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FAMILIES OF COMPLEXITY CLASSES

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ABSTRACT

In a model for a measure of computational complexity, Φ, for a partial recursive function t, let R_t^Φ denote all partial recursive functions having the same domain as t and computable within time t. Let $\Sigma^\Phi = \{R_t^\Phi \mid t \text{ is recursive}\}$ and let $\Omega^\Phi = \{R_{\Phi_i} \mid \Phi_i$ is actually the running time function of a computation $\}$. Σ^Φ and Ω^Φ are partially ordered under set-theoretic inclusion. These partial orderings have been extensively investigated by Borodin, Constable, and Hopcroft in [BCH]. In this paper we present a simple uniform proof of some of their results. For example, we give a procedure for easily calculating models of computational complexity Φ and Φ' for which Σ^Φ is not dense while $\Omega^{\Phi'}$ is dense. In our opinion, our technique is so transparent that it indicates that questions of density are uninteresting for general abstract measures of computational complexity, Φ.

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We let N denote the nonnegative integers and Q the rationals. We assume the reader to be familiar with the notion of a standard indexing $\lambda i \phi_i$ of all partial recursive functions. A measure of computational complexity $\lambda i \Phi_i$ is any sequence of functions for which the domain of Φ_i = the domain of ϕ_i and the relation (of three variables) $\Phi_i(x) \leq y$ is decidable, [B]. If f and g are two functions with domain and range in N, we write $f \leq g$ a.e. to denote that f and g have the same domain and for all but finitely many elements x in the domain $f(x) \leq g(x)$. Similarly $f \leq g$ i.o. means that the functions have the same domain and for infinitely many elements x in the domain $f(x) \leq g(x)$.

Basic Lemma. There exists a measure Φ such that

(i) $\Phi_i(x) < \Phi_j(x) + 2^x$ implies $\Phi_j(x) \leq \Phi_j(x)$

and (ii) $\Phi_j(x) \leq \Phi_i(x)$ a.e. and $\phi_j(y) \neq \phi_i(y)$ for some y implies $\Phi_j(x) + 2^x \leq \Phi_i(x)$ a.e.

Proof. We let Φ' be any measure satisfying $\Phi_i'(x) \geq 1$, and we define a new measure Φ^P by

$$\Phi_i^P(x) = p_i^{\Phi_i'(x)} \quad (\geq 2),$$

where $p_0 = 3$, $p_1 = 5, \ldots$ is an enumeration of all odd prime numbers. We then define the measure Φ by

$$\Phi_i(x) = (\Phi_i^P(x))^{x+1}.$$

Φ has the useful property that

$$\Phi_i(x) < \Phi_j(x) \text{ iff } \Phi_i^P(x) < \Phi_j^P(x),$$

and $\Phi_i(x) < \Phi_j(x) \implies \Phi_i^P(x) < \Phi_j^P(x)$

$$\implies \Phi_j(x) = (\Phi_j^P(x))^{x+1} = (\Phi_i^P(x)+n_x)^{x+1} \text{ for some } n_x>0$$

$$= (\Phi_i^P(x))^{x+1} + (x+1)(\Phi_i^P(x))^x n_x + \ldots$$

$$\geq \Phi_i(x) + 2^x.$$

Therefore

$$\Phi_i(x) < \Phi_j(x) \implies \Phi_i(x)+2^x \leq \Phi_j(x),$$

establishing (i) of the basic lemma.

To prove (ii), we observe that if there is some y such that $\Phi_j(y) \neq \Phi_i(y)$ then $\Phi_j^P(y) \neq \Phi_i^P(y)$ and $i \neq j$ so also $p_j \neq p_i$. Therefore for any x

(a) $$\Phi_j^P(x) = p_j^{\Phi_j^!(x)} \neq p_i^{\Phi_i^!(x)} = \Phi_i^P(x).$$

Therefore

$$\Phi_j(x) \leq \Phi_i(x) \text{ a.e. } \implies \Phi_j^P(x) \leq \Phi_i^P(x) \text{ a.e.}$$

$$\implies \Phi_j^P(x) < \Phi_i^P(x) \text{ a.e. by (a)}$$

$$\implies \Phi_j(x) < \Phi_i(x) \text{ a.e.}$$

$$\implies \Phi_j(x) + 2^x \leq \Phi_i(x) \text{ a.e. by (i)},$$

which proves (ii).

Now the basic lemma guarantees that in the measure Φ, any infinite time Φ_i is essentially isolated: any infinite run time Φ_j which is less than $\Phi_i(x) + 2^x$ a.e. must either be Φ_i or else lie below $\Phi_i(x) - 2^x$ a.e. This enables us more or less at will to add new run times about any infinite Φ_i in virtually any order we please. To this end, we first define a total recursive function t such that

(i) domain $\phi_{t(i,n)}$ = domain ϕ_i,

(ii) $\phi_j = \phi_{t(i,n)}$ implies $\Phi_j(x) \geq \Phi_i(x) + 2^x$ a.e.,

and (iii) $\langle i,n\rangle \neq \langle j,m\rangle$ implies $\phi_{t(i,n)} \neq \phi_{t(j,m)}$.

