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SYMMETRIC VERSUS NONSYMMETRIC DIFFERENCING

FOR SELF-ADJOINT ELLIPTIC PROBLEMS

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ABSTRACT

Consider the self-adjoint elliptic problem $(pu_x)_x + (qu_y)_y + ru = f$ with Dirichlet boundary conditions on the unit square. This problem is symmetric in the sense that if the data is symmetric then so is the solution. The usual finite difference discretization has one expand the derivatives and apply differences to $pu_{xx} + p_x u_x + \cdots$. This discretization is not symmetric which has lead to the derivation of symmetric difference discretizations for this problem. Symmetric discretizations are attractive intuitively and are usually recommended. We have observed that symmetric discretizations are sometimes much less accurate; a simple analysis is made to compare the expected behavior of the two discretizations. Data from a simplified model problem confirms the expectations that non-symmetric differences are more accurate than symmetric differences much more often than vice-versa.

Symmetric versus Nonsymmetric Differencing for Self-Adjoint Elliptic Problems

Wayne R. Dyksen[†] and John R. Rice[†]

1. The Difference Approximations

Consider the finite difference discretizations of the terms

$$(pu_x)_x \text{ and } pu_{xx} + p_x u_x .$$

One introduces a grid $x_i = ih$ for $0 \le i \le n+1 = 1/h$, and uses the variables u_i to approximate $u(x_i)$ and $p_i = p(x_i)$. The symmetric discretization of $(pu_x)_x$ at x_i is

$$\left[p_{i-1/2}u_{i-1} - (p_{i-1/2} + p_{i+1/2})u_i + p_{i+1/2}u_{i+1}\right]/h^2$$

$$+ (h/2)^2\left[p_{i+1/2}u'''_{i+1/2} - p_{i-1/2}u'''_{i-1/2} + p'''_i u'_i + p_i u_i^{(iv)}\right]/3 + O(h^4)$$

where primes indicate differentiation with respect to x. The nonsymmetric discretization of $pu_{xx} + p_x u_x$ at x_i is

$$\left[p_i/h^2 - p'_i/(2h)\right]u_{i-1} - \left[2p_i/h^2\right]u_i + \left[p_i/h^2 + p'_i/(2h)\right]u_{i+1}$$

$$+ h^2\left[p_i u_i^{(iv)}/4 + p'_i u'''_i\right]/3 + O(h^4).$$

The error terms of these two approximations are substantially different and it is clear that one can construct problems (chose $p(x)$ and $u(x)$) so that either approximation is much more accurate than the other. The term $p_{i+1/2}u'''_{i+1/2} - p_{i-1/2}u'''_{i-1/2}$ is $h(pu''')' + O(h^3)$ and thus likely to be small. Intuitively, one would expect the nonsymmetric difference to be more accurate when $p(x)$ is rapidly varying because the derivative p_x is computed symbolically. This is

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indicated also by the presence of the third derivative of $p(x)$ in the leading error term for the symmetric difference while the nonsymmetric difference has only the first derivative of $p(x)$ in the leading error term. On the other hand, if $u(x)$ is rapidly varying and $p(x)$ is not, then one would expect the symmetric difference to be more accurate; it does not have $p'u'''$ in its leading error term.

2. An Experimental Study

We observed substantial differences in the discretization errors of these difference approximations for several problems. To illustrate the nature of the situation, we consider the simplified model problem

$$-(p(x)u_x)_x = f \quad \text{in } [0,1]$$
$$u(0),\ u(1) \text{ given.}$$

The function $f(x)$ is chosen to make the model problem solution be as specified. We choose ten functions for $u(x)$:

$$x^2,\ x^4,\ e^x,\ \sin x,\ \frac{1}{1+x^2},\ e^{10x},\ \frac{1}{1+10x^2},\ \sin 10x,\ \sin 100x,\ x^{10}$$

and ten functions for $p(x)$:

$$1,\ x,\ x^2,\ x^4,\ e^x,\ \sin x,\ \frac{1}{1+x^2},\ e^{10x},\ 1.1+\sin 100x,\ \frac{1}{1+10x^2}.$$

