1984

Optimal Placement for Multi-Terminal Nets

Mikhail J. Atallah
Purdue University, mja@cs.purdue.edu

Susanne E. Hambrusch
Purdue University, seh@cs.purdue.edu

Report Number:
84-502
Optimal Placement for Multi-Terminal Nets

Mikhail J. Atallah
Susanne E. Hambrusch

Department of Computer Sciences
Purdue University
West Lafayette, IN 47907

November 1984

TR - 502

Abstract

In the general multi-terminal placement problem we have to place the terminals of \( n \) multi-terminal nets in any of a number of given positions in a channel. The positions for the terminals must be chosen so that the connections made by a routing algorithm (which follows the placement) require a channel of minimum width. While the general multi-terminal placement problem is NP-hard, we show that a number of variants have efficient placement algorithms. For the variant in which the upper row positions of the nets have already been determined, the efficiency of our algorithm is based on a new combinatorial characterization of the optimal placement.

Key Words

Analysis of algorithms, channel routing, data structures, density, design automation, multi-terminal nets, placement.

This work was supported by the Office of Naval Research Contract N00014-84-K-0502.
1. Introduction

A common approach in dealing with computationally intractable problems arising in layout design is to partition them into simpler subproblems that can be handled more efficiently. One such subproblem is the problem of placing and connecting sets of terminals in a channel [HS,R]. In this paper we consider a number of placement problems in a channel in which we are given the freedom of placing each terminal on any one of a number of given positions. The positions for the terminals must be chosen so that the connections made by a routing algorithm following the placement require a channel of minimum width.

Assume that each one of positions $p_1, \ldots, p_r$ on the upper row of a channel and each one of positions $q_1, \ldots, q_m$ on the lower row can be used to place a terminal, $1 \leq p_1 < p_2 < \ldots < p_r \leq M$, $1 \leq q_1 < q_2 < \ldots < q_m \leq M$. We are given $n$ nets, where the $i$-th net consists of $u_i$ entry terminals and $k_i$ exit terminals. We have to place the entry terminals on distinct positions on the upper row, the exit terminals on distinct positions on the lower row, such that the resulting multi-terminal channel routing problem (CRP) can be routed using minimum channel width. We call this problem the Multi-Terminal Placement (MTP) problem. Problems of this nature arise when modules to be connected can be placed on a number of specified positions along a channel, or when a number of terminals on a functional block must be connected to terminals on a driver module and it does not matter which [GCW,P].

It is not hard to show that the MTP-problem defined above is NP-hard. The main contribution of this paper is to present efficient algorithms for two variants of the MTP-problem. In the first variant (which we call MTP1-problem), all the entry terminals have already been positioned, and all the $k_i$'s are equal. We show how to determine an optimal placement of exit terminals in $O(s + (n+m)\log(n+m))$ time. The efficiency of our algorithm is based on a technique which makes use of an elegant and somewhat surprising combinatorial characterization of an optimal placement. The second variant of the MTP-problem for which we give an efficient solution is that where the $k_i$'s are equal, the $u_i$'s are equal, and both entry and exit terminals have to be placed. We call this the MTP2-problem. We present an algorithm.
for the MTP2-problem that generates an optimal placement in $O((s + m) \log \tilde{d})$ time, $\tilde{d} \leq \eta$ ($\tilde{d}$ will be defined shortly).

We now give a more precise definition of the problems considered in this paper, and make some preliminary observations. In the MTP1-problem we are given, in addition to the lower positions $q_1, \ldots, q_m$ and an integer $k$, sets $P_1, \ldots, P_n$, which are a partition of the set $\{p_1, \ldots, p_s\}$ of positions on the upper row, $m \geq kn$. $P_i$ contains the positions of the entry terminals of net $i$; $P_i = \{p_i^1, p_i^2, \ldots, p_i^k\}$ with $p_i^1 < \cdots < p_i^k$, $1 \leq i \leq n$. The problem is to assign each one of the $k$ exit terminals of each net to a lower position, i.e. we must compute sets $Q_1, \ldots, Q_n$ where set $Q_i = \{q_i^1, \ldots, q_i^k\}$ contains the $k$ lower positions assigned to net $i$, $q_i^1 < \cdots < q_i^k$. The assignment should be such that the multi-terminal CRP consisting of the nets $(P_i, Q_i)$ can be wired using minimum channel width over all other assignments. Since in a number of routing models the width required by a routing algorithm is proportional to the density of the CRP [CL, H, PL, RBM], the density is the cost measure to be minimized by our algorithms. The density $d$ of a multi-terminal CRP is defined as follows. Let $x_i = \min\{p_i^1, q_i^1\}$ and $y_i = \max\{p_i^k, q_i^k\}$, then $d$ is the maximum over all $c$ of the number of pairs $(x_i, y_i)$ for which $x_i < c < y_i$ or $x_i > c \geq y_i$ [SP]. Throughout, we use $\hat{d}$ to denote the optimal density, i.e. the smallest density over all assignments.

For the MTP1-problem we can, without loss of generality, assume that every $P_i$ contains at most two upper positions: The leftmost upper position $p_i^1$ (also called $l_i$), and the rightmost upper position $p_i^k$ (also called $r_i$). Henceforth we consider every $P_i$ to be an interval rather than a set, and initially interval $P_i$ extends from position $l_i$ to position $r_i$. Throughout, we use the words "net $i$" and "interval $i$" interchangeably. Assigning to $P_i$ lower positions that are between $l_i$ and $r_i$ does not increase the initial density. Of course, it will not always be possible to assign lower positions in this fashion. See Figure 1.1 for a solution to an MTP1-problem; the intervals are indicated by solid lines, the assignments by dashed lines.

In this paper we present a characterization of the optimal density $\hat{d}$ which is a generalization of the one introduced for 2-terminal nets in [AH]. This
characterization allows us to compute $\tilde{d}$ in $O(s+m)$ time. We present an algorithm that, given $\tilde{d}$, generates in $O((n+m)\log(n+m))$ time the assignment of lower positions to intervals that results in a CRP of density $\tilde{d}$. We also show that for a special case of the MTP1-problem, which includes 2-terminal nets, an optimal assignment can be produced in $O(n+m)$ time.

