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ABSTRACT

Rearrangeable multistage interconnection networks such as the Benes network realize any permutation, yet their routing algorithms are not cost-effective. On the other hand, non-rearrangeable networks can have inexpensive routing algorithms, but no simple technique exists to characterize all the permutations realized on these networks. This paper introduces the concept of frame and shows how it can be used to characterize all the permutations realized on various multistage interconnection networks. They include any subnetwork of the Benes network, the class of networks that are topologically equivalent to the baseline network, and cascaded baseline and shuffle-exchange networks. The question of how the addition of a stage to any of these networks affects the type of permutations realized by the network is precisely answered. Also, of interest from a theoretical standpoint, a new simple proof is provided for the rearrangeability of the Benes network.

Index Terms — Multistage interconnection network, permutations, rearrangeability, topological equivalence, balanced matrices, frames.

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List of Symbols

IN: interconnection network.

IP: interconnection pattern.

IP_{in}: interconnection pattern formed by input links.

IP_{out}: interconnection pattern formed by output links.

SB: switching box; switch.

BE: baseline network; *see* Definition II.3.

RB: reverse baseline network; see Definition II.3.

SE: shuffle-exchange network; see Definition IL3.

SE^{-1}: inverse shuffle-exchange network; see Definition II.3.

BS: Benes network; see Definition II.3.

CS: Clos network; see Definition 11.3.

N: number of inputs/outputs of a network.

n: $\log_2 N$.

$F^{sa}_{1:k}$: the standard a-type frame with k columns; see Definition 111.2.

$F^{a}_{1:k}$: an a-type frame with A columns; see Definition III.3.

$F^{*}_{1:k}$: the universal frame with k columns; see Definition III.6.

I: the identity permutation matrix; see Definition 11.1.

R: the reverse permutation matrix; see Definition II.1.

r: the reverse permutation represented by R.

$A_{N \times k}$: matrix A with N rows and A columns.

$A_{N \times k}(i)$: the *i*th row of matrix A.

γ: a permutation on the set {0,1, . . . ,N−1); see Definition 111.2.

β: a mapping of the set {1,2, . . . ,k) into {1,2, . . . ,n}; see Definition III.2.

P: a tuple of partitions; see Definition III.2.

I. INTRODUCTION

Interconnection **networks** are utilized to provide communication among processing: elements **and/or** memory modules. Network performance significantly affects the **overall** cost and **performance** of a computational system because **processors** may spend a **considerable** amount of time in processor-processor **and/or** processor-memory **communication.** Therefore, it is important to know exactly the interconnection patterns that **can** be implemented by a network. In particular, it is desirable to know what permuta*tions* can be realized because parallel algorithms often require permutation-type data **transfers.** This paper presents a simple and easily understandable characterization of the permutations realized by any network with $N=2^n$ inputs that is topologically equivalent to one of the following **networks:** first k stages, $1 \leq k \leq n$, of the reverse **baseline** network, the last $n+k-1$ stages of Benes network **[7],** or a **cascade** of baseline **[11]** and k-stage shuffle-exchange **[1,5]** networks. The proposed characterizations are **based** on the notion of *''frame''* (introduced in this paper), balanced matrices [2] and **graph theory [3,4].**

The effectiveness of any interconnection network **depends** on factors such as the efficiency of the routing algorithm, the number and type of **permutations** it realizes, **and** the actual hardware implementation of the network. ***On*** one **hand,** rearrangeable multistage interconnection networks such as Benes and $\Omega\Omega^{-1}$ (the $\Omega\Omega^{-1}$ is a cascade of **omega** and inverse omega **[1])** can realize any permutation. However, there are no known efficient routing algorithms to allow dynamic configuration in an environment **where** the switching permutations change rapidly. ***On*** the other hand, some networks **such** as baseline and omega have efficient routing algorithms and **small** propagation delays, but cannot realize many permutations. In these cases, it is **important** to know which permutations are realizable and this is possible by using the results of this paper.

Different approaches have been proposed in the literature to circumvent inefficient routing algorithms. One approach is to determine certain **types** of permuta**tions** that occur more frequently than others in a parallel processing environment. Such permutations have been classified by Lenfant **[23]** into five families. In **order** to implement these permutations on the Benes network with a small **propagation** &lay, Lenfant proposed a specialized routing algorithm for each family. A **permutation** that fails to be in one of these families still is routed using an inefficient **routing** algorithm. To increase the number of the families of permutations that can be **realized** by a network, Youssef and Arden **[22]** introduced an $O(\log^2 N)$ routing algorithm. which sets the (rxr) crossbar switches of the first stage of 3-stage Benes networks with $N=r^2$ inputs to **a** fixed configuration and acts exactly like a self-routing algorithm in setting the **remaining** switches. Another approach is to provide self-routing **algorithms** for realizing some classes of permutations in various multistage interconnection networks such as 'Benes, 2n-stage shuffle-exchange. Nassimi and Sahni **[24] presented** simple **self-**

routing algorithms to realize some important permutations in Bcnes networks. Raghavendra and Boppana [25] proposed self-routing algorithms to realize the class of linear permutations on Benes and $2n$-stage shuffle-exchange networks.

Although a large number of multistage interconnection networks are extensively studied, there is a relatively small number of basic designs for their underlying topologies, Especially, Benes networks and six topologically equivalent networks, namely, omega, flip, indirect binary cube, modified data manipulator, baseline and reverse baseline have been investigated in depth and are frequently used in research studies and real systems. Characterizations of the topologies of these networks are given in [9,26,27]. However, to our knowledge, the characterization of the permutations performed by these and other networks is done for the first time in this paper. One exception is the work of Lee [10] which characterizes the permutations realized by the inverse omega network in terms of residue classes.

The rest of the paper is organized as follows. Basic definitions and notations used throughout the paper are presented in Section II. Also included in this section is a motivational example for the concept of frame. In Section III, this concept, illustrations of many different frames, notation and terminology are introduced. Permutations realized by the k-stage reverse-baseline, $1 \le k \le n$, and the networks which are topologically equivalent to it are characterized in Section IV. In Section V, the permutations realized by a cascade of reverse baseline and the k-stage shuffle-exchange networks are identified. These cases show how frames can be used to characterize the permutations of some relatively complex networks with more than n stages. Section VI provides new proofs for the rearrangeability of the three-stage Clos and Benes networks. Permutations realized by the last $n+k-1$ stages of Benes network are identified in Section VII. This characterization illustrates how frames can be used to understand why a network is rearrangeable. Section VIII concludes the paper. The Appendix (Section IX) contains the proofs of most of theorems and lemmata in the paper.

II. BASIC DEFINITIONS AND A MOTIVATIONAL EXAMPLE

Throughout this paper, matrices are denoted by single capital letters and columns of a matrix are represented by the lower case of the capital letter denoting that matrix. Matrix A having N rows and k columns is denoted by $A_{N\times k}$. Given a matrix, e.g. $A_{N\times k}$, the jth column is denoted by a_j, $1 \le j \le k$. To be able to refer to a set of specific columns of a matrix, the notation $A_{x:y}$ is used to denote the submatrix that contains those columns of A whose indices are x, $x+1$, ... ,y, where $1 \le x \le y$; if x happens to be greater than y, then $A_{x:y}$ refers to a nil matrix, unless stated otherwise. If $x=y$, then $A_{x:y}$ refers to a single column a_x. Unless specifically stated, the number of the rows of a matrix $A_{x:y}$ is assumed to be equal to N. $A_{N\times n}(i)$ refers to the ith row of the mamx

$A_{N\times n}$, where $0 \le i \le N-1$. A column vector of N entries of which half are 0's and the other half are 1's is called a column permutation. Unless otherwise stated, any column of any matrix in this paper is a column permutation. The binary representation of a positive integer $0 \le b \le N-1$ is $(b_1 b_2 \cdots b_n)$ such that $b = b_1.2^{n-1} + b_2.2^{n-2} + \cdots + b_n.2^0$.

A permutation on a set X is a bijection of X onto itself. A permutation f permutes the ordered list $0, 1, \cdots, N-1$ into $f(0), f(1), \cdots, f(N-1)$. A cyclic notation [20,21] can be used to represent a permutation as the product of cycles, where a cycle $(c_0\, c_1\, c_2\, \cdots\, c_{k-1}\, c_k)$ means $f(c_0) = c_1$, $f(c_1) = c_2$, $\cdots$ $f(c_{k-1}) = c_k$, and $f(c_k) = c_0$. The composition of several permutations $f_1.f_2 \cdots f_k$ is evaluated from left to right, i.e., it maps i into $f_k(\cdots(f_2(f_1(i)))\cdots)$.

Definition II.1. (Permutation matrix, identity permutation matrix, reverse permutation matrix): A permutation h can be represented by a $N \times n$ binary matrix called *permutation* matrix, H, such that its ith row, $H_{N\times n}(i)$, is the binary representation of the integer $h(i)$. The identity permutation matrix denoted by $I_{N\times n}$ is the matrix whose ith row is the binary representation of i (this is called "standard matrix" in [12]). The reverse permutation matrix, denoted $R_{N\times n}$, is the matrix whose jth column is the $(n+1-j)$th column of $I_{N\times n}$.

For instance, the identity permutation matrix $I_{8\times 3}$, the reverse pennutation matrix $R_{8\times 3}$ and a permutation matrix $E_{8\times 3}$ are shown below:

$$I_{8\times 3} = \begin{bmatrix} 0&0&0\\0&0&1\\0&1&0\\0&1&1\\1&0&0\\1&0&1\\1&1&0\\1&1&1 \end{bmatrix} \quad R_{8\times 3} = \begin{bmatrix} 0&0&0\\1&0&0\\0&1&0\\1&1&0\\0&0&1\\1&0&1\\0&1&1\\1&1&1 \end{bmatrix} \quad E_{8\times 3} = \begin{bmatrix} 1&1&1\\1&0&0\\0&0&0\\0&1&0\\1&1&0\\0&0&1\\1&0&1\\0&1&1 \end{bmatrix}.$$

Clearly, there is a one-to-one correspondence between permutations and permutation matrices. For instance, $R_{8\times 3}$ represents the permutation r:

$$r = \begin{bmatrix} 0&1&2&3&4&5&6&7\\0&4&2&6&1&5&3&7 \end{bmatrix}.$$

Using the cyclic notation, r is represented by $r = (0)(1\ 4)(2)(3\ 6)(5)(7)$.

II.1. Networks

In the terminology used in this paper, a k-stage interconnection **network** (IN) consists of k columns of switching boxes **(SBs)**, each followed and **preceded** by links which **form** interconnection patterns *(IPs)* as shown in Figure 11.1. **The** *IPs* formed by the input and output links **are** denoted by IP_{in} and IP_{out}, respectively. **Thus,** an IN contains $(k+1)$ **interconnection** patterns labeled IP_{in}, IP_1, $IP_2, \ldots, IP_{k-1}$, IP_{out}. A **column** of IN contains $N/2$ **(2x2) SBs,** each of which can be set either **straight** or cross. Figures **II.2, II.3, II.4, II.5,** and **III.6** show several networks considered in this paper for $N=16$, namely, reverse baseline, baseline, Benes, the 4-stage shuffle-exchange (SE), and. the 4-stage inverse SE. If some networks **are** placed in parallel to form a new IN, **then** the IN is said to be a "pile of networks". Unless otherwise stated, any IN is **assumed** to have N **inputs/outputs** and its stages are labeled from left to right starting with 1. Network stages **are** defined below and illustrated in the figures.

Definition **II.2** (Stages **of** reverse baseline, baseline, Benes, SE, and inverse SE **networks):** With one exception, a stage in the reverse baseline and SE networks **consists** of a connection pattern and the following column of **SBs.** The exception is the rightmost stage **(i.e.,** the output stage) which consists of the last column of **SBs** and both the preceding and succeeding connection patterns. Stages **are** labeled from left to right in ascending order starting with 1. In the baseline **network** the kth stage corresponds to the $(n-k+1)$th stage of the reverse baseline network. **(Notice** that both the **reverse** baseline and the baseline can have at most n stages, by **definition).** In the inverse SE network with m stages, its kth stage corresponds to the $(m-k+1)$th stage of the m-stage SE network. In this paper, Benes network is **considered** as being composed of the first $n-1$ stages of the n-stage baseline followed by the n-stage reverse baseline. (It could also be considered as being composed of the n-stage baseline followed by the last $n-1$ stages of the n-stage reverse baseline). **Therefore,** the stages of **Benes** network are labeled according to the labeling rules of the baseline and the reverse baseline.

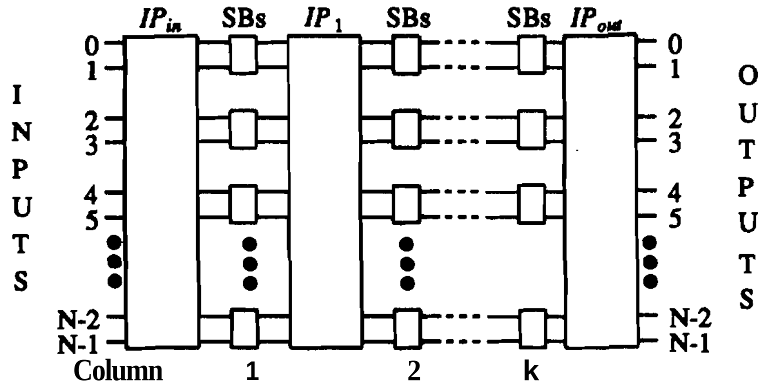


Figure 11.1. An IN with (2x2) SBs and interconnection patterns shown as large boxes.

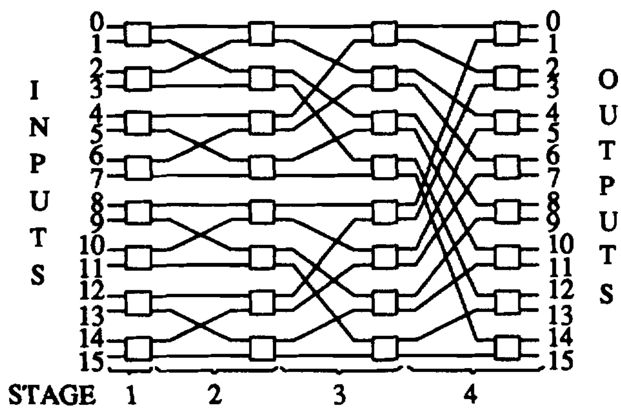


Figure II.2. The 4-stage reverse baseline network with 16 inputs/outputs.

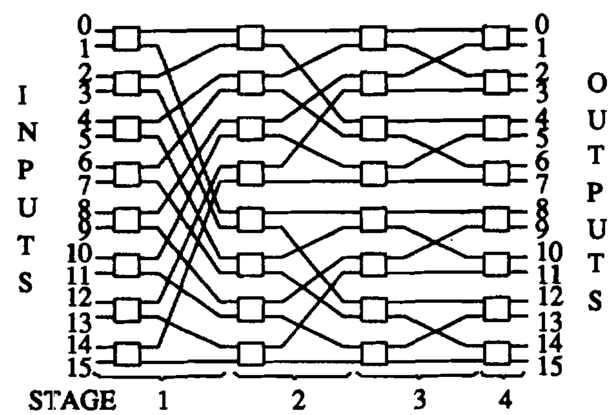


Figure II.3. The 4-stage baseline network with 16 inputs/outputs.

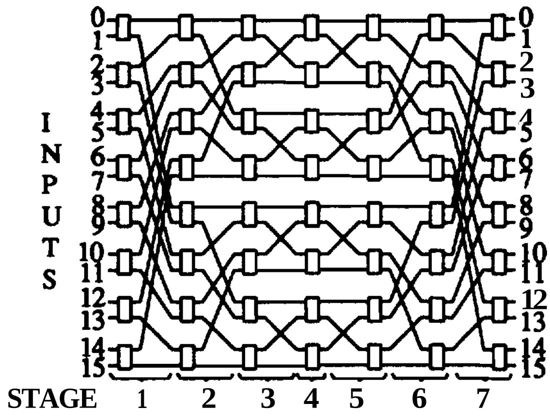


Figure II.4. Benes network with 16 inputs/outputs.

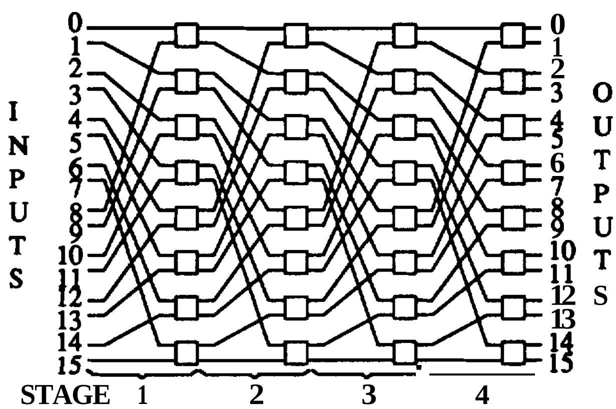


Figure II.5. The omega network (i.e., the 4-stage SE) with 16 inputs/outputs.

