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ABSTRACT

This paper reports on an experimental study of the effectiveness of high order numerical methods applied to linear elliptic partial differential equations whose solutions have singularities or similar difficulties (e.g. boundary layers, sharp peaks). Three specific hypotheses are established with high levels of statistical confidence to support the general conclusion: There is a strong correlation between the order of a method and its efficiency. Higher order is better.

Key words: elliptic partial differential equations, software evaluation numerical methods comparisons, finite differences, bicubic collocation, FFT, Hodie method, Dyakanov method

HIGH ORDER METHODS FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH SINGULARITIES

E.N. HOUSTIS* and J.R. RICE**

1. THE EXPERIMENT. This paper reports on an experimental study of the effectiveness of high order numerical methods applied to elliptic partial differential equations whose solutions have singularities or similar difficulties (e.g. boundary layers, sharp peaks). It is commonly believed that high order methods perform poorly for such problems; we present evidence that shows almost beyond doubt that high order methods are more effective than low order methods for linear, elliptic, nearly singular partial differential equations on rectangular domains.

The experiment performed was as follows: A set of 37 partial differential equations problems (PDEs) was chosen whose solutions exhibit singularities or near singularities. See Appendix One for more details. The PDEs are from the population of [Rice et al, 1980] and are described in Appendix One. The eight programs used and the methods they implemented are described in the following table, the references cited provide further details.

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Program Name	Method Implemented
5-POINT STAR	Ordinary second order finite differenced [Forsythe and Wasow, 1969]; Gauss elimination [Dongarra et al, 1979]
P3C1-COLLOCATION	Fourth order collocation with Hermite bicubics [Houstis et al, 1977]; Gauss elimination [Dongarra et al, 1979]
DYAKANOV CG	Ordinary second order finite differences; iteration with generalized marching algorithm and conjugate gradient method. [Bank, 1977]
DYAKANOV CG-4	Dyakanov CG with Richardson extrapolation to achieve fourth order accuracy [Bank, 1977]
HODIE ACF	Fourth order finite difference method [Lynch and Rice, 1978], [Boisvert, 1979]; Gauss elimination [Dougarra et al, 1979]
FFT9(IORDER=2)	Ordinary second order finite differences plus the Fast Fourier Transform [Houstis and Papatheodorou, 1979], [Hockney, 1971]
FFT9(IORDER=4)	Fourth order finite difference method plus the Fast Fourier Transform [Houstis and Papatheodorou, 1979]
FFT9(IORDER=6)	Sixth order finite difference method plus the Fast Fourier Transform [Houstis and Papatheodorou, 1979]

No one program is applicable to all the PDEs. Each was applied to the PDEs for a sequence of rectangular meshes and statistics collected about the performance and errors achieved. This was carried out using the ELLPACK system [Rice, 1977] which ensures uniform measurements and treatment of all methods. The associated system for the evaluation of software for partial differential equations [Boisvert, Houstis and Rice, 1979] was used for the automatic generation of the problems and collection of the data.

The methodology used in this experiment is that described in [Rice, 1979] and [Houstis and Rice, 1980A]. One can obtain the details that define the experiment from the references cited.

II. THE PERFORMANCE DATA. The basic measure of performance is the amount of computer time required to obtain a given level of accuracy in the numerical solution. The accuracy is measured as the maximum absolute error on the rectangular mesh (ERR-NODES) of the method divided by the maximum absolute value of the PDE solution. Due to the discrete nature of the choice of meshes, one cannot obtain a specified accuracy exactly, so least squares straight line fits to the data (on a log-log plot) are used to interpolate computer times for three levels of accuracy: 5%, .5% and .05%. The slopes of these lines estimate the rate of convergence. The nature of the data and the appropriateness of this approach can be judged from Figures 1, 2 and 3 where one each of the good, fair and poor data sets is plotted. The regularity of the data behavior is rated (subjectively) as good for 30, fair for 6 and poor for 1 PDE. The subjective ratings of the data for distinguishing the programs are good for 26, fair for 9 and poor for 2 PDEs.

The programs are ranked according to their use of computer time to achieve the three specified levels of accuracy. Rankings seem to be the most robust measure of performance because averages are tremendoulsy affected by a few exceptional problems and the problem accuracies are somewhat incomensurate in any case. The pairwise average rankings are given in Tables 1-4 for the slopes and three accuracy levels. The comparisons are made pairwise for all the PDEs where possible (a few data are missing for extraneous reasons as noted in Appendix One). This approach gives the most robust comparative evaluation of the programs. See [Hollander and Wolfe, 1973] for the non-parametric statistical tests to be applied to such data. This data is discussed in the next section, but it is apparent that the standard second order finite difference method is not competitive with the higher order methods.

