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High Order Methods for Elliptic Partial Differential Equations with Singularities

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HIGH ORDER METHODS FOR

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH SINGULARITIES

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ABSTRACT

This paper reports on an experimental study of the effectiveness of high order numerical methods applied to linear elliptic partial differential equations whose solutions have singularities or similar difficulties (e.g. boundary layers, sharp peaks). Three specific hypotheses are established with high levels of statistical confidence to support the general conclusion: There is a strong correlation between the order of a method and its efficiency. Higher order is better.

Key words: elliptic partial differential equations, software evaluation, numerical methods comparisons, finite differences, bicubic collocation, FFT, Hodié method, Dyakanov method
HIGH ORDER METHODS FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS
WITH SINGULARITIES

E.N. HOUSTIS* and J.R. RICE**

1. THE EXPERIMENT. This paper reports on an experimental study of the effectiveness of high order numerical methods applied to elliptic partial differential equations whose solutions have singularities or similar difficulties (e.g. boundary layers, sharp peaks). It is commonly believed that high order methods perform poorly for such problems; we present evidence that shows almost beyond doubt that high order methods are more effective than low order methods for linear, elliptic, nearly singular partial differential equations on rectangular domains.

The experiment performed was as follows: A set of 37 partial differential equations problems (PDEs) was chosen whose solutions exhibit singularities or near singularities. See Appendix One for more details. The PDEs are from the population of [Rice et al, 1980] and are described in Appendix One. The eight programs used and the methods they implemented are described in the following table, the references cited provide further details.

* Supported in part by NSF grants MCS77-01408 and MCS 79-01437

** Supported in part by NSF grants MCS77-01408 and MCS 79-09154
<table>
<thead>
<tr>
<th>Program Name</th>
<th>Method Implemented</th>
</tr>
</thead>
<tbody>
<tr>
<td>5-POINT STAR</td>
<td>Ordinary second order finite differenced [Forsythe and Wasow, 1960]; Gauss elimination [Dongarra et al, 1979]</td>
</tr>
<tr>
<td>P3C1-COLLOCATION</td>
<td>Fourth order collocation with Hermite bicubics [Houstis et al, 1977]; Gauss elimination [Dongarra et al, 1979]</td>
</tr>
<tr>
<td>DYAKANOV CG</td>
<td>Ordinary second order finite differences; iteration with generalized marching algorithm and conjugate gradient method. [Bank, 1977]</td>
</tr>
<tr>
<td>DYAKANOV CG-4</td>
<td>Dyakanov CG with Richardson extrapolation to achieve fourth order accuracy [Bank, 1977]</td>
</tr>
<tr>
<td>HODIE ACF</td>
<td>Fourth order finite difference method [Lynch and Rice, 1978], [Boisvert, 1979]; Gauss elimination [Dongarra et al, 1979]</td>
</tr>
<tr>
<td>FFT9(IORDER=2)</td>
<td>Ordinary second order finite differences plus the Fast Fourier Transform [Houstis and Papatheodorou, 1979], [Hockney, 1971]</td>
</tr>
<tr>
<td>FFT9(IORDER=4)</td>
<td>Fourth order finite difference method plus the Fast Fourier Transform [Houstis and Papatheodorou, 1979]</td>
</tr>
<tr>
<td>FFT9(IORDER=6)</td>
<td>Sixth order finite difference method plus the Fast Fourier Transform [Houstis and Papatheodorou, 1979]</td>
</tr>
</tbody>
</table>

No one program is applicable to all the PDEs. Each was applied to the PDEs for a sequence of rectangular meshes and statistics collected about the performance and errors achieved. This was carried out using the ELLPACK system [Rice, 1977] which ensures uniform measurements and treatment of all methods. The associated system for the evaluation of software for partial differential equations [Boisvert, Houstis and Rice, 1979] was used for the automatic generation of the problems and collection of the data.