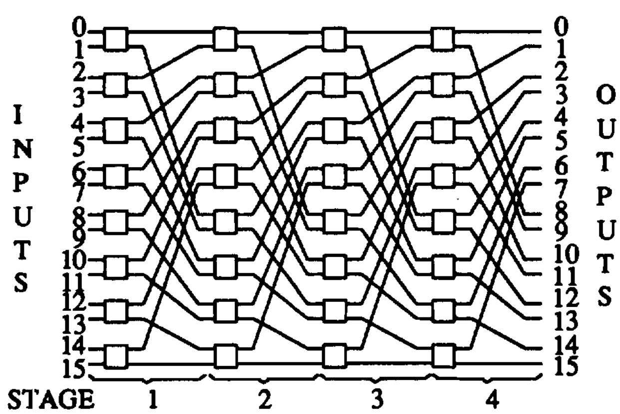


Figure II.6. The inverse omega network (i.e., the 4-stage inverse SE) with 16 inputs/outputs.

An IN having N **inputs/outputs** and k stages is &noted by both $IN_{N\times k}$ and $IN_{1:k}$, **where** $k \geq 1$. The subnetwork that consists of the stages x through y of $IN_{1:k}$ is denoted by $IN_{x:y}$, where $1 \leq x<y \leq k$. If $x>y$, then $IN_{x:y}$ refers to **a** nil network, unless specified otherwise. IN„ $1 \leq j \leq k$, refers to the jth stage of $IN_{1:k}$. The **notation** used for net**works** is different from that used for matrices because **matrices** are always &noted by single letters.

Without loss of generality, it is assumed that routing of a **permutation** through a **network** is done as described in this paragraph. Assuming that the stages of the net**work** are labeled from left to right starting with 1, if the routing tag is $d_1d_2 \cdots d_k$, **then** d_i is examined to set the SB at stage i as follows: if d_i equals **zero** then the output is sent to the upper output of the SB; otherwise, it is sent to the **lower** output. The ith entries of the routing tags of the two inputs entering a SB are also **called** the control **bits** of that SB. So, to set a SB properly to either straight or cross (**or** equivalently not to have any conflict in a SB), the control bits of a SB must constitute: the set $\{0,1\}$. In some networks, the routing tag of an input equals its **destination address,** but this is not always the case.

In this paper the following convention is adopted to &note an IN: if the name of an IN has more than one word, then it is denoted by the upper case form of the first **letters** of those words; otherwise, it is denoted by the upper case **form** of its first and **last** letters. Also, if XX denotes an IN, then the ***inverse XX*** network **may** be denoted by XX^{-1}. The following definition applies this convention to the baseline, reverse baseline, **shuffle-exchange,** inverse shuffle-exchange, Benes and three-stage Clos networks of interest in this paper.

Definition II.3. (BE, RB, SE, SE^{-1}, BS, CS, composite **IN):** The symbols BE, RB, SE, SE^{-1}, BS and CS in this paper refer to the networks baseline, reverse baseline, shuffle-exchange, inverse shuffle-exchange, Benes and three-stage Clos network **$v(2,2,N/2)$ [7,13],** respectively. (If the number of **inputs/outputs** of **three-stage** Clos network **$v(2,2,N/2)$** is equal to N, then each of the outside stages of three-stage Clos network in this paper contains N/**2** (2x2) **SBs** and the middle stage consists of 2 boxes with ***N*/2 inputs/outputs** each). If an IN is a cascade of different **INs,** then it is called a ***composite*** IN and is &noted by the concatenation of symbols that represent the ***INs*** in the order they are cascaded.

As an example for a composite network, the notation $RB_{1:n}SE_{1:m}$, $m \geq 1$, denotes the network consisting of $RB_{1:n}$ followed by $SE_{1:m}$.

Linial and Tarsi [2] introduced the concept of balanced **matrices** to establish a relation between SE networks and their realizable permutations. The following **definition** is equivalent to the one given in **[2].**

Definition II.4. (Balanced matrix): Let $N=2^n$ and call a 0-1 matrix $A_{N\times k}$ balanced if either one of the following conditions is satisfied:

1. For $k \leq n$, it consists of any k columns of the binary representation of a permutation on the set $\{0,1,...,N-1\}$.
2. For $k>n$, every n consecutive columns form the binary representation of a permutation on the set $\{0,1,...,N-1\}$.

As an example, two balanced matrices E and F are shown below. But notice that the matrix [E F] is not balanced.

$$E = [e_1\ e_2\ e_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad F = [f_1\ f_2] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

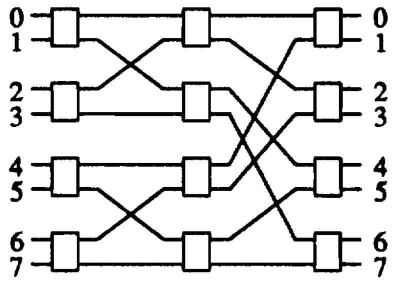
Definition II.5. (Pass, realize): A balanced matrix $A_{N\times k}$ (respectively, an IN) is said to pass a k-stage IN (respectively, a matrix $A_{N\times k}$) if no conflict occurs in the SBs of the IN when $A_{N\times k}(i)$ is used as the routing tag for the *ith* input of the IN. A network IN realizes a permutation represented by $B_{N\times n}$ if there is a network switch setting such that input i is sent to output $B(i)$ for all $i=0,1,\ldots,N-1$.

According to the last definition, in this paper, the phrases "an IN passes a balanced matrix" and "a balanced matrix passes an IN" are used alternatively. It is also assumed that only "one pass" is allowed through a network to realize a permutation. Therefore, the phrase "one pass" is omitted in the sequel. To emphasiu: the distinction between the meaning of the terms "pass" and "realize" as used in this paper, it is important to notice that matrix $A_{N\times k}$ in Definition II.5 does not necessarily correspond to the permutation realized by the network IN. Indeed, the ith row of $A_{N\times k}$ is the routing tag for input i and it is only when it equals the destination of input i that $A_{N\times k}$ is the permutation realized by IN; the cases in which this occurs will become clear in the remainder of the paper.

II.2. A Motivational Example

Consider permutations $\pi_1 = (0\ 6)(1\ 2)(3\ 5\ 4)(7)$, $\pi_2 = (0\ 2)(1\ 4\ 3\ 7)(5\ 6)$, and the reverse baseline network with 8 inputs/outputs, &noted by $RB_{8\times 3}$ and shown in Figure II.7a. A frame is illustrated in Figure II.7b. The binary representations of these permutations are given below:

$$\pi_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad \pi_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



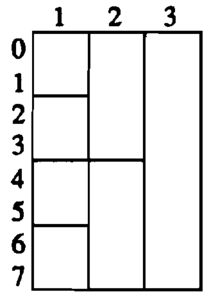


Figure II.7. (a) The reverse baseline network with 8 inputs/outputs.
(b) A *frame*.

When the ith row, $0 \le i \le 7$, of both π_1 and π_2 is used as the routing tag for the ith input of $RB_{8\times3}$, no conflict occurs in the switches and connections are established **between** the input i and the outputs $\Pi_1(i)$ and $\Pi_2(i)$, respectively. **Therefore,** $RB_{8\times3}$ **realizes** π_1 and π_2. Now, let us place the ith row of π_1 and π_2 into the ith row of the frame in Figure **II.7b** with 8 rows as shown in Figure **II.8a** and **Figure II.8b, respectively.**

	1	2	3
0	1	1	0
1	0	1	0
2	0	0	1
3	1	0	1
4	0	1	1
5	1	0	0
6	0	0	0
7	1	1	1

(a)

	1	2	3
0	0	1	0
1	1	0	0
2	0	0	0
3	1	1	1
4	0	1	1
5	1	1	0
6	1	0	1
7	0	0	1

(b)

Figure II.8. (a) Frame with the binary representation of the permutation π_1.
(b) Frame with the binary representation of the permutation π_2.

The first k columns, $1 \le k \le 3$, of any of these two frames consists of 2^{3-k} rectangles of size $2^k \times k$. Note that the matrix enclosed by any rectangle of the frames is balanced (in fact, it represents a permutation on $\{0,1,\ldots,2^k\}$). It is shown in Section IV that, when the rows of any permutation realized by the reverse baseline network are placed into this type of frame, the matrix enclosed by each rectangle is balanced, and vice versa. Different frames are introduced in this paper and it is shown how they are useful to identify the permutations realized by some frequently used networks.

III. FRAMES AND FUNDAMENTAL CONCEPTS

This section introduces the concept of frames to characterize the permutations realized by a network. Different frames are derived from this concept and their graphical representations are presented. In addition, some related fundamental concepts used in the proofs of this paper are introduced. More extended discussion of these concepts appears in [28].

In order to facilitate the understanding of the concept of frame, the following definition is first introduced (a *k-tuple* V with the elements $v_1, v_2, \ldots, v_k$, denoted by $V = <v_1, v_2, \ldots, v_k>$, refers to an ordered collection of k elements).

Definition III.1. (Partition P_i, block, standard partition P_i^*, P^*): Let $X=\{0,1,\ldots,N-1\}$, $N=2^n$ and $i=1,2,\ldots,n$. A *partition* P_i of X is a tuple of 2^{n-i} disjoint ordered subsets of X, called blocks, each of which is a tuple with 2^i distinct elements. The partition $P_i^* = < <h, h+1, \ldots, h+2^i-1>$ such that $h \bmod 2^i = 0$ and $h = 0,1,\ldots,N-1>$ is a *standard partition* of X. The *n-tuple* $<P_i^*, i=1,2,\ldots,n>$ is denoted by P^*.

Example III.1. Let $N=8$. The following are the standard partitions: $P_1^* = < <0,1>, <2,3>, <4,5>, <6,7> >$, $P_2^* = < <0,1,2,3>, <4,5,6,7> >$ and $P_3^* = <0,1,2,3,4,5,6,7>$. Also, $P^* = <P_1^*, P_2^*, P_3^*>$.

The notion of frame is **defined** next and an example (Example **III.2)** is given after tht: definition. Note that the frame of Figures II.7 and **II.8** is characterized by the labeling of its columns, the labeling of its rows and how each column is partitioned. Therefore, the definition of frame is done in terms of two mappings (the column and row labeling) and a tuple of partitions (one for each column). The column labels determine the number and size of the blocks in each partition and the row labeling determines the elements in each block and their order. As precisely stated in the definition, column with label $\beta(i)$ corresponds to a partition with $2^{n-\beta(i)}$ blocks with $2^{\beta(i)}$ elements each and the mth element within the jth block corresponds to the label $\gamma(r)$ of row $r = 2^{\beta(i)}(j-1)+m-1)$ After Example **III.2,** a convenient graphical nzpresentation for frames is introduced and its use is illustrated in Example **III.3** for the frames described in Example III.2.

Definition III.2. (Frame): Let $1 \leq k \leq n$ and $1 \leq i \leq k$. A *frame* $F_{N \times k}$, $1 \leq k \leq n$, is a 3-tuple $<\beta,\gamma,P>$, where

— β is a mapping of the set $\{1,2,\ldots,k\}$ into $\{1,2,\ldots,n\}$,
— γ is a permutation on the set $\{0,1,\ldots,N-1\}$ and
— P is a tuple of partitions $<P_{\beta(1)},P_{\beta(2)},\ldots,P_{\beta(k)}>$ determined by β and γ as follows:
$P_{\beta(i)} = <P_{\beta(i),1},P_{\beta(i),2},\ldots,P_{\beta(i),2^{n-\beta(i)}}>$ where
$P_{\beta(i),j} = <u_{1,j},u_{2,j},\ldots,u_{2^{\beta(i)},j}>$ such that
$u_{m,j} = \gamma(2^{\beta(i)}(j-1)+m-1)$ for $1 \leq j \leq 2^{n-\beta(i)}$ and $1 \leq m \leq 2^{\beta(i)}$.

Definition III.3. (a-frame, **standard** a-frame): Consider the 3-tuple $<\beta,\gamma,P>$ that defines a frame $F_{N \times k}$. If β is the identity permutation, then $F_{N \times k}$ is an ***a-frame*** denoted by $F^{a}_{N \times k}$. If β and γ are the identity permutations (which implies $P=P^*$), then $F_{N \times k}$ is the standard ***a-frame*** denoted by $F^{sa}_{N \times k}$.

By definition of standard a-type frame, column f^{sa}_i, $1 \leq i \leq n$, has 2^{n-i} blocks, each having 2^i rows. Unless otherwise stated, the number of the rows of $F^{sa}_{1:k}$, $k \geq 1$, is assumed to be N. Similar to the notation of matrices, to be able to refer to specific columns of a frame, the notation $F_{x:y}$ is used to denote the **subframe** that contains those columns of F whose indices are $x, x+1,\ldots,y$. Unless specifically stated, the **number** of **rows** of $F_{x:y}$ is assumed to be N.

Example III.2. The following are examples of **frames** for N=8 and k=3.
(*a*) $F_{8 \times 3}=<\beta,\gamma,P>$ where $\beta=(1\ 2)(3)$, γ is the identity **permutation** and $P=<P_2,P_1,P_3>$ such that $P_1=P^*_1$, $P_2=P^*_2$ and $P_3=P^*_3$.
(b) $F^{a}_{8 \times 3}=<\beta,\gamma,P>$ where β = identity permutation, $\gamma=(0)(1\ 2)(3)(4)(5)(6)(7)$, $P=<P_1,P_2,P_3>$, $P_1=<<0,2>, <1,3>, <4,5>, <6,7>>$, $P_2=<<0,2,1,3>, <4,5,6,7>>$ and $P_3=<0,2,1,3,4,5,6,7>$.

(c) $F^{a}_{8\times3}=<\beta,\gamma,P>$ where β = identity permutation, $\gamma=(0)(1\ 3\ 6\ 4)(2\ 5)(7)$, $P=<P_1,P_2,P_3>$, $P_1=< <0,3>, <5,6>, <1,2>, <4,7> >$, $P_2=< <0,3,5,6>, <1,2,4,7> >$ and $P_3=<0,3,5,6,1,2,4,7>$.

(d) $F_{8\times3}=<\beta,\gamma,P>$ where $\beta=\begin{bmatrix}1\\2\end{bmatrix}$, γ is the identity permutation, $P=<P_2,P_2,P_3>$, $P_2=P_2^*$ and $P_3=P_3^*$.

Definition III.4. (Graphical representation of a frame, rectangle of a frame): The graphical representation of a *frame* $F_{N\times k}=<\beta,\gamma,P>$ consists of k columns labeled f_i, $i=1,\ldots,k$, from left to right and N rows labeled $\gamma(j)$, $j=0,1,\ldots,N-1$ starting at the top. The column f_i corresponds to the partition $P_{\beta(i)}$, that is, f_i consists of $2^{n-\beta(i)}$ blocks of $2^{\beta(i)}$ entries each. In the graphical representation of a frame, any polygon with four sides and four right angles is a rectangle of *the frame*.

Example III.3. Figures III.1a, III.1b, III.1c and III.1d show the graphical representation of the frames described in the part (a), (b), (c) and (d) of Example III.2, respectively. Figure III.1e shows the graphical representation of the standard a-frame $F^{sa}_{8\times3}$. The labels of the partitions below each column are implicit by the sizes of the rectangles in the column and can be omitted

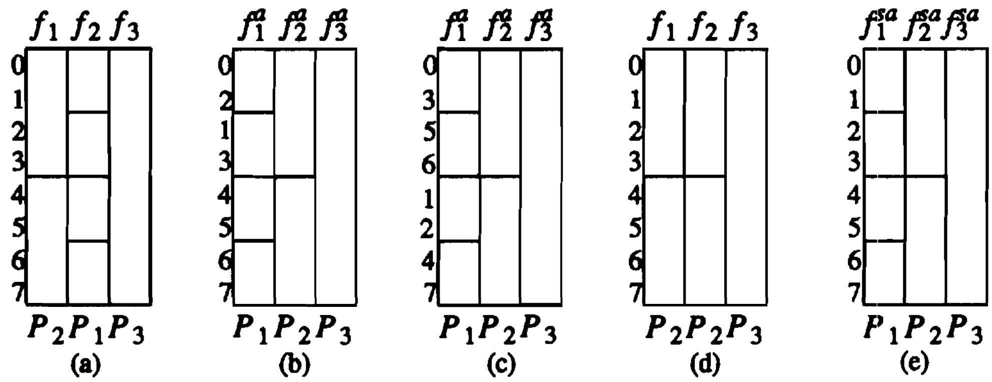


Figure III.1. (a), (b), (c) *and* (d) are the *graphical* representations of the frames described in the part (a), (b), (c) *and* (d) of *Example* III.2, respectively. (e) Graphical representation of the standard a-frame $F^{sa}_{8\times3}$.