In the MTP2-problem the positions of both the entry and the exit terminals have not yet been determined, and every net $i$ consists of $u$ entry terminals and $k$ exit terminals, $s \geq un$, $m \geq kn$. The characterization of the optimal density used in the MTP1-problem for precomputing $\tilde{d}$ cannot be used in the MTP2-problem, but the freedom of selecting the entry terminal positions does allow us to generate nets of a special structure. We present an algorithm for the MTP2-problem that generates a CRP of optimal density $\tilde{d}$ in $O((s+m)\log\tilde{d})$ time.

This paper is organized as follows. In Section 2 we define a quantity $\Delta$, which we show to be a lower bound on the best achievable density of any solution to an MTP1-problem. We describe how to compute $\Delta$ in $O(s+m)$ time. Sections 3 and 4 describe an assignment algorithm, which, given $\Delta$, determines in $O((n+m)\log(n+m))$ time the $k$ lower positions to be assigned to every interval so that the resulting CRP has density $\Delta$. In section 5 we prove the correctness of our assignment algorithm, thus establishing that the optimal density $\tilde{d}$ equals $\Delta$. Section 6 contains the result about the MTP2-problem and the proof of NP-hardness of the MTP-problem.

2. Characterization of the Optimal Density for the MTP1-problem

As discussed in Section 1, we can view the input to an MTP1-problem as consisting of $m$ lower positions and $n$ intervals $P_i = (l_i, r_i)$, where $l_i$ is the left endpoint (which corresponds to the leftmost upper position of net $i$) and $r_i$ is the right endpoint (which corresponds to the rightmost upper position of net $i$), $l_i \leq r_i$, $1 \leq i \leq n$. Preprocessing the problem to put it in this form can be done in $O(s)$ time. We now have to assign to every interval $P_i$ $k$ lower positions (which are the positions for $P_i$'s $k$ exit terminals), such that the resulting multi-terminal CRP has minimum density (the density is defined as in Section 1).
We say that interval \( i \) crosses column \( x \) if and only if it contains the point at \( x \pm 0.5 \). Let \( c(x) \) be the number of intervals that cross column \( x \); i.e., the number of intervals \( P_i \) with \( l_i \leq x < r_i \). Let \( up(x, y) \) be the number of intervals with \( x \leq l_i \leq r_i \leq y \), and let \( down(x, y) \) be the number of lower positions that are \( \geq x \) and \( \leq y \). We next define a quantity \( \Delta \), which we later show to be equal to the optimal density.

**Definition 2.1**

\[
\Delta = \max \left\{ \max_x \left( c(x) - \frac{\text{down}(1,x)/k)}{h} \right), \right. \\
\left. \max_{x,y} \left( \left[ c(x) + c(y) + up(x+1,y) - \frac{\text{down}(x+1,y)/k)}{2} \right], \right) \right. \\
\left. \max_x \left( c(x) + up(x+1,M) - \frac{\text{down}(x+1,M)/k)}{} \right) \right. \\
\right. 
\]

Note that for \( x = y \) the middle term equals \( c(x) \).

**Lemma 2.2** \( \hat{d} \geq \Delta \).

**Proof:** We only go through the argument that \( \hat{d} \) is no smaller than the middle term in Definition 2.1 (the arguments for the other two terms are similar). The number of intervals between positions \( x \) and \( y \) that are crossing neither column \( x \) nor column \( y \) is \( up(x+1,y) \). At least \( up(x+1,y) - \frac{\text{down}(x+1,y)/k)}{2} \) intervals have to be assigned lower positions that are \( \leq x \) or \( > y \). Thus, at least \( c(x) + c(y) + up(x+1,y) - \frac{\text{down}(x+1,y)/k)}{} \) nets have to cross column \( x \) or column \( y \). Dividing this number equally between columns \( x \) and \( y \) is certainly a lower bound on \( \hat{d} \). \( \square \)

The next lemma states that \( \Delta \) can be computed efficiently.

**Lemma 2.3** If the intervals are given both sorted according the left endpoints and sorted according to the right endpoints, then \( \Delta \) can be computed in \( O(n + m) \) time.

(The proof is straightforward and is omitted.)

3. A Special Case

In this section we describe some of important concepts used in the algorithm for the MTP1-problem, whose complete description is given in the next section. We do so by considering a special version of the MTP1-problem, which is defined next.
Note that, throughout, we assume that the intervals \( P_i = (l_i, r_i) \) are given sorted according to their right endpoints; i.e., \( r_1 < r_2 < \cdots < r_n \), \( 1 \leq i \leq n \).

**Definition 3.1** Let \( P_i \) and \( P_j \) be two intervals. We say that \( P_i < P_j \) iff \( l_i < l_j \) and \( r_i < r_j \). We say that \( P_i \subseteq P_j \) iff \( l_j < l_i \leq r_i < r_j \) (i.e., interval \( P_j \) encloses interval \( P_i \)).

Note that if \( r_i < r_j \) then either \( P_i < P_j \) or \( P_i \subseteq P_j \).

The special case of the MTP1-problem which is considered in this section is when \( P_1 < P_2 < \cdots < P_n \). The algorithm we give determines an assignment achieving density \( \Delta \) in \( O(n+m) \) time. Note that this special case includes 2-terminal nets. The assignment algorithm for the general case of the MTP1-problem (i.e., when \( P_i \subseteq P_j \) is possible) runs in \( O((n+m)\log(n+m)) \) time, is considerably more complex, and is left till the next section.

A basic operation in our algorithm is the **left** (resp. **right**) **extension** of an interval. When a lower position \( q_j \) to the left of interval \( P_i \) is assigned to \( P_i \), position \( q_j \) becomes the new left endpoint of interval \( P_i \), and we say that the algorithm has performed a left-extension of \( P_i \) to position \( q_j \). Intuitively, it 'stretches' the interval so that it now starts at \( q_j \). The effect of such a left-extension is that the density in every column between and including column \( q_j \) and the interval's old left endpoint, is increased by 1. A **right-extension** is defined similarly. However, even though intervals may expand during the assignment algorithm, the functions \( e(x) \), \( u_p(x,y) \), and \( d(x,y) \) defined in Section 2 are defined on the initial endpoints; i.e., the endpoints before the assignment algorithm started. The relations \( < \) and \( \subseteq \) of Definition 3.1 are also relative to the initial situation. During the algorithm, the intervals that cross a column \( x \) are those that crossed it initially \( (e(x) \) of them) plus those that cross column \( x \) because they were extended.

