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ON LINEAR-TIME ALGORITHMS
FOR 5-COLORING PLANAR GRAPHS*

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Abstract. Certain properties of planar graphs are established in a particularly straightforward fashion. These properties assure good performance in two linear-time algorithms for five-coloring planar graphs. A new linear-time algorithm, based on a third property, is also presented.

Keywords. graph coloring, Euler's formula, planar graph.

In [CNS], it is shown that in every planar graph of minimum degree five, if S is a subset of vertices such that every vertex of degree five is adjacent to at least two vertices in S, and every vertex of degree six is adjacent to at least one vertex in S, then S contains at least $\frac{1}{12}$ of all vertices. During a stage in their algorithm, a vertex of the following kind will be identified and participate in a contraction. The vertex may either be a vertex of degree five that is adjacent to at most one vertex that has resulted from a contraction in this stage, or a vertex of degree six that is adjacent to no vertex that has resulted from a contraction in this stage. The contractions will continue until no more vertices of the type described can be found. If appropriate data structures are employed, the total work in a stage will be linear. By the desired property, the number of vertices will have been reduced by at least a fixed percentage. Hence the linearity of the complete algorithm will follow.

Coincidentally, the proof of each of the key properties can be improved substantially. The proof of the property in [MST] is rather long and involved, considering among other features the planar embedding of the graph. We present a rather short and essentially algebraic proof of this property. The proof of the key property in [CNS] is reasonably direct, but the analysis is not very tight. We show that the specified set S will have greater than $\frac{1}{7}$ of all the vertices. Again the proof is simple and algebraic.

In addition, we devise a new linear-time algorithm to five-color planar graphs. Our algorithm contracts only vertices of degree 5, as does [MST], and it maintains simpler data structures. The key property for this method is that in every planar graph of minimum degree five, there is a vertex of degree five that has two neighbors that are mutually adjacent and of degree no greater than 7.

2. A key property for sequential contraction

We first consider the key property used in the algorithm of Matula, *et. al.* We first introduce some notation and some well-known identities. Let $G = (V,E)$ be a planar graph with minimum degree five. Let n be the number of vertices in G, and e the number of edges in G. Let n_i be the number of vertices of degree i in G. It follows that

$$\sum_i n_i = n \tag{1}$$

$$\sum_i i\, n_i = 2e \tag{2}$$

From Euler's formula (See [BM]), there is the well-known result for planar graphs:

$$e \le 3n - 6 \tag{3}$$

Theorem 1. Let G be a connected planar graph of minimum degree five. There is a vertex of degree five that is adjacent to at most one vertex of degree greater than eleven.

Proof. Combining equations (2) and (3), we have

$$\sum_{i \ge 5} i\, n_i \le 6n - 12$$

Subtracting six times equation (1), we get

$$-n_5 + \sum_{i>6} (i-6) n_i \le -12$$

Thus

$$\begin{aligned} n_5 &\ge 12 + \sum_{i>6} (i-6) n_i \qquad (4) \\ &> \sum_{i>11} \tfrac{i}{2} n_i \end{aligned}$$

Hence

$$2\, n_5 > \sum_{i>11} i\, n_i$$

The latter inequality means that not every vertex of degree 5 is be adjacent to at least two vertices of degree greater than 11. ▪

We have not been able to generate a planar graph in which every vertex of degree five is adjacent to two or more vertices of degree at least eleven. However, we can establish that the theorem is reasonably close to being tight. We shall produce a planar graph in which every vertex of degree five is adjacent to a vertex of degree twelve and a vertex of degree ten or twelve. The graph may be generated as follows. Start with a planar graph in which every vertex is the juncture of three faces, two hexagonal and three pentagonal. This is the graph representing a truncated icosahedron, and a portion is shown in Figure 1a. Replace each vertex by a triangular face, as shown in Figure 1b. Now add a vertex in the middle of each non-triangular face, connecting it to all other vertices in the face. The final graph, a portion of which is shown in Figure 1c, will have twenty vertices of degree twelve, twelve vertices of degree ten, and 180 vertices of degree five. Every vertex of degree five will be adjacent to a vertex of degree twelve and a vertex of degree ten or twelve.

3. A key property for batch contraction

We now consider the key property used in the analysis of the algorithm of Chiba, *et.al.* We show that the analysis can be tightened.

