^k}{k!(2^{k+s-1}-1)}$$

and by (3.10) the geometric series in the above formula is convergent.

□

The second main result of this section is

Theorem 3.5. If hypotheses of Theorem 3.4 hold, then recurrence (2.1) possesses the following solution

$$l_n=l_0+(1-\delta_{n0})l_s{}^*\frac{n!}{(n+s)!}+b_n+\frac{1}{n+1}\sum_{k=(2-s)^-}^{n}\binom{n+1}{k}\frac{\hat{b}_k}{2^{k+s-1}-1} \tag{3.19}$$

where

$$\begin{aligned} b_0&=0\\ b_n&=2^{-s}a_n-l_0(1-2^{-s})+g_n\chi_{(n\le N)} \quad n>0 \end{aligned} \tag{3.20}$$

$$\hat{b}_k=2^{-s}(\hat{a}_k-a_0B_k)-l_0(1-2^{-s})\delta_{k1}+(1-\delta_{k0})\sum_{i=1}^{\min\{k,N\}}\binom{k}{i} \tag{3.21}$$

and χ_A is a function equal to one if condition A is satisfied, otherwise it is zero.

Proof. Eq. (3.19) follows directly from (3.17) by applying multiplication formula for generating functions.

□

Remarks

i) Assumptions a) and b) from Section 2 are relevant while c) is not relevant for the above derivations. If a constant γ is any number, then (3.4) becomes

$$L(z)=\gamma L(z/2)(e^{z/2}+1)+b(z)$$

and using (3.6) together with $e^z-1=(e^{z/2}-1)(e^{z/2}+1)$ we obtain

$$H(z)=2\gamma H(z/2)+\frac{b(z)z}{e^z-1}$$

instead of (3.7). (Note that $b(z)$ here is defined slightly different than in (3.5a)). Such form of the above functional equation is relevant to get a closed form solution for $H(z)$.

(ii) For (3.19) (more precisely: (3.21)) we must compute $\hat{a}_n$ for a given sequence a_n, $n=0,1\ldots$ For example, if $a_n=\binom{n}{r}q^n$, q is a constant and r is an integer, then using (A4) from Appendix A and (3.13) we find

$$\hat{a}_n=\sum_{k=0}^{n}\binom{n}{k}B_k\binom{n-k}{r}q^{n-k}=\binom{n}{r}q^r\sum_{k=0}^{n-r}\binom{n-r}{k}B_k q^{n-r-k}=\binom{n}{r}q^rB_{n-r}(q)$$

where $B_{n-r}(q)$ is Bernoulli polynomial (see (A1)). Hence

$$a_n=\binom{n}{r}q^n \qquad \hat{a}_n=\binom{n}{r}q^rB_{n-r}(q) \tag{3.22}$$

For more Bernoulli inverse relations see Riodan [17].

□

Example 1b

In that case we must substitute in (2.1) $c_0=c_1=0$, $N=1$, $s=0$, $a_n=1-2\cdot 2^{-n}$. Then, by (3.5b), (3.18) and (3.20) $g_1=0$; $l_0{}^*=0$; $b_k=1-2^{1-k}+\delta_{n0}$. But $B_n(\frac{1}{2})=-(1-2^{1-n})B_n$, (see (A9) in Appendix A) so $\hat{a}_n=3B_n+\delta_{n1}-2^{2-n}B_n$. Then, by (3.22) $\hat{b}_n=4B_n(1-2^n)+\delta_{n1}$ and by (3.19), (A4), (A8) we finally obtain

$$c_n=2-\frac{4}{n+1}+2\delta_{n0}+\frac{2}{n+1}\sum_{k=2}^{n}\binom{n+1}{k}\frac{B_k}{2^{k-1}-1} \tag{3.23}$$

Example 2b

Consider first (2.5). Then, $a_n = 1$, $s = 0$, $N = 1$, and $q_1 = -1$, $l^*_0 = 0$, $b_n = 1 - \delta_{n0} - \delta_{n1}$. Naturally, $\hat{a}_n = B_n + \delta_{n1}$ and $\hat{b}_n = -nB_{n-1}$ for $n \geq 2$. Therefore, by (3.19) we find

$$t_n = 2 - \delta_{n0} - \delta_{n1} - \frac{1}{n+1} \sum_{k=2}^{n} \binom{n+1}{k} \binom{k}{1} \frac{B_{k-1}}{2^{k-1} - 1} \tag{3.24}$$

