1985

# The Aspect Ratio Significant for Finite Element Problems

John R. Rice

*Purdue University*, jrr@cs.purdue.edu

Report Number:

85-535

# IS THE ASPECT RATIO SIGNIFICANT FOR FINITE ELEMENT PROBLEMS?

John R. Rice

CSD-TR-535
September 1985

# IS THE ASPECT RATIO SIGNIFICANT FOR FINITE ELEMENT PROBLEMS?

John R. Rice
Computer Science Department
Purdue University

CSD-TR 535
September 24, 1985

## ABSTRACT

An experiment study is made of the effect of elements with large aspect ratios on the solution of second order elliptic partial differential equations. We use collocation with Hermite bicubics and Galerkin with nine different piecewise polynomial basis functions on rectangular grids. We conclude that large aspect ratios do not imply large losses of accuracy. We also conclude that the condition number of the matrix associated with the method does not give a reliable guide to the effects of round-off.

*This work was supported in part by National Science Foundation grant MS-8301589*

## 1. INTRODUCTION AND SUMMARY

We present some experimental results about the effect of elements with large aspect ratios in finite element method. The *aspect ratio a* of an element is defined by

$$a = \frac{r_2}{r_1}$$

where $r_2$ is the radius of the largest circle containing the element and $r_1$ is the radius of the smallest circle contained in the element. For triangular elements a large aspect ratio is equivalent to a small angle in the element. Our experiment concerns rectangular elements exclusively, so we approximate the aspect ratios by the ratio of the longest to shortest side of the rectangle.

It is widely believed that finite element methods require "moderate" aspect ratios; otherwise the condition of the problem becomes large and computational accuracy is lost. The data presented here indicate that large aspect ratios (say 100-100,000) do not result in loss of computational accuracy and that the condition number (as computed by the LINPACK software [Dongarra et. al., 1979]) provides almost no information about the round-off effects to be expected. This experiment uses a 32-bit (7 decimal digit) machine and condition numbers of $10^{10}$ to $10^{14}$ are observed often to have no effect on the accuracy obtained.

The numerical methods used are all from the ELLPACK system [Rice and Boisvert, 1985]. They are:

(a) Collocation by bicubic Hermite polynomials

(b) Ordinary second order finite differences

(c) Galerkin method using piecewise polynomials of degree D with N continuous derivatives.

The elements with large aspect ratio are created by drawing a base grid (rectangular) and then introducing a new grid line at a small distance ($\Delta x$ or $\Delta y$) from one of the base grid lines. Varying $\Delta x$ or $\Delta y$ gives aspect ratios of varying sizes.

These results do not show conclusively that large aspect ratios are never harmful; it might be a property of other numerical methods or of non-rectangular elements. We note that a few people have reported using triangles with large aspect ratios without observing numerical difficulties [Gelinas et. al., 1982], [Alexander et. al., 1979]. Likewise, these results do not show that large condition numbers can be ignored; there are some problems where these numbers are important warning signs. They do not seem to be reliable indicators of trouble for those linear systems that arise from solving elliptic partial differential equations.

## 2. EXPERIMENTAL METHOD

The experimental approach is to first do extensive experimentation for one single situation. We choose one PDE and explore the effect of having one grid line move progressively closer to another with all the others remaining fixed. This is done for ten finite element methods and ordinary finite differences. We then experiment with variations involving two different PDEs or involving two or three grid lines which are closely spaced.

The problem chosen for extensive study is PDE10-6 of the PDE population of [Rice, Houstis and Dyksen, 1981]

$$u_{xx} + u_{yy} = f \qquad 0 \le x, y \le 1$$

$$u = 0 \qquad \textit{on boundary}$$

The function $f(x,y)$ is chosen to make the true solution

$$u(x,y) = e^{-10[(x-.5)^2 + (y-.117)^2]}(x^2 - x)(y^2 - y)$$

Thus, this problem is simple and well behaved in all respects; its solution appears as a smooth bump with peak at $x = 0.5$, $y = 0.117$. The maximum of $u(x,y)$ is about .04

The numerical methods chosen are all from the ELLPACK system [Rice and Boisvert, 1985] and we use the names and notation of this system. The methods are documented in the ELLPACK book and the references are therein. The methods used are

INTERIOR COLLOCATION (INT.COL.)