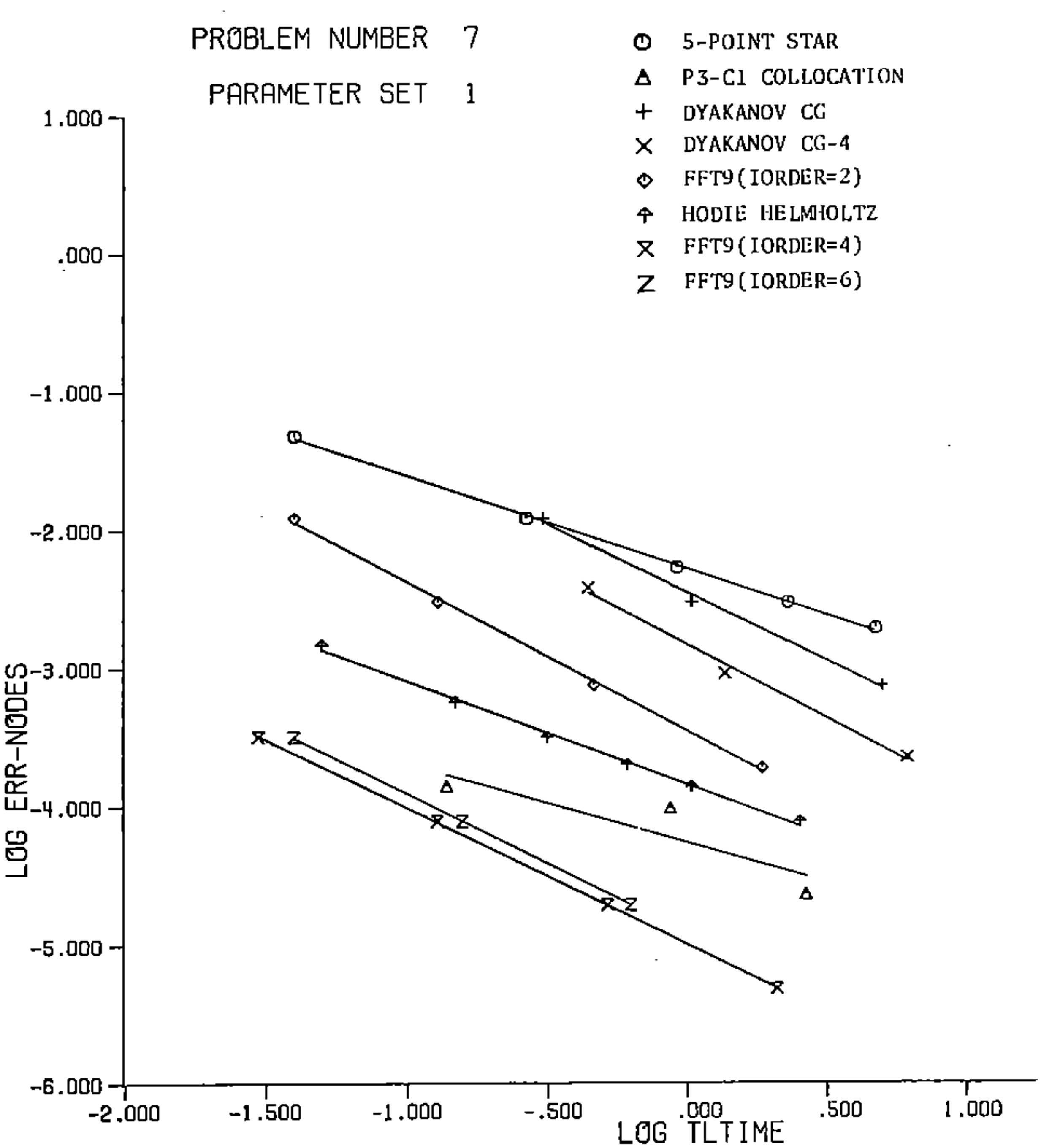

Figure 1: The data obtained for PDE 7-1. This is typical of the regularity of the behavior as observed in most cases.

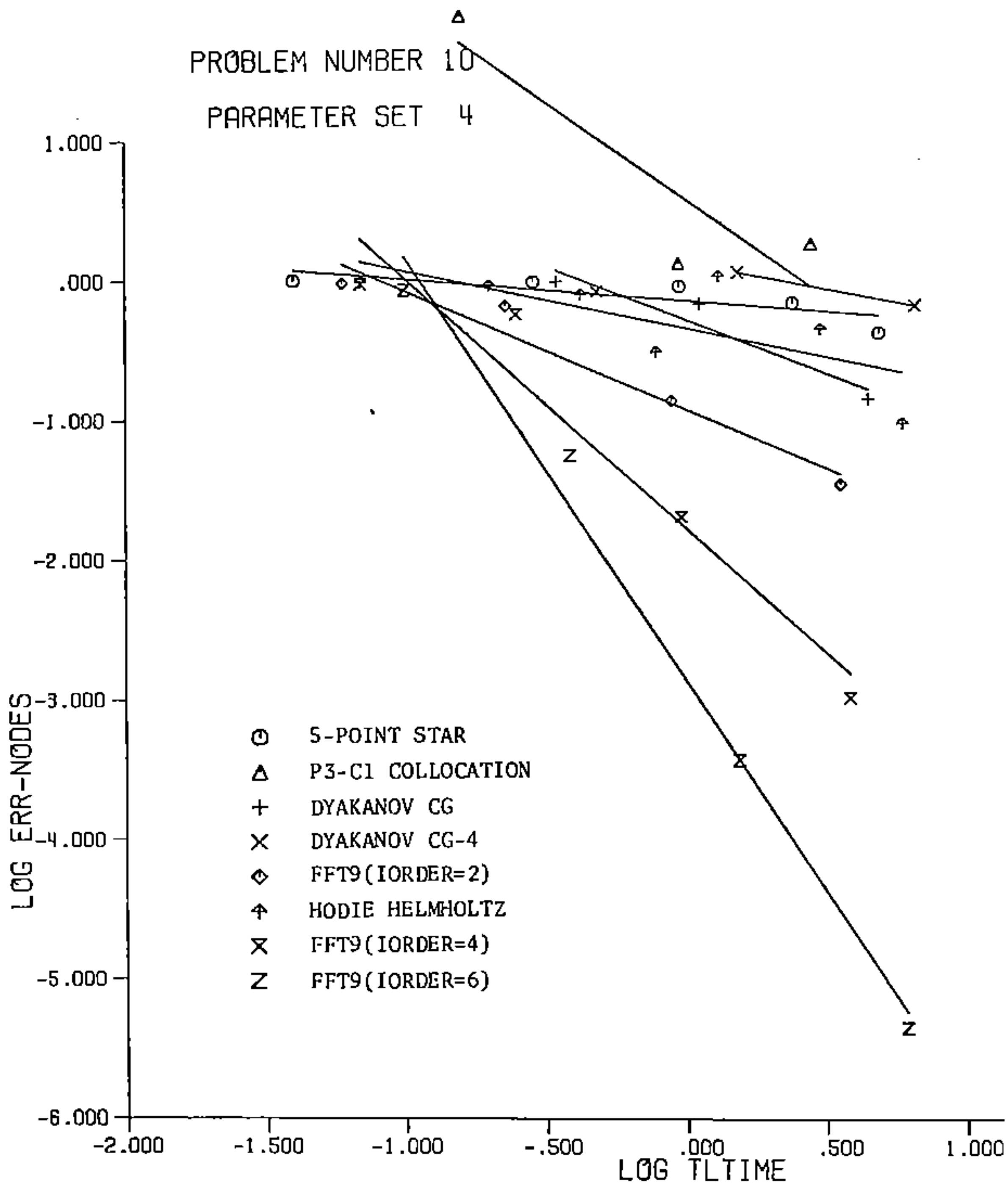

Figure 2: The data for PDE 10-4. This is fair data; note that several methods achieve very little accuracy at all.