On the other hand for (2.6) we have $a_n = 2^{-n}$, $s = 1$, $N = 1$, and then we compute $g_1 = 0.25$, $l^*_1 = 0$, $b_n = 2^{-n-1} - 0.5 + 0.25\delta_{n1}$. Moreover, by (3.22) and (A9) $\hat{a}_n = B_n(2^{1-n} - 1)$, and $\hat{b}_n = B_n(2^{-n} - 1) - 0.5\delta_{n1} + 0.25nB_{n-1}$. Hence, after some algebra

$$w_n = \frac{1}{n+1} + \frac{1}{4}\delta_{n1} + \frac{0.25}{n+1} \sum_{k=1}^{n} \binom{n+1}{k} \binom{k}{1} \frac{B_{k-1}}{2^k - 1} \tag{3.25}$$

Example 3c

For (2.8) we assume $s = 0$, $N = 1$ $a_n = 2 + n2^{-n} - 2^{-n}$. Then $g_1 = -2$, $l^*_0 = 1$ and $b_n = a_n - \delta_{n0} - 2\delta_{n1}$. Using (3.22) we show that $\hat{a}_k = 2(B_n + \delta_{n1}) + 0.5k\, B_{k-1}(1/2) - B_k(1/2)$ and by (A9) we obtain $\hat{b}_k = B_k - 1.5k\, B_{k-1} + 2kB_{k-1}(2^{-k} - 2^{-1}) - B_k(2^{1-k} - 1) + 1.5\delta_{n1}$. Hence, by the above and (A8) we find

$$\begin{aligned} t_n = 4.5 - 2.5\delta_{n0} - 2.5\delta_{n1} - \frac{2}{n+1} + \frac{1}{n+1} \sum_{k=2}^{n} \binom{n+1}{k} \frac{B_k}{2^{k-1} - 1} - \\ \frac{1.5}{n+1} \sum_{k=2}^{n} \binom{n+1}{k} \binom{k}{1} \frac{B_{k-1}}{2^{k-1} - 1} \end{aligned} \tag{3.26}$$

Recurrence (2.9) is equivalent to (2.6) so the solution is given by (3.25).

4. APPROXIMATIONS

In this section we present an approximation of the exponential generating function $l(z)$ for small value of z and an asymptotic approximation of l_n for large value of n.

Let us start with a small value approximation of $l(z)$. Such an approximation might be very useful in practice if one is more interested in $l(z)$ than l_n. For example, determining maximum throughput for a conflict resolution algorithm we must optimize a ratio given by (2.7), where exponential generating functions are involved. It turns out that optimal value of z is rather small, hence the discussed approximation is applied. Assume now $z<\beta$, β is a small real value and consider (3.17). Then, we find

$$l(z)=l_0e^z+z^{(1-s)^{-}-1}l_s^*(e^z-1)+b(z)+(e^z-1)\sum_{k=(2-s)^{-}}^{M}\frac{\hat{b}_kz^{k-1}}{k!(2^{k+s-1}-1)}+O(z^{M+1}) \quad (4.1)$$

where $b(z)$, l_s^*, $\hat{b}_k$ are given in (3.5a), (3.18) and (3.21) respectively, while $M>(2-s)^{-}$. The value of M determines the quality of the approximation.

Hereafter we deal only with asymptotic analysis of l_n. Naturally, the problem is to find an approximation for the sum in (3.19), and further we restrict our considerations to that sum. Let

$$S(n,s,b_n)=\frac{1}{n+1}\sum_{k=t}^{n}\binom{n+1}{k}\frac{\hat{b}_k}{2^{k+s-1}-1} \quad (4.2)$$

where t is an integer. In our case $t=(2-s)^{-}$. According to (3.21) $\hat{b}_k$ consists of three terms , however, for asymptotic analysis of (4.2) the first one is the most difficult to handle, since it includes $\hat{a}_k$. Moreover, we restrict our considerations to such a class of a_k that the other terms of (3.21) will be automatically included in the analysis. Let a_n be given by (3.22), that is

$$a_n=\binom{n}{r}q^n \qquad \hat{a}_n=\binom{n}{r}q^rB_{n-r}(q) \quad (4.3)$$

where r is an integer while $q>0$. Note that for $q=1$ we obtain, as a special case, the other terms of (3.21).

