Best Strategy for Each Team in The Regular Season to Win Champion in The Knockout Tournament

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Abstract

In [5], Schwenk identified a surprising weakness in the standard method of seeding a single elimination (or knockout) tournament. In particular, he showed that for a certain probability model for the outcomes of games it can be the case that the top seeded team would be less likely to win the tournament than the second seeded team. This raises the possibility that in certain situations it might be advantageous for a team to intentionally lose a game in an attempt to get a more optimal (though possibly lower) seed in the tournament. We examine this question in the context of a four or eight team league which consists of a round robin "regular season" followed by a single elimination tournament with seedings determined by the results from the regular season. Using the same probability model as Schwenk we show that there are situations where it is indeed optimal for a team to intentionally lose. Moreover, we show how a team can make the decision as to whether or not it should intentionally lose. We did two detailed analysis. One is for the situation where other teams always try to win every game. The other is for the situation where other teams are smart enough, namely they can also lose some games intentionally if necessary. The analysis involves computations in both probability and (multi-player) game theory.

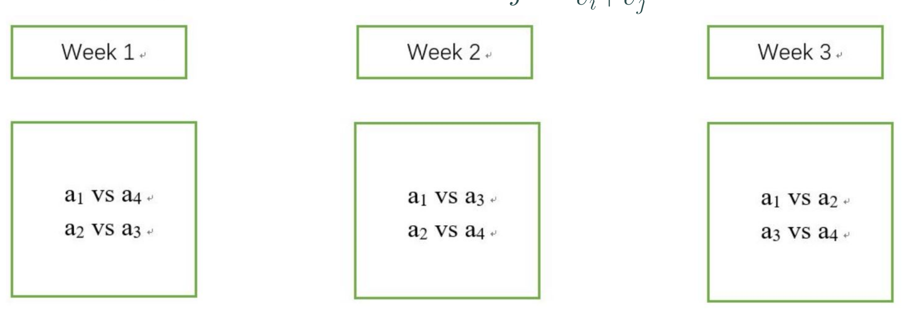
Introduction

In contemporary society, sport competitions such as NBA, NCAA basketball, baseball are more and more prevalent and attracting. In most of these competitions, every team in the knockout tournament has to play head-to-head matches to eliminate the rival and finally tries best to win the champion. Whether the knockout tournament is fair and what strategy each team has under the knockout tournament is sparking argue between fans every day. In this article, we use the single elimination tournament model created by J. Schwenk.(2018).

After we have examined the fairness of the model, we conduct a detailed analysis of the best team's strategy, which can help the best team to decide whether to win or lose intentionally in every week in the regular season. It is noteworthy that this model is based on assumption that other teams try their best to win every match. However, in the real world, every professional team is smart enough. Every team has intelligent people to make decision for them. Thus, we build another model, which assumes that all teams are smart enough and are able to make the most correct decision for them. More importantly, this model can get strategy for every team in the knockout tournament. To our surprise, the final result shows that the stratgy of every team is not mixed strategy, but pure strategy no matter the every team's team weight is.

The probability model used in our research

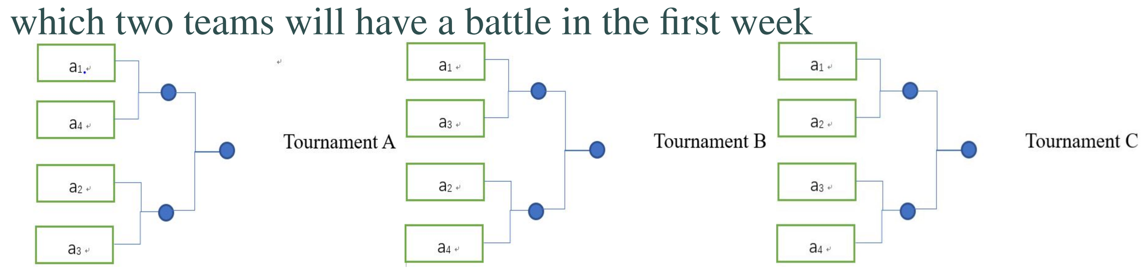
In our model, we have four teams, a_1, a_2, a_3, a_4. Their weight is v_1, v_2, v_3, v_4. Assume that $v_1 \geq v_2 \geq v_3 \geq v_4$, so we will help a1 to get a strategy. In addition, we build a particular schedule for last three weeks. We assume that there is no home advantage. The winning probability in a single game between a_i and a_j is $\frac{v_i}{v_i+v_j}$.



