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Abstract

A single multiaccess channel is studied with the outcome of a transmission being either 'idle', 'success', or 'collision' (ternary channel). Packets involved in a collision must be retransmitted, and an efficient way to solve a collision is known in the literature as Gallager-Tsybakov-Mikhailov algorithm. Performance analysis of the algorithm is quite hard. In fact, it bases on a numerical solution of some recurrence equations and on a numerical evaluation of some series. The obvious drawback of it is lack of insight into the behaviour of the algorithm. We shall present a new approach of looking at the algorithm and discuss some attempts of analyzing its performance. In particular, expected lengths of an resolution interval and a conflict resolution interval as well as throughput of the algorithm will be discussed using asymptotic approximation and "a small input rate" approximation.

1. INTRODUCTION

In a broadcast packet-switching network a number of users share a common communication channel. Since the channel is the only way of communications among the users, packet collisions are inevitable if a central coordination is not provided. The problem is to find an efficient algorithm for retransmitting conflicting packets. There are a number of algorithms, however, in recent years theoreticians as well as practicians paid a lot of attention to *conflict resolution algorithms* (CRA) [2], [3], [5], [11], [12]. The basic idea of CRA is to solve each conflict by splitting it into smaller conflicts. This is possible if each user observes the channel and learns whether in the past it was idle, success or collision transmissions. The partition of the conflict can be made on the basis of a random variable [3], [7], [9], [11] or on the basis of the time a user became active [2], [5], [12]. The former algorithm is known as Capetanakis-Tsybakov-Mikhailov algorithm (stack algorithm) while the latter as Gallager-Tsybakov-Mikhailov algorithm.

Performance analysis of CRA-algorithm is quite hard, since most quantities of interest are involved in quite sophisticated recurrence equations. In fact, previous analyses of CRA-algorithm were restricted to numerical evaluation of some infinite series. This was lately relaxed by Hofri [7], Fayolle et al [4] and Szpankowski [9] for stack algorithms by applying asymptotic approximation technique. We use the same methodology to analyze Gallager-Tsybakov-Mikhailov algorithm, however, since the problem is much more difficult than the previous one, we shall use some numerical analysis as well as asymptotic approximation and so called "small input rate" approximation.

In the next section, we shortly describe the algorithm and formulate problems to solve. In Section 3, we reduce the problem to a simple one with a help of some simple numerical computation. Then a closed form solution for the modified problem will be given. Finally, we apply asymptotic and 'small input rate' approximation to obtain tractable formulas for quantities of interest, e.g. expected length of conflict resolution interval and throughput.

2. PROBLEM STATEMENT

Assume a channel is slotted and a slot duration is equal to a packet transmission time. The algorithm defined below allows the transmission of the packets on the basis of their generation times. Assume packets are generated according to a Poisson point process with rate λ. At each step the algorithm marks a subset (an interval) of time axis, and packets which fall in that subset are transmitted in the next slot. The duration of the subset depends on the past outcome of the channel (idle, success or collision).. This is repeated as long as a collision is solved. The rate of successful transmissions is called *throughput*.

More precisely, access to the channel is controlled by a window based on the current age of packets. This window will be referred to as the *enabled interval* (EI). Let s_i denote the starting point for the i-th EI, and t_i is corresponding starting point for the *conflict resolution interval* (CRI), where CRI represents the number of slots needed to resolve a collision . Initially, the enabled interval is set to be $[s_i, \min\{s_i+\tau, t_i\})$, where τ is a constant which will be further optimized. At each step of the algorithm we compute the endpoints of the EI based on the outcomes of the channel. If at most one packet falls in the initial EI, then the conflict resolution interval ends immediately, and $s_{i+1}=s_i+\min\{\tau,t_i-s_i\}$. Otherwise, the EI is split into two halves, and three cases must be considered:

(i) all users whose current age of packets fall into the first (left) half are allowed to transmit packets. If it causes next *collision*, all knowledge about the second half is erased, and the first half is immediately split into two halves,

(ii) if enabling the first half causes on *idle* slot, the second half is immediately split into two halves,

(iii) if the first half gives a *success*, the entire second half is enabled

A CRI that begins with a collision continues until two consecutive pairs of successes occurs. This algorithm is known in the literature as Gallager [5], Tsybakov- Mikhailov [12] algorithm. To analyze it we introduce below some notations.