Then all 100 combinations of elliptic problems are solved. We compute the maximum relative errors e_N and e_S of the nonsymmetric and symmetric differences, respectively. The results are tabulated in the following manner. A factor R is chosen, the two discretizations are said to **tie** if either

$$\frac{\max(e_N, e_S)}{\min(e_N, e_S)} \le R$$

or

$$\max(e_N, e_S) \le \text{round-off}.$$

The computation is made on a VAX 11/780 (6 decimal digit arithmetic) and the round-off level is determined from those cases where the discretization is theoretically exact. Table 1 has four arrays with entry '-' if the methods tie, 'N' if the nonsymmetric error e_N is smaller and there is no tie, 'S' if the symmetric error e_S is smaller and there is no tie. Data are given for $h = 1/20$ with $R = 1.4$, 4.0 and 10.0 and for $h = 1/100$ with $R = 10.0$

Table 1

Arrays showing the error performance of the two discretizations. The columns correspond to the ten $p(x)$ functions, the rows to the ten $u(x)$ functions. A dash means the discretizations tie, N and S mean that nonsymmetric and symmetric are better, respectively.

-	-	N	N	N	N	N	N	N	N
-	-	-	N	-	-	-	N	N	S
-	-	N	N	N	S	-	N	N	N
-	-	N	N	-	-	N	N	N	N
-	-	-	N	-	-	-	N	N	-
-	-	-	-	-	-	-	N	N	-
-	-	S	N	-	-	-	-	N	-
-	-	S	-	-	-	-	-	N	-
-	-	-	-	-	-	-	-	S	-
-	-	-	-	-	-	-	N	N	-

33 N's 5 S's 62 ties
$h = 1/20$ $R = 1.4$

-	-	N	N	N	N	N	N	N	N
-	-	-	-	-	-	-	-	N	-
-	-	-	N	-	-	-	N	N	-
-	-	-	N	-	-	-	N	N	-
-	-	-	-	-	-	-	-	N	-
-	-	-	-	-	-	-	-	N	-
-	-	-	-	-	-	-	-	N	-
-	-	-	-	-	-	-	-	N	-
-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	N	-

20 N's 0 S's 80 ties
$h = 1/20$ $R = 10.0$

-	-	N	N	N	N	N	N	N	N
-	-	-	-	-	-	-	N	N	S
-	-	-	N	-	-	-	N	N	-
-	-	-	N	-	-	-	N	N	-
-	-	-	N	-	-	-	N	N	-
-	-	-	-	-	-	-	-	N	-
-	-	S	-	-	-	-	-	N	-
-	-	-	-	-	-	-	-	N	-
-	-	-	-	-	-	-	-	S	-
-	-	-	-	-	-	-	-	N	-

23 N's 3 S's 74 ties
$h = 1/20$ $R = 4.0$

-	N	N	N	-	N	N	N	N	N
-	-	-	-	-	-	-	-	N	-
-	-	-	N	-	N	N	N	N	N
-	-	-	N	-	-	N	N	N	-
-	-	-	-	-	-	-	-	N	-
-	-	-	-	-	-	-	-	N	-
-	-	-	-	-	-	-	-	N	-
-	-	-	N	-	-	-	-	-	-
-	-	-	-	-	-	-	-	S	-
-	-	-	-	-	-	-	-	N	-

23 N's 1 S 76 ties
$h = 1/100$ $R = 10.0$

The main observation to be made is that most of the time (at least 2/3's) it does not make any difference which discretization is used. It is probably more "fair" to exclude the first row of the arrays where $u = x^2$ and the nonsymmetric difference is exact. However, the general conclusion is unchanged if this done.

In those cases where it does make a difference which discretization is used, the nonsymmetric one is much more likely to be the best and often by a substantial amount. The eight largest differences in discretization errors are tabulated below for $h = 1/20$ (excluding the case $u = x^2$).

e_N	e_S	u	p	e_S / e_N
.0052	3.7	x^4	$1.1 + \sin 100x$	712
.00021	.69	e^x	$1.1 + \sin 100x$	3286
.072	5.8	e^{10x}	$1.1 + \sin 100x$	81
.013	1.9	$1/(1+10x^2)$	$1.1 + \sin 100x$	146
.076	16.0	$\sin 10x$	$1.1 + \sin 100x$	211
.067	6.5	x^{10}	$1.1 + \sin 100x$	97
.00039	.011	e^x	x^4	28
.00064	.023	$\sin x$	x^4	36

These data give strong experimental support to the conclusion reached by the analysis: **Unless it is known that u varies much more rapidly than p, one should use the nonsymmetric differences in order to expect the best accuracy from the discretization.**