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ABSTRACT

Suppose we are given a set S of n (possibly intersecting) simple objects in the plane, such that for every pair of objects in S, the intersection of the boundaries of these two objects has at most α connected components. The integer α is independent of n, i.e. $\alpha = O(1)$. We consider the problem of determining whether there exists a straight line that goes through every object in S. We give an $O(n \log n \gamma(n))$ time algorithm for this problem, where $\gamma(n)$ is a very slowly growing function of n. If $\alpha < 3$ then our algorithm runs in $O(n \log n)$ time. Previously, only special cases of this problem were considered: In [6] the case when every object is a straight-line segment, in [2] the case when the objects are equal-radius circles and in [5] the case when objects all maintain the same orientation. All these cases follow from our general approach, which places no constraints on the size and/or configuration of the objects in S.

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1. Introduction

Consider being given a set S of n simple objects in the plane. By simple objects we mean those that have an $O(1)$ storage description each, and that are such that, for every pair of such objects, constant time suffices to compute their intersection, common tangents, etc. Typical examples of such objects are polygons with a constant number of edges, discs, ellipses, sectors of discs, etc. We seek straight lines, if they exist, that intersects all members of S. Such straight lines are called common transversals or stabbing lines of the set S. Since there exists a common transversal for n possibly non-convex objects iff there exists a common transversal for the n convex hulls of these objects, we can replace every input object by its convex hull (this takes $O(1)$ time per object since we are considering simple objects). We assume that this has already been done, i.e. from now on we assume that each of the n objects in S is convex.

Throughout the paper, we use α to denote the largest number of connected components that the intersection of two object boundaries can have, and we assume that α is a constant independent of n (i.e. $\alpha = O(1)$).

Algorithms for determining transversals are known, however in special cases only. Straightforward solutions arise from results in combinatorial geometry, Danzer, Gruenbaum, Klee [3] and Hadwiger, Debrunner [9], which give rise to worst case time bounds of $O(n^k)$, $k \geq 3$. Edelsbrunner, Overmars, Wood [7] have a general method for visibility problems in the plane which can be used to determine transversals, however in time $O(n^2 \log n)$. $O(n \log n)$ time algorithms were given for the special cases of line segments [6] and for circles of equal radius [2]. Efficient algorithms were then given by Edelsbrunner [5], who reduced transversal problems for a set of homothets of a simple planar object to convex hull problems. Though this gives $O(n \log n)$ time algorithms to determine transversals for a wide class of objects, it applies to only special constrained configurations of the set S of objects. In particular, homethety which involves only scaling and translation, forces all objects to maintain the same orientation.

In this paper we give efficient $O(n \log n \gamma(n))$, (and, if $\alpha < 3$, $O(n \log n)$), time algorithms to determine transversals of simple planar objects without any constraints on the size of the objects or constraints on the configuration of the set of objects, S. Our algorithm actually computes a description of all transversals of S.

1.1. Some Preliminaries

Let the functions $f_1, \cdots, f_n$ be real-valued, continuous functions of a parameter t, where each f_i has an $O(1)$ storage description. Suppose we want to compute the pointwise *Min* of these functions, defined by $h(t) = \underset{1 \le i \le n}{Min} f_i(t)$. Note that h itself is continuous and is typically made up of "pieces" each of which is a section of one of the f_i's. More formally, a piece of h is the portion of a function f_i over an interval $[t_1, t_2]$ such that (i) h is identical to f_i over that interval, (ii) h is not identical to any f_j over an interval which properly contains $[t_1, t_2]$. The storage representation of such a piece consists of the index i together with the interval $[t_1, t_2]$ (so a piece has an $O(1)$ storage description). (Detail: If f_i and f_j are identical over the interval $[t_1, t_2]$ then we break the tie arbitrarily, e.g. by taking $min(i, j)$.) The desired description of h is a list of the descriptions of the successive pieces that make it up. The next lemma bounds the number of pieces that make up h if no two distinct functions f_i and f_j intersect more than s times (f_i and f_j *intersect p times* iff the set of real values of t for which $f_i(t) = f_j(t)$ consists of p disjoint intervals on the real line).

Lemma 1. Let $f_1, \cdots, f_n$ be continuous, real-valued functions of variable t. Every f_i has an $O(1)$ storage description and can be evaluated at any t in $O(1)$ time. Every two distinct functions f_i and f_j intersect at most s times where $s = O(1)$; furthermore, these (at most s) intersections can all be computed in $O(1)$ time. Let h be the pointwise *Min* of the f_i's, i.e. $h(t) = \underset{1 \le i \le n}{Min} f_i(t)$. Then the description of h can be computed in $O(n \log n)$ time if $s < 3$, in $O(n \log n \gamma(n))$ if $s \ge 3$, where $\gamma(n)$ is an extremely slowly growing function of n.