Definition III.5. (Fit): Let $k\geq1$, $0\leq i\leq N-1$ and $1\leq j\leq k$. Consider a balanced matrix $A_{N\times k}$ and a frame $F_{N\times k}$. The mamx A *fits* $F_{N\times k}$ if and only if, after placing a_{ij} in the ith row and jth column of $F_{N\times k}$, every rectangle of $F_{N\times k}$ contains a balanced matrix.

Example III.4. The **matrix** E, shown just after Definition **II.4,** fits all the frames shown in Figures **III.1** except $F^{sa}_{8\times3}$ shown in Figure **III.1e** because, for example, the **submatrix** in the top **leftmost** rectangle (the 2-tuple $P_{1,1}$) is not **balanced.**

Note that the value of k in Definition **III.5** does not have to equal n. It will become **clear** that frames of any number of columns can be used to characterize permutations (which are represented by balanced matrices of n columns).

In addition to a-frames, other two types of frames are of use in this paper. One is called universal frame and, as suggested by its name, any balanced **matrix** fits it. The other type of frame is a concatenation of frames and is useful in **characterizing** the permutations realized by, for example, composite networks.

Definition 111.6. (Universal frame $F^*_{1:k}$): The universal *frame* $F^*_{1:k}$, $k \geq 1$, is **such** that, for $i=1,2,\ldots,k$, $\beta(i)=n$, γ is the identity permutation **and** $P_i = P^*_n$. The **universal** frame $F^*_{1:k}$ is illustrated in Figure III.2.

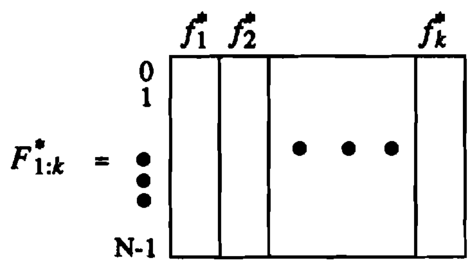


Figure III.2. The universal frame $F^*_{1:k}$.

Definition III.7. ($F^{sa}_{1:n}F^*_{1:m}$): The notation $F^{sa}_{1:n}F^*_{1:m}$, $m \geq 1$, represents the **frame** obtained by concatenating $F^{sa}_{1:n}$ and $F^*_{1:m}$ as shown in Figure 1113.

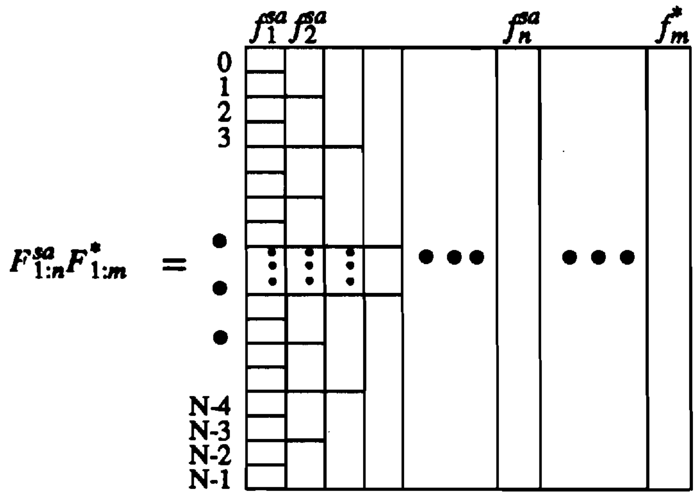


Figure III.3. The frame $F^{sa}_{1:n}F^*_{1:m}$ which is obtained by concatenating $F^{sa}_{1:n}$ and $F^*_{1:m}$.

The following definition states precisely what means to establish a correspondence between a frame and a network.

Definition III.8. (Correspondence between frames and networks): A frame (respectively, an IN) is said to *correspond* to *an* IN (respectively, a frame) if a balanced matrix fits the frame if and only if it passes the network.

When a kame corresponds to a network it suffices to check if a, matrix fits the frame in order to determine whether the network passes the matrix. This does not mean that., when the matrix represents a permutation, the network realizes the permutation. Instead, it means that, when the rows of the matrix are used as routing tags, no conflicts occur in the network.

The complexity of checking that a matrix fits a frame is discussedl next. First, the complexity of testing if a rectangle contains a permutation matrix is considered. Next, the complexity of checking all rectangles of the same size is discussed and, finally, the complexity of checking all rectangles (i.e., the entire frame) is derived. Note that it suffices to consider only those rectangles whose number of columns equals the logarithm of the number of rows.[1] To check whether a given rectangle with x rows and $\log x$ columns contains a balanced matrix, it suffices to verify that the rows of the matrix are distinct. This can be done by building a binary search tree starting with the root which corresponds to the first row of the matrix; each row is then added as a leaf to the tree as long as it is distinct from all previously inserted rows and so that it satisfies the binary-search-tree property [29]. According to this property, if $v(p)$ is the value of the row that corresponds to node p, then $v(y)<v(p)$ for any node y in the left

[1] All logarithms are in base 2 unless stated otherwise.

subtree of p and $v(z)>v(p)$ for any node z in the right subtree of p. In the worst case, this procedure takes $O(x^2)$ steps and has average complexity of $O(x\log x)$ [29]. If several rectangles of the same size exist in a frame, then the total number of rows contained in all the rectangles with the same columns is N. The same procedure can be used for each rectangle and the total worst case and average complexities will be $O(N^2)$ and $O(N\log N)$, respectively. Because there are at most k different types of rectangles in a frame with k columns, the total worst case and average complexities are $O(kN^2)$ and $O(kN\log N)$, respectively. These bounds apply to any frame, but it is possible to do better with particular frames. For example, for a-frames the worst-case complexity becomes $O(N^2+2(N/2)^2+\cdots+(N/2)2^2)=O(N^2)$.

IV. BASELINE-TYPE NETWORKS

Equivalence relations among INs have been extensively studied in the literature using different tools such as graph theory, group theory, and Boolean algebra [6,11,27,26]. Networks can be modeled by directed graphs where vertices and edges represent switches and links, respectively. Two INs are *functionally equivalent* if they realize the same set of permutations while two INs are topologically *equivalent* if their topologies (i.e., directed graphs) are isomorphic. Wu and Feng [11] have shown the topological equivalence of a class of MINs, which include data manipulator [14], omega [1], flip [15], SW-banyan (s=f=2) [16], and indirect binary n-cube [17], baseline and reverse baseline [11]. From [18], "the notion of functional equivalence is more practical than that of topological equivalence because it provides an equivalence basis among networks at their inputs, and thus it does not call for any modification in their internal switching structure". Given a network in a class of isomorphic INs, it is possible to rename its inputs and/or outputs so that this network can directly simulate any network in the class [11]. In this section, all the matrices that pass those networks that are topologically equivalent to the k-stage baseline, $1 \le k \le n$, are identified by a-frames that may differ only in how their rows are labeled. First, the permutations realized by the k-stage reverse baseline are identified. Then, this result is extended to the other networks. These results also show how the addition of a stage to the right of these networks changes the type of their realizable permutations. An algorithm is provided to find whether a network is topologically equivalent to the reverse baseline network, its corresponding frame and how to relabel inputs and outputs to achieve functional equivalence. Omitted proofs are provided in the Appendix.

IV.1. Correspondence between $F^{sa}_{1:k}$ and $RB_{1:k}$

Because $RB_{1:n}$ is functionally and topologically equivalent to $BE_{1:n}$ [11], any permutation that is realized by $RB_{1:n}$ is also realized by $BE_{1:n}$, and vice versa. However, this is not true for $RB_{1:k}$ and $BE_{1:k}$, $1 < k < n-1$, because they are not functionally equivalent (they are only topologically equivalent). But, the set of balanced matrices that pass $RB_{1:k}$ is the same as the set of balanced matrices that pass $BE_{1:k}$ as explained next. The network $RB_{1:k}$ can be obtained by repositioning the **SBs** of the second stage through the last stage of $BE_{1:k}$ and reordering its outputs. It follows that any pair of routing tags that enter a SB at the kth stage of $BE_{1:k}$ also enter a SB at the kth stage of $RB_{1:k}$, and vice versa. So, if the routing tags used in $BE_{1:k}$ do not create any conflict, then they also do not have any conflict in the **SBs** of $RB_{1:k}$, and vice versa. Therefore, a balanced matrix $D_{1:k}$ passes $RB_{1:k}$ if and only if $D_{1:k}$ passes $BE_{1:k}$.

The following theorem shows that there exists a very close relation between $RB_{1:k}$ and $F^{sa}_{1:k}$, $1 \le k \le n$, so that the matrices that pass the network can be identified by $F^{sa}_{1:k}$. It shows that the ith input of $RB_{1:k}$ is sent without conflicts to the output whose value equals ($\lfloor i/2^k \rfloor \times 2^k$) plus the value of $D_{1:k}(i)$ when the ith row of a matrix $D_{1:k}$ that fits $F^{sa}_{1:k}$ is used as the routing tag for the ith input of $RB_{1:k}$.

Theorem IV.1. A matrix $D_{1:k} = [d_1\ d_2\ \cdots\ d_k]$ fits $F^{sa}_{1:k}$ if and only if $D_{1:k}$ passes $RB_{1:k}$, $1 \le k \le n$. Moreover, $RB_{1:k}$ sends its ith input to its jth output, where j is equal to the sum of ($\lfloor i/2^k \rfloor \times 2^k$) and the value of $D_{1:k}(i)$.

Basic idea of *proof* (complete proof appears in Appendix):

(+) $D_{1:k}$ fits $F^{sa}_{1:k} \rightarrow D_{1:k}$ passes $RB_{1:k}$.
Induction on k is used. For $k=1$, each rectangle of $F^{sa}_{1:k}$ has a 0 and a 1. These correspond to the control bits of a switch in $RB_{1:k}$ and, thus, no conflict occurs. For $k>1$, assuming the theorem holds for $k-1$, it is also shown that each switch in the kth stage "has" control bits 0 and 1 and, therefore, no conflicts occur. These control bits must appear as the kth bits at the end of identical (k−1)-bit rows of subframes $F^{\alpha_1}_{2^{k-1}\times k-1}$ and $F^{\alpha_2}_{2^{k-1}\times k-1}$ of $F^{sa}_{1:k}$ so that $D_{1:k}$ fits $F^{sa}_{1:k}$. Each subframe corresponds to a subnetwork of $RB_{1:k}$ which is also a reverse baseline network $RB_{2^{k-1}\times k-1}$.

(←) $D_{1:k}$ passes $RB_{1:k} \rightarrow D_{1:k}$ fits $F^{sa}_{1:k}$.
Induction on k is used. For $k=1$, if d_1 passes RB_1, then each rectangle of $F^{sa}_{1:k}$ contains a 0 and a 1 and d_1 fits f^{sa}_1. For $k>1$, assuming the theorem holds for $k-1$, it is shown that for the outputs of two subnetworks $RB^{\alpha_1}_{2^{k-1}\times k-1}$ and $RB^{\alpha_2}_{2^{k-1}\times k-1}$ to cause no conflict in any switch of the kth stage it must be the case that a 0 and a 1 are added to the $k-1$ entries of identical rows of the frames that correspond to the two subnetworks. This implies that $D_{1:k}$ fits $F^{sa}_{1:k}$. The value of j follows from the topology of $RB_{1:k}$ and how switches are set by control bits. □

Corollary IV.l. A network with k stages and **N inputs/outputs** is topologically **equivalent** to the ***k*-stage** reverse baseline, $RB_{1:k}$, if and only if it **corresponds** to an a-type frame $F^{a}_{1:k}$, where $1 < k < n$.

IV.2. Permutations Realized by Baseline-Type Networks

In this section, a-type frames arc used to characterize all the permutations realized by any network that is topologically equivalent to the baseline network. An algorithm, called **FRAME_IN,** is introduced to **determine** the a-type frame that corresponds to a given network. It is also shown how to construct a network to realize all the permutations that fit an a-type frame.

Let $F^{a}_{1:k}(\alpha^{-1})$ &note a particular a-type frame where $\gamma=\alpha^{-1}$, **i.e.,** whose row labels form the vector α^{-1}. Let Π &note a network with k stages which is the same as $RB_{1:k}$ except that the label of its ith input equals the ith **entry** of α^{-1}. By Corollary IV.**1**, a balanced matrix $D_{1:k}$ fits $F^{a}_{1:k}(\alpha^{-1})$ if and only if $D_{1:k}$ passes Π. If $k=n$, any of **these** balanced matrices represents a permutation, so that Π is a **network** that realizes **all** the permutations characterized by $F^{a}_{1:k}(\alpha^{-1})$. If $k<n$, then the relation between a $D_{1:k}$ that fits $F^{a}_{1:k}(\alpha^{-1})$ and a permutation that passes Π is first determined. By applying this relation to every balanced matrix that fits $F^{a}_{1:k}(\alpha^{-1})$, all the permutations realized by Π are determined. **Theorem** IV.3 determines the relation between a balanced mamx that fits $F^{sa}_{1:k}$ and the permutation realized by $RB_{1:k}$ when this balanced matrix **passes** the network. Corollary IV.3 generalizes this result to the **class** of baseline-type networks.

Theorem IV.3. A **matrix** $D_{1:k}$, $1 \le k \le n$, fits $F^{sa}_{1:k}$ if and only if $RB_{1:k}$ realizes **the permutation** represented by $[I_{1:n-k}\, D_{1:k}]$.

Proof. ($\rightarrow$) Let $D_{1:k}$ fit $F^{sa}_{1:k}$. It **is** shown that $RB_{1:k}$ realizes the permutation **represented by** $[I_{1:n-k}\, D_{1:k}]$.

Theorem IV.1 states that $RB_{1:k}$ sends its ith input, $0 \le i \le N-1$, to its jth output, **where** j is equal to the sum of $(\lfloor i/2^k \rfloor \times 2^k)$ and the value of $D_{1:k}(i)$. Due to the fact **that** $[(\lfloor i/2^k \rfloor \times 2^k)+D_{1:k}(i)]$ equals the ith row **of** $[I_{1:n-k}\, D_{1:k}]$, $RB_{1:k}$ realizes the **permutation** represented by $[I_{1:n-k}\, D_{1:k}]$.

($\leftarrow$) Assume that $RB_{1:k}$ realizes the **permutation** represented by $[I_{1:n-k}\, D_{1:k}]$. It is shown that $D_{1:k}$ fits $F^{sa}_{1:k}$.

Because $RB_{1:k}$ realizes the permutation represented by $[I_{1:n-k}\, D_{1:k}]$, it sends its **ith** input to the output whose value equals the sum of $(\lfloor i/2^k \rfloor \times 2^k)$ and the value of $D_{1:k}(i)$, $D_{1:k}$ passes $RB_{1:k}$. It follows from Theorem IV.l that $D_{1:k}$ fits $F^{sa}_{1:k}$. □

Corollary IV.3. Consider a k-stage, $1 \le k \le n$, network Π which is topologically **equivalent** to $RB_{1:k}$. The network Π is functionally and topologically equivalent to a network $IP_{in}RB_{1:k}IP_{out}$, where IP_{in} and IP_{out} are interconnection patterns that realize **permutations** α_{in} and α_{out}, **respectively**. Also, let $F^a_{1:k}(\alpha^{-1}_{in})$ denote an a-type k-column **frame** whose ith row label equals $\alpha^{-1}_{in}(i)$ for $i=0,1,\ldots,N-1$. A matrix $D_{1:k}$ fits $F^a_{1:k}(\alpha^{-1}_{in})$ if and only if Π realizes the **permutation** $\alpha_{in}.\mu.\alpha_{out}$, where μ is the permutation represented **by** the balanced matrix $[I_{1:n-k}\ D^*_{1:k}]$ and $D^*_{1:k}(i) = D_{1:k}(\alpha^{-1}_{in}(i))$.

Corollary IV.3 implies that the network $IP_{in}RB_{1:k}$ corresponds to the frame $F^a_{1:k}(\alpha^{-1}_{in})$, where IP_{in} **realizes** the permutation α_{in}. Hence, for a given $F^a_{1:k}$, a corresponding network can be constructed easily. The following example shows the **construction** of a network that realizes a set of permutations which **includes** two given permutations.