Theorem 2. Let $G = (V,E)$ be a planar graph with minimum degree 5, and let S be a subset of V. If every vertex of degree 5 is adjacent to at least two vertices in S, and every vertex of degree 6 is adjacent to at least one vertex in S, then

$$|S| \geq \frac{1}{7}(n+12).$$

Proof. Let V_* be the vertices of degree greater than six, and n_* the number of such vertices. Let r_5 be the number of vertices of degree five in S, r_6 the number of vertices of degree six in S, and r_* the number of remaining vertices in S. From the given conditions we have

$$\sum_{v \in S} d(v) \ \geq \ 2n_5 + n_6 \tag{4}$$

where $d(v)$ is the degree of vertex v. Note that

$$\sum_{v \in S} d(v) \ = \ 5r_5 + 6r_6 + \sum_{v \in S_*} d(v) \tag{5}$$

Rewriting (1), we have

$$n_5 + n_6 + n_* \ = \ n$$

and from equations (2) and (3) we get

$$5n_5 + 6n_6 + \sum_{v \in V_*} d(v) \ \leq \ 6n - 12$$

Subtracting seven times the former from the latter, we get

$$\begin{aligned} 2n_5 + n_6 &\geq n + 12 + \sum_{v \in V_*} (d(v) - 7) \\ &\geq n + 12 + \sum_{v \in S_*} (d(v) - 7) \\ &= n + 12 + \sum_{v \in S_*} d(v) - 7r_* \end{aligned}$$

Combining (4) and (5) with the latter yields

$$5r_5 + 6r_6 + \sum_{v \in S_*} d(v) \ \geq \ n + 12 + \sum_{v \in S_*} d(v) - 7r_*$$

Thus

$$5r_5 + 6r_6 + 7r_* \ \geq \ n + 12$$

from which the claimed result follows. ▫

The result of the theorem is tight, as we shall demonstrate. First, consider the graph constructed in the previous section. Suppose that S consists of all vertices other than those of degree five. It is easily seen that S satisfies the adjacency requirements of the theorem. The cardinality of S is 32, while the cardinality of V can be seen to be 212. The formula in the theorem then holds with equality.

Equality is also realized for other planar graphs as well, and we show how to generate many such graphs. Take any planar graph with minimum degree four, a portion of an example of which is shown in Figure 2a. Replace each vertex v with $d(v)$ vertices as shown in Figure 2b. Now replace each vertex by a triangular face, as shown in Figure 2c. As before, place a vertex in each non-triangular face, and connect it to all other vertices in the face, as shown in Figure 2d. Let f be the number of faces in the original graph. The first transformation will generate a graph with $2e$ vertices and $f + n = e + 2$ faces, all non-triangular. The second will generate a graph with $6e$ vertices of degree three, $2e$ triangular faces, and $e + 2$ faces bounded by more than six edges. The final transformation will yield $e + 2$ vertices of degree greater than six, and $6e$ vertices of degree five. If S is chosen as all vertices of degree greater than five, the formula will hold with equality.

Using Theorem 2, it is not hard to show that the fraction of vertices removed during a stage of the algorithm of [CNS] is at least $\frac{3}{17}$, which is larger than the fraction $\frac{1}{9}$ given in the paper.

4. Another sequential method

In this section we present another sequential method for five-coloring a graph in linear time. As in the algorithm of [MST], only vertices of degree 5 are involved in contractions. In addition, only one ready list for vertices of degree 5 need be kept. The algorithm, and the key property on which it is based, were discovered after the author had read an unpublished manuscript by Williams [W]. Although our algorithm differs substantially from that of Williams, it drew some inspiration from his, and we wish to acknowledge this gratefully.

Our approach is motivated by the following observation. In the near-worst case example of Figure 1c, every vertex of degree 5 has two neighbors of degree 5 that are mutually nonadjacent. The result of Theorem 1 guarantees only that there is some vertex of degree 5 which has two neighbors of degree less than 12 that are mutually nonadjacent. This suggests that Theorem 1 establishes a condition that is more than sufficient to find a vertex of degree 5 that can be contracted. We establish a tighter characterization in the following theorem.

Theorem 3. Let G be a planar graph of minimum degree 5. There is a vertex of degree 5 that has two neighbors that are mutually nonadjacent and of degree no greater than 7.