Assume an initial collision of a CRI is of multiplicity n. Then all packets whose generation time fall into an interval $[s_i, s_{i+1})$ are successfully sent in the i-th CRI. The interval $[s_i, s_{i+1})$ is called the i-th *resolved interval* (RI). Let T_n and τW_n denote the expected value for CRI and RI, respectively, assuming you start with n-conflict. In [12], [2] it is proved that they satisfy the following recurrence equations:

$$T_0=T_1=1$$
$$T_n=(2^n-2)^{-1}[2^n+nT_{n-1}+\sum_{k=1}^{n-1}\binom{n}{k}T_k], \quad n\geq 2 \tag{1}$$

$$W_0 = W_1 = 1$$
$$W_n = (2^{n+1}-1)^{-1}[1 + nW_{n-1} + \sum_{k=1}^{n-1} \binom{n}{k} W_k], \quad n \geq 2 \tag{2}$$

Moreover, let T and $W\tau$ denote the unconditional average value of CRI and RI, respectively. For a Poisson arrival process with $x = \lambda\tau$ one finds that

$$W(x) = e^{-x} \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} \quad ; \quad T(x) = e^{-x} \sum_{n=0}^{\infty} T_n \frac{x^n}{n!} \tag{3}$$

In [12] it was also proved that the algorithm is *stable* if and only if the input rate $\lambda < \lambda_{max}$ where

$$\lambda_{max} = \max_x \frac{xW(x)}{T(x)} \tag{4}$$

In the next section we analyze (1) - (4), and give easy computable formulas for the quantities of interests.

3. ANALYSIS

To find a closed form solution and asymptotic approximation for T_n, W_n, we first reduce the problem of solving (1) and (2) to a simple one using some numerical analysis. Then an analytical solution of the modified problem will be given and finally we shall present some approximations.

Reduction to a simple problem

Solution of (1) and (2) are not known, but we are able to present a rigorous solution of the following non-trivial recurrence equations:

$$t_0 = t_1 = 1$$
$$t_n = (2^n - 2)^{-1}[2^n + \sum_{k=1}^{n-1} \binom{n}{k} t_k], \quad n \geq 2 \tag{5}$$

$$w_0 = w_1$$
$$w_n = (2^{n+1} - 1)^{-1}[1 + \sum_{k=1}^{n-1} \binom{n}{k} w_k] \quad n \geq 2 \tag{6}$$

Naturally, $t_n \le T_n$ and $w_n \le W_n$. Before we deal with (5) and (6) let us find a relationship between T_n, W_n and t_n, w_n. We have computed the differences $T_n - t_n$ and $(n+1)\,[W_n - w_n]$ for $n \ge 2$. The results for $2 \le n \le 60$ are presented in Table I. It reveals that for sufficiently large n, $T_n - t_n$ oscillates around 2.644 with quite small amplitude. More precisely, for $n > 13$, $T_n - t_n = 2.644 \pm 0.002$, while for $6 < n \le 13$, $T_n - t_n = 2.644 \pm 0.02$.

Table I.

n	$T_n - t_n$	$(n+1)(W_n - w_n)$	n	$T_n - t_n$	$(n+1)(W_n - w_n)$
2	1.0000000	1.0000000	31	2.6444123	0.9119384
3	2.5000000	1.0000000	32	2.6441737	0.9118818
4	2.8095238	0.9285714	33	2.6439780	0.9118553
5	2.7142857	0.8986175	34	2.6438320	0.9118564
6	2.6362519	0.8990015	35	2.6437381	0.9118812
7	2.6201656	0.9067697	36	2.6436947	0.9119252
8	2.6286303	0.9127764	37	2.6436977	0.9119836
9	2.6399828	0.9154350	38	2.6437410	0.9120515
10	2.6473729	0.9155932	39	2.6438174	0.9121245
11	2.6501643	0.9144902	40	2.6439192	0.9121985
12	2.6497514	0.9130827	41	2.6440387	0.9122701
13	2.6477862	0.9119184	42	2.6441685	0.9123365
14	2.6455221	0.9112107	43	2.6443020	0.9123954
15	2.6436850	0.9109607	44	2.6444331	0.9124454
16	2.6425630	0.9110635	45	2.6445571	0.9124854
17	2.6421563	0.9113823	46	2.6446699	0.9125149
18	2.6423135	0.9117907	47	2.6447684	0.9125339
19	2.6428316	0.9121922	48	2.6448507	0.9125427
20	2.6435167	0.9125247	49	2.6449154	0.9125419
21	2.6442140	0.9127571	50	2.6449620	0.9125324
22	2.6448177	0.9128821	51	2.6449907	0.9125152
23	2.6452682	0.9129087	52	2.6450021	0.9124915
24	2.6455438	0.9128554	53	2.6449972	0.9124624
25	2.6456506	0.9127447	54	2.6449776	0.9124291
26	2.6456120	0.9125994	55	2.6449450	0.9123928
27	2.6454609	0.9124401	56	2.6449012	0.9123547
28	2.6452333	0.9122836	57	2.6448480	0.9123158
29	2.6449635	0.9121426	58	2.6447876	0.9122770
30	2.6446817	0.9120260	59	2.6447218	0.9122393