Example IV.1. Let $N=16$, $k=2$, $0 \le i \le N-1$. Assume that α_{in} and α_{out} &note the permutations realized by the interconnection patterns IP_{in} and IP_{out}. Given two permutations a = (0 9 8 5 1 2 12 10 1 4 6 3 7 11 13)(4)(15) and b = (0 7)(1)(2 3 9 13 11 8 4 5)(6 12)(10 15 14), it is shown how to **construct** a network $IP_{in}RB_{1:2}IP_{out}$ that realizes a set of permutations including a and b. **Let** A and B refer to the binary representations of a and b, respectively. By Theorem IV.3, any permutation that passes $RB_{1:2}$ must be represented by a balanced matrix whose first (leftmost) two columns **form** $I_{1:2}$ (recall that $k=2$ and $n=4$ in this example). If there was only one given permutation, then the balanced matrix representing the **permutation** could be **converted** by IP_{in} to a balanced matrix whose first two columns **form** $I_{1:2}$ because IP_{in} **can** be chosen so **as** to permute the rows in any given way. However, if more than one **permutation** are given, and the first two columns of their binary **representations** do not **form** the same matrix, then IP_{out} is needed to convert the binary **representations** of these permutations into balanced matrices whose first two columns; **form** the same matrix. So, the matrices A and B are first converted by IP_{out} to A and B such that $\dot{A}_{1:2} = \dot{B}_{1:2}$. Specifically, α_{out} converts A and B to A and B, respectively such that $A(i) = \alpha^{-1}_{out}(A(i))$ and $B(i) = \alpha^{-1}_{out}(B(i))$. Then, A and B **are** converted by α_{in} to A and B, respectively such that the first two columns of each of these **matrices form** $I_{1:2}$. **Specifically,** $\ddot{A}(i) = A a\ (i))$ and $\ddot{B}(i) = B(\alpha^{-1}_{in}(i))$. **For** instance, α_{out} = (0 13 12)(1 5 7 2 4)(3 8 6 9 14)(10 15 11) converts a and b to $\dot{a}$ = (0 6 14 8 1 7 15 1093542131211) and $\dot{b}$ = (0 5 7 12 8 2 14 11 3 6 13 15 9 0 5 7), respectively. Similarly, α_{in}= (0 5 4 1 7 15 9 3 6 13 12 11)(2 14 8)(10) converts 6 and b into a = (0)(1 2)(3)(4)(5 6)(7)(8)(9 10)(11)(12)(13 14)(15) and $\ddot{b}$ = (0 3)(1)(2)(4 7)(5)(6)(8 11)(9)(10)(12 15)(13)(14), respectively. The binary representations of a, a, $\ddot{a}$, b, b and *bare* shown below. The network that realizes the pennutations a and b is shown in Figure IV.1.

$$
A: \begin{array}{r} 0\\1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15 \end{array}
\begin{bmatrix} 1001\\0010\\1100\\0111\\0100\\0001\\0011\\1011\\0101\\1000\\1110\\1101\\1010\\0000\\0110\\1111 \end{bmatrix}
\quad
B: \begin{array}{r} 0\\1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15 \end{array}
\begin{bmatrix} 0111\\0001\\0011\\1001\\0101\\0010\\1100\\0000\\0100\\1101\\1111\\1000\\0110\\1011\\1010\\1110 \end{bmatrix}
\quad
\dot{A}: \begin{array}{r} 0\\1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15 \end{array}
\begin{bmatrix} 0110\\0111\\1101\\0101\\0010\\0100\\1110\\1111\\0001\\0011\\1001\\0000\\1011\\1100\\1000\\1010 \end{bmatrix}
$$

$$
\dot{B}: \begin{array}{r} 0\\1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15 \end{array}
\begin{bmatrix} 0101\\0100\\1110\\0110\\0001\\0111\\1101\\1100\\0010\\0000\\1010\\0011\\1000\\1111\\1011\\1001 \end{bmatrix}
\quad
\ddot{A}: \begin{array}{r} 0\\1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15 \end{array}
\begin{bmatrix} 0000\\0010\\0001\\0011\\0100\\0110\\0101\\0111\\1000\\1010\\1001\\1011\\1100\\1110\\1101\\1111 \end{bmatrix}
\quad
\ddot{B}: \begin{array}{r} 0\\1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15 \end{array}
\begin{bmatrix} 0011\\0001\\0010\\0000\\0111\\0101\\0110\\0100\\1011\\1001\\1010\\1000\\1111\\1101\\1110\\1100 \end{bmatrix}
$$

Because a 2x2 switch has two possible settings (cross and straight), the number of balanced matrices that pass a k-stage baseline-type network with N inputs equals $2^{m/2}$. By Corollary IV.l, for any given k-column a-type frame, there exists a corresponding baseline-type network. Therefore, exactly $2^{kN/2}$ balanced matrices fit any $F^a_{1:2}$. For $k=2$, 2^N balanced matrices pass a baseline-type network. Let $D^r_{1:2}$, $1 \le r \le 2^N$, &note one of the 2^N balanced matrices that fit $F^a_{1:2}(\alpha_{in}^{-1})$. Also, assume that $D^{*r}_{1:2}$ is obtained from $D^r_{1:2}$ such that $D^{*r}_{1:2}(i) = D^r_{1:2}(\alpha_{in}^{-1}(i))$. Let μ_r denote the permutation represented by $[I_{1:2}\, D^{*r}_{1:2}]$. So, the network shown in Figure IV.l realizes any of those permutations that result from $\alpha_{in}.\mu_r.\alpha_{out}$. The ith row of $D^{*r}_{1:2}$ is used as the routing tag for the ith input of $RB_{1:2}$ in $IP_{in}RB_{1:2}IP_{out}$. As an example, let $r=1$ and consider the balanced matrix $D^1_{1:2}$, shown in Figure IV.2a, that fits $F^a_{1:2}(\alpha_{in}^{-1})$. The matrix $D^{*1}_{1:2}$ that is obtained from $D^1_{1:2}$, and the matrix $[I_{1:2}\, D^{*1}_{1:2}]$ are also shown in Figure IV.2. When the: ith mw of $D^{*1}_{1:2}$ is used as the muting tag for the ith input of $RB_{1:2}$, $RB_{1:2}$ realizes the: permutation $\mu_1 = (0)(1\ 3\ 2)(4\ 5\ 6)(7)(8\ 10\ 11)(9)(12\ 15\ 14\ 13)$ which is represented by $[I_{1:2}\, D^{*1}_{1:2}]$. On the other hand, the network $IP_{in}RB_{1:2}IP_{out}$ realizes the permutation $(0\ 9\ 4\ 8\ 5\ 7\ 3\ 1\ 2\ 12\ 6)(10)(11\ 13)(14\ 15)$ which results from $\alpha_{in}.\mu_1.\alpha_{out}$. End of example.

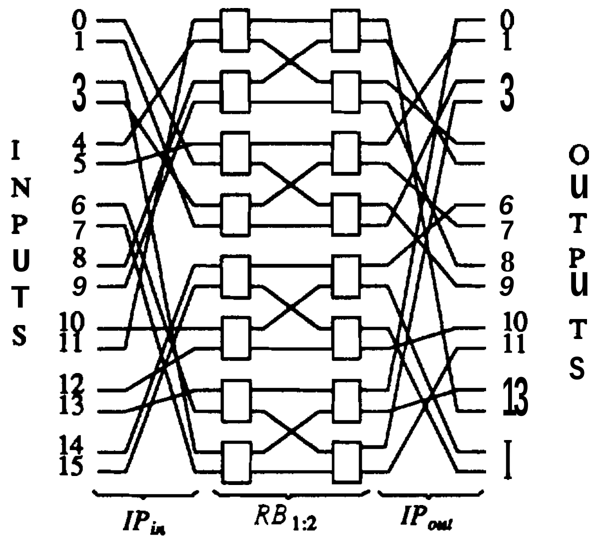


Figure IV.1. The network $IP_{in}RB_{1:2}IP_{out}$ of Example IV.1.

(a) $D^1_{1:2}$

0	10
1	11
2	01
3	00
4	11
5	01
6	00
7	10
8	01
9	10
10	11
11	00
12	00
13	11
14	10
15	01

(b) $F^a_{1:2}(\alpha_{in}^{-1})$

11	0	0
4	1	1
8	0	1
9	1	0
5	0	1
0	1	0
3	0	0
1	1	1
14	1	0
15	0	1
10	1	1
12	0	0
13	1	1
6	0	0
2	0	1
7	1	0

(c) $D^{*;1}_{1:2}$

0	00
1	11
2	01
3	10
4	01
5	10
6	00
7	11
8	10
9	01
10	11
11	00
12	11
13	00
14	01
15	10

(d) $[I_{1:2}\ D^{*;1}_{1:2}]$

0	0000
1	0011
2	0001
3	0010
4	0101
5	0110
6	0100
7	0111
8	1010
9	1001
10	1011
11	1000
12	1111
13	1100
14	1101
15	1110

Figure IV.2. *(a) A balanced matrix $D^1_{1:2}$ which fits $F^a_{1:2}(\alpha_{in}^{-1})$.* (b) $F^a_{1:2}(\alpha_{in}^{-1})$ *with* $D^1_{1:2}$. *(c)* $D^{*;1}_{1:2}$ *whose ith row equals* $D^1_{1:2}(\alpha_{in}^{-1}(i))$. (d) $[I_{1:2}\ D^{*;1}_{1:2}]$.

In the rest of this section, some preliminary results used in the Algorithm FRAME_IN are first presented, then the algorithm is introduced.

Lemma **IV.1.** Let r denote the **reverse** permutation mpresented by the reverse permutation matrix $R_{N \times n}$ described in Definition 11.1. The reverse baseline network $RB_{1:n}$ **realizes** r when all the switches are set straight.

Proof. The permutation **realized** by $RB_{1:n}$ when all the **switches are** set straight is determined by the **interconnection** patterns $IP_{in}, IP_1, \ldots, IP_{n-1}, IP_{out}$. Because $IP_{in} = IP_{out}$ = identity pattern, the permutation is given by $\alpha_1 . \alpha_2 \ldots \alpha_{n-1}$ where α_i is the permutation realized by IP_i. Permutation α_i is **such** that $\alpha_i(x)$ **rotates** left the **right-most** $i+1$ bits of x by one position because IP_i is a pile of 2^{n-i-1} shuffle-exchange **patterns** each with 2^{i+1} links. Applying this operation for all i starting with the initial **matrix** $I_{N \times n}$ yields the reverse permutation matrix $R_{N \times n} = [i_n \, i_{n-1} \cdots i_1]$. □

Because the reverse baseline network can be converted to the **baseline** network by repositioning the switches of the middle stages only, Lemma **IV.1** is also valid for the baseline network. If there exists a unique path between any input and any output of a network, then the network satisfies the Banyan ***property*** [6,26]. **Bermond** et. al [26] **present** a set of properties to determine whether a network is topologically equivalent to baseline network. Their main result is **formally** restated below.

Theorem IV.2. [26] Let G be a directed graph representing a network with n **stages** and N **inputs/outputs** which satisfies the Banyan property. This network is topologically equivalent to the baseline network if and only if both the **first** j stages and the **last** j stages of G contain 2"-J connected components for each j, $1 \leq j \leq n$.

This result is used next as the basis for Algorithm FRAME_IN. The description of the algorithm is followed by a proof of its correctness and analysis of its complexity.

Algorithm ***FRAME***_IN

Input: A network GN with 2x2 switches, n stages and 2" **inputs/outputs.**

Output: An a-type frame that **corresponds** to GN if GN is topologically equivalent to the baseline network, the permutations α_{in} and α_{out} realized by the interconnection patterns IP_{in} **and** IP_{out}, respectively, such that the network $IP_{in} GN_{1:n} IP_{out}$ is functionally equivalent to $RB_{N \times n}$.

Step 1. Let G denote a graph with n "stages" that is **obtained** by representing the switches and links of the given network by vertices and edges that **are** directed from left to right, respectively.

Step 2. Using a **breadth-first** search algorithm check whether there exists a unique path between any input vertex and **any** output vertex of G. If so, go to next step. If not, go to Step 9,

Step 3. Let j andp be integer variables initialized to 0.

Step 4. Increment j by 1. If $j>n$, then go to next step; otherwise, **using** a depth-search algorithm, check whether the last j stages of the G contain 2''-J connected components. If so, go to Step **4.** If not, go to Step 9.

Step 5. Increment p by 1. **If** $p>n$, then go to Step 7; otherwise, **using** a depth-search algorithm, check whether the **first** p stages of G contains 2^{n-p} connected components. If so, go to next step. If not, go to Step 9.

Step 6. If $p=1$, let V_r^1 denote a vector of the input labels of a distinct connected component (a **2x2** switch) for each r, $(1 \le r \le 2^{n-1})$, and then go to Step 5; otherwise, do: let V_r^p, $1 \le r \le 2^{n-p}$, denote a vector that is formed by merging two vectors V_s^{p-1} and V_t^{p-1} for 1 Ss, $t \le 2^{n-p+1}$ and $s \neq t$ such that the set: of entries of V_r^p equals the set of input labels of a distinct connected component determined in Step 5. Go to Step 5.

Step 7. Let $\gamma(i) = V_1^n(i)$ for $i = 1,2,...,N-1$ (note that V_1^n is obtained in Step 6). Write "The a-type frame $F_{1:n}^a$ whose ith row **label** equals $\gamma(i)$ corresponds to the GN".

Step 8. Let σ denote the permutation realized by the given network $GN_{1:n}$ when all the switches **are** set straight. The permutation realized by IP_{in} is $\alpha_{in} = \gamma^{-1}$. The **permutation** realized by IP_{out} is $\alpha_{out} = \sigma^{-1}.\alpha_{in}^{-1}.r$, where r is the reverse permutation represented by the reverse permutation matrix $R_{N \times n}$ (see Definition **II.1**). Stop.

Step 9. Write "The given network is not topologically equivalent to baseline network and no **corresponding** a-type frame exists". Stop.

In Steps 2 through 6, Algorithm FRAME–IN checks whether **the** given network satisfies the set of properties described in Theorem IV.2. Specifically, Step 2 checks the Banyan **property,** while Steps 3 through 6 check whether both the first j stages and the last j stages of the network graph contain 2''-J connected **components,** for each **j.** So, if Algorithm FRAME_IN fails in any of these steps, then it **follows** from Theorem **IV.2** that the given network is not topologically equivalent to baseline network and, by Corollary IV.1, has no corresponding a-type frame.

It is now shown that the given network corresponds to the ***a*-type** frame determined in Step 7, that is, any balanced matrix that fits the a-type **frame** determined in

Step 7 passes the given network, and vice versa. Theorem **IV.1** proves that, for $1 \le k \le n$, the frame $F^{sa}_{1:k}$ corresponds to $RB_{1:k}$, that is, a balanced matrix $D_{1:k}$ fits $F^{sa}_{1:k}$ if and only if $D_{1:k}$ passes $RB_{1:k}$. Note that $RB_{1:j}$ is a pile of 2^{n-j} $RB_{2^j \times j}$s. Recall that the only difference between the standard a-frame $F^{sa}_{1:k}$ and an a-type **frame** $F^{a}_{1:k}$ is the **order** of their row labels. Because Step 7 assigns $\gamma(i)$ to the ith row label of $F^{a}_{1:n}$, this **frame** corresponds to the given network. Step 8 **first** assumes that the permutation **realized** by the given network equals σ when all the switches are set straight. Then, Step 8 **states** that the interconnection pattern IP_{in} **realizes** the permutation $\alpha_{in} = \gamma^{-1}$. **Relabeling** the ith input of the given network by $\gamma(i)$ is equivalent to adding the interconnection pattern IP_{in} to the left of the given network. Thus, any balanced **matrix** that fits the a-type frame obtained in Step 7 passes the network $IP_{in}GN_{1:n}$, and vice versa. Algorithm **FRAME_IN** also adds an interconnection pattern IP_{out} that realizes a permutation called α_{out} to the right of the given network such that the **network** $IP_{in}GN_{1:n}IP_{out}$ realizes the permutation r when all the switches are set straight. By **Lemma IV.1,** the reverse baseline (baseline) realizes the permutation r when all the switches are set **straight.** Therefore, the network $IP_{in}GN_{1:n}IP_{out}$ is functionally **and** topologically **equivalent** to the reverse baseline and baseline networks. This completes the proof of **correctness** of the algorithm.

The graph of Algorithm **FRAME_IN** can have at most $O(N\log N)$ vertices **because** each vertex represents a switch. Algorithm FRAME–IN **uses** a breadth-first **search** to check whether the given network holds the Banyan property. A depth-first **search** is **used** to identify the connected components of G, and that the depth-first forest **contains** as many trees as G has connected components **[29]**. If V and E *are* the sets of **vertices** and edges, respectively, the running time of both a **breadth-first** search and a **depth-first** search is $\Theta(V+E)$. This implies that, for each value of j, Algorithm **FRAME_IN** takes $\Theta(N\log N)$ time. Because there *are* $2\log N$ iterations, the running **time** of Algorithm **FRAME_IN** is $\Theta(N \log^2 N)$.

Algorithm FRAME–IN yields a **frame** that corresponds to the given network. **This** means that any matrix that fits the **frame** also passes the network and vice versa. However, this does not necessarily mean that the permutation **represented** by the **matrix** is realized by the network. When a balanced matrix $D_{N\times n}$ fits an a-frame corresponding to a baseline-type network, the network realizes the permutation $d \cdot \alpha_{out}$, where d is the permutation represented by $D_{N\times n}$ and α_{out} is the permutation realized by IP_{out} determined in Step 8 of Algorithm FRAME–IN. In other words, given a network that is topologically equivalent to the reverse baseline, relabeling its **inputs** and outputs by α_{in}^{-1} and α_{out}, respectively, results in a new network that is functionally equivalent to the reverse baseline.