The methodology used in this experiment is that described in [Rice, 1979] and [Houstis and Rice, 1980A]. One can obtain the details that define the experiment from the references cited.
II. THE PERFORMANCE DATA. The basic measure of performance is the amount
of computer time required to obtain a given level of accuracy in the numerical
solution. The accuracy is measured as the maximum absolute error on the
rectangular mesh (ERR-NODES) of the method divided by the maximum absolute
value of the PDE solution. Due to the discrete nature of the choice of
meshes, one cannot obtain a specified accuracy exactly, so least squares
straight line fits to the data (on a log-log plot) are used to interpolate
computer times for three levels of accuracy: 5%, .5% and .05%. The slopes
of these lines estimate the rate of convergence. The nature of the data
and the appropriateness of this approach can be judged from Figures 1, 2 and 3
where one each of the good, fair and poor data sets is plotted. The regular-
ity of the data behavior is rated (subjectively) as good for 30, fair for 6
and poor for 1 PDE. The subjective ratings of the data for distinguishing
the programs are good for 26, fair for 9 and poor for 2 PDEs.

The programs are ranked according to their use of computer time to
achieve the three specified levels of accuracy. Rankings seem to be the
most robust measure of performance because averages are tremendously affected
by a few exceptional problems and the problem accuracies are somewhat
incommensurate in any case. The pairwise average rankings are given in Tables
1-4 for the slopes and three accuracy levels. The comparisons are made pair-
wise for all the PDEs where possible (a few data are missing for extraneous
reasons as noted in Appendix One). This approach gives the most robust
comparative evaluation of the programs. See [Hollander and Wolfe, 1973] for
the non-parametric statistical tests to be applied to such data. This data
is discussed in the next section, but it is apparent that the standard
second order finite difference method is not competitive with the higher
order methods.
Figure 1: The data obtained for PDE 7-1. This is typical of the regularity of the behavior as observed in most cases.
Figure 2: The data for PDE 10-4. This is fair data; note that several methods achieve very little accuracy at all.
Figure 3: The data for PDE 39-4. This is the least conclusive data for any problem attempted.
The data in Tables 1-4 are completely objective in that they are computed mechanically from the raw data. Table 5 contains rankings based on subjective overall evaluations of the methods for each PDE.

The entries in Tables 1-5 is in the format

\[\begin{array}{ccc}
A & N \\
B & C \\
\end{array}\]

- A = Average rank of method in the column (1.00 = best)
- B = Average rank of method in the row
- N = Number of PDEs in comparison
- C = Confidence level in difference ≥ C% (only C=90,99 done)

The plots for each PDE (such as Figures 1-3) were examined visually and the performance of the programs ranked for each PDE. This data is given in Appendix Two and the average of the subjective ranks in Table 5. Subjective factors that enter this ranking include consistancy of behavior and tentative extrapolation to accuracies not covered by the data. Equal ranks are given when the performance was about the same.

III. ANALYSIS AND CONCLUSIONS

The ideal hypothesis to be tested in this experiment is:

High order methods are more effective than low order methods for the numerical solution of linear, elliptic PDEs on rectangles and which have singularities or near singularities.

We cannot test this hypothesis because we do not have a well-defined population of "high order" and "low order" methods. Indeed, so many things affect performance that it is unlikely that the order of the method is so dominant that all other considerations may be neglected. In other words, if ordinary second order finite differences is worse than (or better than) fourth order collocation with bicubic Hermite polynomials, one cannot assume the difference
<table>
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<td>1.00</td>
<td>1.90</td>
<td>1.10</td>
</tr>
</tbody>
</table>

The columns use the abbreviations:

- S = 5-POINT STAR
- C = P3-C1 COLLOCATION
- D = DYAKANOV CG
- D4 = DYAKANOV CG-4
- F2 = FFT9(IORDER=2)
- H = HODIE HELMHOLTZ
- F4 = FFT9(IORDER=4)
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<tr>
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<td>2.00 19</td>
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<td>1.95 22</td>
<td>1.80 16</td>
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<td>1.26 90</td>
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<td>1.00 99</td>
<td>1.06 99</td>
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</tr>
</tbody>
</table>