In fact, only for $n = 2$ (eventually for $n = 3$) the above is not satisfied. On the other hand, the difference $(n+1)(W_n - w_n)$ oscillates around 0.9122 with quite small amplitude, e.g. for $n > 8$, $(n+1)(W_n - w_n) = 0.9122 \pm 0.004$. Concluding out, we assume that

$$T_n \approx t_n + a \quad n > 3 \tag{7}$$

$$W_n \approx w_n + \frac{b}{n+1} \quad n > 3 \tag{8}$$

where $a = 2.644$ and $b = 0.9122$. Hence, by (7) and (8) we reduce (1), (2) to solution of the recurrence equations (5) and (6).

Closed form solution

Let $t(x)$ and $w(x)$ be exponential generating function for t_n and w_n, that is,

$$t(x) = \sum_{n=0}^{\infty} t_n \frac{x^n}{n!} \quad ; \quad w(x) = \sum_{n=0}^{\infty} w_n \frac{x^n}{n!}$$

where x is a real number. We find explicit formulas for $t(x)$ and $w(x)$.

Let us start with $t(x)$. Multiplying both sides of (5) by $x^n/n!$ and using boundary conditions ($t_0 = t_1 = 1$) one finds that

$$t(x) - t(\frac{x}{2})(e^{x/2} - 1) = e^x - e^{x/2} - x - 1 \tag{9}$$

Define now a new function $H(x)$ as

$$H(x) = \frac{x}{e^x - 1} t(x) \tag{10}$$

then multiplying (9) by $x/(e^x - 1)$ we obtain

$$H(x) = 2H(\frac{x}{2}) + \frac{x}{e^x - 1}(e^x - e^{x/2} - x - 1) \tag{11}$$

Before we solve the functional equation (11) let us derive a similar equation for $w(x)$. Using similar arguments as above we obtain

$$2w(x) - w(\frac{x}{2})(e^{x/2} + 1) = \frac{x}{2} \tag{12}$$

Let now

$$h(x)=\frac{x}{e^x-1}w(x)$$

then (12) is transformed to

$$h(x)=h(\frac{x}{2})+\frac{x}{4}\cdot\frac{x}{e^x-1} \tag{13}$$

Note that (11) and (13) are the same type. The common pattern is:

$$f(x)=a(x)f(px)+b(x) \tag{14}$$

where $0<p<1$ (in (11) and (13) $p=½$). Formally iterating (14) n times we find

$$f(x)=f(p^{n+1}x)\prod_{j=0}^{n}a(p^jx)+\sum_{k=0}^{n}b(p^kx)\prod_{j=0}^{k-1}a(p^jx)$$

Assume now that

$$(i)\quad \lim_{n\to\infty}f(p^{n+1}x)\prod_{j=0}^{n}a(p^jx)\overset{def}{=}f^*(x)\quad exists \tag{15a}$$

$$(ii)\quad \sum_{k=0}^{\infty}b(p^kx)\prod_{j=0}^{k-1}a(p^jx)\qquad is\ convergent \tag{15b}$$

Then, a general solution of (14) is

$$f(x)=f^*(x)+\sum_{k=0}^{\infty}b(p^kx)\prod_{j=0}^{k-1}a(p^jx) \tag{16}$$

To apply (16) successfully we have to compute the products in the right-hand side of (16). Therefore, instead of solving (9) and (12) we deal with (11) and (13). Let us start with (11). Here $a(x)=2$, so $\prod_{j=0}^{k-1}a(\frac{x}{2})=2^k$. To check that (15) are satisfied we shall transforme (11) into another equation. Let us introduce a function $H_1(x)$ defined as

$$H_1(x)=H(x)-\frac{x}{e^x-1} \tag{17}$$

Note that, $H_1(0)=0$ and the derivative of $H_1(z)$ at $z=0$ is $H'_1(0)=1$. Then, (11) is reduced to the following functional equation

$$H_1(x)=2H_1(\frac{x}{2})+\frac{x}{e^x-1}(e^x-x-1) \tag{18}$$

which is of type (14). However, then (15a) is equal to ($u \overset{def}{=} x2^{-k}$)

$$\lim_{n\to\infty} H_1(x2^{-n})2^n = x\lim_{u\to 0}\frac{H_1(u)}{u} = xH'_1(0)=x\,, \tag{19}$$

so (15a) is satisfied. It is also easy to check that the series (15b) is convergent for all $x > 0$. Hence, by (10), (11), (16), (17)-(19) we find

$$t(x)=e^x+(e^x-1)\sum_{k=0}^{\infty}\left[1-\frac{x2^{-k}}{e^{x2^{-k}}-1}\right] \tag{20}$$