As an example, for N=16, Algorithm FRAME–IN can be used to characterize the **permutations** of the following baseline-type networks: **generalized** cube, omega, indirect binary n-cube, banyan (S=F=2), inverse omega, modified data manipulator, flip. The topological equivalence among these networks and baseline and reverse baseline networks is well known and previously studied in [6,11,18,26]. From Corollary IV.1, each of these networks corresponds to an a-frame. Algorithm **FRAME_IN** yields the row labeling γ and α_{out} for each of these networks and frames as follows: $\gamma = \alpha_{out}$ = identity permutation for the reverse baseline and baseline networks, γ = the reverse permutation = (0)(1 8)(2 4)(3 12)(5 10)(6)(7 14)(9)(11 13)(15) and α_{out} = identity permutation for the omega and generalized cube, γ = identity permutation and α_{out} = (0)(1)(2 8)(3 9)(4)(5)(6 12)(7 13)(10)(11)(14)(15) for the indirect binary cube, banyan, inverse omega, and flip networks, γ = (0)(1)(2 8)(3 9)(4)(5)(6 12)(7 13)(10)(11)(14)(15) and α_{out} = identity permutation for the modified data manipulator network.

V. NETWORKS $RB_{1:n}SE_{1:m}$ AND $SE^{-1}_{1:m}RB_{1:n}$

This section illustrates how frames can be used to characterize permutations performed by relatively complex networks with more than n stages. It is first shown that the balanced matrices that pass the network $RB_{1:n}SE_{1:m}$, $m \geq 0$, are identified by the frame $F^{sa}_{1:n}F^{*}_{1:m}$ (Theorem V.1), then it is shown that $RB_{1:n}SE_{1:m}$ is functionally and topologically equivalent to $SE^{-1}_{1:m}RB_{1:n}$ (Theorem V.2). Hence, any balanced matrix passing $RB_{1:n}SE_{1:m}$ also passes $SE^{-1}_{1:m}RB_{1:n}$, and vice versa. Theorem V.1 also shows how the addition of a SE stage to the right of $RB_{1:n}SE_{1:m}$ affects the type of permutations realized by the network. Theorem V.2 proves that the addition of a SE stage to the right of $RB_{1:n}SE_{1:m}$ is equivalent to the addition of an inverse SE stage to the left of $SE^{-1}_{1:m}RB_{1:n}$. All the proofs are provided in the Appendix.

V.1. Balanced Matrices and Shuffle-Exchange Networks

Linial and Tarsi [2] have shown how balanced matrices can be used to determine the number of SE stages (or the number of passes through a single SE stage) necessary to realize a given permutation. Lemma V.1 below restates their result using the notation and assumptions of this paper.

Lemma V.1. [2] Let $M_{N\times m}$ and $C_{N\times k}$ be balanced matrices such that $M_{N\times m} = [I_{N\times n}\ C_{N\times k}]$, $k \geq 1$ and $n+k=m$. The network $SE_{N\times k}$ realizes the permutation represented by $M_{(m+1-n):m}$.

To illustrate Lemma V.1, consider the identity permutation matrix $I_{8\times 3} = [i_1\ i_2\ i_3]$ and the balanced matrices $M_{8\times 4} = [I_{8\times 3}\ i_1]$, $M_{8\times 5} = [I_{8\times 3}\ i_1\ i_2]$ and $M_{8\times 6} = [I_{8\times 3}\ I_{8\times 3}]$. Because $M_{8\times 4}$, $M_{8\times 5}$ and $M_{8\times 6}$ are balanced, the permutations represented in binary by [i_2 i_3 i_1], [i_3 i_1 i_2] and [i_1 i_2 i_3] are **realized** by the **single**-stage SE, 2-stage SE and 3-stage SE with $N=8$ **inputs/outputs,** respectively.

V.2. Permutations Realized by $RB_{1:n}SE_{1:m}$

The following theorem shows how the concatenated frame $F^{sa}_{1:n}F^{*}_{1:m}$ can be used to characterize the permutations **realized** by $RB_{1:n}SE_{1:m}$.

Theorem V.1. A balanced matrix $D_{1:(n+m)}$, $m \geq 0$, **fits** the frame $F^{sa}_{1:n}F^{*}_{1:m}$ if **and** only if $D_{1:(n+m)}$ passes the network $RB_{1:n}SE_{1:m}$. Moreover, $RB_{1:n}SE_{1:m}$ realizes the **permutation represented by** $D_{(m+1):(n+m)}$.

V.3. Permutations Realized by $SE^{-1}_{1:m}RB_{1:n}$

It is shown that the network $SE^{-1}_{1:m}RB_{1:n}$ constructed by **appending** the network $SE^{-1}_{1:m}$ to the left of $RB_{1:n}$ is functionally and topologically equivalent to the network $RE_{1:n}SE_{1:m}$ constructed by appending $SE_{1:m}$ network to the right of $RB_{1:n}$. Also, **because** $RB_{1:n}$ is functionally and topologically equivalent to $BE_{1:n}$, Theorem V.2 **remains** valid when $RB_{1:n}$ is **replaced** by $BE_{1:n}$.

Theorem V.2. The network $RB_{1:n}SE_{1:m}$, $m \geq 1$, is topologically and functionally **equivalent** to the network $SE^{-1}_{1:m}RB_{1:n}$.

VI. NEW PROOFS FOR REARRANGEABILITY OF BENES AND THREE-STAGE CLOS NETWORKS

Rearrangeability of Benes and three-stage Clos networks is **proven** in **[7,13]** using the Slepian-Duguid **theorem** which applies only to symmetric **networks.** In this section, new simpler proofs **are** provided for **rearrangeability** of these **networks** using balanced **matrices** and the properties of graph **theory.** These proofs directly **lead** to routing **algorithms [19]** and provide an insight into the proofs of Section VII that identify the permutations realized by subnetworks of the Benes network. In what follows, some known results from [2] and definitions used in the proofs are presented **first.** Lemma VI.1 from **[2]** is self-explanatory.

Lemma VI.1. [2] For $n \geq 2$, let A and B be two $N\times(n-1)$ **balanced** matrices. **Then** there exists a column vector x such that both [A x] and [x B] **are** balanced **matrices.**

Note that, when the order of columns in a balanced **matrix** with **at** most n columns is changed, the matrix remains balanced. Therefore, the position of x in the matrices A and B in Lemma VI.1 is immaterial. Because the possible choices of vector x increase as the number of columns of A or B is reduced, Lemma VI.1 remains valid when A and B have less than $n-1$ columns.

Some properties of balanced matrices can be captured by graphs. Therefore, some basic definitions of graph theory are given below. A graph $G=(V,E)$ consists of a set of vertices V and a set of edges E, each of which is a pair of vertices. The union of two graphs $G_1=(V,E_1)$ and $G_2=(V,E_2)$ is the graph $G=G_1 \cup G_2=(V,E_1 \cup E_2)$. In other words, an edge is present in $G=G_1 \cup G_2$ if and only if it is present in either G_1 or G_2. A subset M of edges in a graph G is called independent or a matching if no two edges of M have a vertex in common. A matching M is said to be a *perfect* matching if it covers all vertices of G. More extended discussion of these basic concepts can be found in [3,4].

Definition **VI.1.** (Perfect **matching** graph of a matrix): Let A be an $N \times k$ ($1 \le k \le n-1$, $n \ge 2$) balanced matrix. A *perfect* matching graph of A, denoted by PG_A, is a graph whose vertices are in one-to-one correspondence with the rows of A, have degree one and vertices v_i and v_j are joined by an edge only if the ith row and jth row of A are identical.

If the number of columns in a balanced matrix $A_{N \times k}$ is less than $n-1$ (i.e., if $k<n-1$), then its perfect matching graph is not unique because each distinct row in A appears 2^{n-k} times. If $k=n-1$, then PG_A is unique because each distinct row in A appears twice. As an example, consider the balanced matrix $F_{8 \times 2}$ presented just after Definition 11.4. Its perfect matching graph is unique and shown in Figure VI.1a.

Definition VL2. (Labeling): 2-labeling or 2-coloring of a graph is the assignment of integers 0 and 1 to its vertices such that the labels of the vertices incident with an edge are different.

Fact VI.l. [2]. The union of two perfect matching graphs with the same set of vertices is a union of disjoint even cycles and, therefore, it can be 2-labeled.

Definition VL3. (Perfect matching graph of a frame column): Let f, &note a column of a frame $F_{N \times k}$. A *perfect* matching graph off„ &noted by PG_{f_m}, is a graph whose vertices are in one-to-one correspondence with the row labels of $F_{N \times k}$, have degree one and vertices v_i and v_j are joined by an edge only if i and j belong to the same block off,.

Example **VI.1.** One possible perfect matching graph for frame column f_2^3 in Figure III.1b is shown in Figure VI.1a. The graph in Figure VI.1b is the unique perfect matching graph of frame column f_1^a in Figure III.1b.

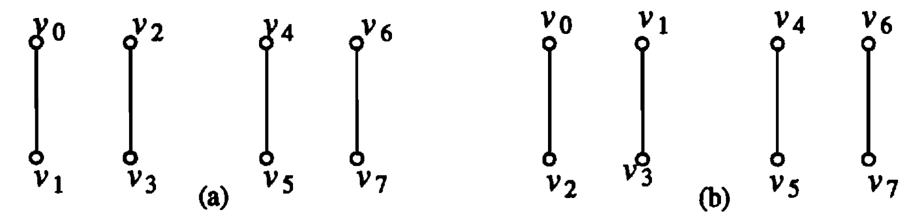


Figure VI.1. **(a) The perfect matching graph of $F_{8\times 2}$; it is also one possible perfect matching graph for f_2^a shown in Figure III.1b.**
(b) The unique perfect matching graph for f_1^a shown in Figure III.1b.

From Definition VL3 and Example **VI.1**, it is clear that the **perfect** matching **graph** of the frame column that consists of only the blocks of size two is unique and is also a perfect matching graph for all the other columns in the same **frame**.

Let the black box, called P(N!) and shown in Figure VI.2, &note a **rearrangeable** (permutation) network on N elements, **i.e.**, it realizes all N! distinct permutations in **a** single pass.

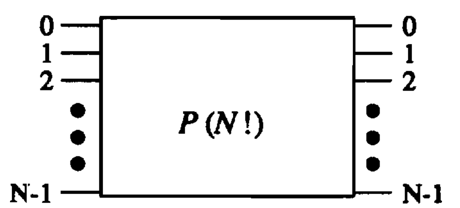


Figure V1.2. **A black box P(N!) *which realizes* all N! permutations.**

This black box P(N!) can be expanded recursively using Algorithm CONSBENES **presented** below until all of its black boxes are **identical** to (2x2) switching boxes **(SBs),** each of which can be set both straight and cross. This **expansion** results in the **Benes** network. Algorithm CONSBENES substitutes the three-stage Clos network **with** R **inputs/outputs,** &noted by $CS_{R\times 3}$ and shown in Figure VI.3, **for** the black box P(R!).

Algorithm CONS–BENES

Input: A black box called P(N!).

Output: Benes Network

Step 1. **Let** R be an integer variable and be **initialized** to N. Relabel the black **box** P(N!) by P(R!) and let BS denote a network consisting of P(R!).

Step 2. Replace each and every black **box** called P(R!) of BS by $CS_{R\times 3}$ shown in Figure VI.3.

Step 3. If all the **SBs** of BS are **(2x2),** then call BS Benes network and stop; otherwise first relabel each of its non-(2x2) **SBs by** P(R!) and halve the value of R, then go to Step 2.

Using the notions of balanced matrices and frames, it is first **shown** in the following **theorem** that $CS_{R\times 3}$ is functionally equivalent to P(R!). Then, it follows that the **Benes** network constructed by Algorithm CONS–BENES is rearrangeable because, **due to** the recursive structure of the algorithm, only the correctness of Step 2 needs to **be** proven.

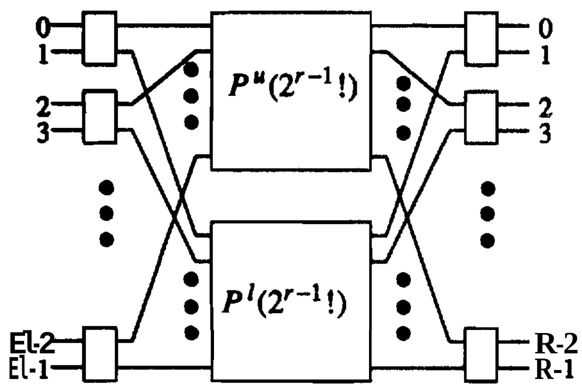


Figure VI.3. **Three-stage Clos network with R inputs/outputs which is denoted by $CS_{R\times 3}$, where R = 2^r.**

Theorem VI.1. Three-stage Clos network with R inputs is **rearrangeable.**

Proof. As it is shown in Fig. VI.4, the network $CS_{R\times 3}$ is composed of three components, namely, a) an inverse SE stage with 2^r **inputs/outputs,** b) a pile of two permutation networks $P^u(2^{r-1}!)$ and $P^l(2^{r-1}!)$, and c) a SE stage with 2^r **inputs/outputs.** It is assumed in this proof that, unless otherwise stated, any balanced matrix has $R=2^r$ rows. Recall that $P(2^r!)$ refers to a **rearrangeable** network on 2^r elements. Because $P(2^r!)$ passes any balanced matrix $B_{1:r}$ **corresponding** to a **permutation** on 2^r elements, $CS_{R\times 3}$ must also pass $B_{1:r}$ in order to state that $CS_{R\times 3}$ is functionally equivalent to $P(2^r!)$.

It is now shown that the inverse SE stage with 2^r **inputs/outputs** partitions $B_{1:r}$ into $B^u_{2^{r-1}\times r}$ and $B^l_{2^{r-1}\times r}$ such that the submamces $B^u_{2^{r-1}\times(r-1)}$ and $B^l_{2^{r-1}\times(r-1)}$ are balanced, where $B^u_{2^{r-1}\times(r-1)}$ and $B^l_{2^{r-1}\times(r-1)}$ are the first $(r-1)$ columns of $B^u_{2^{r-1}\times r}$ and $B^l_{2^{r-1}\times r}$, **respectively**. Both $B^u_{2^{r-1}\times(r-1)}$ and $B^l_{2^{r-1}\times(r-1)}$ pass the **permutation** network $P(2^{r-1}!)$ because it realizes any **permutation** on 2^{r-1} elements. **Because** the control bits of each SB must constitute the set $\{0,1\}$ to avoid conflict, any vector that fits f_1^a can be used as the vector of control bits of the **SBs** of the inverse **SE** stage. Let the **perfect** matching graph of f_1^a denote a graph with R vertices such that the vertices v_{2j} **and** v_{2j+1}, $0 \le j \le 2^{r-1}-1$, **are** connected by **an** edge, where v_{2j} **and** v_{2j+1} correspond to the $2j$th and $(2j+1)$th rows of f_1^a, respectively. Let x be a column vector obtained by a 2-labeling of the union of the perfect matching graphs of f_1^a and $B_{1:(r-1)}$. By Fact **VI.1**, the matrix $[x\ B_{1:(r-1)}]$ is balanced. This implies that x "**partitions**" the balanced **matrix** $B_{1:(r-1)}$ into two balanced **submatrices** $B^u_{2^{r-1}\times(r-1)}$ and $B^l_{2^{r-1}\times(r-1)}$ in **such** a **way** that row i of $B_{1:(r-1)}$ belongs to $B^u_{2^{r-1}\times(r-1)}$ if the ith entry of x equals zero, and belongs to $B^l_{2^{r-1}\times(r-1)}$ otherwise, **where** $0 \le i \le 2^r-1$. Without loss of generality, **assume** that the **SBs** of the inverse SE stage with 2^r **inputs/outputs** are labeled in ascending order starting with **0** and that the control bit for the ith **input** is the ith entry of x. So, when the 2jth and $(2j+1)$**th** enmes of x are used as control biits for the jth SB of the inverse SE stage, no **conflict** occurs and, hence, the matrix $B_{1:(r-1)}$ is partitioned into $B^u_{2^{r-1}\times(r-1)}$ and $B^l_{2^{r-1}\times(r-1)}$. Because both $P^u(2^{r-1}!)$ and $P^l(2^{r-1}!)$ can pass any balanced matrix of order $2^{r-1}\times(r-1)$, the mamces $B^u_{2^{r-1}\times(r-1)}$ and $B^l_{2^{r-1}\times(r-1)}$ pass $P^u(2^{r-1}!)$ and $P^l(2^{r-1}!)$, respectively.