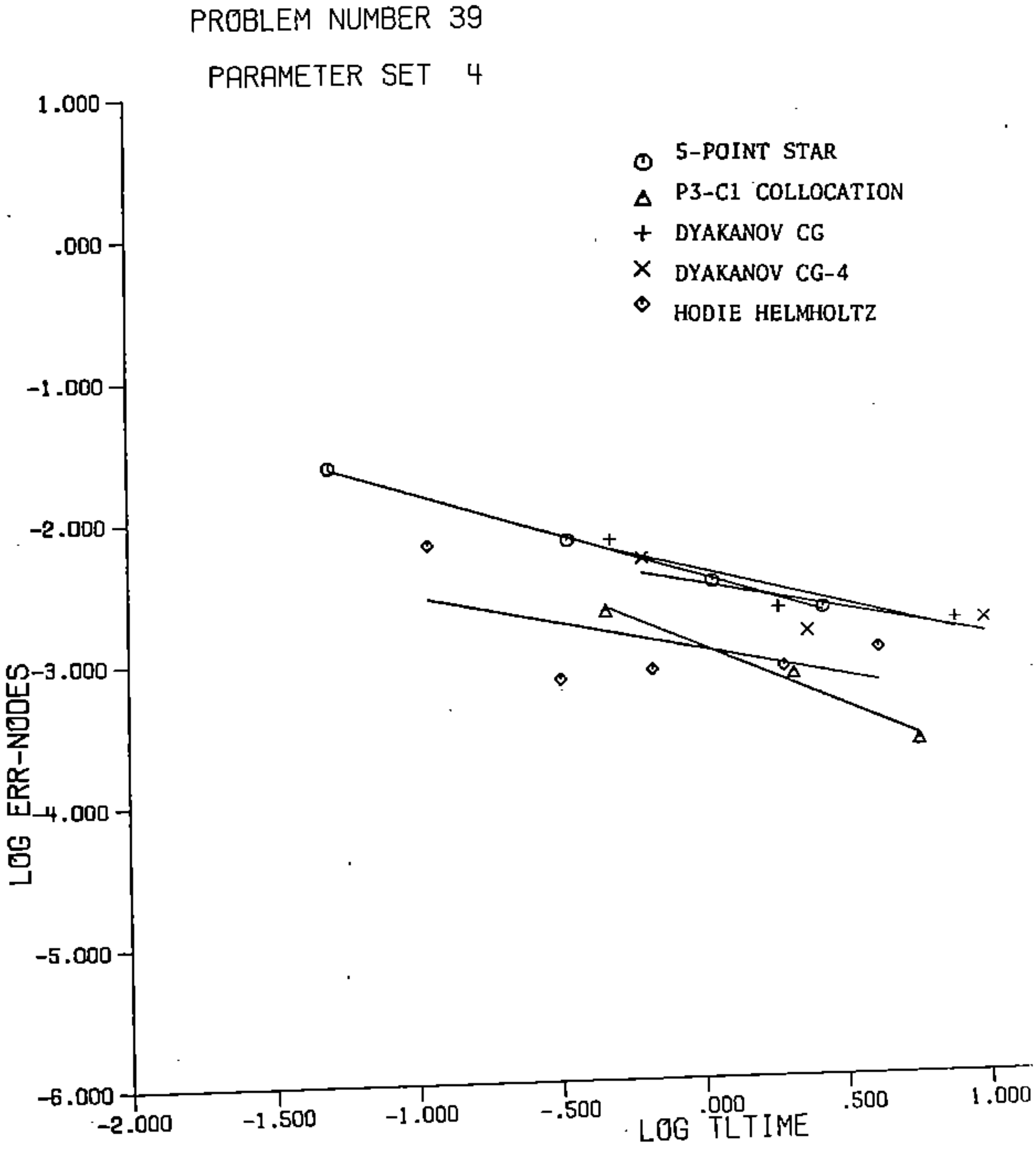

Figure 3: The data for PDE 39-4. This is the least conclusive data for any problem attempted.

The data in Tables 1-4 are completely objective in that they are computed mechanically from the raw data. Table 5 contains rankings based on subjective overall evaluations of the methods for each PDE.

The entries in Tables 1-5 is in the format

A	N
B	C

A = Average rank of method in the column (1.00 = best)

B = Average rank of method in the row

N = Number of PDEs in comparison

C = Confidence level in difference $\geq$ C% (only C=90,99 done)

The plots for each PDE (such as Figures 1-3) were examined visually and the performance of the programs ranked for each PDE. This data is given in Appendix Two and the average of the subjective ranks in Table 5. Subjective factors that enter this ranking include consistancy of behavior and tentative extrapolation to accuracies not covered by the data. Equal ranks are given when the performance was about the same.

III. ANALYSIS AND CONCLUSIONS

The ideal hypothesis to be tested in this experiment is:

High order methods are more effective than low order methods for the numerical solution of linear, elliptic PDEs on rectangles and which have singularities or near singularities.

We cannot test this hypothesis because we do not have a well-defined population of "high order" and "low order" methods. Indeed, so many things affect performance that it is unlikely that the order of the method is so dominant that all other considerations may be neglected. In other words, if ordinary second order finite differences is worse than (or better than) fourth order collocation with bicubic Hermite polynomials, one cannot assume the difference

Table 1: SLOPE COMPARISONS FOR SINGULAR PROBLEMS: ERR-NODES vs. TIME

	S	C	D	D4	F2	H	F4
Collocation	1.75 28 1.25 99						
Dyakanov	1.83 24 1.17 99	1.45 20 1.55					
Dyakanov-4	1.79 24 1.21 99	1.80 20 1.20 99	1.67 27 1.25 90				
FFT-2	2.00 16 1.00 99	1.60 15 1.40	1.74 19 1.26 90	1.68 19 1.32			
Hodie	1.90 21 1.10 99	1.78 18 1.22 90	1.59 22 1.41	1.73 22 1.27 90	1.61 18 1.39		
FTT-4	2.00 16 1.00 99	1.93 15 1.07 99	1.89 19 1.11 99	1.68 19 1.32	1.95 19 1.05 99	1.94 18 1.06 99	
FFT-6	2.00 14 1.00 99	1.92 12 1.08 99	2.00 16 1.00 99	1.75 16 1.25 90	1.94 16 1.06 99	2.00 15 1.00 99	1.88 16 1.15 99

The colums use the abbreviations

S = 5-POINT STAR
C = P3-C1 COLLOCATION
D = DYAKANOV CG
D4 = DYAKANOV CG-4
F2 = FFT9(IORDER=2)
H = HODIE HELMHOLTZ
F4 = FFT9(IORDER=4)