Under the above assumption we have to find an asymptotic approximation of the following

$$S(n,r,s)=\frac{q^r}{n+1}\sum_{k=t}^{n}\binom{n+1}{k}\binom{k}{r}\frac{B_{k-r}(q)}{d^{k+s-1}-1} \quad (4.4)$$

where $d>1$, $t>1-s$. In our case $d=2$. Note that after some algebra (4.4) is reduced to

$$S(n,r,s) = \binom{n+1}{r} \frac{q^r}{n+1} \sum_{j=1}^{\infty} d^{-j(r+s)} \sum_{k=m}^{n-r} \binom{n+1-r}{k} B_k(q)\, d^{-j(k-1)} \tag{4.5}$$

where $m = \max\{t, r\} - r$. In our case $t = (2-s)^-$, and

$$m = \begin{cases} 2-s-r & s \le \max\{1, 2-r\} \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

Let us now consider the inner sum in (4.5) divided by n. Then

$$\frac{1}{n} \sum_{k=m}^{n-r} \binom{n+1-r}{k} B_k(q)\, d^{-j(k-1)} = \sum_{k=m}^{n-r} \binom{n+1-r}{k} \frac{1}{n^k} B_k(q) \left[\frac{n}{d^j}\right]^{k-1} \tag{4.7}$$

But [13]

$$\binom{n+1-r}{k} \frac{1}{n^k} = \frac{1}{k!} [1 + O(n^{-1})] \tag{4.8}$$

and let $x = nd^{-j}$. Eq. (4.8) suggests the following approximation of (4.7) for large values of n

$$\sum_{k=m}^{n-r} \binom{n+1-r}{k} \frac{1}{n^k} B_k(q) x^{k-1} \simeq \sum_{k=m}^{\infty} \frac{B_k(q) x^{k-1}}{k!}$$

From (A1) we know that for $|x| < 2\pi$

$$\sum_{k=m}^{\infty} \frac{B_k(q)}{k!} x^{k-1} = \frac{e^{qx}}{e^x - 1} - \sum_{k=0}^{m-1} \frac{B_k(q) x^{k-1}}{k!} \tag{4.9}$$

Therefore, we approximate $S(n,r,s)$ by $T(n,r,s)$ where

$$T(n,r,s) = \binom{n+1}{r} \frac{nq^r}{n+1} \sum_{j=1}^{\infty} d^{-j(r+s)} \left[\frac{e^{qx}}{e^x - 1} - \sum_{k=0}^{m-1} \frac{B_k(q) x^{k-1}}{k!} \right] \tag{4.10}$$

We prove that

Theorem 4.1. For any values of r and s

$$\delta(n,s) \overset{def}{=} T(n,r,s) - S(n,r,s) = O(n^{-s-1}) \tag{4.11}$$

Proof. Let $\delta(n,s)=\delta_1(n,s)+\delta_2(n,s)$, where $\delta_1(n,s)$ is computed for $x \le 1$ $(nd^{-j}<1)$ while $\delta_2(n,s)$ for $x>1$. We first evaluate $\delta_1(n,s)$. Then for $d^j > n$ one finds

$$\delta_1(n,s)=O(n^r)\sum_{j=\log_d n}^{\infty} d^{-j(r+s)} | \sum_{k=m}^{n-r} B_k(q)x^{k-1}\left[\frac{1}{k!}-\binom{n+1-r}{k}\frac{1}{n^k}\right]+\sum_{k=n-r+1}^{\infty}\frac{B_k(q)x^{k-1}}{k!} | \le \tag{4.12}$$

$$O(n^r)\sum_{j=\log_d n}^{\infty} d^{-j(r+s)}O(n^{-1}x^{m-1})$$

The inequality in (4.12) comes from the fact that the second term in (4.12) represents a reminder of a convergent series and we can make it as small as we want for large values of n, so the first term dominates in (4.12). Therefore,

$$\delta_1(n,s)\le \sum_{j=\log_d n}^{\infty} d^{-j(r+s)}O(n^{r-1}n^{m-1}d^{-j(m-1)})=\sum_{j=\log_d n}^{\infty} d^{-j(r+s+m-1)}O(n^{r+m-2})=O(n^{-s-1})$$

since under (4.6) the above geometric series is convergent.

Assume now $x>1$, i.e. $d^j<n$. For simplicity of our considerations we also assume that $m=0$ and $q=1$. Then the finite sum in (4.5) may be rewritten in the presence of (A8) as

$$A \overset{def}{=} \frac{1}{n+1-r}\sum_{k=0}^{n-r}\binom{n+1-r}{k}B_k d^{-j(k-1)}=\frac{1}{d^{j(n-r)}}\sum_{k=1}^{d^j-1}k^{n-r}$$