In order for $B_{1:r}$ to pass $CS_{R\times3}$, $CS_{R\times3}$ must send its ith input to the output whose value equals $B_{1:r}(i)$. So far, this proof showed that $CS_{R\times3}$ sends its ith input with the **row** $B_{1:r}(i)$ to either the hth output of $P^u(2^{r-1}!)$ or the hth output of $P^l(2^{r-1}!)$, **where** h equals the value of $B_{1:(r-1)}(i)$. Because $B_{1:r}$ is a balanced mamx, the last enmes of the routing tags of the rows that are sent to the jth outputs of $P^u(2^{r-1}!)$ and $P^l(2^{r-1}!)$ **constitute** the set $\{0,1\}$. Due to the fact that the third component of $CS_{R\times3}$ is an **SE** stage, the rows that are sent to the jth outputs of $P^u(2^{r-1}!)$ and $P^l(2^{r-1}!)$ enter the jth **SB** of the SE stage. Because the connections of the SE stage implement the perfect **shuffle** permutation and the last **entries** of the routing tags of the rows entering a SB **constitute** the set $(0,1\}$, no conflict occurs in the **SBs**. It follows **that** $CS_{R\times3}$ sends its ith input to the output whose value equals $B_{1:r}(i)$. Therefore, the **theorem** holds. □

Corollary VI.1. The Benes network obtained by Algorithm **CONS_BENES** is **rearrangeable**.

Proof. Because Steps 1 and 3 of Algorithm **CONS_BENES** **are** relabelings and **the** network is constructed recursively, it suffices to show that $CS_{R\times3}$ is functionally equivalent to $P(2^r!)$. Because this is proven in Theorem VI.1, the **corollary** holds. □

VII. PERMUTATIONS REALIZED BY $BS_{(n-r):(2n-1)}$

Recall that Benes network can be considered as being $BE_{N\times(n-1)}RB_{N\times n}$. Theorem IV.1 identified the permutations passing $RB_{N\times n}$ in the sense that a balanced matrix $D_{N\times n}$ passes $RB_{N\times n}$ if and only if $D_{N\times n}$ fits $F^{sa}_{N\times n}$. Likewise, the following theorem and corollary determine the set of permutations that pass the network $BS_{(n-r):(2n-1)}$ which consists of the subnetwork $BS_{(n-r):(n-1)}$ followed by $RB_{N\times n}$, where $1 \leq r \leq n-1$. (Recall that $IN_{x:y}$ denotes the stages x through y of an IN and that $IN_{x:y}$ refers to a nil network if $x>y$). The permutations that pass $BS_{(n-r):(2n-1)}$ are characterized by the frames defined next. This characterization illustrates how frames can be used to gain insight into why the Benes network is rearrangeable. All the proofs are provided in the Appendix. An example is presented to illustrate the results of these proofs. For $N=16$, this example clearly shows how the addition of the stage BS_{n-r-1} to the left of $BS_{(n-r):(2n-1)}$ converts the frame that corresponds to $BS_{(n-r):(2n-1)}$ into a new frame that corresponds to the resulting network.

Definition VII.1. ($F^{sa,r}_{1:k}$): The frame $F^{sa,r}_{1:k}$, $r \in \{0,1,\ldots,k-1\}$ and $k \in \{1,2,\ldots,n\}$, is a frame $<\beta,\gamma,P>$ where

$$\beta(i)=\begin{cases} r+1 & \text{if } 1 \leq i \leq r+1 \\ i & \text{if } r+1<i \leq k, \end{cases}$$

γ is the identity permutation on the set $\{0,1,\ldots,N-1\}$ and

$$P_i=\begin{cases} P^*_{r+1} & \text{if } 1 \leq i \leq r+1 \\ P^*_i & \text{if } r+1<i \leq k. \end{cases}$$

Note that $F^{sa,0}_{1:k}$ and $F^{sa,n-1}_{1:k}$ are identical to $F^{sa}_{1:k}$ and $F^*_{1:k}$, respectively. As examples of $F^{sa,r}_{1:k}$, the frames $F^{sa,0}_{1:4}$, $F^{sa,1}_{1:4}$, $F^{sa,2}_{1:4}$ and $F^{sa,3}_{1:4}$ for $N=16$ are illustrated in Figure VII.1.

Theorem VII.1. Consider the frame $F^{sa,r}_{N\times n}$, $0 \leq r \leq n-1$. Let S be a pile of 2^{n-r-1} copies of a rearrangeable network $P(2^{r+1}!)$. Let T be an IN that consists of the network S followed by $RB_{(r+2):n}$. A balanced matrix $D_{N\times n}$ fits $F^{sa,r}_{N\times n}$ if and only if $D_{N\times n}$ passes T.

Corollary VII.1. A balanced matrix $D_{N\times n}$ fits the frame $F^{sa,r}_{N\times n}$ if and only if $D_{N\times n}$ passes the network $BS_{(n-r):(2n-1)}$, where $0 \leq r \leq n-1$.

Example **VII.1.** Let $N=16$ and $n=4$. The frames $F^{sa,0}_{16\times4}$ $F^{sa,1}_{16\times4}$ $F^{sa,2}_{16\times4}$ and $F^{sa,3}_{16\times4}$ are shown in Figure VII.1. By Theorem IV.1, all balanced matrices that fit $F^{sa}_{16\times4}$ pass $RB_{16\times4}$. If the stage BE_3 is added *to* the left of $RB_{1:4}$, the network $BS_{3:7}$ shown in **Figure VII.2a** is obtained. While $RB_{1:4}$ passes all balanced matrices that fit $F^{sa,0}_{16\times4}$ (the same as $F^{sa}_{16\times4}$), a balanced matrix $D_{1:4}$ passes $BS_{3:7}$ if and only if $D_{1:4}$ fits $F^{sa,1}_{16\times4}$. *If* the stage BE_2 is added *to* the left of $BS_{3:7}$, then $BS_{2:7}$ shown in **Figure VII.2b** is obtained. *A* balanced matrix $D_{1:4}$ passes $BS_{2:7}$ if and only if $D_{1:4}$ fits $F^{sa,2}_{16\times4}$. *If* the stage BE_1 is added to the left of $BS_{2:7}$, then Benes network, $BS_{1:7}$, shown in **Figure** II.4 is obtained. It is obvious that a balanced matrix $D_{1:4}$ passes $BS_{1:7}$ if and only if $D_{1:4}$ fits $F^{sa,3}_{16\times4} = F^{*}_{16\times4}$. Notice that, when the stage BE_j, $1 \le j \le n-1$, is added *to* the left of $BE_{(j+1):(n-1)}RB_{1:n}$, the subnetwork $BE_{j:(n-1)}RB_{1:(n-j+1)}$ becomes a pile of 2^{j-1} copies of Benes network with 2^{n-j+1} inputs/outputs and $2n-2j+1$ stages. Because Benes network with 2^{n-j+1} inputs/outputs and $2n-2j+1$ stages is a rearrangeable network, it corresponds to the universal frame with 2^{n-j+1} rows and $n-j+1$ columns. **Therefore,** the **first** $n-j+1$ columns of $F^{sa,n-j}_{16\times4}$ is a pile of 2^{j-1} copies of the universal **frame** with 2^{n-j+1} rows and $n-j+1$ columns. End of example.

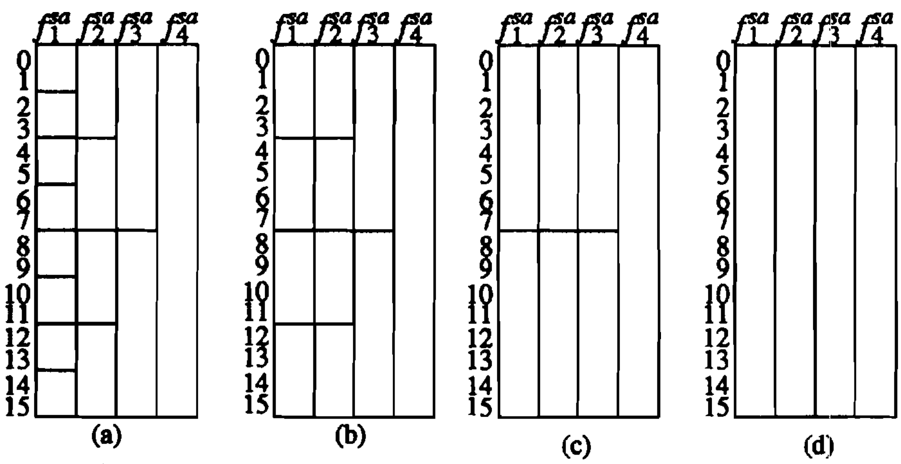
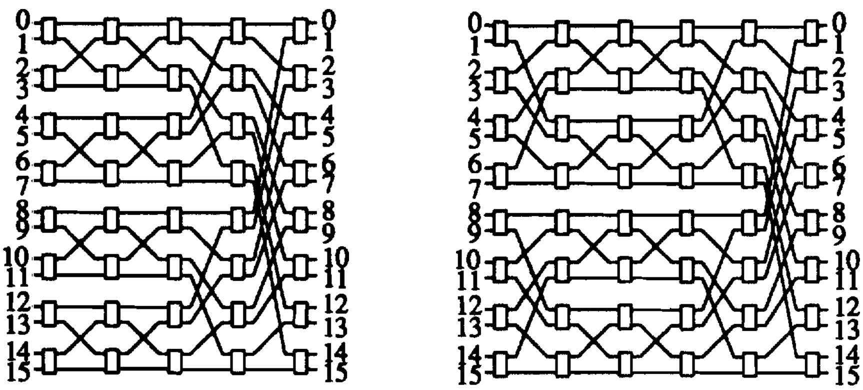


Figure VII.1. (a) $F^{sa,0}_{16\times4}$, (b) $F^{sa,1}_{16\times4}$, (c) $F^{sa,2}_{16\times4}$ and (d) $F^{sa,3}_{16\times4}$.



(a) (b)

Figure W.2. (a) $BS_{3:7}$ (BE_3 followed by $RB_{1:4}$). (b) $BS_{2:7}$ ($BE_{2:3}$ followed by $RB_{1:4}$).

VIII. CONCLUSIONS

In this paper, a new approach has been developed to characterize permutations realized by some frequently used networks. The concept of *frame* has been introduced and different frames have been illustrated. It is simple to check whether a given permutation is realized by a given network once the corresponding frame and the output interconnection pattern are known.

The permutations of the following three classes of networks have been characterized: the class of k-stage baseline-type networks that are topologically equivalent to the k-stage baseline network, the class of those networks that are constructed by appending shuffle-exchange stages to the left or right of a baseline-type network., and the class of those networks that form a part of Benes network.

The proof that Benes network is rearrangeable was first presented in [7]. This proof is based on the Slepian-Duguid theorem which applies only to symmetric networks. In this paper, a new simple proof has been presented for rearrangeability of Benes and three-stage Clos networks using the notion of balanced matrices and graph theory. The technique used in this proof can also be applied to nonsymmetric networks.

In practice, the results presented in this paper can be used to design networks that realize classes of permutations that fit the same frame. In addition, engineers and/or compilers may use frames to test if the corresponding networks realize a given permutation. Debuggers and programming environment can also use frames to detect when and why a permutation cannot be realized by the network. The definitions, theorems and lemmata that are presented in this paper to characterize the permutations realized in the aforementioned networks can also be used to address the issues of routing and counting permutations. But, to limit the size of this paper, these issues are addressed in

[19,28].

It is clear that frames, as defined in this paper, cannot characterize the permutations of every network. Conceivably, extensions of the **definitions may** be possible to **characterize** a larger class of networks. In particular, the concepts should be extensible to networks not considered in this paper including those **constructed** with $(k \times k)$ switches for k>2. Future research will address these issues.

IX. APPENDIX

Proof Theorem IV.1: (+) It is shown that if $D_{1:k}$ fits $F^{sa}_{1:k}$ **then** $D_{1:k}$ passes $RB_{1:k}$. Proof is by induction on k. Also, it is proven that $RB_{1:k}$ sends its ith input to its jth output, where j is equal to the sum of $\left[\left\lfloor i/2^k \right\rfloor \times 2^k\right]$ and the value of $D_{1:k}(i)$.

Basis *Step:* Let $k=1$. Label the **SBs** of RB_1 in ascending order starting with 0. (Recall that RB_1 refers to the first stage of a reverse baseline network with N **inputs/outputs).** By definition, f_1^{sa} contains 2^{n-1} blocks of size 2 **each.** The fact that $D_{1:k}$ fits $F^{sa}_{1:k}$ implies that d_1 fits f_1^{sa}. Therefore, the **2rth** and **(2r+1)th** entries of d_1 **constitute** the set $\{0,1\}$, where $0 \le r \le 2^{n-1}-1$. Hence, when the **2rth** and **(2r+1)th entries** of d_1 are used as the **control** bits to set the rth SB of RB_1, no conflict occurs and RB_1 **sends** its ith input to its jth output, where j is equal to the sum of $\left[\left\lfloor i/2 \right\rfloor \times 2\right]$ **and** the value of the ith entry of d_1, where $0 \le i \le N-1$. (Recall that, if the control bit of the routing tag of an input equals zero, then the input is sent to the upper output of **the** SB that it enters; otherwise it is sent to the the lower output of the SB).

Induction *Step:* Assume that, for $2 \le k \le n$, if $D_{1:(k-1)}$ fits $F^{sa}_{1:(k-1)}$, then $D_{1:(k-1)}$ **passes $RB_{1:(k-1)}$ and $RB_{1:(k-1)}$ sends its ith input to its jth output, where j is equal to the sum of $\left[\left\lfloor i/2^{k-1} \right\rfloor \times 2^{k-1}\right]$ and the value of $D_{1:(k-1)}(i)$. Now, show that, if $D_{1:k}$ fits** $F^{sa}_{1:k}$, then $D_{1:k}$ passes $RB_{1:k}$ and $RB_{1:k}$ sends its ith input to its jth output, where j is **equal** to the sum **of** $\left[\left\lfloor i/2^k \right\rfloor \times 2^k\right]$ and the value **of** $D_{1:k}(i)$.

The frame $F^{sa}_{1:m}$, $(m=k-1,k)$, can be considered as being composed of 2^{n-m} **copies** of $F^{sa}_{2^m \times m}$ in parallel if the **row** labels of the **αth,** $0 \le \alpha \le 2^{n-m}-1$, $F^{sa}_{2^m \times m}$ consists of the **numbers** $(\alpha \times 2^m)$ to $[(\alpha+1) \times 2^m - 1]$ inclusive. Let $F^{\alpha}_{2^m \times m}$ denote the **αth** $F^{sa}_{2^m \times m}$. $RB_{1:m}$ can also be considered as being the pile of 2^{n-m} distinct $RB_{2^m \times m}$**s.** Label these $RB_{2^m \times m}$s in ascending order starting with 0 at the top and denote the **αth** one by $RB^{\alpha}_{2^m \times m}$. By hypothesis, $D_{1:m}$ fits $F^{sa}_{1:m}$. Let $D^{\alpha}_{2^m \times m}$ denote the **submatrix** of $D_{1:m}$ that fits $F^{\alpha}_{2^{n}\times m}$. Thus, the induction hypothesis also implies that $D^{\alpha}_{2^{k-1}\times k-1}$ (which fits $F^{\alpha}_{2^{k-1}\times k-1}$) passes $RB^{\alpha}_{2^{k-1}\times k-1}$ and that $RB^{\alpha}_{2^{k-1}\times k-1}$ sends its pth input to the output whose **value** is equal to the value of $D^{\alpha}_{2^{k-1}\times k-1}(p)$, where $0 \leq p \leq 2^{k-1}-1$.

Let $F^{\alpha_1}_{2^{k-1}\times k-1}$ and $F^{\alpha_2}_{2^{k-1}\times k-1}$ &note the 21th and $(2l+1)$th $F^{\alpha}_{2^{k-1}\times k-1}$s, respectively, where 0 5 1 $\leq 2^{n-k}-1$. Similarly, let $D^{\alpha_1}_{2^{k-1}\times k-1}$ and $D^{\alpha_2}_{2^{k-1}\times k-1}$ **denote** the 21th and $(2l+1)$th $D^{\alpha}_{2^{k-1}\times k-1}$s. Likewise, assume that $RB^{\alpha_1}_{2^{k-1}\times k-1}$ and $RB^{\alpha_2}_{2^{k-1}\times k-1}$ &note the $2l$th and $(2l+1)$th $RB^{\alpha}_{2^{k-1}\times k-1}$s.