The same arguments may be applied to find $w(x)$. Note that in that case the product in (16) is equal to one, so (13) and (16) directly imply that

$$w(x)=\frac{e^x-1}{x}+\frac{1}{4}(e^x-1)\sum_{k=0}^{\infty}\frac{x2^{-2k}}{e^{x2^{-k}}-1} \tag{21}$$

We use now (20) and (21) to derive explicit formula for t_n and w_n. Note that [1], [6] and [8]

$$\frac{z}{e^z-1}=\sum_{k=0}^{\infty}B_k\frac{z^k}{k!}$$

where B_k are Bernoulli numbers. Then, expanding the expression under the sum of (20) and using the above we get

$$\begin{aligned} t(x)&=e^x+(e^x-1)\left[1-\frac{x}{e^x-1}+\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{B_j}{j!}x^j2^{-kj}\right]= \\ &2e^x-1-x+(e^x-1)\sum_{k=1}^{\infty}\frac{B_kx^k}{k!(2^k-1)} \end{aligned} \tag{22}$$

Applying the rule of multiplication for series to the last component of (22) and comparing coefficients on both sides of (22) we find

$$t_n = 2-\delta_{n0}-\delta_{n1}+\sum_{k=1}^{n-1}\begin{Bmatrix} n \\ k \end{Bmatrix}\frac{B_k}{2^k-1} \tag{23}$$

where δ_{nk} is Kronecker delta. Using the same arguments to (21) we transform it into

$$w(x)=\frac{e^x-1}{x}+\frac{1}{4}x+\frac{1}{4}(e^x-1)\sum_{k=0}^{\infty}\frac{B_k x^k}{k!(2^{k+1}-1)} \tag{24}$$

and then one gets immediately

$$w_n=\frac{1}{n+1}+\frac{1}{4}\delta_{n1}+\frac{1}{4}\sum_{k=0}^{n-1}\begin{Bmatrix} n \\ k \end{Bmatrix}\frac{B_k}{2^{k+1}-1} \tag{25}$$

The values of T_n and W_n are obtained through the approximate formulas (7) and (8).

Approximations

Formulas (20), (21) and (23), (25) are one step forward to have better insight into the behavior of the system. However, from the qualitative point of view they are still to complicated to judge of the real nature of the algorithm. Therefore, we apply asymptotic approximation to discover the nature of t_n and w_n, and we use "small input rate" approximation to find properties of W and T.

Let us consider first $w(x)$ given by (21). Note that the function under the sum in (21) is of type $1/(e^u-1)$ where $u=x2^{-k}$. In Appendix we prove that

$$\frac{1}{e^u-1}=\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\zeta(z)\Gamma(z)u^{-z}dz \qquad ; \quad u-real$$

where $\zeta(z)$ and $\Gamma(z)$ are zeta and gamma functions [1], [6]. Hence, the sum in (21) becomes

$$\begin{aligned}\sum_{k=0}^{\infty}2^{-2k}\frac{x}{e^{x2^{-k}}-1}&=\frac{1}{e^x-1}+\sum_{k=1}^{\infty}x2^{-2k}\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\zeta(z)\Gamma(z)x^{-z}2^{kz}dz=\\ &\frac{1}{e^x-1}+\frac{x}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\zeta(z)\Gamma(z)x^{-z}dz\sum_{k=1}^{\infty}2^{-k(2-z)}=\\ &\frac{1}{e^x-1}+\frac{x}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\frac{\zeta(z)\Gamma(z)x^{-z}}{2^{2-z}-1}dz\end{aligned} \tag{26}$$

The evaluation of the counter integral is routine, and it is equal minus the sum of residues of the function under the integral, right to the line of integration [6], [8]. The poles of the function are roots of the equation $2^{2-z}-1=0$, that is, $z_k=2-2\pi ik/ln\,2$, $k=0,\pm 1,\pm 2,\ldots$. Noting that $\zeta(2)=\pi^2/6$ [1], [13], then after some algebra we finally obtain

$$w(x)=\frac{e^x-1}{x}+\frac{x}{4}+\frac{\pi^2}{24\ln 2}\frac{e^x-1}{x}+\frac{e^x-1}{4x}P_2(x) \tag{27}$$

where

$$P_2(x)=\frac{1}{\ln 2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\zeta(2-2\pi ik/ln\,2)\Gamma(2-2\pi ik/\ln 2)\exp\left[2\pi ik\,lgx\right]$$

and $lgx=\log_2 x$. Formula (27) might be used twofold. At first, to compute $w(x)$ for sufficiently large real values of x, and secondly to find asymptotic approximation for w_n. To take advantage of (27) we must somehow evaluate $P_2(x)$. In fact we show that $P(x)=O(1)$. This is a consequence of the following well known facts:

(i) $|\exp(iy)|\le 1$, y - real

(ii) $\zeta(s+iy)=O(1)$ for s,y–real and $s>1$ [13],

(iii) for any nonnegative integer s, and real-valued y [13], [8].

$$|\Gamma(s+iy)|^2=\frac{\pi}{y\sinh y}\prod_{j=0}^{s-1}(j^2+y^2)$$

Thus, the series is uniformly bounded, i.e. the bound does not depend on x. Then (27) may be rewritten as

$$w(x)=\frac{e^x-1}{x}+\frac{x}{4}+\frac{\pi^2}{24\ln 2}\cdot\frac{e^x-1}{x}+\frac{e^x-1}{4x}O(1) \tag{28}$$

Comparing coefficients of the power of x on both sides of (28) we find that

$$w_n=\frac{1}{n+1}+\frac{1}{4}\delta_{n,1}+\frac{\pi^2}{24\ln 2}\cdot\frac{1}{(n+1)}+O(n^{-1}) \tag{29}$$

The last approximation might be also derived from (25). In [10] we proved that

$$w_n = \frac{1}{n+1} + \frac{1}{4}\delta_{n,1} + \frac{\pi^2}{24\ln 2}\cdot\frac{1}{n} + \frac{1}{4n} f_2(n) + O(n^{-1}) \qquad (30a)$$

where

$$f_2(n) = \frac{1}{\ln 2}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \zeta(2-2\pi ik/\ln 2)\Gamma(2-2\pi ik/\ln 2)\exp(2\pi ik\lg n) \qquad (30b)$$

Noting that $f_2(n)$ is bounded (the proof repeats the same arguments used above for $P(x)$) we conclude that (29) and (30) are of the same nature.

Consider now (20) and assume $u = x2^{-k}$. In Appendix we prove that

$$\frac{1}{e^u-1} - \frac{1}{u} = \frac{1}{2\pi i}\int_{½-i\infty}^{½+i\infty} \zeta(z)\Gamma(z)u^{-z}dz\,, \quad u - \text{real}$$

Then

$$\sum_{k=1}^{\infty}\left[1-\frac{u}{e^u-1}\right] = \sum_{k=1}^{\infty} u\left[\frac{1}{u}-\frac{1}{e^u-1}\right] = -\frac{1}{2\pi i}\int_{½-i\infty}^{½+i\infty}\zeta(z)\Gamma(z)x^{1-z}\sum_{k=1}^{\infty}2^{-k(1-z)} =$$
$$-\frac{1}{2\pi i}\int_{½-i\infty}^{½+i\infty}\frac{\zeta(z)\Gamma(z)x^{1-z}}{2^{1-z}-1}dz \qquad (31)$$

The same idea as before might be used to evaluate the integral. However, now in addition to the poles $z_k = 1-2\pi ik/\ln 2$ $k = 0, \pm 1, \pm 2, \ldots,$ (roots of the denominator) there is one simple pole at $z_0 = 1$ of zeta function. This double pole is the most difficult to handle, however using series expansions of the functions under the integral we prove that the residue at $z_0 = 1$ is equal to $\lg x - 0.5$ [10]. Then, $t(x)$ may be evaluated as

$$t(x) = 2e^x - x - 1 + (e^x-1)\lg x - \frac{1}{2}(e^x-1) + (e^x-1)P_1(x) \qquad (32)$$

where

$$P_1(x) = \frac{1}{\ln 2}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\zeta(1-2\pi ik/\ln 2)\Gamma(1-2\pi ik/\ln 2)\exp(2\pi ik\lg x)$$

Since $P_1(x)$ is uniformly bounded we might use (32) as an asymptotic approximation for $t(x)$ (for $x > 7$ it is a good approximation). However, it is much more complicated to determine an asymptotic approximation for t_n, because it is not easy to find coefficients in an expansion of $(e^x - 1)lgx$. But, we may use alternatively (23) to get such an approximation.

Therefore, let us consider the sum in (32). In [10] we proved that

$$S_n = \sum_{k=1}^{n-1} \binom{n}{k} \frac{B_k}{2^k - 1} = n \sum_{j=1}^{\infty} 2^j \left[\frac{1^{n-1} + 2^{n-1} + \cdots + (2^j - 1)^{n-1}}{2^{j(n-1)}} - \frac{2^j}{n} \right]$$

Let $u = n\, 2^{-j}$. Then a simple algebra reveals that the sum in the square bracket is approximated by $(e^u - 1)^{-1} + O(n^{-1})$ [10], hence

$$S_n = u \sum_{j=1}^{\infty} \left[\frac{1}{e^u - 1} - \frac{1}{u} \right] + O(1) \qquad (33)$$

To evaluate (33) we might use the derivation from (31) replacing x by n. Finding the integral as above we finally obtain

$$t_n = 1.5 - \delta_{n0} - \delta_{n1} + lgn + f_1(n) + O(1) \qquad (34)$$

where

$$f_1(n) = \frac{1}{\ln 2} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \zeta(1 - 2\pi ik/\ln 2)\Gamma(1 - 2\pi ik/\ln 2)\exp(2\pi ik lgn)$$

But, by the same arguments as before $f_1(n)$ is bounded and numerical analysis reveals that the value of the function is very small in comparison with the other terms of (34). Thus, we may safely ignore $f_1(n)$ for practical purposes.