Table 3: 1/2% LEVEL COMPARISONS FOR SINGULAR PROBLEMS: ERR-NODES vs. TIME

<table>
<thead>
<tr>
<th>Collocation</th>
<th>Dyakanov</th>
<th>Dyakanov-4</th>
<th>FFT-2</th>
<th>FFT-4</th>
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<td>1.87 15</td>
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<td>1.00 99</td>
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<td>1.83 18</td>
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<td>1.05 99</td>
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<td>1.83 18</td>
</tr>
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<td>1.00 99</td>
<td>1.16 99</td>
<td>1.17 99</td>
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<td>1.00 99</td>
<td>1.00 99</td>
<td>1.06 99</td>
<td>1.06 99</td>
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<tr>
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<td>1.06 99</td>
<td>1.06 99</td>
<td>1.13 99</td>
<td>1.06 99</td>
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<tr>
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<td>1.17</td>
<td>1.00 99</td>
<td>1.00 99</td>
<td>1.06 99</td>
<td>1.06 99</td>
</tr>
</tbody>
</table>

|      | C | D | D4 | F2 | H | F4 |
Table 4: 0.05% LEVEL COMPARISONS FOR SINGULAR PROBLEMS: ERR-NODE vs. TIME

<table>
<thead>
<tr>
<th>Method</th>
<th>Error Rate</th>
<th>Nodes</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Collocation</td>
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</tbody>
</table>

Table 5: SUBJECTIVE COMPARISONS FOR SINGULAR PROBLEMS: ERR-NODES vs. TIME

<table>
<thead>
<tr>
<th>Method</th>
<th>Error Rate</th>
<th>Nodes</th>
<th>Time</th>
</tr>
</thead>
<tbody>
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<td>1.4</td>
</tr>
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<td>27</td>
<td>1.74</td>
</tr>
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</table>
in performance is simply due to the difference in order of the methods. There is also a width difference in the generality of the methods. 5-POINT STAR and P3-C1 COLLOCATION implement methods that are applicable to any PDE while FFT9 requires a very special PDE. The extent to which the programs could be applied is seen from Table 6 in the Appendix. There are several ways to factor this aspect into a performance evaluation; we give evaluation in Tables 1-5 which ignore this aspect completely and then we test hypotheses below which remove most (but not all) of this aspect from the study.

There is also the difficulty that "numerical method" is not a precisely defined object. The same method can lead to vastly different performances just due to variations in the implementations. Thus, one can only compare computer programs, not methods. One intuitively feels that there is a "good" implementation for any particular method even though it is unclear that this is true in any precise sense. So, it is natural to ask if the programs used in this experiment are "good" implementations. All the programs have been developed with considerable care for efficiency and some have been shown to be superior to competing implementations (e.g., the FFT methods and the LINPACK Gauss eliminations subroutines). The only known lack of efficiency in these programs is for P3-C1 COLLOCATION. This program was developed for the more general situation of non-uniform meshes on general domains and thus it does not take any advantage of the uniform mesh or the tensor product nature of the bicubic Hermite polynomial basis. A recent analysis [Eisenstat, Schultz and Weiser, 1980] suggests that this lack degrades the execution speed by a factor of perhaps 2 to 5, depending on the nature of the PDE. While this improvement would obviously improve collocation is position in this evaluation, it would only strengthen the general conclusion of this experiment and not affect the actual hypotheses tested (as they do not involve collocation).
In order to eliminate as many irrelevant variables as possible, we break
the ideal hypothesis into three separate hypotheses (all involve the same
problem population as above):

H1: The fourth order finite difference program HODIE ACF is more efficient
than the ordinary second order finite differences, 5-POINT STAR.

H2: The efficiency ranks of the three FFT methods are: First: FFT9(IORDER=6)
second: FFT9(IORDER=4) and last FFT9(IORDER=2).

H3: The fourth order DYAKANOV CG -4 program is more efficient than the second
order DYAKANOV CG program.

The data from Tables 1-4 shows that H1 is established with a confidence
level greater than 99% in all four measures of efficiency. These data shows
that H2 is established with a confidence level greater than 99% in six of the
eight measures, greater than 90% in one other measure and inconclusive in the
eighth. Even in the eighth measure (FFT9(IORDER=6) verses FFT9(IORDER=4) at
the 5% error level) the higher order program FFT-6 has average rank 1.69
compared to average rank of 1.31 for the lower order program FFT-4. This
difference in rank of .38 just barely misses being large enough (.41) to have
significance at the 90% level. H3 is established with confidence level greater
than 90% in three out of the four measures of efficiency and in the fourth the
average ranks are 1.41 and 1.59, respectively, for the fourth and second order
method. Thus, in one performance measure, time required to achieve an error of 5%,
the ranks are close but tending to contradict H3.