**Definition 3.2** Let \( P_i \) and \( P_j \) be two intervals with \( P_i < P_j \). We say that the **ordered property** holds for \( P_i \) and \( P_j \) if for every lower position \( q_a \) assigned to \( P_i \) and for every lower position \( q_b \) assigned to \( P_j \), \( q_a < q_b \) holds.
Among all assignments achieving density \( \Delta \) there exists one in which the ordered property holds for every pair of intervals \( P_i \) and \( P_j \) with \( P_i < P_j \) (the proof of this depends on the fact that \( k_i = k_j = k \), and is left to the reader). When \( P_1 < \cdots < P_n \) (as is the case in this section), this implies that the \( k \) lower positions assigned to an interval \( P_i \) can be chosen so that they are consecutive (i.e., the \( k \) lower positions assigned to \( P_i \) are \( q_j, \cdots, q_{j+k-1} \) for some \( j \)). We call an interval that has \( k \) lower positions assigned to it a satisfied interval. The algorithm terminates when all \( n \) intervals become satisfied.

In the assignment algorithm we simultaneously scan, from left to right, the list of intervals and that of the lower positions. Recall that the intervals are sorted according to their right endpoints and that \( q_1 < \cdots < q_m \). Assume the scan is currently at interval \( P_{up} \) in the list of intervals and at \( q_{down} \) in the list of lower positions. We distinguish two cases:

(i) If \( q_{down} < l_{up} \) and density \( \Delta \) would be exceeded if \( P_{up} \) were left-extended to \( q_{down} \), the algorithm continues with \( P_{up} \) and \( q_{down}+1 \). It is not hard to show that, if density \( \Delta \) were exceeded, this would happen in column \( q_{down} \), and possibly other columns to the right of column \( q_{down} \) (the proof of this is omitted).

(ii) If \( q_{down} \geq l_{up} \), or if \( q_{down} < l_{up} \) and \( P_{up} \) can be left-extended to position \( q_{down} \), without causing the density to exceed \( \Delta \), then the algorithm assigns the \( k \) lower positions, \( q_{down}, \cdots, q_{down+k-1} \), to \( P_{up} \). It then sets \( up = up + 1 \) and \( down = down + k \), and continues with the new \( P_{up} \) and \( q_{down} \).

In (ii), assigning \( q_{down}, \cdots, q_{down+k-1} \) to \( P_{up} \) would not be possible if there are either not enough lower positions (i.e., \( down + k - 1 > m \)) or if, in the case that \( q_{down+k-1} > r_{up} \), a right-extension of \( P_{up} \) to \( q_{down+k-1} \) would result in density \( > \Delta \). As we will show in Section 5, none of these two failure possibilities can occur.

We next describe how to determine in \( O(1) \) time whether or not density \( \Delta \) would be exceeded if \( P_{up} \) were left-extended to \( q_{down} \). To do so, the algorithm uses a variable \( dens \) which contains the current density of column \( q_{down} \). Note that this density is made up by intervals that crossed column \( q_{down} \) initially, and by intervals that cross column \( q_{down} \) because they were left-extended. The algorithm also uses a
pointer that is positioned in the list of intervals at the interval $P_s$ with $r_s > q_{down}$ and $r_s -1 \leq q_{down}$.

If $dens = \Delta$, then no left-extension of $P_{up}$ to $q_{down}$ can be done, and the algorithm updates $dens$ before continuing with $q_{down+1}$ and $P_{up}$; i.e., as long as $r_s \leq q_{down+1}$, it sets $s = s + 1$ and $dens = dens - 1$.

If $dens < \Delta$ (i.e., the left-extension can be done), the algorithm sets $dens = dens + 1$ and an updating of variable $dens$ similar to the one described above computes the current density in column $q_{down+k-1}$.

Recall that we need a 'density check' only for left-extensions, since we show in section 5 that right-extensions are always safe. This concludes the description of our $O(n + m)$ time algorithm for the MTP1-problem when $P_1 < P_2 < \cdots < P_n$.

4. An Assignment Algorithm for the MTP1-Problem

We now return to the general case of the MTP1-problem in which intervals can enclose other intervals (i.e., $P_i \subset P_j$ is possible). This section describes an algorithm that generates an assignment of density $\Delta$ in $O((n + m) \log(n + m))$ time (the correctness of the algorithm is proved in Section 5).

As we pointed out in the previous section, there exists an optimal assignment in which the ordered property holds for every $P_i$ and $P_j$ with $P_i < P_j$. But if $P_i \subset P_j$, this is no longer true, since now it is possible that lower positions that are $< l_i$ or $> r_i$ cannot be assigned to $P_i$, while they can be assigned to $P_j$.

**Definition 4.1** Let $P_i$ and $P_j$ be two intervals with $P_i \subset P_j$. The mixed property holds for $P_i$ and $P_j$ if there exist two lower positions $q_\alpha$ and $q_\gamma$ assigned to $P_j$, and two lower positions $q_\beta$ and $q_\delta$ assigned to $P_i$, and $q_\alpha < q_\beta < q_\gamma < q_\delta$.

There always exists an optimal solution in which, for every pair of intervals, the mixed property does not hold. This can be seen by noting that, if the mixed property holds for $P_i$ and $P_j$, then we can always assign $q_\alpha$ and $q_\beta$ to $P_j$, and $q_\gamma$ and $q_\delta$ to $P_i$ without increasing the density. Doing this repeatedly gives an optimal solution of the desired type. Our algorithm produces a solution in which
- for every pair of intervals $P_i \subset P_j$, the mixed property does not hold, and
- for every pair of intervals with $P_i \subset P_j$, the ordered property holds.