Because $D^{\alpha}_{2^{k-1}\times(k-1)}$ is a balanced matrix of order $2^{k-1}\times(k-1)$, it has 2^{k-1} distinct **rows**. Therefore, the matrix

$$H = \begin{bmatrix} D^{\alpha_1}_{2^{k-1}\times(k-1)} \\ D^{\alpha_2}_{2^{k-1}\times(k-1)} \end{bmatrix}$$

contains 2^{k-1} distinct rows, each being repeated twice. Assume that the rows of H are **partitioned** into 2^{k-1} classes, each of which contains 2 identical rows, that is, each class contains the two copies of a distinct row of H. After adding a column permutation of length 2^k to the right of H, **call** the resultant matrix $D^{\beta}_{2^{k}\times(k)}$. This implies that the number of the entries of the rows of a class is incremented **by** 1. In order for $D^{\beta}_{2^{k}\times(k)}$ to fit $F^{\alpha}_{2^{k}\times k}$ the *kth* entries of the rows of each class of H **must** constitute the set $\{0,1\}$, which is true because $D_{1:k}$ fits $F^{sa}_{1:k}$ by the induction hypothesis.

By **definition,** the kth stage of **reverse** baseline, RB_k, consists of a pile of 2^{n-k} copies of the SE stage with 2 **inputs/outputs**. Assume that the network consisting of the pile of two networks $RB^{\alpha_1}_{2^{k-1}\times(k-1)}$ and $RB^{\alpha_2}_{2^{k-1}\times(k-1)}$ followed by the SE stage with 2^k **inputs/outputs** is called $RB^{\beta}_{2^{k}\times k}$. Because $RB^{\alpha}_{2^{k-1}\times k-1}$ sends its pth input to the output whose value is equal to the value of $D^{\alpha}_{2^{k-1}\times k-1}$ (p), the first $(k-1)$ **entries** of the row that is sent to the pth output **of** the **network** $RB^{\alpha_1}_{2^{k-1}\times(k-1)}$ is the same as the first $(k-1)$ entries **of** the row that is sent to the **pth** output of the network $RB^{\alpha_2}_{2^{k-1}\times(k-1)}$. The kth entries of **those** two rows sent to the **pth** outputs **of** $RB^{\alpha_1}_{2^{k-1}\times(k-1)}$ and $RB^{\alpha_2}_{2^{k-1}\times(k-1)}$ constitute the set $\{0,1\}$ because $D^{\beta}_{2^{k}\times k}$ fits the frame $F^{\alpha}_{2^{k}\times k}$ by induction hypothesis. **Because** the rows that are sent to the pth outputs of $RB^{\alpha_1}_{2^{k-1}\times(k-1)}$ and $RB^{\alpha_2}_{2^{k-1}\times(k-1)}$ enter the pth SB of the SE stage following these **networks** such that the kth entries of these rows are the control bits for the SB, no conflict occurs in the pth SB. This **amounts** to stating that $RB^{\beta}_{2^{k}\times k}$ sends its hth input to the output whose value is equal to the value of $D^{\beta}_{2^{k}\times k}(h)$, where $0 \leq h \leq 2^k-1$. Therefore, the balanced matrix $D_{1:k}$ passes $RB_{1:k}$ and $RB_{1:k}$ **sends** its ith input to its jth output, where $0 \leq i \leq N-1$ and j is equal to the sum of $\left\lceil \left\lfloor i/2^k \right\rfloor \times 2^k \right\rceil$ and the value of $D_{1:k}(i)$.

(C) is shown that, if $D_{1:k}$ passes $RB_{1:k}$, then $D_{1:k}$ fits $F^{sa}_{1:k}$. Proof is by induction on k.

Basis Step: Let $k=1$. The fact that d_1 passes RB_1 implies that no conflict occurs in the SBs of RB_1 when the ith enay of d_1 is used as the control bit for the ith input of RB_1 in setting its rth SB. Because the control bits of the rth SB of RB_1 constitute the set $\{0,1\}$ and **fit the rth block of f_1^{sa}, d_1 fits f_1^{sa}.**

Induction Step: Assume that the **theorem** holds for $k-1$. It is **shown** that it also **holds for** k, where $2 \le k \le n$.

By induction hypothesis, if $D^{\alpha}_{2^{k-1}\times k-1}$ passes $RB^{\alpha}_{2^{k-1}\times k-1}$ fits $F^{\alpha}_{2^{k-1}\times k-1}$. Notice that **the** last stage of $RB^{\beta}_{2^k\times k}$ is the SE stage with 2^k **inputs/outputs.** Recall that the network consisting of the pile of two networks $RB^{\alpha_1}_{2^{k-1}\times(k-1)}$ and $RB^{\alpha_2}_{2^{k-1}\times(k-1)}$ followed by the SE stage with 2^k **inputs/outputs** is called $RB^{\beta}_{2^k\times k}$. As it is also explained above, the rows **that are** sent to the pth outputs of $RB^{\alpha_1}_{2^{k-1}\times(k-1)}$ and $RB^{\alpha_2}_{2^{k-1}\times(k-1)}$ enter the pth SB of the SE stage that follows these networks. If $D^{\beta}_{2^k\times k}$ passes $RB^{\beta}_{2^k\times k}$, then the kth entries of the **rows** of a class of H must constitute the set $\{0,1\}$ to avoid having a conflict in the pth **SB. Therefore,** $D^{\beta}_{2^k\times k}$ fits $F^{\beta}_{2^k\times k}$. **It** follows that $D_{1:k}$ fits $F^{sa}_{1:k}$. □

Proof of Corollary IV.1: (→) Let Φ be topologically equivalent to $RB_{1:k}$. When interconnection networks are modeled by directed graphs in which **vertices** represent **the** switches and edges the links, two networks are said to be topologically equivalent if **the** graphs representing them **are** isomorphic. Two graphs G and H **are** said to be isomorphic if there exist bijections from the vertices and edges of G to the vertices and **edges** of H, respectively such that the relationship of adjacency is preserved. So, if two **networks** are topologically equivalent to each other, one of them can be made identical to the other network by relabeling the inputs **and/or** outputs. This implies that Φ can be **made** identical to $RB_{1:k}$ by relabeling the inputs and/or outputs of Φ, and vice versa. Because (1) $F^{sa}_{1:k}$ corresponds to $RB_{1:k}$ such that there exists a one-to-one **correspondence** between the row labels of $F^{sa}_{1:k}$ and $RB_{1:k}$ (Theorem N.1), (2) the only difference between $F^{sa}_{1:k}$ and an a-type frame $F^{a}_{1:k}$ is the order of their row labels, and (3) @ is **topologically** equivalent to $RB_{1:k}$, there exists an a-type frame $F^{a}_{1:k}$ corresponding to Φ such that no conflict occurs in the switches of Φ when the contents of the ith row, $0 \le i \le N-1$, of $F^{a}_{1:k}$ **are** used as the routing tag for the ith input of Φ.

(←) Let γ denote the vector of input labels of Φ such that the ith enay of γ equals **the** ith input label of Φ. Let $F^{a}_{1:k}(\gamma)$ &note the frame corresponding to Φ such that the ith entry of γ equals the ith row label of the frame. By definition of "**correspondence**" (Definition **III.8),** no conflict occurs in the switches of Φ when the **contents** of the ***i*th row** of $F^{a}_{1:k}(\gamma)$ **are** used as the routing tag for the ith input of Φ. Note that there exists a **one-to-one correspondence** between the input labels of Φ and **the** row labels of $F^{a}_{1:k}(\gamma)$. Therefare, when both the ith row label of $F^{a}_{1:k}(\gamma)$ and the ith input label of Φ are replaced by the integer i, the resulting frame $F^{sa}_{1:k}$ and **network** still remain **correspondent** to each other. By Theorem N.1, $F^{sa}_{1:k}$ corresponds to $RB_{1:k}$. It follows that Φ can be converted to $RB_{1:k}$ by relabeling the input and/or output labels of Φ. Thus, Φ is topologically equivalent to $RB_{1:k}$. □

Definition IX.1. (forward-routing, **reverse-routing**): Given an $IN_{N\times k}$ and a setting of its SBs that realizes $h: i \rightarrow h(i)$, forward-routing of a matrix A means that $A(i)$ is sent from input i to output $h(i)$, where $0 \leq i \leq N-1$. Likewise, reverse-routing of A means that $A(i)$ is sent from the output i to the input $h^{-1}(i)$. The matrix $A^F = A(h^{-1}(i))$, $i=0,1,\ldots,N-1$, is obtained by forward-routing of A. Similarly, the matrix $A^R = A(h(i))$, $i=0,1,\ldots,N-1$, is obtained by reverse-routing of A.

Proof of **Corollary** IV.3: Because the network Π is a k-stage baseline-type network, it is topologically equivalent to $RB_{1:k}$. This implies that $RB_{1:k}$ can be made identical to Π by relabeling its inputs and/or outputs. Because relabeling the inputs (respectively, outputs) of $RB_{1:k}$ is equivalent to adding an interconnection pattern to the left (respectively, right) of $RB_{1:k}$, there exist two interconnection patterns IP_{in} and IP_{out} such that Π is topologically and functionally equivalent to $IP_{in}RB_{1:k}IP_{out}$.

($\rightarrow$) Assume that $D_{1:k}$ fits $F^{a}_{1:k}(\alpha_{in}^{-1})$. It is shown that the network Π realizes the permutation $\alpha_{in}.\mu.\alpha_{out}$.

Adding the interconnection pattern IP_{in} to the left of $RB_{1:k}$ is equivalent to relabeling the ith input of $RB_{1:k}$ by $\alpha_{in}^{-1}(i)$. Because the only difference between two a-type frames with k columns is the order of their row labels and IP_{out} is just an interconnection pattern, it follows from Theorem IV.1 that $D_{1:k}$ passes Π. By Definition IX.1, when $D_{1:k}$ is forward-routed through the interconnection pattern IP_{in}, $D_{1:k}$ is mapped to $D^{*}_{1:k} = D_{1:k}(\alpha_{in}^{-1}(i))$, $i=0,1,\ldots,N-1$. By Theorem IV.3, the subnetwork $RB_{1:k}$ of Π realizes the permutation μ represented by $[I_{1:n-k}\, D^{*}_{1:k}]$. Therefore, the network Π realizes the permutation $\alpha_{in}.\mu.\alpha_{out}$.

($\leftarrow$) Assume that the network Π realizes the permutation $\alpha_{in}.\mu.\alpha_{out}$. It is shown that $D_{1:k}$ fits $F^{a}_{1:k}(\alpha_{in}^{-1})$.

The fact that Π realizes the permutation $\alpha_{in}.\mu.\alpha_{out}$ implies that the permutation μ is realized by $RB_{1:k}$ of Π. Because μ is the permutation represented by the balanced matrix $[I_{1:n-k}\, D^{*}_{1:k}]$ such that $D^{*}_{1:k}(i) = D_{1:k}(\alpha_{in}^{-1}(i))$, it follows from Theorem IV.3 that $D^{*}_{1:k}$ passes $RB_{1:k}$. By Definition IX.1, when $D_{1:k}$ is reverse-routed through the interconnection pattern IP_{in}, $D^{*}_{1:k}$ is mapped to $D_{1:k}$. Thus, $D_{1:k}$ passes $IP_{in}RB_{1:k}$. Note that the network $IP_{in}RB_{1:k}$ is identical to the network obtained by relabeling the ith input of $RB_{1:k}$ by $\alpha_{in}^{-1}(i)$. In addition, because $F^{a}_{1:k}(\alpha_{in}^{-1})$ is the same as $F^{sa}_{1:k}$ except that the ith row label of $F^{a}_{1:k}(\alpha_{in}^{-1})$ equals $\alpha_{in}^{-1}(i)$ instead of i, $D_{1:k}$ fits $F^{a}_{1:k}(\alpha_{in}^{-1})$. □

Proof of Theorem **V.1:** (→) It is shown that if $D_{1:(n+m)}$ fits $F^{sa}_{1:n}F^{*}_{1:m}$, then $D_{1:(n+m)}$ passes $RB_{1:n}SE_{1:m}$ and the permutation represented in binary by $D_{(m+1):(n+m)}$ is realized by $RB_{1:n}SE_{1:m}$.

Recall that by definition, $RB_{1:n}SE_{1:m}$ consists of $RB_{1:n}$ **followed** by $SE_{1:m}$. **Because, by** hypothesis, $D_{1:(n+m)}$ fits $F^{sa}_{1:n}F^{*}_{1:m}$, $D_{1:n}$ **fits** $F^{sa}_{1:n}$. $RB_{1:n}$ maps the matrix $D_{1:(n+m)}$ into the matrix denoted **by** $D^{*}_{1:(n+m)}$ when $D_{1:(n+m)}(i)$, $0 \le i \le N-1$, is used **as** the **routing** tag for the ith input of $RB_{1:n}$. Theorem **IV.1** has shown that any balanced **matrix** $D_{1:n}$ fitting the frame $F^{sa}_{1:n}$ passes the network $RB_{1:n}$. So, when $D_{1:n}(i)$ is used *as* **the** muting tag for the ith input of $RB_{1:n}$, $RB_{1:n}$ sends its ith **input** to the output **whose** value equals $D_{1:n}(i)$. So, $RB_{1:n}$ maps any $D_{1:n}$ fitting the **frame** $F^{sa}_{1:n}$ to $I_{1:n}$. **This** implies that, when $D_{1:(n+m)}(i)$ is used as the muting tag for **the** ith input of $RB_{1:n}SE_{1:m}$, the **submatrix** $D^{*}_{1:n}$ of $D^{*}_{1:(n+m)}$ is the same as the **identity** permutation matrix $I_{1:n}$. Therefore, $D^{*}_{1:(n+m)}$ is **equal to** the **balanced** matrix $[I_{1:n}\, D^{*}_{(n+1):(n+m)}]$. **By Lemma** V.1, $SE_{1:m}$ realizes the permutation **represented** by $D^{*}_{(m+1):(n+m)}$ and no conflict **occurs** in the **SBs** of $SE_{1:m}$ when $D^{*}_{(n+1):(n+m)}(i)$ is used as **the** routing tag for the ith input of $SE_{1:m}$. Therefore, $D_{1:(n+m)}$ passes $RB_{1:n}SE_{1:m}$. Now, it remains to **show** that $RB_{1:n}SE_{1:m}$ realizes the permutation represented by $D_{(n+1):(n+m)}$.

Let the entries of $D_{1:(n+m)}(i)$ be &noted in binary by $(x^i_1 x^i_2 \cdots x^i_n \cdots x^i_{n+m})$. The **fact** that $D^{*}_{1:n}$ of $D^{*}_{1:(n+m)}$ is **identical** to $I_{1:n}$ implies that $RB_{1:n}$ of $RB_{1:n}SE_{1:m}$ sends the routing tag $D_{1:(n+m)}(i)$ to the output of $RB_{1:n}$ whose value **equals** the value **of** $(x^i_1 x^i_2 \cdots x^i_n)$. Because the jth output of $RB_{1:n}$ is the same as the jth input of $SE_{1:m}$ **when** $RB_{1:n}SE_{1:m}$ is considered, $D_{1:(n+m)}(i)$ is sent to the jth input of $SE_{1:m}$ by $RB_{1:n}$, **where** j equals $(x^i_1 x^i_2 \cdots x^i_n)$. Hence, the bit x^i_{n+p}, $1 \le p \le m$, of $(x^i_1 x^i_2 \cdots x^i_n \cdots x^i_{n+m})$ **is** used as the control bit to set a SB at the pth stage of $SE_{1:m}$, **where** $(x^i_1 x^i_2 \cdots x^i_n)$ and $(x^i_{m+1} x^i_{m+2} \cdots x^i_{n+m})$ *are* the **addresses of** the input and the destination, respectively. Due to the fact that $D_{1:(n+m)}$ passes $RB_{1:n}SE_{1:m}$ and a SE **stage** performs the shuffle operation followed by the exchange operation, $RB_{1:n}SE_{1:p}$ sends $D_{1:(n+m)}(i)$ to the output of $RB_{1:n}SE_{1:p}$ whose value equals $(x^i_{p+1} x^i_{p+2} \cdots x^i_{n+p})$. Therefore, the permutation represented by $D_{(m+1):(n+m)}$ is **implemented** by $RB_{1:n}SE_{1:m}$.

(←) It is shown that, if $D_{1:(n+m)}$ passes $RB_{1:n}SE_{1:m}$, then $D_{1:(n+m)}$ **fits** $F^{sa}_{1:n}F^{*}_{1:m}$ and $RB_{1:n}SE_{1:m}$ realizes the permutation **represented** by $D_{(m+1):(n+m)}$.

Because, by hypothesis, $D_{1:(n+m)}$ passes $RB_{1:n}SE_{1:m}$, the submatrix $D_{1:n}$ of $D_{1:(n+m)}$ passes $RB_{1:n}$. So, by Theorem **IV.1,** the submatrix $D_{1:n}$ fits $F^{sa}_{1:n}$. By **definition,** any column of the universal frame $F^{*}_{1:m}$ is a single block of size N. Therefore, **any** balanced matrix of order $(N \times m)$ **fits** $F^{*}_{1:m}$. It **follows** that $D_{(n+1):(n+m)}$ fits $F^{*}_{1:m}$. Hence, $D_{1:(n+m)}$ **fits** $F^{sa}_{1:n}F^{*}_{1:m}$.