Analysis for the regular season with a four teams' model

In this part, we suppose that except a_1, other teams will try their best to win for every match. We want to analyze the last several weeks in the regular season to get a strategy for the best team to decide whether to try to win or lose intentionally in each game under the probability model. Our logic is that first analyze the last game in the regular season, second analyze the last two games

Because there are four teams, there has 4!=24 situations in the knockout tournament. Hence, there exists three kinds of knockout tournaments, we ignore the exact rank of each team and we only care that which two teams will have a battle in the first week

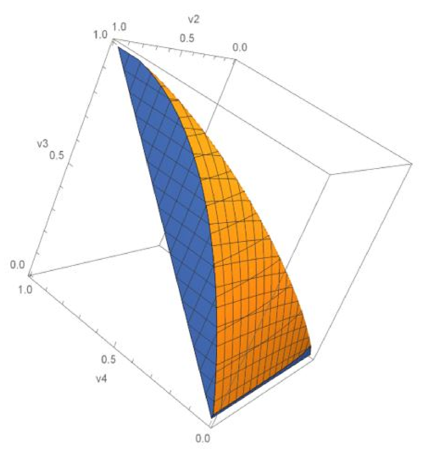


An interesting idea is that we find that the probability for a_1 to win the champion in tournament A is always bigger than the one in tournament B and tournament C, no matter what value v_1, v_2, v_3, v_4 are. We have introduced that our logic is to first analyze the strategy in last week. We build a probability winning vector which represents the number of winning games of each team in first two weeks,$[x_1, x_2, x_3, x_4]$. There are fifteen different situations of $[x_1, x_2, x_3, x_4]$. Next step, we consider the game a_3 vs a_4 in the last week. There are two possible situations: a_3 wins and a_4 wins.

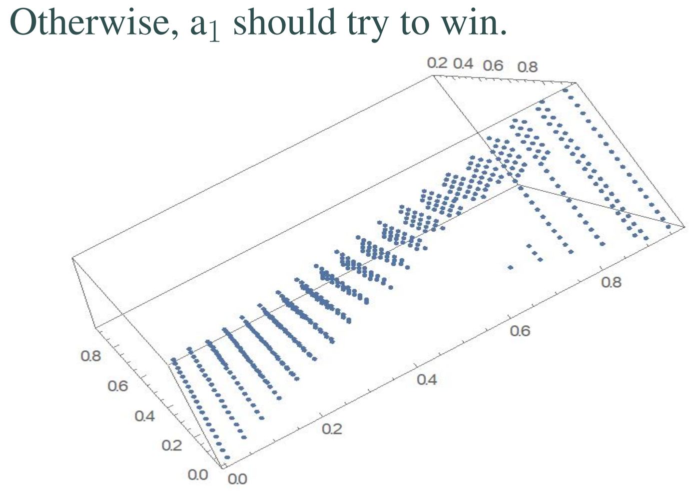
Then we will analyze the match a_1 vs a_2. If a_1 tries to win, there are two possibilities: a_1 wins and a_1 loses. If a_1 wants to lose intentionally, the only possible result is a_1 loses. Because a_1 likes tournament A best, a_1 will try to meet a_4 in the first round. We display the first several rows of the strategy table of a_1.

Winning vector	$a_3$wins $a_4$wins	TW	TL	S
2,2,0,0	2,2,1,0 2,2,0,1	3,2,1,0 A 3,2,0,1 B	2,3,1,0 B 2,3,0,1 A	W L
2,1,1,0	2,1,2,0 2,1,1,1	3,1,2,0 A 3,1,1,1 A/B/C	2,2,2,0 A/B/C 2,2,1,1 A/B	W L
2,1,0,1	2,1,1,1 2,1,0,2	3,1,1,1 A/B/C 3,1,0,2 B	2,2,1,1 A/B 2,2,0,2 A/B/C	L W/L
2,0,1,1	2,0,2,1 2,0,1,2	3,0,2,1 C 3,0,1,2 C	2,1,2,1 A/C 2,1,1,2 B/C	L L

Then we compare the difference of probability of winning the champion between different strategy of a_1. For example, if the winning probability before the week 3 is [1,2,0,1], then if the point (v_2, v_3, v_4) is in the shaded region of the figure below, a_1 should lose intentionally in week 3.