But for any $a>1$

$$\sum_{k=1}^{a-1}(ka^{-1})^n \le a^{-n}\int_1^{a-1}x^n dx \le (n+1)^{-1}a^{-1}(1-a^{-1})^{n+1}$$

hence $A \le O(x^{-1}(1-d^{-j})^n)$. Using the following well known inequalities [13]

$$\frac{1}{e^x-1}>\frac{e^{-x}}{x}, \qquad x>0 \quad ; \qquad 1-d^{-j}<e^{-d^{-j}}$$

we may evaluate $\delta_2(n,r)$ as follows

$$\delta_2(n,r)\le O(n^r)\sum_{j=1}^{\log_d n} d^{-j(r+s)}\left[\frac{1}{x}(1-d^{-j})^n-\frac{1}{e^x-1}\right]\le$$

$$O(n^r)\sum_{j=1}^{\log_d n} d^{-j(r+s)}O\left(\frac{e^{-x}}{x}\right)\le O\left(\exp\{-[nd+(r+s)\ln d-r\ln n-\ln\log_d n]\}\right)$$

which may be made as small as we need for large n, e.g. $O(n^{-s-1})$ as required.

□

By Theorem 4.1 the problem of computing $S(n,r,s)$ is reduced to finding $T(n,r,s)$ given by (4.10). We apply Mellin transform method [15]. In Appendix B we prove that for an odd integer c, real x, $q>0$ and $\mathrm{Re}\, z > c/2$

$$I(c,q) \overset{def}{=} \frac{1}{2\pi i}\int_{c/2-i\infty}^{c/2+i\infty} \zeta(z,q)\Gamma(z)x^{-z}dz = \frac{e^{x(1-q)}}{e^x-1} - \sum_{k=0}^{(1+c)/2}\frac{B_k(1-q)}{k!}x^{k-1} \tag{4.13}$$

where $\zeta(z,q)$ is generalized zeta function (see Appendix A) and $\Gamma(z)$ is gamma function [1] [5]. Then, (4.10) is equal to

$$T(n,r,q) = \binom{n+1}{r}\frac{nq^r}{n+1}\sum_{j=1}^{\infty} d^{-j(r+s)} I(3-2m,\, 1-q+\delta_{q1}) \tag{4.14}$$

and we restrict the range of q to $0 \le q \le 1$. The case $q=1$ needs some additional considerations, therefore δ_{q1} appears in (4.14) as shown in Appendix B. Note that $x = nd^{-j}$, hence for $\mathrm{Re}\, z < r+s$ we obtain

$$T(n,r,q) = [1+O(n^{-1})]\frac{q^r}{r!}\int_{(3/2-m)} \frac{\zeta(z,1-q+\delta_{q1})\Gamma(z)n^{r-z}}{d^{r+s-z}-1}dz \tag{4.15}$$

where $\int_{(c)}$ stands for $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}$.

The calculation of the counter integral in (4.15) is routine, and is equal minus the sum of residues of the function under integral right to the line of integration [9]. Three types of singularities must be taken into account:

(i) *zeros of the denominator*, that is, $d^{r+s-z}-1=0$. The roots of this equation are equal to $z_k = r+s+2\pi i k/\ln d$, $k=0,\pm 1,\ldots,$

(ii) *singular point of zeta function* at $z=1$,

(iii) *singular points of gamma function* at $z=-m$, where m is nonnegative integer.

The number of singularities we must consider for evaluation of (4.15) depends on the position of the line of integration, that is, it depends on the value of r and s. The most difficult to handle is a double pole which might occur if a zero of the denominator coincides with singular point of zeta function or with gamma function. We prove that

Proposition 4.2. Let $M=r+s$.

(i) if $M=r+s<1$ then

$$S(n,r,s)=-\frac{n^{-s}q^r}{r!}\{\frac{(-1)^{-M+1}}{(-M)!\ln d}\left[\frac{B_{1-M}(1-q+\delta_{q1})}{(1-M)}(\ln n-\psi(1-M)-0.5\ln d)+\zeta'(M,1-q+\delta_{q1})\right]$$
$$+\sum_{\substack{l=-1\\ l\neq -M}}^{m-2}\frac{(-1)^{l+1}}{(l+1)!}\frac{B_{l+1}(1-q+\delta_{q1})}{d^{M+l}-1}n^{l+M}-f_M(n)\}+O(n^{-s-1}) \tag{4.16}$$

(ii) if $M=r+s=1$, then

$$S(n,r,s)=-\frac{n^{-s}q^r}{r!}\{\ln^{-1}d[\ln n+\gamma+\psi(1-q+\delta_{q1})-0.5\ln d]+$$
$$+\sum_{l=0}^{m-2}\frac{(-1)^{l+1}}{(l+1)!}\frac{B_{l+1}(1-q+\delta_{q1})}{d^{M+l}-1}n^{M+l}-f_M(n)\}+O(n^{-s-1}) \tag{4.17}$$