Proof. Consider graph G', which is G augmented by additional edges to make it maximal planar. A desired vertex in G' will also be a desired vertex in G. We first examine the case in which G' is 4-connected. Consider a vertex v of degree 5, with neighbors $w_0, \cdots, w_4$, listed in cyclic order around v in some planar embedding of G'. Any pair w_i and $w_{(i+2) mod 5}$ cannot be adjacent, since otherwise those two vertices plus v would be a separation triple contradicting the claim that G' is 4-connected. If v is not a vertex that satisfies the theorem, then there must be a cyclic indexing of the w_j, such that w_0, w_1 and w_2 are of degree greater than 7. Vertex v claims a total degree of 5 from the set of vertices of degree greater than 7, since edges (v,w_0), (v,w_1) and (v,w_2) are used directly, and at worst (w_0,w_1) and (w_1,w_2) are each "shared" with at most one other vertex of degree 5. Thus if the theorem is not true then

$$5n_5 \le \sum_{i>7} i\, n_i$$

However, from (4) we have

$$n_5 \ge 12 + \sum_{i>6} (i-6) n_i$$

$$> \sum_{i>7} \frac{i}{4} n_i$$

This is at variance with the above derived inequality. Thus the theorem follows for 4-connected graphs.

Suppose that G' is not 4-connected. Consider a vertex v of degree 5. If v is not a desired vertex, then additional cases are possible. In the first case there is a cyclic indexing of the w_j such that w_2 and w_4 are adjacent, and w_0, w_1 and w_3 are of degree greater than 7. In the second case, edges (w_0,w_2) and (w_2,w_4) are present, and w_1 and w_3 are of degree greater than 7. Consider an embedding of G' in the plane. Let v be a vertex, such that no similar vertex is embedded in one of the regions bounded by (v,w_2), (v,w_4) and (w_2,w_4). Such a region surely exists. We consider the subgraph in this region, including the vertices v, w_2 and w_4. All vertices in the subgraph are of degree at least 5, except for v, w_2 and w_4, which are each of degree at least 3 with respect to the subgraph. Note that the subgraph is itself maximal planar. In a fashion similar to the derivation of (4), we may deduce that

$$n_5 + 2n_4 + 3n_3 \geq 12 + \sum_{i>6} (i-6)n_i$$

Since the subgraph is such that $n_3 + n_4 \leq 3$, we get

$$n_5 + 9 \geq 12 + \sum_{i>6} (i-6)n_i$$

Hence

$$n_5 \geq 3 + \sum_{i>7} (i-6)n_i$$

Applying arguments similar to those in the first half of the proof, a vertex of the desired type is shown to exist in the subgraph. ▫

Our algorithm for finding a five-coloring of a planar graph is the following. First determine the degree $DEG(v)$ of each vertex v. Vertices of degree less than five are placed on a ready list $R(1)$, and vertices of degree five are placed on a ready list $R(2)$. While at least one list is not empty, do the following. If $R(1)$ is not empty, remove a vertex v from $R(1)$, push v and a pointer to its adjacency list onto a stack, and delete v from the graph. Otherwise, remove a vertex v from $R(2)$: If v has two neighbors x and y of degree at most seven that are mutually nonadjacent, then identify x and y, push v and a pointer to its adjacency list onto a stack, and delete v from the graph.

To delete a vertex v, remove v from the adjacency list of each neighbor w, and decrement the degree of w. If $DEG(w) = 5$, insert w on $R(2)$. If $DEG(w) = 4$, remove w from $R(2)$ and insert it in $R(1)$. If $DEG(w) = 7$, then for each neighbor x of w, if $DEG(x) = 5$ and x is not on $R(2)$, place x on $R(2)$.

To identify two vertices u and v, do the following. For each neighbor w of v, mark w with v. For each neighbor w of u, delete w from the adjacency list of u. If w has no mark v, then add w and v to each other's adjacency lists, increment $DEG(v)$, and remove v from $R(2)$ if $DEG(v) = 6$. Otherwise decrement $DEG(w)$, and adjust the ready lists on the basis of $DEG(w)$ in a fashion similar to that discussed in the previous paragraph. When all neighbors of u have been handled, remove u from $R(2)$ if it is present on it, and push (u,v) onto the stack.

When both ready lists are empty, the stack is popped repeatedly and colors assigned in a fashion similar to that in [MST]. For each pair (v,t) popped, if t is a pointer to an adjacency list, color v with a color different from those vertices on the list. Otherwise color v the same color as t.