Our algorithm consists of a procedure MAIN_SCAN, which simultaneously scans, from left to right, the list of the lower positions and that of the intervals (which, we recall, are given sorted according to their right endpoints). This algorithm differs from the one described in Section 3 in a number of ways:

1) When $P_i \subset P_j$, the scan reaches $P_i$ before it reaches $P_j$. Suppose the algorithm is unable to assign $q_i$ to $P_i$: Unlike Section 3, we cannot discard $q_i$ from further consideration, since at a later stage the scan may be able to assign it to $P_j$.

2) In the density check (i.e., deciding whether or not density $\Delta$ would be exceeded if interval $P_i$ were left-extended to $q_j$) the density $\Delta$ can now be exceeded in a column between $q_j$ and $l_i$ without being exceeded in column $q_j$.

3) When the algorithm of Section 3 assigns a lower position $q_i$ to an interval, the assignment is final. Under some circumstances, the algorithm of this section 'steals' lower positions that were already assigned to an interval $P_i$ from $P_j$ and assigns these lower positions to another interval $P_j$, $i < j$, in order to achieve the optimal density (the situation in which such stealing occurs is discussed later on).

We start by describing how to determine the lower positions that are $\leq r_i$ and that can be assigned to interval $P_i$ (but could not be assigned to intervals reached earlier in the scan). In order to do so, MAIN_SCAN calls a procedure TRY_LEFT. Intuitively, one purpose of TRY_LEFT is to scan the list of still available lower positions from left to right and to find the leftmost $q_j$ so that $P_i$ can be left-extended to $q_j$ without exceeding density $\Delta$. However, in the actual implementation, $q_j$ is determined without performing a scan (since this would be too time-consuming). We later show how position $q_j$ can be found in $O(\log(n+m))$ time. We next describe procedure TRY_LEFT in more detail.

Procedure TRY_LEFT

**Input:** interval $P_i$. 

- 9 -
Output: an integer \( \mu_i \) and, if \( \mu_i < k \), a lower position \( q_a \).

Effect: TRY_LEFT assigns to \( P_i \) as many lower positions \( \leq r_i \) as it can without causing the density to exceed \( \Delta \). The lower positions assigned to \( P_i \) are as far to the left of \( r_i \) as possible. Integer \( \mu_i \), which it returns, is the number of lower positions \( P_i \) still needs in order to be satisfied. The rightmost lower position assigned to \( P_i \) by TRY_LEFT is \( q_a \) (note that such a \( q_a \) exists only if \( \mu_i < k \)).

Method: In order to do the above, TRY_LEFT first determines the leftmost available lower position \( q_j \), \( q_j \leq r_i \), such that \( P_i \) can be left-extended to \( q_j \) without exceeding density \( \Delta \). Note that such a \( q_j \) may not exist, or that we may have \( q_j = l_i \). The integer \( \mu_i \) to be returned is determined as follows. If no such lower position \( q_j \) exists, set \( \mu_i = k \) and then return \( \mu_i \). Otherwise let \( k^* \) be the number of unassigned lower positions between (and including) \( q_j \) and \( r_i \). Assign the leftmost \( \min(k, k^*) \) of these lower positions to \( P_i \) and let \( q_a \) be the rightmost one of the lower positions assigned to \( P_i \). TRY_LEFT returns the value \( \mu_i = k - \min(k, k^*) \) and \( q_a \) to MAIN_SCAN. We will later describe how TRY_LEFT can be implemented to run in \( O(\log(n+m)+k^*) \) time.

This completes the description of TRY_LEFT.

We continue to outline algorithm MAIN_SCAN. Assume that MAIN_SCAN is currently at interval \( P_{up} \) in the list of intervals and at lower position \( q_{down} \) in the list of lower positions. No lower positions have yet been assigned to \( P_{up} \), and \( q_{down} \) is still available. During MAIN_SCAN, \( q_{down} \leq r_{up} \) will hold. Every interval whose right endpoint is to the left of \( q_{down} \) has either been satisfied, or, if such an interval still requires lower positions, then it has been put in a set \( S \). Set \( S \) is an auxiliary structure used by MAIN_SCAN; and the intervals in \( S \) will eventually be satisfied by being assigned lower positions that are \( \geq q_{down} \).

The strategy used by MAIN_SCAN is motivated by the following observation.

Let \( P_i \) be an interval in set \( S \) that still requires \( \mu_i \) lower positions. Assume, for example, that \( \mu_i < k \), \( r_{up} = q_{down} \) and \( P_i \subset P_{up} \). Since it is possible that, in the optimal
solution, interval $P_{up}$ uses lower positions that MAIN_SCAN was unable to assign to $P_l$, MAIN_SCAN calls TRY_LEFT($P_{up}$). Observe that if $P_{up}$ still requires some lower positions after TRY_LEFT (i.e., $\mu_{up}>0$), then the density is minimized and the mixed property avoided by letting $P_{up}$ 'steal' from $P_l$ some (or all) lower positions already assigned to $P_l$. After this lower position 'redistribution' $P_l$ requires more lower positions than before, while $P_{up}$ is closer to being satisfied (or is already satisfied).

We now give a more formal description of MAIN_SCAN. The entries in set $S$ are pairs $(P_i, \mu_i)$, where $P_i$ is an interval that still requires $\mu_i$ lower positions in order to be satisfied. Initially set $S$ contains all the intervals $P_i$ with $r_i < q_1$, and $\mu_i = k$. Pointer $up$ is initially positioned at the first interval in the list whose right endpoint is $= q_1$, while $down$ is initially equal to 1. Thus, the condition $q_{down} \leq r_{up}$ is satisfied. Assume that, when MAIN_SCAN is at $P_{up}$ and $q_{down}$, $S = \{(P_{i_1}, \mu_{i_1}), (P_{i_2}, \mu_{i_2}), \ldots, (P_{i_h}, \mu_{i_h})\}$, where $P_{i_1}, \ldots, P_{i_h}$ are intervals whose right endpoints are to the left of $q_{down}$ (and thus to the left of $r_{up}$). The intervals $P_{i_1}, \ldots, P_{i_h}$ still require (respectively) $\mu_{i_1}, \ldots, \mu_{i_h}$ lower positions. $P_{i_1}$ is always the interval of $S$ with the smallest left endpoint among all intervals in $S$. During the algorithm the following condition holds: Only $P_{i_1}$ can require fewer than $k$ lower positions, all the other intervals in $S$ require exactly $k$ lower positions; i.e., $\mu_{i_1} = k$ and $\mu_{i_2} = \cdots = \mu_{i_h} = k$. Note that this condition is satisfied after the initialization of $S$.