Table 2: 5% LEVEL COMPARISONS FOR SINGULAR PROBLEMS: ERR-NODES vs. TIME

Collocation	1.21	28												
	1.79	99												
Dyakanov	1.58	24	1.75	20										
	1.42		1.25	90										
Dyakanov-4	1.46	24	1.55	20	1.41	27								
	1.54		1.45		1.59									
FFT-2	1.81	16	1.80	15	2.00	19	2.00	19						
	1.19	90	1.20	90	1.00	99	1.00	99						
Hodie	1.90	21	1.78	18	1.95	20	2.00	22	1.61	18				
	1.10	99	1.22	90	1.05	99	1.00	99	1.39					
FFT-4	1.75	16	1.80	15	2.00	19	2.00	19	1.79	19	1.67	18		
	1.25	90	1.20	90	1.00	99	1.00	99	1.21	90	1.33			
FFT-6	1.86	14	1.83	12	2.00	16	2.00	16	1.94	16	1.80	15	1.69	16
	1.14	99	1.17	90	1.00	99	1.00	99	1.06	99	1.20	90	1.31	
	5		C		D		D4		F2		H		F4	

Table 3: 1/2% LEVEL COMPARISONS FOR SINGULAR PROBLEMS: ERR-NODES vs. TIME

Collocation	1.61	28												
	1.39													
Dyakanov	1.79	24	1.65	20										
	1.21	99	1.35											
Dyakanov-4	1.78	24	1.70	20	1.67	27								
	1.21	99	1.30	90	1.33	90								
FFT-2	1.94	16	1.87	15	1.74	19	2.00	19						
	1.06	99	1.13	99	1.26	90	1.00	99						
Hodie	1.95	21	1.83	18	1.95	22	2.00	22	1.72	18				
	1.05	99	1.17	99	1.05	99	1.00	99	1.28	90				
FFT-4	1.94	16	1.87	15	2.00	19	2.00	19	1.84	19	1.83	18		
	1.06	99	1.13	99	1.00	99	1.00	99	1.16	99	1.17	99		
FFT-6	2.00	14	1.83	12	2.00	16	2.00	16	1.94	16	1.87	15	1.94	16
	1.00	99	1.17	90	1.00	99	1.00	99	1.06	99	1.13	99	1.06	99
	5		C		D		D4		F2		H		F4	

Table 4: 0.05% LEVEL COMPARISONS FOR SINGULAR PROBLEMS: ERR-NODE vs. TIME

Collocation	1.71 28 1.29 90						
Dyakanov	1.83 24 1.17 99	1.55 20 1.45					
Dyakanov-4	1.79 24 1.21 99	1.75 20 1.25 90	1.67 27 1.33 90				
FFT-2	2.00 16 1.00 99	1.87 15 1.13 99	2.00 19 1.00 99	1.74 19 1.26 90			
Hodie	2.00 21 1.00 99	1.83 18 1.17 99	1.91 22 1.09 99	1.91 22 1.09 99	1.72 18 1.28 90		
FFT-4	2.00 16 1.00 99	2.00 15 1.00 99	2.00 19 1.00 99	2.00 19 1.00 99	1.89 19 1.11 99	1.89 18 1.11 99	
FFT-6	2.00 14 1.00 99	1.92 12 1.08 99	2.00 16 1.00 99	2.00 16 1.00 99	1.94 16 1.06 99	1.93 15 1.07 99	1.94 16 1.06 99
	5	C	D	D4	F2	H	F4

Table 5: SUBJECTIVE COMPARISONS FOR SINGULAR PROBLEMS: ERR-NODES vs. TIME

Collocation	1.6 33 1.4						
Dyakanov	1.89 27 1.11 99	1.74 23 1.26 90					
Dyakanov-4	1.81 27 1.19 99	1.76 23 1.24 90	1.72 27 1.28 90				
FFT-2	2.00 19 1.00 99	1.94 18 1.06 99	1.84 19 1.16 99	1.84 19 1.16 99			
Hodie	2.00 23 1.00 99	1.90 21 1.10 99	1.96 23 1.04 99	2.00 23 1.00 99	1.74 19 1.26 90		
FFT-4	2.00 19 1.00 99	2.00 18 1.00 99	2.00 19 1.00 99	2.00 19 1.00 99	1.89 19 1.11 99	2.00 19 1.00 99	
FFT-6	2.00 16 1.00 99	2.00 16 1.00 99	2.00 16 1.00 99	2.00 16 1.00 99	2.00 16 1.00 99	2.00 16 1.00 99	2.00 16 1.00 99
	5	C	D	D4	F2	H	F4

in performance is simply due to the difference in order of the methods. There is also a width difference in the generality of the methods. 5-POINT STAR and P3-C1 COLLOCATION implement methods that are applicable to any PDE while FFT9 requires a very special PDE. The extent to which the programs could be applied is seen from Table 6 in the Appendix. There are several ways to factor this aspect into a performance evaluation; we give evaluation in Tables 1-5 which ignore this aspect completely and then we test hypotheses below which remove most (but not all) of this aspect from the study.

There is also the difficulty that "numerical method" is not a precisely defined object. The same method can lead to vastly different performances just due to variations in the implementations. Thus, one can only compare computer programs, not methods. One intuitively feels that there is a "good" implementation for any particular method even though it is unclear that this is true in any precise sense. So, it is natural to ask if the programs used in this experiment are "good" implementations. All the programs have been developed with considerable care for efficiency and some have been shown to be superior to competing implementations (e.g., the FFT methods and the LINPACK Gauss eliminations subroutines). The only known lack of efficiency in these programs is for P3-C1 COLLOCATION. This program was developed for the more general situation of non-uniform meshes on general domains and thus it does not take any advantage of the uniform mesh or the tensor product nature of the bicubic Hermite polynomial basis. A recent analysis [Eisenstat, Schultz and Weiser, 1980] suggests that this lack degrades the execution speed by a factor of perhaps 2 to 5, depending on the nature of the PDE. While this improvement would obviously improve collocation is position in this evaluation, it would only strengthen the general conclusion of this experiment and not affect the actual hypotheses tested (as they do not involve collocation).

In order to eliminate as many irrelevant variables as possible, we break the ideal hypothesis into three separate hypotheses (all involve the same problem population as above):

H1: The fourth order finite difference program HODIE ACF is more efficient than the ordinary second order finite differences, 5-POINT STAR.

H2: The efficiency ranks of the three FFT methods are: first: FFT9(IORDER=6) second: FFT9(IORDER=4) and last FFT9(IORDER=2).

H3: The fourth order DYAKANOV CG -4 program is more efficient than the second order DYAKANOV CG program.

The data from Tables 1-4 shows that H1 is established with a confidence level greater than 99% in all four measures of efficiency. These data shows that H2 is established with a confidence level greater than 99% in six of the eight measures, greater than 90% in one other measure and inconclusive in the eighth. Even in the eighth measure (FFT9(IORDER=6) verses FFT9(IORDER=4) at the 5% error level) the higher order program FFT-6 has average rank 1.69 compared to average rank of 1.31 for the lower order program FFT-4. This difference in rank of .38 just barely misses being large enough (.41) to have significance at the 90% level. H3 is established with confidence level greater than 90% in three out of the four measures of efficiency and in the fourth the average ranks are 1.41 and 1.59, respectively, for the fourth and second order method. Thus, in one performance measure, time required to achieve an error of 5%, the ranks are close but tending to contradict H3.