Collocation at the Gauss points by Hermite bi-cubic basis functions

SPLINE GALERKIN (DEGREE=D, NDERV=M) (SG(D.M))

Galerkin method using a piecewise polynomial basis with polynomial degree D and M continuous derivatives. The values of D and M used are:

(2.0), (2.1) (= quadratic splines)

(3.0), (3.1) (= Hermite bicubics), (3.2) (= cubic splines)

(4.1), (4.2), (4.3) (= quartic splines)

5 POINT STAR (5 PT)

Ordinary second order finite differences.

The problem PDE10-6 is solved first on a uniform rectangular grid which gives substantial accuracy. Then one additional $x$ grid line is introduced at a distance $\Delta x$ from one of the uniformly spaced grid lines. Then $\Delta x$ is varied through a range of values until it approaches round-off error level.

The Ridge 32 computer was used for the computations; its arithmetic is essentially identical to the VAX 11/780 and other common 32-bit machines. All computations were in single precision which implies that $\Delta x = 10^{-7}$ is the lower limit of values where one could hope to obtain good results.

---

The linear system of equations was solved by standard Gauss elimination with partial pivoting (BAND GE) or Cholesky factorization (LINPACK BAND). The condition numbers of the matrices were computed using the LINPACK routines.

The variations used in the experiment are described next. None of the variations produced results significantly different from those of the main experiment.

## A. Change the PDE

We use PDE 9-3

$$u_{xx} + u_{yy} - 100u = 1200 \frac{cosh(50y)}{cosh(50)} \quad 0 \le x, y \le 1$$

$$u = g(x,y) \quad on\ boundary$$

where $g(x,y)$ is chosen to make

$$u(x,y)=0.5\left(\frac{cosh(10x)}{cosh(10)}+\frac{cosh(20y)}{cosh(20)}\right)$$

This problem has a strong boundary layer effect along the lines $x=1$ and $y=1$. The initial grid chosen is somewhat skewed toward the boundary layer; it is a 10 by 14 grid with minimum $x$ spacing of about 0.05 and minimum $y$ spacing of about 0.03. INTERIOR COLLOCATION is applied to PDE 9-3

We also use PDE 14-3

$$u_{xx}+2u_{yy}+3u_x-4u_y-u=f \quad 0 \le x,y \le 1$$

$$\begin{aligned} u&=0 \quad on \quad y=0 \\ u&=y \quad on \quad x=0 \\ u&=g(x,y) \; on \quad x=1 \\ u&=0.2+|x-0.8| \; on \; y=1 \end{aligned}$$

The function $g(x,y)$ is chosen to make

$$u(x,y)=y(.2+|x-0.8|^{2-y})+xy\,e^{-xy}(y-1)$$

This problem has a variable jump discontinuity in the first derivative above the line $x=0.8$ which prevents most methods from obtaining much accuracy. The initial grid chosen is 11 by 11 graded somewhat toward $x=0.8$ and somewhat toward $y=1$. The minimum $x$ spacing is about 0.015 (actual $x$ lines included are .6, .7651376, .8, .8238532, .8385321 and .9), the minimum $y$ spacing is about 0.05. The extra grid line is inserted at $x=0.8+\Delta x$. INTERIOR COLLOCATION is applied.

**B. Change the number and location of close grid line pairs**

For PDE 10-6 and INTERIOR COLLOCATION we use grids with

$\Delta x=.0001$ , $\Delta y=.0001$ (two different cases)

$\Delta x_1=.0001$ , $\Delta x_2=.0005$ (one triple of close lines)

$\Delta y=.0045$

$\Delta x_1=.0001$ twice (two pairs of close lines)

$\Delta y=.0001$

For PDE 10-6 and 5 POINT STAR we use a grid with

$$\Delta x = .000001 \,, \Delta y = .00001$$

For PDE 9-3 and INTERIOR COLLOCATION we use grids with

$$\Delta y = .000222$$

$$\Delta y = .000022$$

## 3. EXPERIMENTAL RESULTS

Our first and most important observation is that large aspect ratios are not indicators of large computational errors. We present two tables from the data, Table 1 shows the effect of $\Delta x$ (or $\Delta y$) = .0001 which corresponds to aspect ratios of 667 to 2500. There are 18 cases presented; in only 3 of these does the introduction of a new grid line with $\Delta x$ = .0001 produce a change of more than 10% from the base grid or from the grid with $\Delta x$ much larger than .0001. In no case does it have an effect of 100% and in 15 of 18 cases there is no effect at all compared to much larger $\Delta x$'s.