The algorithm may be seen to perform correctly by the following argument. Initially all vertices that are candidates for deletion are on ready lists. When an identification or deletion is performed, the ready lists are updated correctly. Specifically in the case of decrementing the degree of some vertex, a vertex is placed on $R(1)$ if its degree becomes less than five, and is placed on $R(2)$ if its degree becomes equal to five. In addition, a vertex is placed on $R(2)$ if its degree is currently equal to five and a neighbor has its degree decrease to seven. Thus any vertex that is suitable for deletion must be on the appropriate ready list. Since Theorem 3 guarantees that there will always be a vertex in the graph that is suitable for deletion, the main while loop will not terminate until all vertices have been deleted from the graph.

To establish the linear time complexity of the algorithm, we note the following. A vertex will be deleted from the graph when it is removed from $R(1)$. This is not true for vertices removed from $R(2)$. However, a vertex removed from $R(2)$ may be tested for the crucial property in constant time, since only adjacency lists of length no greater than seven need be scanned. For each subsequent time that a vertex is removed from $R(2)$ and tested, we charge the cost of the removal and the test to the *identify* or *delete* operation that last placed it on the list.

The time to delete a vertex v from the graph is constant, since at most four neighbors must be handled. In worst case, each neighbor w will have its degree changed to seven, and its neighbors must be checked to place any neighbor of degree five on $R(2)$. The cost of testing any such x will accrue to the *delete* operation, but the total time is still constant per *delete*. In a similar fashion, the time for an *identify* operation is seen to be constant.

Thus the time expended in the main while loop can be accounted as a constant amount for each vertex deleted from the graph. The time to pop a vertex off the stack is bounded by some constant. The linearity of the algorithm then follows.

Relative tightness of the result in Theorem 3 can be demonstrated in the sense that the theorem does not hold if the 7 is replaced by a 6. We generate a graph in which each vertex of degree 5 will have all pairs of neighbors that are mutually nonadjacent be such that one member of every pair is of degree 7. Consider any maximal planar graph of minimum degree 5, a portion of an example of which is shown in Figure 3a. Replace each edge with two vertices and three edges as shown in Figure 3b. Now embed the subgraph of Figure 3c into each (triangular) face, yielding the graph of which a portion is shown in Figure 3d. It is not hard to verify that every mutually nonadjacent pair of vertices that are neighbors of some vertex of degree 5 includes a vertex of degree 7.

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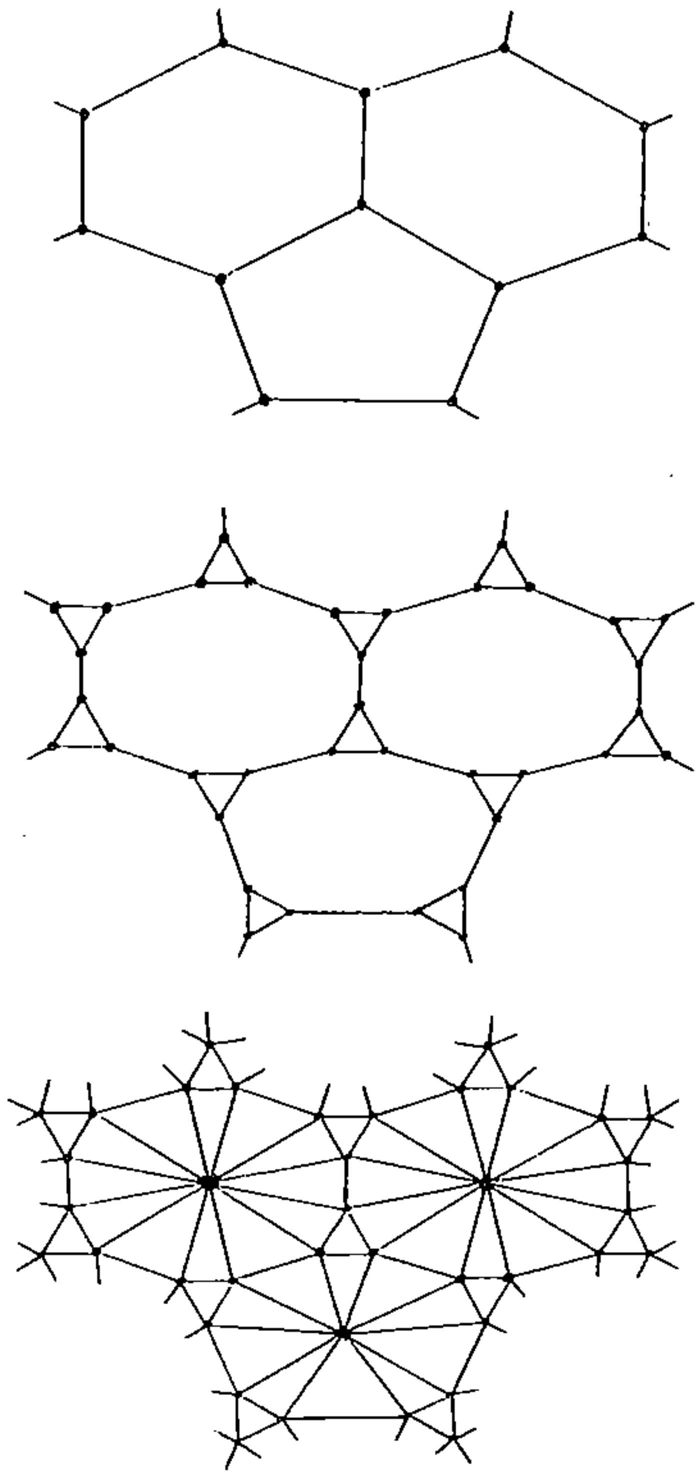