(iii) if $M>1$ and $m>\frac{1}{2}$ then

$$S(n,r,s)=-\frac{n^{-s}q^r}{r!}\{\sum_{l=-1}^{m-2}\frac{(-1)^{l+1}}{(l+1)!}\frac{B_{l+1}(1-q+\delta_{q1})}{d^{M+1}-1}n^{M+1}-f_M(n)$$
$$-\zeta(M,1-q+\delta_{q1})(M-1)!\ln^{-1}d\}+O(n^{-s-1}) \tag{4.18}$$

(iv) if $M>1$ and $m<\frac{1}{2}$ then

$$S(n,r,s)=\frac{n^{-s}q^r}{r!}\{\zeta(M,1-q+\delta_{q1})(M-1)!\ln^{-1}d+f_M(n)\}+O(n^{-s-1}) \tag{4.19}$$

where

$$f_M(n) = \frac{1}{\ln d} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \zeta(M + 2\pi ik/\ln d)\Gamma(M + 2\pi ik/\ln d)\exp(-2\pi ik \log_d n) \qquad (4.20)$$

and $\psi(x)$ is psi function [1].

Proof: see Appendix C.

□

The function $f_M(n)$ may be safely ignored for practical purposes, since numerical analysis shows that values of the function are very small in comparison with the leading component of Eq. (4.16)-(4.19). In fact, it is not difficult to prove that $f_M(n) = f_M(dn)$ (e.g. $f_M(n)$ is a periodic function of $\log_d n$) and $f_M(n)$ is bounded. The last follows from the following well known formula [19]

(i)

$$|\exp(iy)| \le 1 \quad y - \text{real}$$

(ii)

$$\zeta(s+iy, q) = \begin{matrix} O(1) & \text{for} \quad s > 1 \\ O(y^{1/2 - s}) & \text{for} \quad s < 0 \end{matrix}$$

(iii) for any nonnegative integer, s,

$$|\Gamma(s+iy)|^2 = \frac{\pi}{y \sin hy} \prod_{j=0}^{s-1} (j^2 + y^2)$$

$$|\Gamma(-s+iy)|^2 = \frac{\pi}{y \sin hy} \left[\prod_{j=1}^{s} (j^2 + y^2) \right]^{-1}$$

and $\sinh y = O(e^{|y|})$.

Example 1c.

To evaluate (3.23) we put $s = 0$, $r = 0$, $q = 1$, $m = 2$ in (4.16) and after some algebra we obtain

$$c_n = \frac{1}{2} lg(n/\pi) + \frac{\gamma}{2 ln\, 2} + \frac{5}{4} + f_0(n) + O(n^{-1}), \quad n \ge 1 \qquad (4.21)$$

where $lg\ n = \log_2 n$ and $f_0(n)$ is given by (4.20) with $M = 0$ (see also [12]).

Example 2c.

For (3.24) we assume $s=0$, $r=1$, $m=1$ and by (4.17) we find

$$t_n = 1.5 - \delta_{n1} + lg\ n + f_1(n) + O(n^{-1}), \quad n \geq 1 \tag{4.26}$$

For (3.25) we assume $s=1$, $r=1$, $m=0$, hence by (4.19) and (A16) we obtain

$$w_n = \frac{1}{n+1} + \frac{1}{4}\delta_{n1} + \frac{\pi^2}{24\ln 2} \cdot \frac{1}{n} + \frac{1}{4n} f_2(n) + O(n^{-2}) \tag{4.25}$$

However, in order to compute the maximum throughput of the algorithm (Eq. (2.7)) we need exponential generating function of t_n and w_n for small values of z. But by (4.7) one finds

$$t(z) = \frac{z}{2} - \frac{z^2}{36} + \frac{z^4}{24 \cdot 450} + O(z^6) \tag{4.26a}$$

$$w(z) = 1 - \frac{z}{6} + \frac{z^2}{84} - \frac{z^4}{180 \cdot 31} + O(z^8) \tag{4.26b}$$

where z is a real number. For details see [16].

Example 3c.

For by (3.26), the two sums might be evaluated either as in Example 1c or as in Example 2c. We immediately obtain

$$t_n = 3 + 0.5 lg\ (n^4/\pi) + \frac{\gamma}{2\ln 2} + O(n^{-1}) \tag{4.27}$$

and w_n is given by (4.25).