Note, however, that to determine the maximum throughput λ_{max} given by (4) we must evaluate $W(x)$ and $T(x)$ (or $w(x)$ and $t(x)$) for small values of x. We deal now with such an approximation which is called a "small input rate" approximation. In fact we may use previously derived formulas. Taking in (24) only first five terms of the series we obtain for small values of x

$$w(x)=\frac{e^x-1}{x}+\frac{x}{4}+\frac{e^x-1}{4}\bar{w}(x)+O(x^6) \qquad (35a)$$

where

$$\bar{w}(x)=1-\frac{x}{6}+\frac{x^2}{84}-\frac{x^4}{180\cdot 31} \qquad (35b)$$

On the other hand, considering the first five terms of the series in (22) we might approximate $t(x)$ for small values of x by

$$t(x)=2e^x-1-x+(e^x-1)\bar{t}(x)+O(x^6) \qquad (36a)$$

where

$$\bar{t}(x)=\frac{x}{2}-\frac{x^2}{36}+\frac{x^4}{24\cdot 450} \qquad (36b)$$

Numerical analysis reveals that (35) and (36) very well approximate $t(x)$ and $w(x)$ for all $x \leq 3$. As we shall see below the small rate approximation is much more useful than asymptotic approximation to evaluate W, T and $\lambda_{\max}$.

4. NUMERICAL ANALYSIS AND DISCUSSIONS

In the previous section we have obtained explicit expressions for $t(x)$, $w(x)$, t_n, w_n, and we have found approximate formulas for W_n and T_n. Now, some numerical results will be presented, however, we restrict our consideration to analysis of the maximum throughput $\lambda_{\max}$ given by (4).

Let us study (4). To determine maximum throughput we have to find maximum over $xW(x)/T(x)$ where x is a real-valued variable defined as $x=\lambda\tau$. Table I of the previous section shown that for $n>3$ the relationship between T_n, W_n and t_n, w_n are given by (7) and (8). Since $T_0=T_1=W_0=W_1=t_0=t_1=w_0=w_1=1$ the only values of n which must be reconsidered are $n=2$ and $n=3$. Therefore, the generating function for T_n and W_n are given as:

$$\begin{aligned} T(x) = T_0 + T_1 x + T_2 \frac{x^2}{2} + T_3 \frac{x^3}{6} + \sum_{k=4}^{\infty} (t_n + a) \frac{x^n}{n!} = \\ t(x) + \frac{x^2}{2}(T_2 - t_2) + \frac{x^3}{6}(T_3 - t_3) + a(e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{3}) \\ = t(x) + \frac{x^2}{2} + \frac{5x^3}{12} + a(e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{6}) \end{aligned} \tag{37}$$

and

$$\begin{aligned} W(x) = w(x) + \frac{x^2}{2}(W_2 - w_2) + \frac{x^3}{6}(W_3 - w_3) + \frac{b}{x}(e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24}) \\ = w(x) + \frac{x^2}{6} + \frac{x^3}{24} + \frac{b}{x}(e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24}) \end{aligned} \tag{38}$$

where $a = 2.644$ and $b = 0.9122$.

Note now that we have found above three different expressions for $t(x)$ and $w(x)$: exact formulas given by (20) and (21), asymptotic approximations (32), (28), and "small input rate approximation" given by (36) and (35). Asymptotic approximation is not useful for us since it is valid for $x > 7$, while we are interested in $0 < x < 2$. (However, using (30) and (34) instead of (28) and (32) we may tightly approximate $w(x)$ and $t(x)$ - for details see [10]).

In Table II we have shown for $0.5 < x < 2.0$ the exact values of $W(x)$ and $T(x)$, as well as their two approximations: series approximation, $W_s(x)$, $T_s(x)$, given by (20)-(21), (37)-(38) and small input rate approximation, $W_{sa}(x)$, $T_{sa}(x)$ given by (35)-(36), (37)-(38). Note that $W_s(x)$ and $T_s(x)$ reflects how good (7) and (8) approximate W_n and T_n, while $W_{sa}(x)$ and $T_{sa}(x)$ determine the quality of the "small input approximation". The table shows that $W_{sa}(x)$ and $T_{sa}(x)$ approximate $W_s(x)$ and $T_s(x)$ very well, almost to the third decimal digit. On the other hand, $W_s(x)$ approximates $W(x)$ a little better than $T_s(x)$ evaluates $T(x)$, however, both approximations are pretty well.