Table 1. Effect of introducing a grid line with $\Delta x$ or $\Delta y$ = .0001. Solution effect indicates the change in the error compared to using the base grid. Perturbation effect indicates the change in the error compared to having $\Delta x$ or $\Delta y$ much larger.

| PDE 10-6 | Aspect ratio | Solution effect | Perturbation effect | PDE 9-3 | Aspect ratio | Solution effect | Perturbation effect |
|---|---|---|---|---|---|---|---|
| INT. COL. | 1250 | +3% | 0 | INT. COL. | 1250 | 0 | 0 |
| 5 PT. | 667 | 0 | 0 | | 2500 | 0 | 0 |
| SG(1.0) | 1250 | 0 | 0 | | 769 | 0 | 0 |
| SG(2.0) | 1250 | 0 | 0 | | 2222 | +80% | +83% |
| SG(2.1) | 1250 | +4% | 0 | | 2222 | 0 | +2% |
| SG(3.0) | 1429 | -3% | 0 | PDE 14-3 | | | |
| SG(3.1) | 1429 | -1% | -3% | INT. COL. | 2000 | -10% | 0 |
| SG(3.2) | 1429 | +5% | 0 | | 2000 | -21% | 0 |
| SG(4.1) | 1429 | +88% | +92% | | | | |
| SG(4.2) | 1429 | +2% | 0 | | | | |
| SG(4.3) | 1429 | +4% | 0 | | | | |

The cases in Table 1 correspond to several different sizes of problems of widely varying condition numbers. An examination of the data does not reveal any pattern of relationship between those cases when the $\Delta x$ = .0001 line makes a difference and either the size or condition number. The condition number is discussed later.

We next tabulate the value of $\Delta x$ or $\Delta y$ when the introduction of the grid line first makes a noticeable effect on the solution obtained. "Noticeable" was measured by having the maximum error change by more than a percent or so; we also obtained contour plots of the errors in all cases and in almost all instances no change in the maximum error coincided with no observable difference in the contour plot.

Note first that $\Delta x$ is usually only a few units in the last place (ulps) when the round-off due to the added grid line becomes noticeable. These $\Delta x$ values correspond to very large aspect ratios. Note second that the condition number of the linear algebra problem is not correlated with the point where round-off becomes noticeable. Note third that the condition numbers are, for the most part, very much larger than linear algebra theory says they can be and still have any accuracy in the computations. According to the standard theory, a condition number of $10^7$ would obliterate all accuracy; a condition number of "distance in solution" would produce noticeable round-off effects.

The five different cases for PDE9-3 come from four different base grids, two are uniform (9×9and 10×14) and two are skewed to adapt to the boundary layer nature of the solution. It is difficult to discern any pattern in the behavior for this problem.

Table 2. Data for the $\Delta x$ values where the new line produced a noticeable effect on the solutions obtained. A dash for $\Delta x$ means that no effect was observed until the new line coincided with a previous one to within machine accuracy. The "distance" measure is in units in the last place (ulps) from round-off in the grid($x$) accuracy and the solution accuracy. The system size is N/BW where N=number of unknowns and BW=bandwidth of the coefficient matrix.