(Conditions for accomplishing (i) and (ii) are implicit in [B] and in [R], and (iii) is easily accomplished, e.g., by requiring domain $\phi_{t(i,n)} \subseteq \{2\langle i,n\rangle, 2\langle i,n\rangle + 1\}$.) Our strategy is to now make up a new measure in which each of the functions $\phi_{t(i,n)}$ has a running time of approximately Φ_i (which is of course exponentially less than $\Phi_{t(i,n)}$).

For example, to interpose orderings of the type $\omega^* + \omega$ of the negative and positive integers, we let I be any effective one-one map from all of the integers {..., -2, -1, 0, 1, 2,...} onto N. We may then define a new standard indexing ϕ' and measure Φ' by

$$\phi'_{\langle i,o\rangle} = \phi_i \text{ and } \phi'_{\langle i,I(m)+1\rangle} = \phi_{t(i,I(m))},$$

$$\Phi'_{\langle i,o\rangle} = \Phi_i \text{ and } \Phi'_{\langle i,I(m)+1\rangle}(x) = \max\{0, \Phi_i(x) + m\}.$$

Now the fact that the infinite run times of the form $\phi'_{<i,n>}$ differ only additively from $\phi'_{<i,o>} = \phi_i$ while if $i \neq j$, the infinite run times $\phi'_{<i,m>}$ and $\phi'_{<j,m>}$ must differ exponentially, together with the basic lemma yields Corollary 1, which asserts that in the measure ϕ' the infinite run times are <u>totally discrete</u>. It is a generalization of Theorem 3.3 of [BCH].

<u>Corollary 1</u>. In the measure ϕ', for every infinite run time ϕ'_i, there exists k_o, k_1, and k_2 such that

(i) $\phi'_{k_1} = \phi'_i$ and $\phi'_{k_o} < \phi'_{k_1} < \phi'_{k_2}$ a.e.

(ii) $(\forall j)[\phi'_j = \phi'_{k_2} \Rightarrow \phi'_j > \phi'_{k_1}$ i.o.$]$

$\quad$ & $(\forall j)[\phi'_j = \phi'_{k_1} \Rightarrow \phi'_j > \phi'_{k_o}$ i.o.$]$

and (iii) $(\forall j)[\phi'_j \leq \phi'_{k_2}$ a.e. & $\phi_{k_1} < \phi_j$ i.o $\Rightarrow \phi'_j = \phi_{k_2}]$

$\quad$ & $(\ j)[\phi'_j \leq \phi'_{k_1}$ a.e. & $\phi'_{k_o} < \phi_j$ i.o. $\Rightarrow \phi'_j = \phi_{k_1}]$

Similarly, if we wish to distort ϕ to a measure ϕ'' for which the run times are <u>totally dense</u>. we let R be any effective one-one mapping from the set, Q, of all rational numbers onto N. Proceeding as before, we define a new standard indexing ϕ'' and measure ϕ'' by

$$\phi''_{<i,o>} = \phi_i \quad \text{and} \quad \phi''_{<i,R(q)+1>} = \phi_{t(i,R(q))}$$

$$\phi''_{<i,o>} = \phi_i \quad \text{and} \quad \phi''_{<i,R(q)+1>}(x) = \max\{0,\ \phi_i(x) + [q \cdot x]\},$$

where $[qx]$ denotes the greatest integer in the rational $q \cdot x$.
This time infinite run times of the form $\Phi''_{<i,n>}$ differ linearly from $\Phi''_{<i,o>} = \Phi_i$, while if $i \neq j$ infinite run times of the form $\Phi''_{<i,m>}$ and $\Phi''_{<j,m>}$ must still differ exponentially. This, together with the basic lemma yields Corollary 2, which, in spite of the complexity of its statement, merely asserts that in the measure Φ'' the infinite run times are totally dense.

Corollary 2. In the measure Φ'', for every infinite run time Φ''_i, there exists a collection of run times

$$\{ \Phi''_{i_q} \mid q \varepsilon Q \}$$

(for which, using an abuse of notation, we write Φ''_q instead of Φ''_{i_q} whenever $q \varepsilon Q$ is considered as a member of Q rather than of N) satisfying

(i) $\Phi''_i = \Phi''_o = (\Phi''_{i_o})$ and $p < q \varepsilon Q \Rightarrow \Phi''_p < \Phi''_q$ a.e.

(ii) $\Phi''_j = \Phi''_{i_q}$ and $p < q \varepsilon Q \Rightarrow \Phi''_{i_p} < \Phi''_j$ i.o.

and (iii) for $p < q \varepsilon Q$

$$(\forall j)[\Phi''_j \leq \Phi''_q \text{ a.e. } \& \ \Phi''_p < \Phi''_j \text{ i.o.} \Rightarrow (\exists r \varepsilon Q)\ \Phi''_j = \Phi''_r].$$

Theorem 4.4 of [BCH] asserts the existence of measures whose <u>run-time</u> classes are dense but whose larger collection of classes determined by <u>arbitrary</u> recursive functions is not dense. In view of the fact that Corollary 2 asserts that in the measure Φ'' the run-time classes are dense, our next Corollary generalizes Theorem 4.4 of [BCH].