The subjective data in Table 5 establishes H1 and H2 with confidence level greater than 99% and establishes H3 with confidence level greater than 90%.

We conclude that these three hypotheses are established with overwhelming confidence (H1), very strong confidence (H2) and strong confidence (H3).

The data in this experiment produces considerable evidence that these eight methods are in the following linear order of performance for this population of singular problems:

FFT9(IORDER=6)	Best
FFT9(IORDER=4)	
HODIE ACF	
FFT9(IORDER=2)	
DYAKANOV CG-4	
DYAKANOV	
P3-C1 COLLOCATION	
5-POINT STAR	Worst

The evidence for this is displayed graphically in Figure 4 where Tables 1-5 are summarized by replacing numerical values by symbols as follows:

- ⬤ Ordering is as stated with 99% confidence
- ● Ordering is as stated with 90% confidence
- ○ Ordering tends to be as stated, but less than 90% confidence
- – Ordering is opposite that stated, but less than 90% confidence
- — Ordering is opposite that stated with 90% confidence
- ▬ Ordering is opposite that stated with 99% confidence

Of the 140 pairwise comparisons we see that 94 support the ordering with 99% confidence, 28 support the ordering with 90% confidence, 11 support the ordering with no confidence level, 6 contradict the ordering with no confidence level, none contradict the ordering with 90% confidence and one contradicts the ordering with 99% confidence.

It is no surprise that the most uncertainty is in the performance measure of time to achieve the 5% error level. Indeed, the support for the linear ordering is surprisingly strong considering that one is achieving a very rough accuracy on PDEs with a wide variety of erratic or singular behaviors. The fact that P3-C1 COLLOCATION is definitely worse that 5-POINT STAR at the rough accuracy level stands out as the one definite point contradicting this ordering. This conclusion is consistant with the previous comparison [Houstis et al, 1978] of these two methods. There, it was argued that collocation the better of the two overall because (i) P3-C1 COLLOCATION is rarely much worse than 5-POINT STAR while 5-POINT STAR is sometimes very much worse than

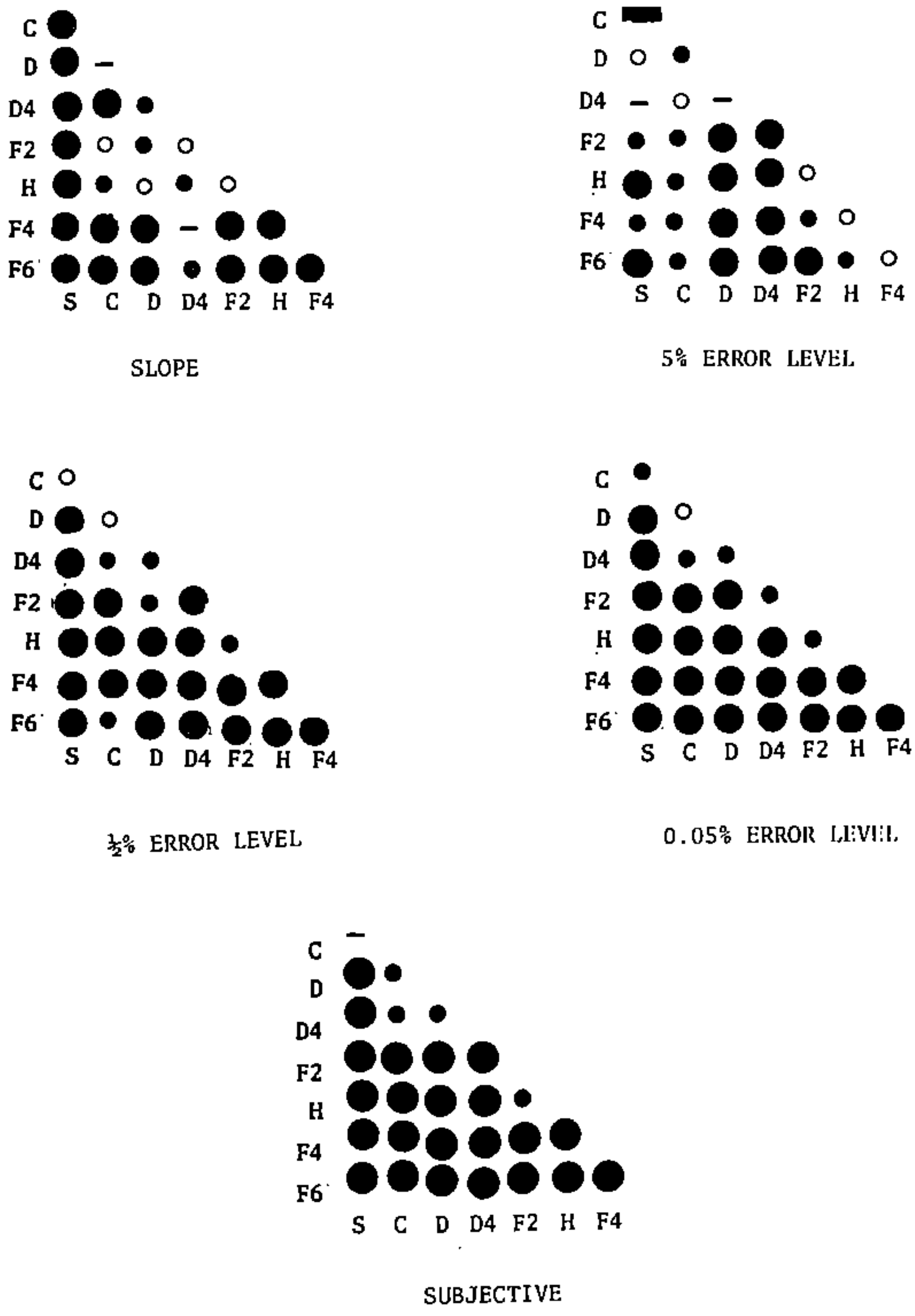

Figure 4: Graphical display of the evidence for the linear ordering in performance of the eight methods. See text for explanation of the symbols.

PS-CI COLLOCATION and (ii) the situation where 5-POINT STAR is better is one where little computation is required in either case.

The intuitive explanation for the relatively poor performance of collocation on coarse meshes is that the span of the basis functions is then much wider than the span of difference formulas which prevents the solution from "isolating" a region of difficulty in the problem.