The purpose of the above condition is to ensure that we get an assignment in which for every column $x$ at most one interval $P_i$ with $r_i \leq x$ is assigned both lower positions that are $> x$, and lower positions that are $\leq x$. We will show in Section 5 that among all the assignments achieving density $\Delta$ there is one with this property.

When MAIN_SCAN is positioned at $P_{up}$ and $q_{down}$ it first decides what to do with lower position $q_{down}$, and it distinguishes 2 cases:

A1) Set $S$ is empty: $P_{up}$ requires exactly $k$ lower positions, and MAIN_SCAN calls TRY_LEFT($P_{up}$). If it returns $\mu_{up}$ and $q_{\alpha}$ with $q_{\alpha} > q_{down}$, then set $down = q_{\alpha}$ (since all the lower positions between and including $q_{down}$ and $q_{\alpha}$ have been
A2) If S is not empty: MAIN SCAN assigns $q_{down}$ to $P_{1'}$. Recall that $r_{1'} < q_{down}$ holds. (In our proof we will show that this assignment never causes the density to exceed $\Delta$.) MAIN SCAN then sets $\mu_{1'}$ to $\mu_{1'} - 1$. If $\mu_{1'}$ is now zero (i.e., $P_{1'}$ is now satisfied), it removes $P_{1'}$ from S and chooses the interval with the smallest left endpoint currently in S as the new $P_{1'}$ (choosing the new $P_{1'}$ in this way helps insure that the generated solution satisfies the ordered property).

TRY_LEFT does not alter the position of pointer $up$, but it possibly advances the position of pointer $down$ to position 0 in case $A1$. After having executed either case $A1$ or case $A2$ described above, MAIN SCAN checks whether there are any right endpoints of intervals between $q_{down}$ and $q_{down+1}$. Assume there are $I$ intervals, namely $P_{up}, P_{up+1}, \ldots, P_{up+i-1}$, whose right endpoints fall between $q_{down}$ and $q_{down+1}$; i.e., $q_{down} \leq r_{up+i} < q_{down+1}$ for $i = 0, 1, \ldots, I - 1$. (None of these $P_{1}$'s is in $S$.) Each one of these intervals is considered in turn, and for every $P_{up+i}, i = 0, 1, \ldots, I - 1$, MAIN SCAN distinguishes 3 cases.

B1) The set $S$ is empty: Call TRY_LEFT($P_{up+i}$) and, if the returned value $\mu_{up+i}$ is $> 0$, add $(P_{up+i}, \mu_{up+i})$ to $S$.

B2) Set $S$ contains an interval $P_{j}$ with $P_{j} < P_{up+i}$: Add $(P_{up+i}, k)$ to $S$. Since $P_{j}$ is not yet satisfied, TRY_LEFT($P_{up+i}$) cannot assign any lower positions to $P_{up+i}$ (otherwise $P_{j}$ would already have taken them).

B3) Set $S$ is not empty, and $P_{j} \subseteq P_{up+i}$ for every interval $P_{j} \in S$: Call TRY_LEFT($P_{up+i}$), and if it returns $\mu_{up+i} > 0$ then do the following. If $\mu_{j} = k$, no 'stealing' of lower positions can occur, and the algorithm adds $(P_{up+i}, \mu_{up+i})$ to $S$. Otherwise, $P_{j}$ has been assigned lower positions that are $\leq q_{down}$ and will be assigned at least one which is $\geq q_{down}$. In order to guarantee that the mixed property does not occur, $P_{up+i}$ 'steals' from $P_{j}$ min$(\mu_{up+i}, k - \mu_{j})$ lower positions assigned to $P_{j}$. This causes $\mu_{j}$ to increase to $\mu_{j} + min(\mu_{up+i}, k - \mu_{j})$. If, after assigning these lower positions to $P_{up+i}, \mu_{up+i} > 0$ (in which case $\mu_{j} = k$), then add
(P_{up+i},P_{up+i+1}) to S as new first element. Again, only one element of S, namely (P_i,\mu_i), has \mu_i < k; i.e., has already been assigned some lower positions.

After the intervals P_{up}, P_{up+1}, \ldots, P_{up+t-1} have been processed as described above, P_{up+t} is the interval with the smallest right endpoint that is \geq q_{down+1}. Set up = up + t, down = down + 1, and continue MAIN_SCAN with the new P_{up} and q_{down}.

To illustrate how the algorithm works, we describe how it produces the solution shown in Figure 1.1. TRY_LEFT assigns q_1 and q_2 to P_1 and q_3 and q_4 to P_2. It then fails to assign q_5 and q_6 to P_3, assigns q_7 to P_3, and puts (P_3,1) into S. When reaching r_4, q_6 is assigned to P_4 in the TRY_LEFT(P_4), and a stealing of lower positions occurs: q_7 is now assigned to P_4 and S = \{(P_3,2)\}. Then, q_8 and q_9 are assigned to P_3. At q_9, TRY_LEFT assigns q_5 to P_5. Finally, q_{10} is assigned to P_5, and q_{11} and q_{12} to P_6.

We now outline how the assignment algorithm described above can be implemented to run in \(O((n+m)\log(n+m))\) time. Recall that the input consists of the intervals sorted according to the right endpoints and the list of lower positions, in sorted order. One of the auxiliary data structures used during the algorithm is a 2-3 tree [AHU] T whose leaves contain, in sorted order, the endpoints of the intervals and the available lower positions. Thus, initially, T contain 2n+m leaves. The leaves corresponding to lower positions are joined together by a doubly linked list. The nodes of T have, besides the standard 2-3 tree entries, additional entries that allow us to perform in \(O(\log(n+m))\) time each one of the following operations.

(i) Determine the leftmost available \(q_j\) so that \(P_i\) can be left-extended to \(q_j\), \(q_j \leq r_i\) without exceeding density \(A\). Recall that this operation is needed in TRY_LEFT(P_i).

(ii) Add/delete the interval (x,y) to/from the tree.

(iii) Delete lower position \(q_j\) from the tree.