<u>Corollary 3.</u> In the measure Φ'', for any infinite run-time Φ''_i, there exist recursive functions $\underline{t}$ and $\overline{t}$, each with the same domain as Φ''_i, and satisfying

$$\text{(i)}\quad \{\phi''_j \mid \Phi''_j \leq \underline{t} \text{ a.e.}\} \subsetneqq \{\phi''_j \mid \Phi''_j \leq \Phi''_i \text{ a.e.}\} \subsetneqq \{\phi''_j \mid \Phi''_j \leq \overline{t} \text{ a.e.}\},$$

but (ii)

$$(\nexists t)[\{\phi''_j \mid \Phi''_j \leq \underline{t} \text{ a.e.}\} \subsetneqq \{\phi''_j \mid \Phi''_j \leq t \text{ a.e.}\} \subsetneqq \{\phi_j \mid \Phi''_j \leq \Phi_i \text{ a.e.}\}]$$

$$\& \; (\nexists t)[\{\phi''_j \mid \Phi''_j \leq \Phi_i \text{ a.e.}\} \subsetneqq \{\phi''_j \mid \Phi''_j \leq t \text{ a.e.}\} \subsetneqq \{\phi_j \mid \Phi''_j \leq \overline{t} \text{ a.e.}\}].$$

<u>Proof.</u> The result follows simply by defining $\underline{t}$ to be any recursive function $<\Phi''_i$ everywhere but satisfying $\underline{t}(x) \geq \Phi''_i(x) + [q\cdot x]$ a.e. for every <u>negative</u> rational number q. Similarly $\overline{t}$ should satisfy the reverse inequalities for every <u>positive</u> rational number q.

Finally, there are several remarks we wish to make about the construction.

If one wishes to obtain about each infinite run time order-types of greater complexity than the order-type of the integers or rationals, then it might be unseful to increase the "global gap-size" 2^x to some larger function.

Each of the infinite run times $\Phi'_{<i,n>} \neq \Phi'_{<i,o>}$ or $\Phi''_{<i,n>} \neq \Phi''_{<i,o>}$ determines only one function in the sense, e.g., that in this case

$$\{\phi'_k \mid \Phi'_k \leq \Phi'_{<i,n>} \text{ a.e.}\} - \{\phi'_k \mid \Phi'_k \leq \Phi'_{<i,n>} - 1 \text{ a.e.}\}$$

contains only the function $\phi'_{<i,n>}$. If one wants these collections to be infinite, one, e.g., simply assigns infinitely many functions the complexity $\Phi'_{<i,n>}$ which have, in the measure Φ, run times exponentially greater than Φ_i (= $\Phi'_{<i,o>}$).

It also might be hoped that by imposing further conditions on the measures of computational complexity that the property of density might become measure-invariant. It is not clear how this might be done, but for one condition used elsewhere, namely properness as formulated in [M M], the condition is seen to be inadequate by the technique of this paper. This follows from the following proposition.

Definition. A measure Φ is proper if for all Φ_i there exists a ϕ_j such that $\phi_j = \Phi_i$ and $\Phi_j \leq \Phi_i$ a.e. (I.e., it is no more difficult to compute the run time of a function then to compute the function.) The measure is strongly proper if there exists an effective procedure σ such that $\phi_{\sigma(i)} = \Phi_i$ and $\Phi_{\sigma(i)}(x) \leq \Phi_i(x)$ for all x.

Proposition. A. For every measure Φ, there exists a strongly proper measure Φ^H such that the run-times of Φ and Φ^H are exactly the same.

B. In the Basic Lemma and Corollaries 1, 2, and 3, each of the measures Φ, Φ', and Φ'' can be assumed to be strongly proper.

<u>Proof of A</u>. By the S_m^n-theorem, for every measure Φ there is a total recursive function d such that $\Phi_i = \Phi_{d(i)}$. We define the measure Φ^H and standard indexing ϕ^H by

$$\phi_{2i}^H = \phi_i \quad \text{and} \quad \phi_{2i+1}^H = \phi_{d(i)}$$

$$\Phi_{2i}^H = \Phi_i \quad \text{and} \quad \Phi_{2i+1}^H = \Phi_{d(i)}$$

Clearly the desired function σ is simply $\sigma(2i) = 2i+1$ while $\sigma(2i+1) = 2i+1$.

<u>Proof of B</u>. In the construction of the function t after the proof of the basic lemma , since we have an <u>a priori</u> bound on the range of $\phi_{t(i,n)}$, we may introduce a diagonalization to guarantee that if $\phi_{t(i,n)}$ is infinite, it is not equal to <u>any</u> run-time Φ_j. Once this has been done, we may use the technique of A to make the measures Φ' and Φ'' highly proper. This will not affect the orderings, since the orderings are now forced by functions which aren't run-times.

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