We close by stating our general conclusions from this study: For the population of singular PDEs considered there is a strong correlation between the order of a method and its efficiency. Higher order is better.

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APPENDIX ONE: THE 37 PDE PROBLEMS

The PDEs in this study taken from [Rice, Houstis and Dyksen, 1980] are:

3-1, 3-2, 7-1, 8-2, 9-1, 9-2, 9-3, 10-2, 10-3, 10-4, 10-7, 11-2, 11-3, 11-4, 11-5, 13-1, 15-1, 15-2, 17-1, 17-2, 17-3, 20-1, 20-2, 28-2, 30-4, 30-8, 34-1, 35-1, 36-2, 39-2, 39-4, 44-2, 44-3, 47-2, 49-3, 51-1, 54-1

The first number refers to the basic problem and the second to the selection of the parameters. The 21 basic problems are presented below.

PROB 3 Artificial [13]

$$u_{xx} + u_{yy} = f$$

DOMAIN unit square
BC $u = 0$
TRUE $c(x^{\alpha/2} - x)(y^{\alpha/2} - y),\ c = 1/(\alpha^{\alpha/(1-\alpha)} - \alpha^{1/(1-\alpha)})^2$

Operator: Laplace
Right side: singular for $\alpha \leq 3$
Boundary condition: Dirichlet, homogeneous
Parameter: $1 < \alpha \leq 5$ adjusts singularity strength

PROB 7 Artificial [6]

$$u_{xx} + u_{yy} = 1$$

DOMAIN unit square
BC $u = 0$
TRUE Approximate series solution gives 10^{-9} accuracy

Operator: Laplace
Right side: Constant
Boundary condition: Dirichlet, homogeneous
Solution: Has logarithmic singularities at corners in second derivatives; approximate solution is a polynomial.
Parameter: None

PROB 8 Artificial [13]

$$u_{xx} + u_{yy} = f$$

DOMAIN unit square

BC $u = g$

TRUE $\varphi(x)\varphi(y)$ where $\varphi(x) = 1$ for $x \le .5 - \alpha$, $= 0$ for $x \ge .5 + \alpha$ and $\varphi(x)$ is a quintic polynomial for $.5 - \alpha \le x \le .5 + \alpha$ so φ has two continuous derivatives.

Operator: Laplace
Right side: Just continuous with a right angle ridge.
Boundary conditions: Dirichlet
Solution: Wave front along a right angle joining two regions where it is constant.
Parameter: α adjusts width and sharpness of wave front.

PROB 9 Artificial [13]

$$u_{xx} + u_{yy} - 100u = .5(\alpha^2 - 100)\cosh(\alpha y)/\cosh\alpha$$

DOMAIN unit square

BC $u = g$

TRUE $.5(\cosh 10x/\cosh 10 + \cosh \alpha y/\cosh \alpha)$

Operator: Helmhotz, constant coefficients, somewhat singular.
Right side: Entire but nearly singular for $\alpha \neq 10$.
Boundary conditions: Dirichlet
Solution: Boundary layer, nearly singular.
Parameter: α adjusts strength of y-side boundary layer.

PROB 10 Artificial [13]

$$u_{xx} + u_{yy} = f$$

DOMAIN unit square

BC $u = 0$

TRUE $e^{-\alpha[(x-.5)^2+(y-\beta)^2]}(x^2 - x)(y^2 - y)$

Operator: Laplace
Right side: Strongly peaked if α large, but entire.
Boundary condition: Dirichlet, homogeneous
Solution: Strongly peaked for large α.
Parameters: α adjusts strength of the peak, β moves it in the y-direction.

PROB 11 Artificial

$$u_{xx} + u_{yy} = f$$

DOMAIN unit square

BC $u = g$

TRUE $\sin[\alpha(x - y + 2)^5/(1 + (x - y + 2)^4)]$

Operator: Laplace
Right side: Oscillatory, analytic
Boundary conditions: Dirichlet
Solution: Oscillatory
Parameter: α adjusts frequency of oscillations

PROB 13 Artificial

$$((1 + (x - .4)^0_+ u_x)_x + u_{yy} = f$$

DOMAIN unit square

BC $u = g$

TRUE $\min[x+.3, .7+.5(x-.4)+(x-.4)^2/(1+x^2)](1+(y-1)^2 e^{-y})$

Operator: Self-adjoint, discontinuous coefficients.
Right side: Line of singularities along $x = 0.4$
Boundary conditions: Dirichlet
Solution: Derivative in x is singular.
Parameter: None

PROB 15 Artificial

$$u_{xx} + u_{yy} + \alpha/(y + \gamma)u_y = f$$

DOMAIN unit square

BC $u = g$

TRUE $[y^\beta + \cos(xy^2) - 1]x^2(x - 1)^2$

Operator: Laplace plus nearly singular derivative term
Right side: Singularity in $\beta - 1$ y-derivative, nearly singular for small γ.
Boundary conditions: Dirichlet
Solution: Boundary layer at $y=0$, derivative singular.
Parameters: α, γ adjust operator singularity, β adjusts unrelated derivative singularity in solution.

PROB 17 Artificial

$$u_{xx} + u_{yy} = f$$

DOMAIN unit square

BC $u = g$

TRUE $e^{[y^2+(\alpha(\beta x)^3/(1+(\beta x)^3))^2]} + \sin(x - y + .5)$

Operator: Laplace
Right side: Large values for x near .15
Boundary conditions: Dirichlet
Solution: Sharp wave front near $x = .15$, entire.
Parameters: α, β adjust the strength and shape of the wave front.

PROB 20 From $u_{xx} + u_{yy} = e^u$ [1]

$$u_{xx} + u_{yy} - wu = f, \quad w = e^T$$

DOMAIN $[0,.5] \times [0,.75]$

BC $u = g$

TRUE $10\varphi(x)\varphi(y) + \alpha$ where $\varphi(x) = e^{-100(x-.5)^2}(x^2 - x)$

Operator: Helmholtz type, approximates nonlinear operator.
Right side: Sharp, large values near $x = y = .5$.
Boundary conditions: Dirichlet, homogeneous.
Solution: T has a peak at $x = y = .5$.
Parameter: α adjusts singularity of operator.

PROB 28 Artificial

$(wu_x)_x + (wu_y)_y = 1$ where $w = \alpha$ if $0 \le x,y \le .5$
$= 1$ otherwise

DOMAIN $[-1,1] \times [-1,1]$
BC $u = 0$
TRUE unknown

Operator: Self-adjoint, discontinuous coefficients.
Right side: Constant
Boundary conditions: Dirichlet, homogeneous
Solution: Approximate solutions given for $\alpha = 1, 10, 100$. Strong wave fronts for $\alpha >> 1$.
Parameter: α adjusts size of discontinuity in operator coefficients which introduces large, sharp jumps in solution.