□

5. CONCLUSIONS

In this paper a *linear recurrence with full history* was considered. We have found a closed-form solution of the recurrence, and in addition generating function of the solution was computed. Then we have established two approximations: small value approximation for the the generating function and asymptotic approximation for l_n . The analysis was illustrated by three examples of a great importance in practice. In the future research assumption a) from Sec. 2 should be relaxed. Note also that for exact solution of (2.3) and (2.4) a little more sophisticated recurrence than (2.1) must be studied. Namely, such a one which includes some additional terms in RHS of (2.1) e.g. nl_{n-1} as in (2.3) and (2.4).

APPENDIX A: Bernoulli polynomials and Riemann zeta function.

We list below some properties of Bernoulli polynomials and Riemann zeta function which are often used in this paper. Details may be found in [1],[5],[9],[11],[14].

Bernoulli numbers B_n and *Bernoulli polynomials* $B_n(x)$.

Definition:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \quad |t|<2\pi \tag{A1}$$

$$B_n=B_n(0) \tag{A2}$$

Properties:

$$B_n(x+1)=B_n(x)+nx^{n-1} \tag{A3}$$

$$B_n(x+h)=\sum_{k=0}^{n}\binom{n}{k}B_k(x)h^{n-k} \tag{A4}$$

$$B_n(1-x)=(-1)^nB_n(x) \tag{A5}$$

$$B_n=\sum_{k=0}^{n}\binom{n}{k}B_k-\delta_{n1} \tag{A6}$$

$$\sum_{k=1}^{m}k^n=\frac{B_n(m+1)-B_{n+1}}{n+1}, \quad n,m=1,2\ldots \qquad \int_a^x B_n(t)dt=\frac{B_{n+1}(x)-B_{n+1}(a)}{n+1} \tag{A8}$$

$$B_n(\tfrac{1}{2})=B_n(2^{1-n}-1) \tag{A9}$$

Generalized zeta function

$$\zeta(z,q)=\sum_{n=0}^{\infty}\frac{1}{(q+n)^z} \quad \text{Re } z>1, \quad q\neq 1,-1,-2,\ldots \tag{A9}$$

$$\zeta(z,q)=-\frac{\Gamma(1-z)}{2\pi i}\int_{\infty}^{0+}(-t)^{s-1}e^{vt}(1-e^{-t})^{-1}dt \quad \text{Re } q>0$$

$$z\neq 1,2,3\ldots \tag{A10b}$$

Riemann zeta function: $\zeta(z)=\zeta(z,1)$

Properties

$$\zeta(0,q)=\tfrac{1}{2}-q \tag{A12}$$

$$\lim_{z\to 1}[\zeta(z,q)-\frac{1}{s-1}]=-\psi(q) \tag{A13}$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$ is psi function.

$$\frac{d\,\zeta(z,q)}{dz}\,|_{z=0}=ln\;\Gamma(q)-\frac{1}{2}\,ln\;2\pi \tag{A14}$$

$$\zeta(-n,q)=-\frac{B_{n+1}(q)}{n+1}\quad n=0,2,3... \tag{A15}$$

$$\zeta(2m)=\frac{(2\pi)^{2m}}{2(2m)!}\,|B_{2m}|\quad m=1,2... \tag{A16}$$

APPENDIX B: Mellin transform

Let us compute the following integral

$$I(c,q)=\frac{1}{2\pi i}\int_{c/2-\infty}^{c/2+i\infty}\zeta(z,q)\Gamma(z)x^{-z}dz\;,\quad \mathrm{Re}\;z>c/2 \tag{B1}$$

where $q>0$ and c is an odd integer, while x is real. To evaluate the integral we use residue method. A path of integration goes from $(c/2+iN)$ to $(c/2+iN-M)$ to $(c/2-iN-M)$ to $(c/2-iN)$ to $(c/2+iN)$. Using properties of zeta and gamma functions [9] [19] one easily proves that the integral over horizontal lines and left vertical line vanishes when $N, M\to\infty$. Therefore, $I(c,q)$ is equal to the sum of residues left to the line $(c/2-i\infty,\, c/2+i\infty)$.

CASE A: $c\le 0$

Then the only singularities of the integrand are poles of the gamma function, that is, nonpositive integers smaller than $c/2$. Hence, noting that for $z=-k$ $(k\ge 0)$ the residue of the gamma function is equal to $(-1)^k/k!$ [9] we obtain

$$I(c,q)=\sum_{k=(1-c)/2}^{\infty}\zeta(-k,a)\frac{(-1)^k x^k}{k!}$$

But, by (A15), (A5) and (A1) for $|x|<2\pi$ we find

$$I(c,q)=\frac{e^{x(1-q)}}{e^x-1}-\sum_{k=0}^{(1-c)/2}\frac{B_k(1-q)}{k!}\,x^{k-1}\quad |x|<2\pi \tag{B2}$$

where $B_k(x)$ is Bernoulli polynomial.