(a)

(b)

(c)

Figure 1. A graph close to the bound in Theorem 1.

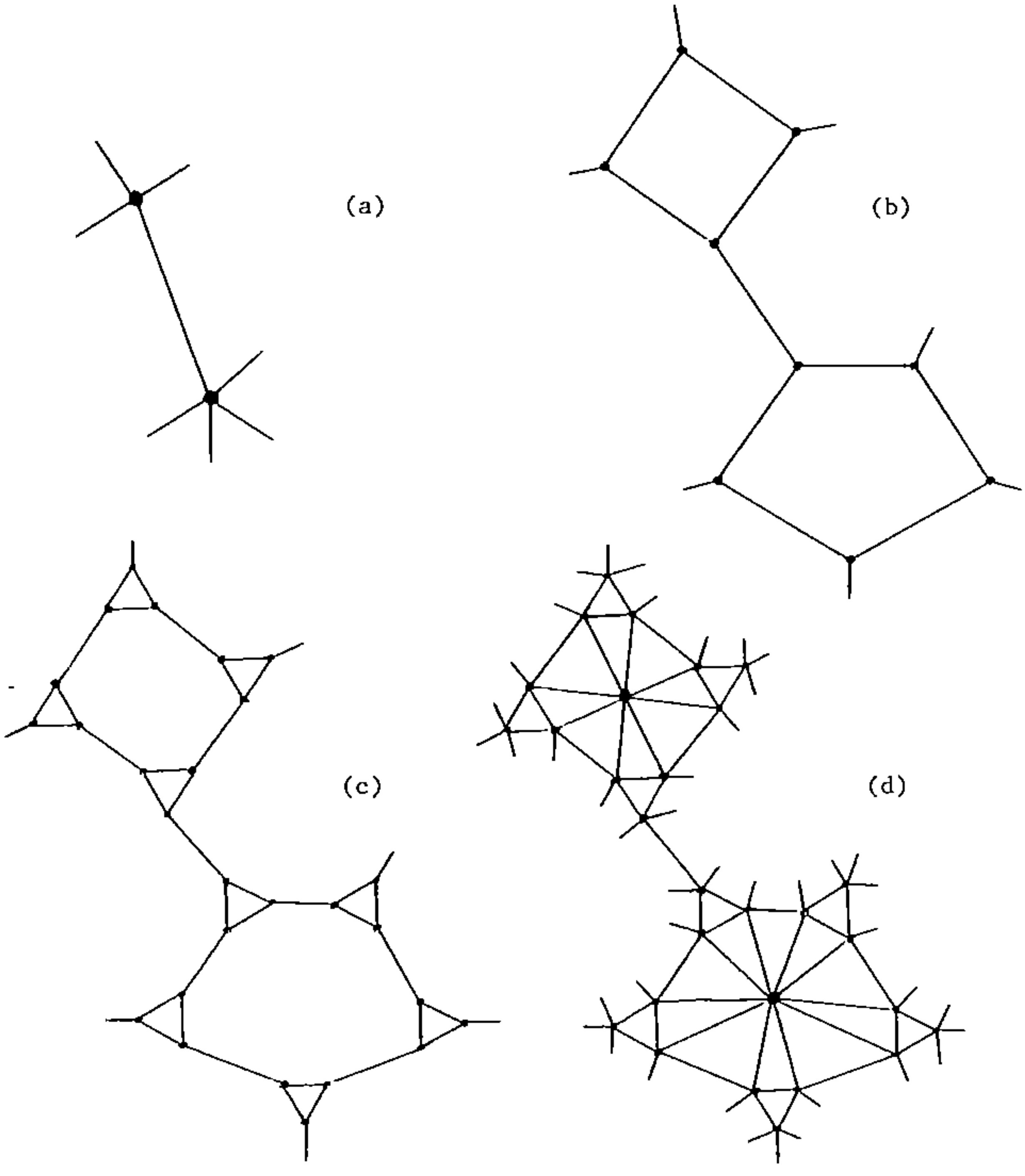

Figure 2. A graph illustrating the bound of Theorem 2.