PROB 30 Artificial

$$[2+(y-1)e^{-\alpha y^4}]u_{xx}+[1+\frac{1}{1+(2x)^\beta}]u_{yy}+\gamma[x(x-1)+(y-.3)(6-.7)]u = f$$

DOMAIN unit square
BC $u = g$
TRUE $\dfrac{x + y^2}{1+(2x)^{\beta-1}} + (y-1)(1+x)e^{-\alpha y^4} + \gamma(x+y)\cos(xy)$

Operator: Coefficients may be widely varying, singular.
Right side: Complicated behavior
Boundary conditions: Dirichlet
Solution: Complicated behavior, with wave fronts, etc.
Parameters: α, β, γ adjust the contribution of 3 independent complexities of the problem.

PROB 34 From infinite region problem [5]

$$u_{xx} + u_{yy} = -1$$

DOMAIN $[-1,1] \times [-1,1]$
BC $u = g$
TRUE $.295776 - (x^2+y^2)/4 - 14476(x^4-6x^2y^2+y^4)/319424 + 429(x^8-28x^6y^2+70x^4y^4-28x^2y^6+y^8)/319424$

Operators: Laplace
Right side: Constant
Boundary conditions: Dirichlet
Solution: Harmonic polynomial expansion for homogeneous boundary conditions.
Parameter: None

PROB 35 Torsion for a beam [5]

$$u_{xx} + u_{yy} = 0$$

DOMAIN $[-1,1] \times [-1,1]$
BC $u = g$ for $y = \pm 1$, $(1+\alpha)u + \alpha u_N = g$ for $x = \pm 1$
TRUE $1.1786 - .1801p + (.006)q$

$p(x,y) = x^4-6x^2y^2+y^4$, $q(x,y) = x^8-28x^6y^2+70x^4y^4-28x^2y^6+y^8$

Operator: Laplace, homogeneous
Right side: Zero
Boundary conditions: Mixed, Dirichlet for $\alpha = 0$.
Solution: Harmonic polynomial combination.
Parameter: α adjusts contribution of mixed boundary condition; $\alpha = 0$ is the physical problem.

PROB 36 Adapted from Problem 27

$$(1+\beta)u_{xx} + \frac{2}{x+\alpha} u_x + \frac{1}{(x+\alpha)^2} u_{yy} + \frac{\cot y}{x+\alpha} u_y = f$$

DOMAIN unit square
BC $u = g$
TRUE $(1-\beta)e^{x+y} + \beta \log_e(x+\alpha)$

Operator: Possibly singular coefficients for $\alpha = 0$.
Right side: Analytic except for $\alpha = 0$; then singular.
Boundary conditions: Dirichlet
Solution: Logarithmic singularity for $\alpha = 0$.
Parameters: α adjusts distance of singularity from domain, β adjusts relative size of exponential and logarithmic terms in solution.

PROB 39 From nonlinear problem [4]

$$u_{xx} + u_{yy} + [1-h(x)^2 w(x,y)^2]/\beta u = 0$$

DOMAIN unit square
BC $u = 1$
TRUE unknown
Operator: Helmholtz type, homogeneous
Right side: Zero
Boundary conditions: Dirichlet, constant
Solution: Approximate solution $w(x,y)$ calculated and tabulated for 5 cases.
Parameters: $h(x) = 1/x$ for $\beta = .5, 1$ (Cases 1 and 2)
$h(x) = e^x$ for $\beta = .25, .5, 1$ (Cases 3, 4 and 5)

PROB 44 From nonlinear problem [20]

$$u_{xx} + u_{yy} + wu = w$$

DOMAIN unit square
BC $u = 0$
TRUE unknown

Operator: Helmholtz type
Right side: Complicated
Boundary conditions: Dirichlet, homogeneous
Solution: Approximate solution given for $r = r(x,y)$ tabulated from a solution to the nonlinear problem; r should be u,
$w(x,y) = -\alpha^2(1-r)^{\beta-1} e^{[\gamma\delta r/(1+\gamma r)]}$.
Parameters: α, β, γ and δ are phycical parameters. Four cases are given:

1. $\alpha = 1.425$ $\beta = 1$ $\gamma = .5$ $\delta = 2$
2. $\alpha = 10$ $\beta = 1$ $\gamma = .5$ $\delta = 2$
3. $\alpha = 1.425$ $\beta = 2$ $\gamma = .04$ $\delta = 25$
4. $\alpha = 1.425$ $\beta = 2$ $\gamma = .5$ $\delta = 2$

PROB 47 Artificial

$$u_{xx} + u_{yy} = f$$

DOMAIN unit square
BC $u = g$
TRUE $(xy)^{\alpha/2}$

Operator: Laplace
Right side: Variable singularities
Boundary conditions: Dirichlet
Solution: Singularity of variable strength.
Parameter: α adjusts singularity strength.

PROB 49 Nonlinear diffusion in catalysts [2]

$$u_{xx} + u_{yy} + wu = f$$

DOMAIN unit square
BC $u = 1$
TRUE Unknown

Operator: Helmholtz type
Right side: Complicated
Boundary conditions: Dirichlet
Solution: Approximate solution given for $r = r(x,y)$ tabulated from a solution to the nonlinear problem; r should be u,

$$w(x,y) = -(1.425)^2[(1+\beta-r)/\beta]^{\alpha-1}e^{\gamma(r-1)/r}.$$

Parameters: (α,β,γ) are physical parameters. Four cases are given: $(1,.5,2)$, $(1,.5,25)$, $(2,.04,2)$ and $(2,.5,2)$.

PROB 51 Fluid flow [18]

$$u_{xx} + \frac{1}{x}u_x + \frac{1}{x^2}u_{yy} = -10$$

DOMAIN unit square
BC $u=0$ for $x=1$; $u_N=0$ for $x,y=0$; $Au_y+Bu=0$ for $y=1$
TRUE Unknown

Operator: Singular coefficients
Right side: Constant
Boundary conditions: Mixed

$$A(x) = \begin{cases} 0 & x > \alpha \\ 1 & x \le \alpha \end{cases} \qquad B(x) = \begin{cases} 1 & x > \alpha \\ 0 & x \le \alpha \end{cases}$$

Solution: Has singularity, unusual behavior.
Parameters: α adjusts position of change in boundary condition for $y = 1$.