The first operation is implemented by simulating the effect on the density when the interval (1,l_i) is inserted into T. If the density increases to \(A+1\), we find the rightmost column \(y\) in which the density would be \(A+1\), and let \(q_j\) be the leftmost available lower position at a position \(\geq y+1\). If no such column \(y\) exists (i.e., \(P_i\) can
be left-extended to position 1 without exceeding density \( \Delta \), \( q_j \) is the leftmost available lower position currently in the list. It should be clear that, in a 2-3 tree in which every interior node \( v \) contains additional entries to record information about the density created by the intervals with both endpoints in \( v \)'s subtree, this operation can be implemented to run in \( O(\log(n+m)) \) time. When performing the second operation we need to update entries in the interior nodes of the tree to record the changes in the density caused by the added/deleted interval. Note that when an interval \( P_i \) is left-extended (resp. right-extended) to position \( x \), we delete the interval \((l_i, r_i)\) and add the interval \((x, r_i)\) (resp. \((l_i, x)\) in order to maintain a tree of height \( O(\log(n+m)) \). The details of the implementation of all three operations are straightforward and are left to the reader.

The data structure used to implement set \( S \) is a heap in which the interval with the smallest left endpoint currently in \( S \), which is \( P_{il} \), is the top element of the heap. Thus, the query 'Is there an interval \( P_j \in S \) with \( P_j < P_{up+i} \)', which is needed in case B2, is answered in constant time by inspecting the top of the heap. Note that if the above query is false and \( S \) is not empty, then \( P_j \subset P_{up+i} \) holds for every interval \( P_j \in S \). Insertion and deletions of intervals are done in \( O(\log n) \) time in a heap, and every interval is inserted and deleted at most once. Thus, the \( O((n+m)\log(n+m)) \) time bound of the assignment algorithm follows. In the next section we prove that this algorithm always generates an assignment of density \( \Delta \).

5. The Assignment Algorithm Achieves Density \( \Delta \)

In this section we show that the assignment algorithm for the MTPI-problem described in Section 4 always generates a solution of density \( \Delta \). We start off with two Lemmas that state properties of the solution generated by the algorithm and the following definition.

**Definition 5.1** Let \( P_i = (l_i, r_i) \) be the initial interval. Then \( P_i \) is to the left of a position \( x \) if \( x \leq r_i \), and it is to the right of position \( x \) if \( x < l_i \). A lower position \( q_j \) is to the left of \( x \) if \( x \leq q_j \), and it is to the right of \( x \) if \( x < q_j \).
Lemma 5.2 For every column \( x \), at most one interval to the left of \( x \) is assigned by the algorithm lower positions both left and right of \( x \).

**Proof:** Suppose to the contrary that \( P_i \) and \( P_j \) are to the left of \( x \), that \( P_i \) is assigned \( q_a \) and \( q_b \), \( q_a \leq x < q_b \), and that \( P_j \) (\( j \neq i \)) is assigned \( q_\gamma \) and \( q_\delta \), \( q_\gamma \leq x < q_\delta \). W.l.o.g. we assume that \( l_i > l_j \). Assume first that \( P_i < P_j \). Then, \( q_a \leq x < q_b \) and \( q_\gamma \leq x < q_\delta \) contradicts the fact that the algorithm maintains the ordered property. (To put it differently, it cannot happen because the algorithm would have assigned \( q_\gamma \) to \( P_i \).

We now consider the case when \( P_j \subset P_i \). If \( q_\gamma \leq r_i \), then having \( q_\gamma \) being assigned to \( P_j \) would have caused a stealing of \( q_\gamma \) by \( P_i \) (at B3 in MAIN_SCAN), which contradicts the fact that \( q_\gamma \) is now assigned to \( P_j \). If \( q_\gamma > r_i \), then the algorithm would assign \( k \) lower positions to \( P_i \) before it assigns any to \( P_j \), and thus \( q_\gamma \) would not be assigned to \( P_j \). \( \Box \)

The following Lemma gives another property of the solution produced by the assignment algorithm. Although it is not needed in the proof, it is stated for the sake of completeness.

**Lemma 5.3** For every column \( x \), at most one interval to the right of \( x \) is assigned by the algorithm lower positions both left and right of \( x \).

**Proof:** Suppose to the contrary that \( P_i \) and \( P_j \) are to the right of \( x \), that \( P_i \) is assigned \( q_a \) and \( q_b \), \( q_a \leq x < q_b \), while \( P_j \) (\( j \neq i \)) is assigned \( q_\gamma \) and \( q_\delta \), \( q_\gamma \leq x < q_\delta \). W.l.o.g. we assume that \( l_i < l_j \). We cannot have \( P_i < P_j \) because it would contradict the fact that the algorithm maintains the ordered property. So suppose that \( P_j \subset P_i \). The algorithm would initially assign \( q_a \) and \( q_\gamma \) to \( P_i \). In order to have \( q_a \) assigned to \( P_i \) at least one stealing process must have occurred. But then the stealing process (at B3 in MAIN_SCAN) would also have taken \( q_\gamma \) away from \( P_j \) and given it to \( P_i \), a contradiction. \( \Box \)

**Lemma 5.4** Algorithm MAIN_SCAN succeeds in satisfying all intervals without exceeding density \( \Delta \).
Proof: We first show that when MAIN_SCAN assigns to interval \( P_i \) a lower position \( q_j \) which is to the right of \( P_i \) (i.e., \( q_j > r_i \)), density \( \Delta \) is never exceeded. Suppose to the contrary that it is, and consider the first time during the algorithm it happens. Then, in some column \( x \) between \( r_i \) and \( q_j \) the density was already \( \Delta \) before we assigned \( q_j \) to \( P_i \) (if there is more than one such column \( x \) then choose the leftmost one). We distinguish two cases: (In either case, we are looking at the situation just before \( q_j \) was assigned to \( P_i \).)