| | | Aspect | Condition | Distance | | System |
|---|---|---|---|---|---|---|
| *PDE10-6* | $\Delta x$ | ratio | number | in x | in solution | size |
| INT. COL. | .000005 | 25000 | 3.4E+11 | 25 | $10^4$ | 288/21 |
| 5 PT. | .000001 | 66667 | 1.7E+6 | 5 | $10^5$ | 210/15 |
| SG(1.0) | .0000001 | 1.2E+6 | 1.9E+14 | 1 | $10^6$ | 90/11 |
| SG(2.0) | .000001 | 125000 | 4.7E+14 | 5 | $10^5$ | 323/40 |
| SG(2.1) | - | 6.2E+6 | 1.5E+9 | 1 | $10^5$ | 100/22 |
| SG(3.0) | .00001 | 14286 | 5.6E+14 | 50 | $10^4$ | 550/78 |
| SG(3.1) | .000002 | 71429 | 5.3E+14 | 10 | $10^5$ | 225/57 |
| SG(3.2) | - | 285714 | 7.6E+9 | 1 | $10^5$ | 110/36 |
| SG(4.1) | .000002 | 71429 | 8.1E+14 | 10 | $10^3$ | 598/108 |
| SG(4.2) | - | 1.4E+6 | 8.3E+10 | 1 | $10^3$ | 223/80 |
| SG(4.3) | - | 285714 | 5.2E+10 | 1 | $10^4$ | 132/52 |
| *PDE9-3* | | | | | | |
| INT. COL. | - | 625000 | 2.4E+14 | 2 | $10^6$ | 256/21 |
| | .000001 | 250000 | 7.3E+12 | 5 | $10^4$ | 256/21 |
| | .000001 | 76923 | 8.6E+13 | 5 | $10^5$ | 504/33 |
| | .0001 | 2222 | 6.3E+9 | 500 | $10^3$ | 520/78 |
| | .000222 | 835 | 1.3E+10 | 1000 | $10^3$ | 520/78 |
| *PDE14-3* | | | | | | |
| INT. COL. | .0000002 | 1E+6 | 9E+13 | 1 | $10^5$ | 440/25 |

Figure 1 shows a scatter plot of

d = size of PDE error in ulps (point where round-off should become apparent)

versus

c = condition number when round-off becomes apparent.

using a logarithmic scale. If the linear algebra theory of condition numbers were perfect, we should have d=c. However, the observed values of c are many orders of magnitude bigger than c and there is little apparent correlation between them.

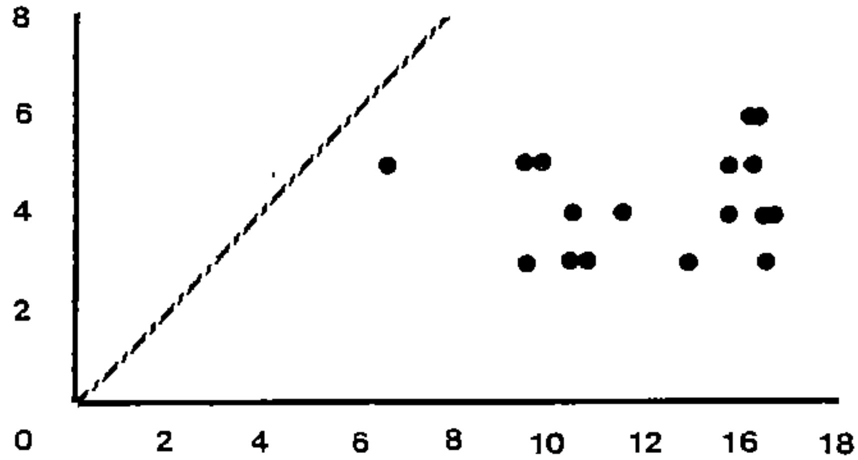


*Figure 1.* Scatter plot of log d = size of PDE error in ulps versus log c = condition number when round-off becomes evident.

## 4. REFERENCES

Alexander, R., Manselli, P., and Miller, K., Moving Finite Elements for the Stefan Problem in 2-D. Rend. Accad. Naz., Lincei (Rome), Ser VIII, 67(1979), 57-61.

Dongarra, J., Moler, C., Bunch, T. and Stewart, G., *LINPACK User's Guide.* SIAM Publications, Philadelphia (1979).

Galinas, R., Doss, S., Vajk, J., Djomehri, J. and Miller, K., Moving Finite Elements in 2-D. Proc. 10th World Congress, IMACS, 1(1982), 58-60.

Rice, J.R. and Boisvert, R.F., *Solving Elliptic Problems Using ELLPACK.* Springer Verlag, New York (1985).