Finally, in Table III for $1.2 < x < 1.3$ the values of $xW(x)/T(x)$, $xW_s(x)/T_s(x)$ and $xW_{sa}(x)/T_{sa}(x)$ are plotted. This table enables to compute maximum input rate $\lambda_{\max}$ given by (4). It is known that direct search over (4) using recurrences (2) and (3) reveals $x_{op} = 1.266$ and

Table II.

x	$W(x)$	$W_s(x)$	$W_{sa}(x)$	$T(x)$	$T_s(x)$	$T_{sa}(x)$
0.500	1.6190	1.6190	1.6190	2.1403	2.1399	2.1398
0.550	1.6960	1.6960	1.6960	2.3445	2.3439	2.3439
0.600	1.7762	1.7761	1.7761	2.5698	2.5698	2.5688
0.650	1.8597	1.8596	1.8596	2.8174	2.8162	2.8161
0.700	1.9466	1.9466	1.9466	3.0888	3.0872	3.0871
0.750	2.0372	2.0371	2.0371	3.3856	3.3834	3.3833
0.800	2.1314	2.1314	2.1314	3.7093	3.7064	3.7063
0.850	2.2296	2.2295	2.2295	4.0617	4.0579	4.0578
0.900	2.3318	2.3317	2.3316	4.4444	4.4396	4.4395
0.950	2.4381	2.4380	2.4380	4.8594	4.8535	4.8534
1.000	2.5489	2.5488	2.5487	5.3087	5.3014	5.3013
1.050	2.6642	2.6640	2.6639	5.7944	5.7854	5.7853
1.100	2.7842	2.7840	2.7839	6.3186	6.3077	6.3076
1.150	2.9091	2.9089	2.9088	6.8837	6.8707	6.8705
1.200	3.0391	3.0389	3.0387	7.4921	7.4765	7.4764
1.250	3.1745	3.1742	3.1740	8.1464	8.1280	8.1278
1.300	3.3154	3.3150	3.3148	8.8493	8.8276	8.8275
1.350	3.4620	3.4616	3.4613	9.6037	9.5785	9.5783
1.400	3.6146	3.6142	3.6138	10.4127	10.3833	10.3831
1.450	3.7735	3.7730	3.7725	11.2793	11.2454	11.2452
1.500	3.9388	3.9383	3.9377	12.2069	12.1679	12.1677
1.550	4.1109	4.1104	4.1096	13.1991	13.1544	13.1543
1.600	4.2901	4.2894	4.2885	14.2596	14.2088	14.2085
1.650	4.4765	4.4758	4.4747	15.3922	15.3345	15.3344
1.700	4.6706	4.6697	4.6685	16.6013	16.5359	16.5358
1.750	4.8725	4.8716	4.8701	17.8910	17.8174	17.8172
1.800	5.0828	5.0818	5.0800	19.2659	19.1832	19.1831
1.850	5.3016	5.3005	5.2984	20.7310	20.6385	20.6383
1.900	5.5293	5.5281	5.5257	22.2912	22.1880	22.1878
1.950	5.7664	5.7651	5.7622	23.9520	23.8371	23.8369
2.000	6.0132	6.0118	6.0083	25.7189	25.5914	25.5912

$\lambda_{max} = 0.48711$. Our approximations give either $x_{op} = 1.28$ and $\lambda_{max} = 0.48822$ or $x_{op} = 1.277$ and $\lambda_{max} = 0.48819$. Naturally, from the conceptual and numerical point of view the small input rate approximation is preferable. In fact, in some situations we may simplify a little formulas (37), (38) assuming that approximations (7) and (8) are good also for $n = 3$. Then the terms with x^3 in (37) and (38) disappear , and numerical analysis shows that either $x_{op} = 1.267$ and $\lambda_{max} = 0.48425$ or $x_{op} = 1.254$ and $\lambda_{max} = 0.48246$.