Proof: Recursively compute the description of the pointwise *Min* of $f_1, \cdots, f_{n/2}$, and that of the pointwise *Min* of $f_{n/2+1}, \cdots, f_n$. Each of these two descriptions, as well as the description of the desired h, has $O(n)$ pieces if $s < 3$ [4], $O(n\gamma(n))$ pieces if $s \ge 3$ (for details about $\gamma(n)$, see the note that follows). These two descriptions are then combined to obtain that of h, giving the following recurrence for the time complexity $T(n)$: $T(n)=2T(n/2)+$(number of pieces). Thus $T(n)=\log n$.(number of pieces). □

Note: Let $\log^* n$ denote the smallest integer i for which $\exp_i(1)>n$, where $\exp_1(x)=e^x$ and $\exp_i(x)=e^{\exp_{i-1}(x)}$. The function $\log^* n$ grows extremely slowly with n and is "almost" a constant for all practical values of n, e.g. $\log^*(10^{100})=4$. Szemeredi proved an upper bound of $O(\log^* n)$ for $\gamma(n)$ [15], and sharper upper bounds were later given by Hart and Sharir [10] and Sharir and Livne [14].

2. Common Transversals

Consider the 1−1 geometric transformation which transforms a line l_0 in the $x-y$ plane into a pair (ρ_0,θ_0), a point in the $\rho-\theta$ parameter space, [Figure 1].

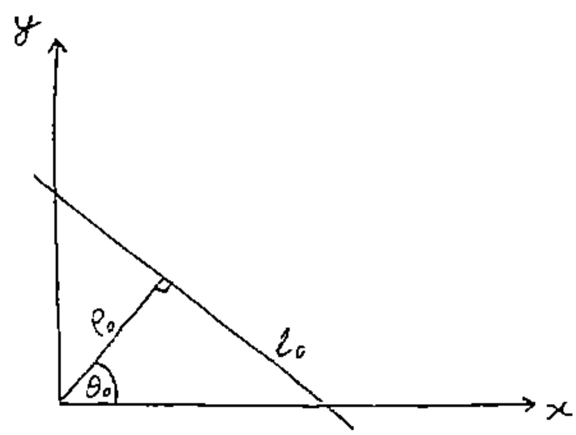

Figure 1 Geometric Transformation

We illustrate our method by first giving an $O(n\log n)$ time algorithm for the case of a set S consisting of n arbitrary circles in the plane (in [2] only the case where all the radii are equal was considered). To determine whether they permit a common transversal or stabbing line we use the above geometric transformation as follows. Each circle C_i is defined by a radius r_i and a center whose polar coordinates are (ρ_i,θ_i). To obtain all possible stabbing lines for C_i consider a general line defined by the pair (ρ,θ). As shown in Figure 2, this line stabs C_i *iff* $\rho_i Cos(\theta-\theta_i)-r_i \le \rho \le \rho_i Cos(\theta-\theta_i)+r_i$.

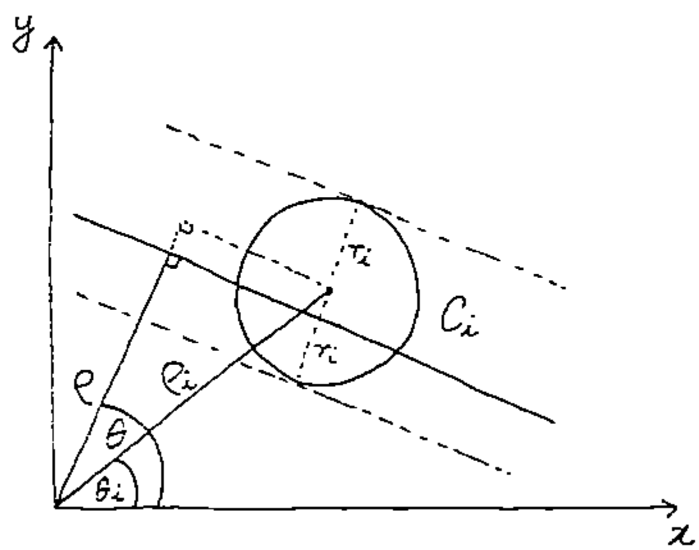

Figure 2 Stabbing Line

Furthermore, the line defined by (ρ,θ) is a stabbing line to all n circles or a common transversal *iff*