## 5. THE EXPERIMENTAL DATA

We present the data in the format of PDE problem, method, base grid. For each such entry we give the $\Delta x$ , $\Delta y$ values, the aspect ratios the maximum error in solving the PDE and the condition number. We also give the size of the linear algebra problem (number N of unknowns and band width BW).

| PDE10-6 INTERIOR COLLOCATION | | | Aspect ratio | Error | Condition |
|---|---|---|---|---|---|
| Base | = | 9 by 9 uniform (N=256, BW=21) | 1 | 5.75E-5 | 1.71E+3 |
| $\Delta x$ | = | .0084862 (at .325) (N=288, BW=21) | 15 | 5.87E-5 | 1.13E+5 |
| | | .0048165 | 26 | 5.83E-5 | 3.63E+5 |
| | | .0005 | 250 | 5.75E-5 | 3.52E+7 |
| | | .0001 | 1250 | 5.91E-5 | 8.88E+8 |
| | | .0001 and .0005 | 1250 | 5.88E-5 | 1.14E+9 |
| | | .00002 | 6250 | 5.23E-5 | 2.17E+10 |
| | | .000005 | 25000 | 8.13E-5 | 3.34E+11 |
| $\Delta x$ | = | .0001 (at .375), .0001 (at .5) | 1250 | 6.07E-5 | 9.63E+9 |
| $\Delta y$ | = | .004505 (at .25) | 28 | 5.92E-5 | 4.30E+5 |
| | | .0001 | 1250 | 6.24E-5 | 8.69E+8 |
| $\Delta x$ | = | .0001 (at .375), $\Delta y$ = .0001(at .251) (N=384, BW = 23) | 1250 | 5.97E-5 | 1.92E+9 |
| Base | = | 11 by 11 uniform (N=400, BW=25) | | 2.19E-5 | 2.50E+3 |
| $\Delta x$ | = | .0001 (at .9)(N=440, BW=25) | 1000 | 2.19E-5 | 7.38E+8 |
| $\Delta y$ | = | .0001 (at .2) | 1000 | 2.34E-5 | 7.91E+8 |
| $\Delta x$ | = | .0001 (at .9), $\Delta y$ = .0001 (at .2) | 1000 | 2.33E-5 | 1.66E+9 |

| PDE 10-6 5 POINT STAR | | | | | |
|---|---|---|---|---|---|
| Base | = | 16 by 16 uniform (N=196, BW=14) | 1 | 7.62E-4 | 8.78E+1 |
| $\Delta x$ | = | .001835 (at .4) (N=210, BW=15) | 36 | 7.62E-4 | 9.51E+2 |
| | | .0001 | 667 | 7.62E-4 | 1.66E+4 |
| | | .00001 | 6667 | 7.63E-4 | 1.66E+5 |
| | | .000001 | 66667 | 9.03E-4 | 1.71E+6 |
| $\Delta x$ | = | .000001 (at .4), $\Delta y$ = .000001 (at .4) | 66667 | 6.39E-4 | 1.90E+6 |

| PDE 10-6 SPLINE GALERKIN (1.0) | | | | | |
|---|---|---|---|---|---|
| Base | = | 9 by 9 uniform (N=81, BW=10) | 1 | 3.84E-3 | 1.77E+9 |
| $\Delta x$ | = | .0001 (at .375)(N=90, BW=11) | 1250 | 3.84E-3 | 1.74E+11 |
| | | .00001 | 12500 | 3.88E-3 | 1.72E+12 |
| | | .000001 | 125000 | 3.80E-3 | 1.70E+13 |
| | | .0000001 | 1.2E+6 | 5.13E-3 | 1.93E+14 |
| | | .00000001 | 1.2E+7 | machine | stop |

| PDE 10-6 | SPLINE GALERKIN (2.0) | Aspect ratio | Error | Condition |
|---|---|---|---|---|
| Base = | 9 by 9 uniform (N=289, BW=36) | 1 | 2.13E-4 | 5.55E+11 |
| Δx = | .0001 (at .375) (N=323, BW=40) | 1250 | 2.13E-4 | 4.79E+12 |
| | .000001 | 125000 | 9.65E-4 | 4.66E+14 |
| | .0000001 | 1.2E+6 | Division by | zero occurred |