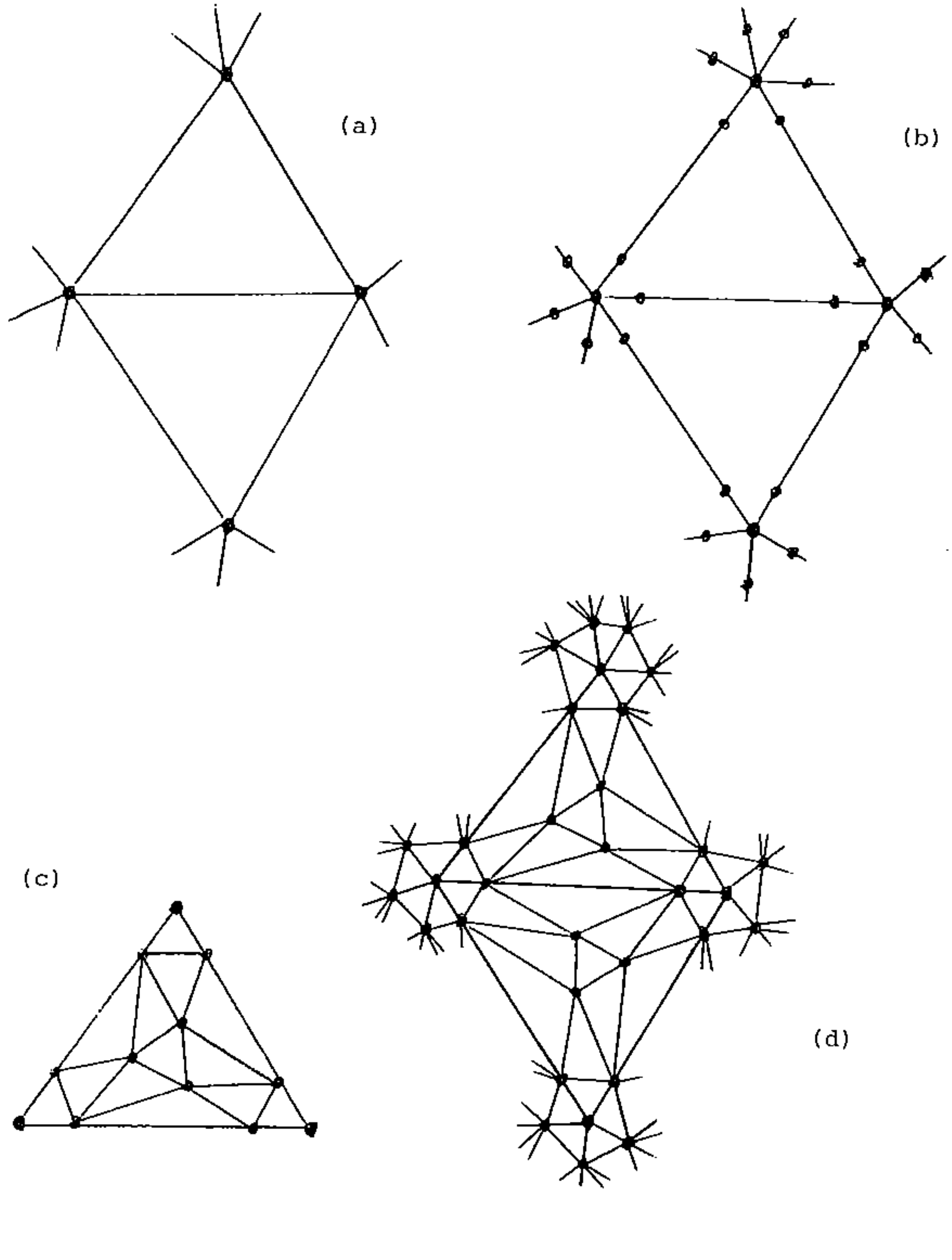

Figure 3. A graph illustrating the bound of Theorem 3.