Case A: There are no available lower positions to the left of \( x \). Let \( \Gamma \) (resp. \( \Theta \)) be the set of satisfied intervals that are to the left of \( x \) and that were not assigned any lower position to the right (resp. left) of \( x \). Intervals that cross column \( x \) have not yet been assigned any lower positions (they have not even been considered yet). By Lemma 5.2 at most one interval to the left of \( x \) can be assigned lower positions both left and right of \( x \). If there is such an interval \( P_i \) then it is satisfied, and \( P_i \) has not been assigned any lower position to the left of \( x \) (because of Lemma 5.2). Therefore \( down(1,x) \leq k \mid \Gamma \mid + k - 1 \), where the term \( k - 1 \) is due to the fact that \( P_i \) may have been assigned up to \( k - 1 \) lower positions to the left of \( x \). Moreover, we then have \( \Delta = c(x) + |\Theta| + 1 \), and \( up(1,x) \geq |\Gamma| + |\Theta| + 2 \). If no such \( P_i \) exists (i.e. no interval to the left of \( x \) has yet been assigned lower positions both left and right of \( x \)) then \( P_i \) may have been assigned up to \( k - 1 \) lower positions to the left of \( x \) and therefore we again have \( down(1,x) \leq k \mid \Gamma \mid + k - 1 \), while \( \Delta = c(x) + |\Theta| + 1 \) and \( up(1,x) \geq |\Gamma| + |\Theta| + 1 \). In either of the above two sub-cases it is easy to verify that

\[
c(x) + up(1,x) - \lfloor down(1,x)/k \rfloor \geq \Delta + 1,
\]

which contradicts the definition of \( \Delta \).

Case B: There are available lower positions left of \( x \). Let \( q_l \) be the rightmost one of them. Obviously \( q_l \) cannot be below \( P_i \) since otherwise it would have been assigned to \( P_i \) at an earlier stage of the algorithm. Also note that \( q_l \) cannot be between \( r_i \) and \( x \) since otherwise there is a column between \( r_i \) and \( x \) where the density is already \( \Delta \), contradicting the fact that \( x \) is the leftmost such column to the right of \( r_i \). Therefore, \( q_l < l_i \) must hold. Since \( q_l \) was not assigned to \( P_i \) (at an earlier stage of the algorithm), there exists a column \( y \) between \( q_l \) and \( l_i \) with density \( \Delta \).
We claim that lower positions to the right of \( y \) could not have been assigned to intervals whose left endpoint is to the left of \( y \). The proof of this claim follows. First, we observe that no interval crossing column \( y \) has been assigned a lower position to the right of \( y \) (actually, to the right of \( q_f \)), because that would imply that a column \( \mathcal{F} \) between \( q_l \) and \( y \) has density \( \Delta \), which contradicts our choice of \( y \) as leftmost. Next, we show that no interval to the left of \( y \) has been assigned a lower position to the right of \( y \). To prove this, assume to the contrary that interval \( P_r \) is left of \( y \) and has been assigned a lower position to the right of \( y \). \( P_r \) cannot be to the left of \( q_f \) because in that case \( q_f \) would have been assigned to it, a contradiction. For the same reason, \( P_r \) cannot cross column \( q_f \). \( P_r \) cannot be between \( q_f \) and \( l \), either, since that would contradict our choice of \( y \) as leftmost (by a now familiar argument). Therefore no such \( P_r \) exists, i.e., no interval left of \( y \) has been assigned a lower position to the right of \( y \). This completes the proof of the claim.

An additional consequence of our choice of \( y \) as leftmost is that no interval between \( y \) and \( x \) was assigned any lower positions left of \( y \) (the argument proving this should now be familiar to the reader and is omitted). These observations imply that every lower position which is \( > y \) and \( \leq x \) (and which is not available) has been assigned to an interval which is either

(i) a satisfied interval whose \( k \) lower positions are \( > y \) and \( \leq x \), or
(ii) a satisfied interval \( P_i \) which has been assigned lower positions both to the right and left of \( x \), or
(iii) interval \( P_i \) itself.

Let there be \( \Gamma \) intervals as described in case (i). Because of Lemma 5.2 there can only be one interval \( P_i \) as described in case (ii), and cases (ii) and (iii) are mutually exclusive. If case (ii) holds then \( P_i \) has not been assigned any position between \( y \) and \( x \), whereas if case (ii) does not hold then \( P_i \) may have been assigned up to \( k - 1 \) lower positions that are \( > y \) and \( \leq x \). In either situation we have \( \text{down}(y + 1, x) \leq k |\Gamma| + k - 1 \). If we let \( A \) (resp. \( B \)) be the set of intervals between \( y \) and \( x \) that have been assigned all of their lower positions from the left of \( y \) (resp.}
right of $x$) then we have the following:

$$\Delta = c(y) + |A|,$$
$$\Delta = c(x) + |B| + 1 \quad \text{if case (ii) holds,}$$
$$\Delta = c(x) + |B| \quad \text{if case (iii) holds,}$$

$$up(y, x) = |A| + |B| + 2 \quad \text{if case (ii) holds,}$$
$$up(y, x) = |A| + |B| + 1 \quad \text{if case (iii) holds.}$$

In either case (ii) or (iii) the following holds:

$$c(y) + c(x) + up(y + 1, x) - \lfloor down(y + 1, x)/k \rfloor \geq 2\Delta + 1,$$

which contradicts the definition of $\Delta$. This completes the proof that whenever MAIN_SCAN joins an interval to an lower position to its right then this does not cause the density to exceed $\Delta$.

This in itself does not guarantee the success of MAIN_SCAN since it does not rule out the possibility that MAIN_SCAN may run out of lower positions. We now show that this cannot happen. Assume to the contrary that MAIN_SCAN runs out of lower positions (i.e. $down$ becomes $m + 1$) at a time when $c, 0 \leq c < k$, lower positions have been assigned to $P_i$. When this happens, all the remaining available lower positions are to the left of $P_i$. Let $q_j$ be the rightmost one of them (i.e. the rightmost lower position still available). Let $x$ be the leftmost column in which density $\Delta$ is exceeded when $P_i$ is left-extended to $q_j$.

First, we claim that no interval whose left endpoint is to the left of $x$ was assigned any lower position to the right of $x$. Suppose to the contrary that $P_i, l_i < x$, was assigned a lower position to the right of $x$. Then we must have $l_i > q_j$ since otherwise $P_i$ would have been assigned $q_j$, which is not the case. This in turn implies that there is a column $x$ between $q_j$ and $x$ whose density is already $\Delta$. This contradicts the fact that $x$ is leftmost. This proves the claim that only intervals to the right of $x$ were assigned lower positions to the right of $x$.