| PDE 10-6 | SPLINE GALERKIN (2.1) | Quadratic | splines | |
|---|---|---|---|---|
| Base = | 9 by 9 uniform (N=100, BW=22) | 1 | 3.30E-4 | 1.37E+9 |
| Δx = | .0001 (at .375)(N=110, BW=24) | 1250 | 3.42E-4 | 1.57E+9 |
| | .00001 | 12500 | 3.42E-4 | 1.57E+9 |
| | .000001 | 125000 | 3.42E-4 | 1.57E+9 |
| | .0000001 | 1.2E+6 | 3.43E-4 | 1.57E+9 |
| | .00000002 | 6.2E+6 | 3.42E-4 | 1.57E+9 |

| PDE 10-6 | SPLINE GALERKIN (3.0) | | | |
|---|---|---|---|---|
| Base = | 8 by 8 uniform (N=484, BW=69) | 1 | 2.95E-5 | 5.40E+11 |
| Δx = | .001 (at .285714) (N=550, BW=78) | 143 | 2.93E-5 | 5.70E+12 |
| | .0001 | 1429 | 2.86E-5 | 5.46E+13 |
| | .00005 | 2857 | 3.15E-5 | 1.12E+14 |
| | .00001 | 14286 | 5.90E-5 | 5.60E+14 |

| PDE 10-6 | SPLINE GALERKIN (3.1) | Hermite | bi-cubics | |
|---|---|---|---|---|
| Base = | 8 by 8 uniform (N=256, BW=51) | 1 | 4.90E-5 | 1.14E+10 |
| Δx = | .001 (at .285714) (N=288, BW=57) | 143 | 5.00E-5 | 1.16E+11 |
| | .0001 | 1429 | 4.85E-5 | 1.05E+12 |
| | .00005 | 2857 | 5.16E-5 | 2.09E+12 |
| | .00001 | 14286 | 5.68E-5 | 1.04E+13 |
| | .000002 | 71429 | 9.57E-5 | 5.30E+13 |

| PDE 10-6 | SPLINE GALERKINE (3.2) | cubic | splines | |
|---|---|---|---|---|
| Base = | 8 by 8 uniform (N=110, BW=33) | 1 | 2.13E-4 | 6.59E+9 |
| Δx = | .001 (at .285714) (N=100, BW=36) | 143 | 2.22E-4 | 7.63E+9 |
| | .0001 | 1429 | 2.23E-4 | 7.64E+9 |
| | .00005 | 2857 | 2.23E-4 | 7.64E+9 |
| | .00001 | 14286 | 2.23E-4 | 7.64E+9 |
| | .000002 | 71429 | 2.23E-4 | 7.64E+9 |
| | .0000005 | 285714 | 2.23E-4 | 7.64E+9 |

| PDE 10-6 | SPLINE GALERKIN (4.1) | | | |
|---|---|---|---|---|
| Base = | 8 by 8 uniform (N=529, BW=96) | 1 | 2.57E-6 | 1.73E+11 |
| Δx = | .001 (at .285714) (N=598, BW=108) | 143 | 2.52E-6 | 1.76E+12 |
| | .0001 | 1429 | 4.84E-6 | 1.61E+13 |
| | .00005 | 2857 | 6.36E-6 | 3.20E+13 |
| | .00001 | 14286 | 5.94E-6 | 1.59E+14 |
| | .000002 | 71429 | 1.00E+0 | 8.10E+14 |

| PDE 10-6 | SPLINE GALERKIN (4.2) | Aspect ratio | Error | Condition |
|---|---|---|---|---|
| Base = | 8 by 8 uniform (N=289, BW=72) | 1 | 6.37E-6 | 6.68E+10 |
| $\Delta x$ = | .001 (at .285714) (N=323, BW=80) | 143 | 6.50E-6 | 8.29E+10 |
| | .0001 | 1429 | 6.50E-6 | 8.31E+10 |
| | .00005 | 2857 | 6.50E-6 | 8.31E+10 |
| | .00001 | 14286 | 6.50E-6 | 8.31E+10 |
| | .000002 | 71429 | 6.49E-6 | 8.31E+10 |
| | .0000005 | 285714 | 6.50E-6 | 8.31E+10 |
| | .0000001 | 1.4E+6 | 6.50E-6 | 8.31E+10 |
| $\Delta x$ = | .0000005, .000002 | 285714 | 2.46E-4 | 6.66E+14 |