We can use the similar method to analyze for week 2 and week 1. Finally, we get the result is that a_1 can investigate the team weight of v_2, v_3, v_4 at the beginning of the first week. If the point (v_2, v_3, v_4) is in the figure below, a_1 should lose intentionally in the first match. Otherwise, a_1 should try to win.



Analysis with four teams' model where other teams are smart enough

In this part, we assume that other three teams are smart enough to lose some games intentionally to maximize their winning probability for the champion. We still analyze the last week and We define the payoff function as the probability to win the champion under given strategy.

a_1 \ a_2	Win	Lose
Win	Q_{1a}, Q_{2a}	Q_{1b}, Q_{2b}
Lose	Q_{1c}, Q_{2c}	Q_{1d}, Q_{2d}

a_3 \ a_4	Win	Lose
Win	Q_{3a}, Q_{4a}	Q_{3b}, Q_{4b}
Lose	Q_{3d}, Q_{4d}	Q_{3d}, Q_{4d}

We first find the nash equilibriums in the two tables. Then, Let E_1, E_2, E_3, E_4 be expectation of team a_1, a_2, a_3, a_4 to win the champion.

$$\begin{cases} E_1 = xyQ_{1a} + (1-x)yQ_{1b} + x(1-y)Q_{1c} + (1-x)(1-y)Q_{1d}, \\ E_2 = xyQ_{2a} + (1-x)yQ_{2b} + x(1-y)Q_{2c} + (1-x)(1-y)Q_{2d}, \\ E_3 = zwQ_{3a} + (1-z)wQ_{3b} + z(1-w)Q_{3c} + (1-z)(1-w)Q_{3d}, \\ E_4 = zwQ_{4a} + (1-z)wQ_{4b} + z(1-w)Q_{4c} + (1-z)(1-w)Q_{4d}, \end{cases} \tag{1}$$

We calculate the partial derivative for the four equations and solve (x,y,z,w) for situation where all partial derivatives equal to 0.

$$\begin{cases} \frac{\partial E_1}{\partial x} = 0, \\ \frac{\partial E_2}{\partial y} = 0, \\ \frac{\partial E_3}{\partial z} = 0, \\ \frac{\partial E_4}{\partial w} = 0, \end{cases} \tag{2}$$

By solving the equation system, we get the only solution is that

$$v_1 = v_2 = 1, v_3 = v_4 \leq 1 \tag{3}$$

Then,

$$\begin{aligned} E_1 &= xyQ_{1a} + (1-x)yQ_{1b} + x(1-y)Q_{1c} + (1-x)(1-y)Q_{1d} \\ &= x^2(Q_{1a} - Q_{1b} - Q_{1c} + Q_{1d}) + x(Q_{1b} + Q_{1c} - 2Q_{1d}) + Q_{1d}. \end{aligned}$$

Due to $P_{1a} = P_{1b}$, we can get

$$\begin{cases} Q_{1a} - Q_{1b} - Q_{1c} + Q_{1d} = 0 \\ Q_{1b} + Q_{1c} - 2Q_{1d} = 0 \end{cases} \tag{4}$$

Thus, $E_1 = Q_{1d}$. By using the same method, we can get

$$\begin{cases} E_2 = Q_{2d} \\ E_3 = Q_{3d} \\ E_4 = Q_{4d} \end{cases} \tag{5}$$

We find that E_i is independent with the strategy variable. Thus, if $v_1 = v_2 = 1$, $v_3 = v_4 \leq 1$, then all x=y, z=w are nash equilibriums. Otherwise, all strategies are pure strategy.

Then, we analysis for the second week. In the second week, there are two games which is a_1 vs a_3 and a_2 vs a_4. Now, we draw the game theory table and use the same algorithm again to find the nash equilibriums. In addition, we can apply our algorithm on week one to obtain the strategy and expectation of winning the champion of each team.

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