The **first** part (→) of the proof has shown that the permutation represented by $D_{(m+1):(n+m)}$ is implemented by $RB_{1:n}SE_{1:m}$ if $D_{1:(n+m)}$ fits $F^{sa}_{1:n}F^{*}_{1:m}$. Because it is shown above that $D_{1:(n+m)}$ fits $F^{sa}_{1:n}F^{*}_{1:m}$, $RB_{1:n}SE_{1:m}$ realizes the permutation corresponding to $D_{(n+1):(n+m)}$. □

Proof of Theorem V.2: Proof is by induction on m.

Basis Step: Let $m=1$. In this step it is proven that $RB_{1:n}SE_1$ is functionally and topologically equivalent to $SE_1^{-1}RB_{1:n}$. Recall that $RB_{1:n}$ is functionallly and topologically equivalent to $BE_{1:n}$. Therefore, $RB_{1:n}SE_1$ is functionally and topologically equivalent to $BE_{1:n}SE_1$. $BE_{2:n}$ consists of 2 copies of $BE_{2^{n-1}\times(n-1)}$ in parallel, while $RB_{1:(n-1)}$ consists of 2 copies of $RB_{2^{n-1}\times(n-1)}$ in parallel. Because $BE_{2^{n-1}\times(n-1)}$ is functionally and topologically equivalent to $RB_{2^{n-1}\times(n-1)}$, $BE_{2:n}$ is functionally and topologically equivalent to $RB_{1:(n-1)}$. Therefore, $BE_{1:n}SE_1$ is functionally and topologically equivalent to $BE_1RB_{1:(n-1)}SE_1$. Because the last stage of $RB_{1:n}$ is identical to the SE stage, $RB_{1:(n-1)}SE_1$ is identical to $RB_{1:n}$. Therefore, $BE_1RB_{1:(n-1)}SE_1$ is functionally and topologically equivalent to $BE_1RB_{1:n}$. Due to the fact that BE_1 is identical to the inverse SE stage, $BE_1RB_{1:n}$ is functionally and topologicallly equivalent to $SE_1^{-1}RB_{1:n}$. It follows that $RB_{1:n}SE_1$ is functionally and topologically equivalent to $SE_1^{-1}RB_{1:n}$.

Induction Step: Assume that, for $m \geq 2$, the theorem holds for $m-1$, and show that it also holds for m.

Because $RB_{1:n}$ is functionally and topologically equivalent to $BE_{1:n}$, $RB_{1:n}SE_{1:m}$ is functionally and topologically equivalent to $BE_{1:n}SE_{1:m}$. As it is explained in the Basis Step above, $BE_{2:n}$ is functionally and topologically equivalent to $RB_{1:(n-1)}$. Therefore, $RB_{1:n}SE_{1:m}$ is functionally and topologically equivalent to $BE_1RB_{1:(n-1)}SE_{1:m}$. Because the last stage of $RB_{1:n}$ is identical to the SE stage, $BE_1RB_{1:(n-1)}SE_{1:m}$ is identical to $BE_1RB_{1:n}SE_{1:(m-1)}$. By the induction hypothesis, $RB_{1:n}SE_{1:(m-1)}$ is functionally and topologically equivalent to $SE^{-1}_{1:(m-1)}RB_{1:n}$. So, $BE_1RB_{1:n}SE_{1:(m-1)}$ is functionally and topologically equivalent to $BE_1SE^{-1}_{1:(m-1)}RB_{1:n}$. Because BE_1 is identical to the inverse SE stage, $BE_1SE^{-1}_{1:(m-1)}RB_{1:n}$ is functionally and topologically equivalent to $SE^{-1}_{1:m}RB_{1:n}$. Thus, the theorem holds. □

Proof of Theorem VII.1: *Case 1:* Let $r=n-1$. When $r=n-1$, T consists of only a rearrangeable network $P(2^n!)$ and $F^{sa,r}_{N\times n}$ is identical to the universal frame $F^{*}_{N\times n}$. By definition, any balanced matrix of order $N\times n$ fits $F^{*}_{N\times n}$ and $P(2^n!)$ passes any balanced matrix of order $N\times n$. Therefore, a $D_{N\times n}$ fits $F^{sa,r}_{N\times n}$ if and only if $D_{N\times n}$ passes T.

Case 2: Let $r=0$. When $r=0$, $F^{sa,r}_{N\times n}$ and T *are* identical to $F^{sa}_{N\times n}$ and $RB_{N\times n}$, respectively. Because Theorem IV.1 shows that a $D_{N\times n}$ fits $F^{sa}_{N\times n}$ if and only if $D_{N\times n}$ passes $RB_{N\times n}$, *Theorem* VII.1 holds for this case.

Case 3: **Let** $1 \le r \le n-2$. Assume that $D_{N\times n}(i)$, $0 \le i \le N-1$, is **used as** the routing **tag** for the ith input of T.

($\leftarrow$) It is shown that, **if** $D_{N\times n}$ fits $F^{sa,r}_{N\times n}$, then $D_{N\times n}$ passes T.

In what follows, it is first shown that the submatrix $D_{1:(r+1)}$ of a $D_{N\times n}$ passes S. By the **definition** of **rearrangeability,** any of the 2^{n-r-1} rearrangeable networks $P(2^{r+1}!)$ of S can pass any balanced matrix of order $2^{r+1}\times(r+1)$. Label these **rearrangeable** networks in ascending order starting with 0. Let $P^{\alpha}(2^{r+1}!)$ denote the α**th** **rearrangeable** network $P(2^{r+1}!)$ of S, where $0 \le \alpha \le 2^{n-r-1}-1$.

Consider the universal frame $F^{*}_{2^{r+1}\times(r+1)}$. Any column of $F^{*}_{2^{r+1}\times(r+1)}$ is just a single block of length 2^{r+1}. Because a column of $F^{*}_{2^{r+1}\times(r+1)}$ requires a column vector of **length** 2^{r+1} to have only 2^r **zeros** and 2^r ones, any column of a balanced matrix of **order** $2^{r+1}\times(r+1)$ fits it. It follows that any balanced matrix of order $2^{r+1}\times(r+1)$ **fits** $F^{*}_{2^{r+1}\times(r+1)}$. Therefore, $P^{\alpha}(2^{r+1}!)$ corresponds to the universal **frame** $F^{*}_{2^{r+1}\times(r+1)}$. **The subframe** $F^{sa,r}_{1:(r+1)}$ can be **considered** as being a pile of 2^{n-r-1} $F^{*}_{2^{r+1}\times(r+1)}$s. Label these universal frames in ascending order starting with 0.

Partition the balanced submatrix $D_{1:(r+1)}$ of $D_{N\times n}$ into 2^{n-r-1} **balanced submatrices** of order $2^{r+1}\times(r+1)$ such that the set of the row indices of the α**th** submatrix **consists** of the numbers $(\alpha\times 2^{r+1})$ to $[(\alpha+1)\times 2^{r+1}-1]$ inclusive. Label these submatrices of **order** $2^{r+1}\times(r+1)$ in ascending order starting with 0. Denote the α**th** submatrix of $D_{1:(r+1)}$ **by** $D^{\alpha}_{1:(r+1)}$.

By hypothesis, $D_{N\times n}$ fits $F^{sa,r}_{1:n}$. This implies that $D_{1:(r+1)}$ fits $F^{sa,r}_{1:(r+1)}$. Therefore, $D^{\alpha}_{1:(r+1)}$ fits the α**th** $F^{*}_{2^{r+1}\times(r+1)}$. **Because** $P^{\alpha}(2^{r+1}!)$ is a **rearrangeable** network, it **passes** $D^{\alpha}_{1:(r+1)}$, that is, $P^{\alpha}(2^{r+1}!)$ sends its *k*th input to the output whose value equals $D^{\alpha}_{1:(r+1)}(k)$ where $0 \le k \le 2^{r+1}-1$. This implies that the network S sends its ith input to its jth output, where j equals the sum of $\left[\left\lfloor i/2^{r+1}\right\rfloor\times 2^{r+1}\right]$ and the value of the leftmost $(r+1)$ bits of the $D_{1:n}(i)$. Hence, $D_{1:(r+1)}$ passes S.

Theorem **IV.1** shows that a balanced matrix $C_{1:n}$ that fits $F^{sa}_{1:n}$ passes $RB_{1:n}$. **Theorem** IV.1 also shows that $RB_{1:(r+1)}$ sends its ith input to its *h*th output, where h is **equal** to the sum of $\left[\left\lfloor i/2^{r+1}\right\rfloor\times 2^{r+1}\right]$ and the value of $C_{1:(r+1)}(i)$. Thus, the networks $RB_{1:(r+1)}$ and S send their ith **inputs** to their jth outputs, where j **equals** the sum of $\left[\left\lfloor i/2^{r+1}\right\rfloor\times 2^{r+1}\right]$ and the value of the ith row of the matrix **passing** the corresponding network that is either $RB_{1:(r+1)}$ or S. By **definition,** $F^{sa,r}_{(r+1):n}$ is the same as $F^{sa}_{(r+1):n}$. This implies that $F^{sa,r}_{(r+2):n}$ is also the same as $F^{sa}_{(r+2):n}$. It follows **from** this paragraph **that** the argument given in the ($\rightarrow$) part of the proof of **Theorem** IV.1 applies to $RB_{(r+2):n}$ **of** T and $F^{sa,r}_{(r+2):n}$. (If in Theorem IV.1 $RB_{1:(r+1)}$ and $F^{sa}_{1:(r+1)}$ are replaced by

S and $F^{sa,r}_{1:(r+1)}$, respectively, Theorem IV.1 becomes identical to Theorem VII.1). Therefore, $D_{N\times n}$ passes T.

($\Leftarrow$) It is shown that, if $D_{N\times n}$ passes T, then $D_{N\times n}$ fits $F^{sa,r}_{N\times n}$.

First, consider the submatrix $D_{1:(r+1)}$ of $D_{1:n}$. By hypothesis, $D_{N\times n}$ passes T. This implies that $D_{1:(r+1)}$ passes S because S consists of 2^{n-r-1} copies of a rearrangeable network $P(2^{r+1}!)$ in parallel and $D^{\alpha}_{1:(r+1)}$ passes $P^{\alpha}(2^{r+1}!)$. Because any balanced matrix of order $2^{r+1}\times(r+1)$ fits a universal frame $F^{*}_{2^{r+1}\times(r+1)}$, $D^{\alpha}_{1:(r+1)}$ also fits $F^{*}_{2^{r+1}\times(r+1)}$. Recall that $F^{sa,r}_{1:(r+1)}$ can be considered as a pile of 2^{n-r-1} copies of $F^{*}_{2^{r+1}\times(r+1)}$. Therefore, $D_{1:(r+1)}$ fits $F^{sa,r}_{1:(r+1)}$.

Now, it is shown by induction on β, $1 \le \beta \le n-r-1$, that $D_{1:(r+1+\beta)}$ fits $F^{sa,r}_{1:(r+1+\beta)}$, assuming that $D_{N\times n}$ passes T. (The proof presented below is analogous to part ($\Leftarrow$) of the proof of Theorem IV.1).

Basis step: Let β=1. For $0 \le l \le 2^{n-r-2}-1$, let $D^{\alpha_1}_{1:(r+1)}(k)$ and $D^{\alpha_2}_{1:(r+1)}(k)$ denote the 2lth and (2l+1)th $D^{\alpha}_{1:(r+1)}$s, respectively. Similarly, let $P^{\alpha_1}(2^{r+1}!)$ and $P^{\alpha_2}(2^{r+1}!)$ denote the 2lth and (2l+1)th rearrangeable networks of S, respectively. Because the stage $RB_{(r+2)}$ consists of a pile of 2^{n-r-2} copies of the SE stage with 2^{r+2} inputs/outputs, the subnetwork that consists of the pile of $P^{\alpha_1}(2^{r+1}!)$ and $P^{\alpha_2}(2^{r+1}!)$ is followed by the SE stage with 2^{r+2} inputs/outputs. Because $D^{\alpha}_{1:(r+1)}$ passes $P^{\alpha}(2^{r+1}!)$, $P^{\alpha}(2^{r+1}!)$ sends its kth input to its mth output, where m equals the contents of $D^{\alpha}_{1:(r+1)}(k)$. Hence, the rows that are sent to the kth outputs of $P^{\alpha_1}(2^{r+1}!)$ and $P^{\alpha_2}(2^{r+1}!)$ enter the kth SB of the succeeding SE stage with 2^{r+2} inputs/outputs. By hypothesis, $D_{N\times n}$ passes T. This implies that $D_{1:(r+2)}$ passes the network consisting of S followed by the stage RB_{r+2} without having any conflict in the SBs. Therefore, the $(r+2)$th entries of the rows that are sent to the kth outputs of $P^{\alpha_1}(2^{r+1}!)$ and $P^{\alpha_2}(2^{r+1}!)$ constitute the set $\{0,1\}$. Notice that these rows have the same first $k-1$ entries. Therefore, the $(r+2)$th entries of any two identical rows of the submatrix

$$\begin{bmatrix} D^{\alpha_1}_{1:(r+1)} \\ D^{\alpha_2}_{1:(r+1)} \end{bmatrix}$$

constitute the set $\{0,1\}$. Therefore, by definition of fit, $D_{1:(r+2)}$ fits $F^{sa,r}_{1:(r+2)}$.

Induction step: Assume that, for $2 \le \beta \le n-r-1$, $D_{1:(r+\beta)}$ fits $F^{sa,r}_{1:(r+\beta)}$. Then, show that $D_{1:(r+1+\beta)}$ also fits $F^{sa,r}_{1:(r+1+\beta)}$.

Let $2 \le \beta \le n-r-1$. By the induction hypothesis, $D_{1:(r+\beta)}$ fits $F^{sa,r}_{1:(r+\beta)}$. It is also known that $D_{N\times n}$ passes T. So, as $D_{1:(r+\beta)}$ passes the network consisting of S followed by $RB_{(r+2):(r+\beta)}$, $D_{1:(r+1+\beta)}$ passes the network consisting of S followed by $RB_{(r+2):(r+1+\beta)}$.

Partition the matrix $D_{1:(r+\beta)}$ into $2^{n-r-\beta}$ submatrices $D^{\gamma}_{2^{r+\beta}\times(r+\beta)}$, $0 \le \gamma \le 2^{n-r-\beta}$, which are labeled in ascending order starting with 0. Let $0 \le u \le 2^{n-r-\beta-1}-1$, $\gamma_1=2u$ and $\gamma_2=2u+1$. The stage $RB_{(r+1+\beta)}$ consists of $2^{n-r-\beta-1}$ copies of the SE stage with $2^{r+1+\beta}$ inputs/outputs. The rows that are sent to the sth, $0 \le s \le 2^{r+\beta}-1$, outputs of the subnetworks that pass $D^{\gamma_1}_{2^{r+\beta}\times(r+\beta)}$ and $D^{\gamma_2}_{2^{r+\beta}\times(r+\beta)}$ enter the *sth* SB of the SE stage with $2^{r+1+\beta}$ inputs/outputs. Because no conflict occurs in the *sth* SB by hypothesis, the $(r+1+\beta)$th entries of the rows entering the *sth* SB must constitute the set $\{0,1\}$. Therefore, by definition of fit, $D_{1:(r+1+\beta)}$ fits $F^{sa,r}_{1:(r+1+\beta)}$. □

Proof of Corollary VII.1: Consider the network T, that is defined in Theorem VII.1, and its components S and $RB_{(r+2):n}$. Recall that S consists of 2^{n-r-1} copies of a rearrangeable network $P(2^{r+1}!)$ in parallel. If the Benes network $BS_{2^{r+1}\times(2r+1)}$ substitutes for each rearrangeable network $P(2^{r+1}!)$ of S, then S consists of 2^{n-r-1} copies of the rearrangeable network $BS_{2^{r+1}\times(2r+1)}$ in parallel and hence S is made identical to the subnetwork $BS_{(n-r):(n+r)}$ of $BS_{N\times(2n-1)}$. Because $BS_{1:(2n-1)}$ can be considered as being composed of $BE_{1:(n-1)}$ followed by $RB_{1:n}$, $BS_{(n-r):(n+r)}$ is the same as $BE_{(n-r):(n-1)}RB_{1:(r+1)}$. So, T is functionally equivalent to the network consisting of $BS_{(n-r):(n+r)}$ followed by $RB_{(r+2):n}$. Because the network that consists of $BS_{(n-r):(n+r)}$ followed by $RB_{(r+2):n}$ is identical to $BS_{(n-r):(2n-1)}$ and the fact that a balanced matrix $D_{N\times n}$ fits $F^{sa,r}_{N\times n}$ if and only if $D_{N\times n}$ passes T (Theorem VII.1), $D_{N\times n}$ fits $F^{sa,r}_{1:n}$ if and only if $D_{N\times n}$ passes $BS_{(n-r):(2n-1)}$. Therefore, the corollary holds. □

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