| PDE 10-6 | SPLINE GALERKIN (4.3) | Quartic | splines | |
|---|---|---|---|---|
| Base = | 8 by 8 uniform (N=121, BW=40) | 1 | 5.50E-5 | 4.72E+10 |
| $\Delta x$ = | .001 (at .285714) (N=132, B=52) | 143 | 6.06E-5 | 5.17E+10 |
| | .0001 | 1429 | 6.06E-5 | 5.18E+10 |
| | .00001 | 14286 | 6.06E-5 | 5.18E+10 |
| | .000002 | 71425 | 6.00E-5 | 5.18E+10 |
| | .0000005 | 285714 | 6.06E-5 | 5.18E+10 |
| $\Delta x$ = | .0000005, .000002 (at .285714) | 285714 | 6.15E-5 | 6.32E+10 |
| $\Delta y$ = | .00001 (at .428571) (N=156, BW=56) and $\Delta x$ = .0000005, .000002 (at .285714) | 285714 | 3.74E-5 | 7.54E+10 |

| PDE 9-3 INTERIOR COLLOCATION | | | Aspect ratio | Error | Condition |
|---|---|---|---|---|---|
| Base | = | 9 by 9 uniform (N=256, BW=21) | 1 | 1.37E-1 | 1.72+3 |
| $\Delta x$ | = | .0001 (at .375) (N=288, BW=21) | 1250 | 1.37E-1 | 8.16E+8 |
| | | .00001 | 12500 | 1.37E-1 | 8.14E+10 |
| | | .000001 | 125000 | 1.37E-1 | 1.02E+13 |
| | | .0000002 | 625000 | 1.37E-1 | 2.44E+14 |
| | | .0000001 | 1.2E+6 | machine | stop |
| Base | = | 9 by 9 skewed (N=256, BW=21) | 7 | 2.13E-3 | 1.29E+5 |
| $\Delta x$ | = | .0001 (at .52) | 2500 | 2.13E-3 | 7.30E+8 |
| | | .00001 | 25000 | 2.13E-3 | 7.22E+10 |
| | | .000001 | 250000 | 8.24E-3 | 7.27E+12 |
| Base | = | 10 by 14 uniform (N=468, BW=31) | 1.4 | 4.28E-2 | 2.91E+4 |
| $\Delta x$ | = | .0001 (at .555555) (N=520, BW=78) | 769 | 4.28E-2 | 7.36E+8 |
| | | .00001 | 7692 | 4.32E-2 | 7.52E+10 |
| | | .000001 | 76923 | 4.62E-2 | 8.68E+13 |
| $\Delta y$ | = | .0001 (at .923077) (N=504, BW=33) | 909 | 4.26E-2 | 7.18E+8 |
| | | .00001 | 9090 | 4.31E-2 | 7.22E+10 |
| Base | = | 10 by 14 skewed (N=468, BW=31) | 23 | 6.46E-5 | 1.44E+5 |
| $\Delta x$ | = | .0002 (at .5358) (N=520, BW=78) | 1111 | 6.35E-5 | 1.58E+8 |
| | | .0001 | 2222 | 1.16E-4 | 6.32E+9 |
| | | .0000201 | 11055 | 1.70E-4 | 1.56E+10 |
| $\Delta x$ | = | .0001 (at .941784) | 2222 | 6.42E-5 | 6.10E+8 |
| $\Delta y$ | = | .000222 (at .222222) | 835 | 6.45E-5 | 1.20E+8 |
| | | .000022 | 8348 | 1.06E-4 | 1.30E+10 |

| PDE 14-3 INTERIOR COLLOCATION | | | | | |
|---|---|---|---|---|---|
| Base | = | 11 by 11 uniform (N=400, BW=25) | 1 | 3.86E-2 | 2.59E+3 |
| Base | = | 11 by 11 graded toward x=.8, y=1.0 | 14 | 3.52E-2 | 3.50E+5 |
| $\Delta x$ | = | .0001 (at .8) (N=440, BW=25) | 2000 | 3.20E-2 | 5.22E+8 |
| | | .00001 | 20000 | 3.30E-2 | 5.17E+10 |
| | | .000001 | 200000 | 3.28E-2 | 5.26E+12 |
| | | .0000005 | 400000 | 3.64E-2 | 2.02E+13 |
| | | .0000002 | 1E+6 | Division by | zero occurred |