PROB 54 Artificial

$$(1+x^2)u_{xx}+(1+A^2)u_{yy}+2xu_x+16yAu_y-(1+(8y-x-4)^2)u = f$$
$$A(y) = 4y^2 + \alpha$$

DOMAIN unit square
BC $u = g$
TRUE $B = \max[0,(3-x/A(y))^3]$, $C = \max[0,x-A(y)]$

$D = 0$ if $C < .02$, $D = e^{-B/C}$ if $C \ge .02$

$$u(x,y) = 2.25x(x-A(y))^2(1-D)/(4A(y)^3)+1/(1+(8y-x-4)^2)$$

Operator: Expanded form of self-adjoint operator. Analytic.
Right side: Complicated with possible wild behavior.
Boundary conditions: Dirichlet
Solution: Wildly behaving for α possible, has singularities for $x - 4y^2 = \alpha$ or $4y^2 = -\alpha$.

APPENDIX TWO: THE COMPARISONS MADE AND SUBJECTIVE RANKINGS

Table 6 presents the information about the comparisons made. The programs are abbreviated as follows:

5 = 5-POINT STAR	F2 = FFT9(IORDER=2)
C = P3-C1 COLLOCATION	F4 = FFT9(IORDER=4)
D = DYAKANOV CG	F6 = FFT9(IORDER=6)
D4 = DYAKANOV CG-4	H = HODIE ACF

The columns correspond to all the pairs of evaluations made with the two programs indicated by the abbreviations. An asterisk shows the comparison was used in the study. P shows that the comparison was not made because one of the methods was doing so poorly that the comparison values would not be computed reliably from the raw data. I shows that the comparison was not made because of problems with the raw data, but it is inconclusive as to the performance situation. The blanks shows that the comparison has not been made for reasons such as (i) one of the methods is inapplicable, (ii) the program has a limitation that makes it inapplicable (iii) chance behavior ruins the comparison (e.g. symmetry produces the exact answer). The number N of PDEs in each comparison is given at the bottom of the table.

The raw data for this experiment is saved in the ELLPACK data base of elliptic PDE software performance data and may be obtained by persons interested in giving it serious study.

Table 7 presents the rankings made by a subjective examination of the plots of error versus computer time. The comparisons presented in Table 5 are based on this data. A ranking of 1 corresponds to the best performance.

Table 6: PDE's IN THE COMPARISONS

PDE	Method Pair																											
	5 C	5 D2	5 D4	5 F2	5 H	5 F4	5 F6	C D	C D4	C F2	C H	C F4	C F6	D D4	D F2	D H	D F4	D F6	D4 F2	D4 H	D4 F4	D4 F6	F2 H	F2 F4	F2 F6	H F4	H F6	F4 F6
3 - 1	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
3 - 2	I	*	*	*	*	*	*	I	I	I	I	I	I	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
7 - 1	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
8 - 2		*	*	*	*	*	*							*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
9 - 1	*	*	*	*	*	*		*	*	*	*	*		*	*	*	*		*	*	*		*	*		*		
9 - 2	*	*	*	*	*	*		*	*	*	*	*		*	*	*	*		*	*	*		*	*		*		
9 - 3	P	P	P	P	P	P		*	*	*	*	*		*	*	*	*		*	*	*		*	*		*		
10 - 2	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
10 - 3	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
10 - 4	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
10 - 7	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
11 - 2	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
11 - 3	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
11 - 4	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
11 - 5	P	P	P	P	P	P	P	P	P	P	P	P	P	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
13 - 1		*	*											*														
15 - 1	*																											
15 - 2	*																											
17 - 1	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
17 - 2	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
17 - 3	P	P	P	P	P	P	P	P	P	P	P	P	P	*	*	I	*	*	*	I	*	*	I	*	*	I	I	*
20 - 1	*																											
20 - 2	*																											
28 - 2		*	*											*														
30 - 4	*																											
30 - 8	*																											
34 - 1	*	*	*		*			*	*		*			*		*				*								
35 - 1	*	*	*		*			*	*		*			*		*				*								
36 - 2	*																											
39 - 2	P	P	P		*			P	P		P			P		P				P								
39 - 4	*	*	*		*			*	*		*			*		*				*								
44 - 2	*	*	*		I			*	*		I			*		I				I								
44 - 3	*	*	*		I			*	*		I			*		I				I								
47 - 2	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
49 - 3	I	*	*		*			I	I		I			*		*				*								
51 - 1	*																											
54 - 1	*																											
N	28	24	24	16	21	16	14	20	20	15	18	15	12	27	19	22	19	16	19	22	19	16	18	19	16	18	15	16

Table 7: SUBJECTIVE RANKINGS

PDE	S	C	D2	D4	F2	H	F4	F6
3 - 1	8	5	7	6	2	4	3	1
3 - 2	7	1	6	4	5	3	2	1
7 - 1	8	3	7	6	5	4	1	1
8 - 2	8	6	5	7	2	4	3	1
9 - 1	7	6	5	4	3	2	1	-
9 - 2	7	6	5	4	3	2	1	-
9 - 3	6	5	4	4	3	2	1	-
10 - 2	7	8	6	5	4	3	2	1
10 - 3	7	8	6	5	3	3	2	1
10 - 4	6	8	5	7	3	4	2	1
10 - 7	7	8	6	4	4	3	2	1
11 - 2	8	7	6	4	4	3	2	1
11 - 3	7	7	6	5	4	3	2	1
11 - 4	7	8	6	5	3	3	2	1
11 - 5	7	8	5	5	3	3	2	1
13 - 1	3	-	1	1	-	-	-	-
15 - 1	2	1	-	-	-	-	-	-
15 - 2	2	1	-	-	-	-	-	-
17 - 1	8	7	6	4	5	3	2	1
17 - 2	7	8	5	6	3	3	2	1
17 - 3	6	6	4	6	3	5	2	1
20 - 1	1	2	-	-	-	-	-	-
20 - 2	1	2	-	-	-	-	-	-
28 - 2	2	-	1	2	-	-	-	-
30 - 4	1	2	-	-	-	-	-	-
30 - 8	2	1	-	-	-	-	-	-
34 - 1	5	3	3	2	-	1	-	-
35 - 1	5	3	3	2	-	1	-	-
36 - 2	1	1	-	-	-	-	-	-
39 - 2	1	-	3	3	-	1	-	-
39 - 4	3	1	3	3	-	2	-	-
44 - 2	2	1	3	3	-	-	-	-
44 - 3	2	1	2	2	-	-	-	-
47 - 2	7	7	6	5	4	3	2	1
49 - 3	2	1	-	-	-	-	-	-
51 - 1	1	2	-	-	-	-	-	-
54 - 1	1	2	-	-	-	-	-	-