CASE B: $c > 0$

In that case all nonpositive integers are singularities of the gamma function, and in addition for $c = 1$ there is a simple singularity at $z = 1$ of zeta function. Therefore,

$$I(c,q) = \sum_{k=0}^{\infty} \zeta(-k,q) \frac{(-1)^k x^k}{k!} + (1-\delta_{c1})x^{-1}$$

since residue at $z = 1$ of zeta function is equal to one. As above using (A15), (A5) and (A1) we finally obtain

$$I(c,q) = \frac{e^{x(1-q)}}{e^x - 1} - x^{-1}\delta_{c1} \qquad |x| < 2\pi \tag{B3}$$

By analytical continuation we prove that (B2) and (B3) hold for all real x. Hence, for any odd integer c we find

$$I(c,q) = \frac{e^{x(1-q)}}{e^x - 1} - \sum_{k=0}^{(1-c)/2} \frac{B_k(1-q)}{k!} x^{k-1} \tag{B4}$$

where the sum in (B4) is assumed to be zero if the upper index is smaller than the lower index in the sum symbol. Moreover, for q=0 it is easy to show (using the fact: $B_n(0)=B_n(1)$ $n > 1$ $B_1(1)=-B_1(0)$) that

$$I(c,1) = \frac{1}{e^x - 1} - \sum_{k=0}^{(1-c)/2} \frac{B_k}{k!} x^{k-1} \tag{B5}$$

and then $\zeta(z,q)$ in (B1) becomes Riemann zeta function $\zeta(z)=\zeta(z,1)$.

APPENDIX C: Proof of Proposition 4.2.

Let us evaluate the following integral

$$J(n,N,M) = \frac{1}{2\pi i} \int_{c/2-i\infty}^{c/2+i\infty} \frac{\zeta(z,q)\Gamma(z)n^{N-z}}{d^{M-z} - 1} dz \tag{C1}$$

where c is an odd integer, $q > 0$, N, M-integers, and $\mathrm{Re}\, z < M$, $c/2 < M$.

In the evaluation of the integral we use the same method as in Appendix B, however, this time the path of integration is right to the line $(c/2 - i\infty,\ c/2 + i\infty)$. By the same arguments as above show that the integral is minus the sum of residues right to the line of integration.

Let $g(z)$ be a function under integral. We must consider four cases depending on the value of M and c (note that $c/2 < M$).

CASE A: $M < 1$

In that case singularities of $g(z)$ are:

- *for gamma functions*, all nonpositive integers in the interval $[(c+1)/2, 0]$, that $z = -m$, $m = 0, 1, \ldots, -\frac{c+1}{2}$,
- *for zeta function* at $z = 1$,
- *zeros of the denominator* in (C1), that is, $z_k = M + 2\pi ik/\ln d$, $k = 0, \pm 1, \pm 2 \ldots$

Then residues of $g(z)$ is equal to:

(i) for $z = -m$, $m \neq -M$, $m = 0, 1, \ldots, -\frac{c+1}{2}$ by (C1) and (A15)

$$\operatorname*{res}_{z=-m\neq M} g(z) = \zeta(-m, q)\,\frac{(-1)^m}{m!}\,\frac{n^{N+m}}{b^{M+m} - 1} = \frac{(-1)^{m+1}}{(m+1)!}\,\frac{B_{m+1}(q)\,n^{N+m}}{b^{M+m} - 1} \tag{C2}$$

(ii) for $z = 1$ ([9] [19])

$$\operatorname*{res}_{z=1} g(z) = \frac{n^{N-1}}{b^M - 1} \tag{C3}$$

(iii) for $z_k = M + 2\pi ik/\ln d$, $k = \pm 1, \pm 2, \ldots$, $(z_k \neq M)$

$$\begin{aligned}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \operatorname{res} g(z_k) &= -n^{N-M}\,\frac{1}{\ln d} \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \zeta(M + 2\pi ik/\ln d)\Gamma(M + 2\pi ik/\ln d)\exp[-2\pi ik \log_d n] \\ &= -n^{N-M} f_M(n)\end{aligned} \tag{C4}$$

where $f_M(n)$ is defined as in (4.20).