$$\rho_1 Cos(\theta - \theta_1) - r_1 \le \rho \le \rho_1 Cos(\theta - \theta_1) + r_1$$

$$\rho_2 Cos(\theta - \theta_2) - r_2 \le \rho \le \rho_2 Cos(\theta - \theta_2) + r_2$$

$$\cdots$$

$$\rho_n Cos(\theta - \theta_n) - r_n \le \rho \le \rho_n Cos(\theta - \theta_n) + r_n$$

which implies that (ρ,θ) is a common transversal *iff*

$$\underset{1\le i\le n}{Max} f_i(\theta) \le \rho \le \underset{1\le i\le n}{Min} g_i(\theta)$$

where

$$f_i(\theta) = \rho_i Cos(\theta - \theta_i) - r_i$$

$$g_i(\theta) = \rho_i Cos(\theta - \theta_i) + r_i$$

Now, observe that every point (ρ,θ) in the intersection of f_i and f_j (i.e. $\rho=f_i(\theta)=f_j(\theta)$) defines a line which is tangent to both C_i and C_j, and is such that C_i and C_j are on the same side of that common tangent. If C_i and C_j are distinct circles, then there are at most two such common tangents, and hence f_i and f_j intersect at most twice. If C_i and C_j coincide, then $f_i=f_j$ and hence f_i and f_j intersect once. Hence by Lemma 1, the description of the pointwise *Max* of the f_i's (call it $\hat{f}$) can be computed in $O(n\log n)$ time. Similar remarks holds for g_i and g_j, and the pointwise *Min* of the g_i's (call it $\hat{g}$). Once $\hat{f}$ and $\hat{g}$ are known, we have a complete description of all the stabbing lines of the C_i's, viz., every point (ρ,θ) in the region below the graph of $\hat{g}$ and above that of $\hat{f}$ defines a stabbing line of the C_i's (if that region is empty then there is no stabbing line).

The above method generalizes for planar objects such as ellipses, ovals,etc., whose boundaries consist of a single smooth closed curve. The method also generalizes for a larger variety of planar objects whose boundary consists of piecewise smooth curves, such as sectors of discs, $k-gons$ etc. The only restriction is that the intersection of any pair of object boundaries must have no more than α connected components, where $\alpha=O(1)$. When $\alpha\geq 3$, we obtain $O(n\log n\gamma(n))$ time performance (rather than $O(n\log n)$). The rest of this section sketches this generalization when each object is a convex $k-gon$, where $k=O(1)$.

For a set S of n convex $k-gons$ consider again the i^{th} object of the set, O_i. We need to obtain the functions f_i and g_i for every object O_i (as for the circles before). These functions are still continuous, but they are no longer smooth everywhere; instead they are piecewise smooth, with angular points separating the smooth pieces. The descriptions of f_i and g_i are computed as follows. We first compute, for every O_i, the set P_i of all antipodal pairs of vertices [13]. This takes $O(1)$ time per object. Corresponding to each antipodal pair $(p,q)\in P_i$ there exists a range of angles $[\theta_1,\theta_2]$ such that any line $L=(\rho,\theta)$ for which $\theta_1\leq\theta\leq\theta_2$ stabs O_i iff it stabs the straight-line segment pq. Therefore within each such range $[\theta_1,\theta_2]$ the functions f_i and g_i are smooth and easily defined. Since O_i has $O(k)$ antipodal pairs, each of f_i and g_i consists of $O(k)$ such smooth pieces.

As before, a straight line defined by (ρ,θ) in parameter space is a stabbing line for the object O_i *iff* $f_i(\theta) \le \rho \le g_i(\theta)$. Further the line (ρ, θ) intersects all n objects *iff* $\forall i, i = 1, \ldots, n$, we have $f_i(\theta) \le \rho \le g_i(\theta)$. Again, this implies that line (ρ,θ) is a transversal of the n objects *iff*

$$\underset{1 \le i \le n}{Max} f_i(\theta) \le \rho \le \underset{1 \le i \le n}{Min} g_i(\theta)$$

The piecewise smooth envelope $\underset{1 \le i \le n}{Max} f_i(\theta)$ is computed using Lemma 1. However, in order to be able to use this lemma, we must first show that f_i and f_j intersect $O(1)$ times. Actually, they intersect at most $2k$ times. To see this, note that there are as many such intersections as there are common tangents between O_i and O_j, and that there are at most $2k$ such common tangents (where by common tangent we mean one, such that both objects are on the same side of it).

The other piecewise continuous envelope $\underset{1 \le i \le n}{Min} g_i(\theta)$ is computed analogously. The region below the *Min* envelope and above the *Max* envelope describes all the transversals of S.

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