Table III.

x	$\frac{xW(x)}{T(x)}$	$\frac{xW_s(x)}{T_x(x)}$	$\frac{xW_{sa}(x)}{T_{sa}(x)}$
1.200	0.48677	0.48775	0.48773
1.205	0.48682	0.48781	0.48779
1.210	0.48687	0.48{86	0.48785
1.215	0.48691	0.48792	0.48790
1.220	0.48695	0.48796	0.48795
1.225	0.48699	0.48801	0.48799
1.230	0.48702	0.48805	0.48803
1.235	0.48704	0.48808	0.48806
1.240	0.48707	0.48811	0.48809
1.245	0.48708	0.48814	0.48812
1.250	0.48710	0.48816	0.48814
1.255	0.48711	0.48818	0.48816
1.260	0.48711	0.48819	0.48817
1.265	0.48712	0.48820	0.48818
1.270	0.48712	0.48821	0.48819
1.275	0.48711	0.48821	0.48819
1.280	0.48710	0.48822	0.48819
1.285	0.48709	0.48822	0.48819
1.290	0.48708	0.48821	0.48818
1.295	0.48706	0.48820	0.48817

5. CONCLUSIONS

In this paper we considered an alternative method of analyzing Gallager-Tsybakov-Mikhailov algorithm. After reducing the problem to a simple one we presented three approximation: series approximation, asymptotic approximation and small input rate approximation. In particular, we showed that conditional expected length of a conflict resolution interval is $O(lgn)$ (compare to $O(n)$ in Capetenakis-Tsybakov-Mikhailov algorithm [7] [9]), and we gave easy computable formulas for average value of resolution interval and conflict resolution interval. Finally, maximum throughput was computed using the above approaches.

Further research should try to analyze exactly the recurrence equations (1) and (2) instead of reducing the problem to solution of recurrence equations (5) and (6).

APPENDIX

We evaluate the following integral

$$I(s)=\frac{1}{2\pi i}\int_{s/2-i\infty}^{s/2+i\infty}\zeta(z)\Gamma(z)x^{-z}\,dz \qquad x-real \tag{A1}$$

where x is a positive real number, and s is an *odd* integer. The evaluation of the integral is routine. We choose a path of integration which goes from $(s/2+iN)$ to $(s/2+iN-M)$ to $(s/2-iN'-M)$ to $(s/2-iN')$ to $(s/2+iN)$. Using properties of zeta and gamma functions one proves that the integral over horizontal lines and left vertical line vanishes when $N, M, N'\to\infty$. Therefore, the integral $I(s)$ is equal to the sum of residues left to the line $(s/2-i\infty, s/2+i\infty)$.

We consider two cases

A. Case $s\le 0$, s odd integer.

The only poles of the function under integral are poles of the gamma function, that is, negative integers left to the line $(s/2-i\infty, s/2+i\infty)$. Hence,

$$I(s)=\sum_{k=(1-s)/2}^{\infty}\zeta(-k)\frac{(-1)^k x^k}{k!} \tag{A2}$$

But $\zeta(-2n)=0$ and $\zeta(1-2n)=-\frac{B_{2n}}{2n}$ for $n=1,2,\ldots$, where B_n is the n-th Bernoulli number [6].

Since $B_{2k+1}=0$ for $k=1,2,\ldots$ we may write $\zeta(-k)=-\frac{B_{k+1}}{(k+1)}$ for $k=1,2,\ldots$, and (A2) is equal to

$$I(s)=\frac{1}{x}\sum_{k=(1-s)/2}^{\infty}\frac{B_{k+1}x^{k+1}}{(k+1)!}=\frac{1}{x}\left[\sum_{k=0}^{\infty}\frac{B_k x^k}{k!}-\sum_{k=0}^{(1-s)/2}\frac{B_k x^k}{k!}\right] \tag{A3}$$

The first series in (A3) is equal to $x/(e^x-1)$ [1] [6], so finally we have

$$I(s)=\frac{1}{e^x-1}-\sum_{k=0}^{(1-s)/2}\frac{B_k x^{k-1}}{k!}, \tag{A4}$$

For example

$$I(-1)=\frac{1}{e^x-1}-\frac{1}{x}+\frac{1}{2}$$

$$I(-3)=\frac{1}{(e^x-1)}-\frac{x}{6}-\frac{1}{x}+\frac{1}{2}$$

B. Case $s > 0$ odd integer

Then all nonpositive integers are poles of the gamma function, and in addition, for $s=1$ there is a simple pole at $z=1$ of zeta function. Thus,

$$I(s)=\sum_{k=0}^{\infty}\zeta(-k)\frac{(-1)^k x^k}{k!}+(1-\delta_{s1})x^{-1} \tag{A5}$$

Combining (A4) and (A5) we find finally

$$\frac{1}{2\pi i}\int_{s/2-i\infty}^{s/2+i\infty}\zeta(z)\Gamma(z)x^{-z}dz=\frac{1}{e^x-1}-\sum_{k=0}^{(1-s)/2}\frac{B_k x^{k-1}}{k!}; \qquad s-\text{odd integer} \tag{A6}$$

where the sum in (A7) is assumed to be zero if the upper index is smaller than the lower index of the sum symbol.

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