Our next claim is that no interval to the right of $x$ has been assigned lower positions both left and right of $x$. To show this, simply note that the existence of such an interval would contradict the fact that $x$ is leftmost.
Let \( A \) be the set of satisfied intervals to the right of \( x \) that were assigned all of their lower positions from the left of \( x \). Note that \( c(x) + |A| = \Delta \). Let \( B \) be the set of satisfied intervals that are to the right of \( x \) and that were assigned all \( k \) of their lower positions to the right of \( x \). We then have

\[
c(x) + \text{up}(x+1,M) - \left\lfloor \text{down}(x+1,M)/k \right\rfloor \leq \\
c(x) + |A| + |B| + 1 - \left\lfloor (k|B|+k-1)/k \right\rfloor = \Delta + 1,
\]

a contradiction. This completes the proof of Lemma 5.4. \( \square \)

**Corollary 5.5** \( \hat{\Delta} = \Delta \).

**Proof:** Lemma 5.4 implies that \( \hat{\Delta} \leq \Delta \). This and Lemma 2.2 together imply that \( \hat{\Delta} = \Delta \). \( \square \)

We are now ready to state the main result about the MTP1-problem:

**Theorem 5.6** The placement achieving the optimal density of an MTP1-problem can be computed in \( O((n+m)\log(n+m)) \) time.

**6. Other Variants and NP-Hardness of MTP**

In this section we describe other multi-terminal placement problems that can be solved efficiently, and we also show that the most general version of the problem is NP-hard. We first consider the MTP2-problem, in which the \( i \)-th net must be assigned \( u \) of the \( s \) available upper positions and \( k \) of the \( m \) available lower positions, \( 1 \leq i \leq n \), so that the resulting CRP has minimum density. We describe a verification algorithm that, for a given integer \( \hat{d} \), decides in \( O(s+m) \) time whether or not a solution of density \( \leq \hat{d} \) exists. A solution achieving the optimal density \( \hat{d} \) is then obtained using binary search in time \( O((s+m)\log \hat{d}) \) (by \( \log \hat{d} \) applications of the verification algorithm).

We start by making the following observation about the MTP2-problem. If a solution of density \( \hat{d} \) exists, then there always exists one in which for every \( i < j \), the rightmost upper position assigned to net \( i \) is to the left of the leftmost upper position of net \( j \). Thus, the intervals formed by the entry terminals alone have density 1. This allows us to choose the upper positions assigned to net \( i \) so that they are con-
secutive. The verification algorithm for the MTP2-problem is then similar to the algorithm given in Section 3. Assume that the scan is currently at upper position \( p_{\text{up}} \) and at lower position \( q_{\text{down}} \), both terminals have not yet been assigned to a net, and nets 1, \( \ldots \), \( i-1 \) have already been satisfied.

- If \( q_{\text{down}} < p_{\text{up}} \), and density \( d \) would be exceeded when net \( i \) contains \( p_{\text{up}} \) and \( q_{\text{down}} \), continue with \( q_{\text{down}}+1 \) and \( p_{\text{up}} \).
- If \( q_{\text{down}} > p_{\text{up}} \), and density \( d \) would be exceeded when net \( i \) contains \( p_{\text{up}} \) and \( q_{\text{down}} \), continue with \( q_{\text{down}} \) and \( p_{\text{up}}+1 \).
- If neither of the above two cases holds, assign the next \( u \) upper and the next \( k \) lower positions to net \( i \), and continue with \( p_{\text{up}}+u \), \( q_{\text{down}}+k \), and net \( i+1 \).

The implementation details and the proof of correctness are left to the reader.

**Theorem 6.1** The MTP2-problem can be solved in \( O((s+m)\log d) \) time.

A result similar to the above Theorem holds when the \( u_j \)'s are arbitrary and we know in what order the upper row positions are assigned to the nets (i.e., we know to which net the \( i \)-th assigned upper row position is given). Here too, we leave the details to the interested reader.

Recall that the MTP-problem is the one where both the \( u_j \)'s and \( k_j \)'s are arbitrary, and we are free to assign both upper and lower row positions in order to minimize the density.

**Theorem 6.2** The MTP-problem is NP-hard.

**Proof:** The proof is by a transformation from 3-Partition, which is NP-hard in the strong sense [GJ] and which is defined as follows. Let \( A = \{a_1, a_2, \ldots, a_{3q}\} \) with \( \sum_{i=1}^{3q} a_i = qB \) and \( B/4 < a_i < B/2 \). This instance of 3-Partition has a solution iff set \( A \) can be partitioned into \( q \) disjoint subsets \( A_1, \ldots, A_q \) so that \( \sum_{a_i \in A_i} a_i = B \), \( 1 \leq i \leq q \).

Given set \( A \) and the bound \( B \), we construct a corresponding MTP-problem consisting of \( 4q-1 \) nets, as follows. For \( 1 \leq i \leq 3q \), net \( i \) requires \( a_i \) upper positions and no lower positions (i.e., \( u_i = a_i \) and \( k_i = 0 \)). For \( 3q+1 \leq i \leq 4q-1 \), net \( i \) requires no upper positions and \( B \) lower positions. The channel contains \( qB \) upper and \( (q-1)B \) lower
positions. Each one of columns \(2iB + 1, 2iB + 2, \ldots, 2iB + B\), \(0 \leq i \leq q - 1\), contains an upper position and each one of columns \((2i + 1)B + 1, (2i + 1)B + 2, \ldots, (2i + 1)B + B\), \(0 \leq i \leq q - 2\), contains a lower position. It is not hard to see that 3-Partition has a solution if and only if the corresponding MTP-problem has a solution of density 1.

References


\[
\begin{array}{cccccccc}
\ell_1 & \ell_2 & \ell_3 & \ell_4 & \ell_5 & \ell_6 \\
\end{array}
\]

\[
\begin{array}{cccccc}
r_1 & r_2 & r_3 & r_4 & r_5 & r_6 \\
\end{array}
\]

An optimal solution to an MTP1-problem with \(k=2\)

Figure 1.1