(iv) for $z=M$

This is the most difficult to handle, since $z=M$ is double pole of $g(z)$ (gamma function and the denominator of (C1)). To find the residue we use the following expansions of the functions under the integral at $z=M$ (let $w=z+M$) [5] [9]:

$$\zeta(z,q)=-\frac{B_{1-M}(q)}{1-M}+w\,\zeta'(M,q)+O(w^2) \tag{C5a}$$

$$\Gamma(z)=w^{-1}\frac{(-1)^{-M}}{(-M)!}+\frac{(-1)^{-M}}{(-M)!}\psi(1-M)+O(w) \tag{C5b}$$

$$n^{N-z}=n^{N-M}-w\,n^{N-M}\ln n+O(w^2) \tag{C5c}$$

$$\frac{1}{d^{M-z}-1}=-w^{-1}\frac{1}{\ln d}-\frac{1}{2}+O(w) \tag{C5d}$$

The residue at $z=M$ is the coefficient of w^{-1} in the product of (C5a) - (C5d). After some algebra we find that

$$\operatorname*{res}_{z=M} q(z)=n^{N-M}\frac{(-1)^{1-M}}{(1-M)!}\{B_{1-M}(q)(\log_d n-\tfrac{1}{2}-\psi(1-M)/\ln d)+\zeta'(M,q)/\ln d\} \tag{C6}$$

where $\psi(x)$ is psi function [1],[5] and $\zeta'(x,q)$ denotes the derivative of zeta function for $z=x$. For example, [1],[5]

$$\zeta'(0,q)=\ln\Gamma(q)-\frac{1}{2}\ln 2\pi$$

For other values of $\zeta'(x,q)$ see [3].

Finally, taking into account (C2) - (C6) we find that for $c/2<M<1$

$$J(n,N,M)n^{M-N}=-\frac{(-1)^{1-M}}{(1-M)!}\{B_{1-M}(q)[\log_d n-\tfrac{1}{2}-\psi(1-M)/\ln d]+\zeta'(M,q)/\ln d\}$$
$$-\sum_{\substack{m=-1\\ m\neq -M}}^{-(c+1)/2}\frac{(-1)^{m+1}}{(m+1)!}\frac{B_{m+1}(q)n^{m+M}}{b^{M+m}-1}+f_M(n) \tag{C7}$$

where $f_M(n)$ is defined in (C4).

CASE B: $M = 1$

In that case we have the same singularities as above, however, now $z = 1$ is double pole of the denominator and zeta function. Hence, (C2) holds for all nonnegative integers $m \in [0, -\frac{(c+1)}{2}]$, and (C4) holds for $M = 1$. The only problem is double pole at $z = 1$. But denoting $w = z - 1$ and using expansions (C5c), (C5d) together with [5],[9]

$$\zeta(z,q) = w^{-1} - \psi(q) + O(w)$$
$$\Gamma(z) = 1 - \gamma w + O(w^2)$$

we find that

$$\operatorname*{res}_{z=1} q(z) = n^{N-1}(\log_d n + \frac{\gamma + \psi(q)}{\ln d} - \frac{1}{2})$$

Therefore,

$$J(n,N,M)n^{1-M} = -(\log_d n + \frac{\gamma+\psi(q)}{\ln d} - \frac{1}{2}) - \sum_{m=0}^{-(c+1)/2} \frac{(-1)^{m+1}}{(m+1)!} \frac{B_{m+1}(q)n^{m+1}}{b^{M+m}-1} + f_1(n) \qquad \text{(C8)}$$

where $f_1(n)$ is defined in (C4) for $M = 1$.

CASE C: $M > 1$ and $c/2 < 1$

We have here the same singularities as before, but there is no double pole. Therefore, by the same arguments as above we find

$$J(n,N,M)n^{M-N} = -\sum_{m=-1}^{-(c+1)/2} \frac{(-1)^{m+1}}{(m+1)!} \frac{B_{m+1}(q)n^{m+M}}{b^{M+m}-1} + \frac{\zeta(M,q)(M-1)!}{ln\ d} + f_M(n) \qquad \text{(C9)}$$

CASE D: $M > 1$ and $c/2 > 1$

In that case only zeros of the denominator are poles of the function under the integral. Noting that for $z_0 = M$

$$\operatorname*{res}_{z_0=M} q(z) = -\frac{\zeta(M,q)(M-1)!\, n^{N-M}}{ln\ d}$$

we find that

$$J(n,N,M)n^{M-N}=\zeta(M,q)(M-1)!/\ln d+f_M(n) \qquad \text{(C10)}$$

where $f_M(n)$ is defined in (C4).

In order to prove (4.16) - (4.19) we must assume in (C7), (C8), (C9) and (C10) $N=r$ $M=r+s$, and take into account (4.15).

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