The subjective data in Table 5 establishes H1 and H2 with confidence level
greater than 99% and establishes H3 with confidence level greater than 90%.

We conclude that these three hypotheses are established with overwhelming
confidence (H1), very strong confidence (H2) and strong confidence (H3).

The data in this experiment produces considerable evidence that these eight
methods are in the following linear order of performance for this population of
singular problems:

<table>
<thead>
<tr>
<th>Method</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>FFT9 (IORDER=6)</td>
<td>Best</td>
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<tr>
<td>FFT9 (IORDER=4)</td>
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<td>HODIE ACI</td>
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</tr>
<tr>
<td>FFT9 (IORDER=2)</td>
<td></td>
</tr>
<tr>
<td>DYAKANOV CG-4</td>
<td></td>
</tr>
<tr>
<td>DYAKANOV</td>
<td></td>
</tr>
<tr>
<td>P3-CI COLLOCATION</td>
<td></td>
</tr>
<tr>
<td>5-POINT STAR</td>
<td>Worst</td>
</tr>
</tbody>
</table>

The evidence for this is displayed graphically in Figure 4 where Tables 1-5 are summarized by replacing numerical values by symbols as follows:

- Ordering is as stated with 99% confidence
- Ordering is as stated with 90% confidence
- Ordering tends to be as stated, but less than 90% confidence
- Ordering is opposite that stated, but less than 90% confidence
- Ordering is opposite that stated with 90% confidence
- Ordering is opposite that stated with 99% confidence

Of the 140 pairwise comparisons we see that 94 support the ordering with 99% confidence, 28 support the ordering with 90% confidence, 11 support the ordering with no confidence level, 6 contradict the ordering with no confidence level, none contradict the ordering with 90% confidence and one contradicts the ordering with 99% confidence.

It is no surprise that the most uncertainty is in the performance measure of time to achieve the 5% error level. Indeed, the support for the linear ordering is surprisingly strong considering that one is achieving a very rough accuracy on PDEs with a wide variety of erratic or singular behaviors. The fact that P3-CI COLLOCATION is definitely worse than 5-POINT STAR at the rough accuracy level stands out as the one definite point contradicting this ordering. This conclusion is consistent with the previous comparison [Houstis et al, 1978] of these two methods. There, it was argued that collocation the better of the two overall because (1) P3-CI COLLOCATION is rarely much worse than 5-POINT STAR while 5-POINT STAR is sometimes very much worse than.
Figure 4: Graphical display of the evidence for the linear ordering in performance of the eight methods. See text for explanation of the symbols.
PS-C1 COLLOCATION and (ii) the situation where 5-POINT STAR is better is one where little computation is required in either case.

The intuitive explanation for the relatively poor performance of collocation on coarse meshes is that the span of the basis functions is then much wider than the span of difference formulas which prevents the solution from "isolating" a region of difficulty in the problem.

We close by stating our general conclusions from this study: For the population of singular PDEs considered there is a strong correlation between the order of a method and its efficiency. Higher order is better.

REFERENCES


APPENDIX ONE: THE 57 PDE PROBLEMS

The PDEs in this study taken from [Rice, Houstis and Dyksen, 1980] are:

3-1, 3-2, 7-1, 8-2, 9-1, 9-2, 9-3, 10-2, 10-3, 10-4, 10-7, 11-2, 11-3, 11-4, 11-5, 13-1, 15-1, 15-2, 17-1, 17-2, 17-3, 20-1, 20-2, 28-2, 30-4, 30-8, 34-1, 35-1, 36-2, 39-2, 39-4, 44-2, 44-3, 47-2, 49-3, 51-1, 54-1

The first number refers to the basic problem and the second to the selection of the parameters. The 21 basic problems are presented below.

**PROB 3** Artificial [13]

\[ u_{xx} + u_{yy} = f \]

**DOMAIN** unit square

**BC** \[ u = 0 \]

**TRUE** \[ c(a/2 - x)(a/2 - y), c = 1/(a/(1-a) - (1-a)/2) \]

Operator: Laplace

Right side: singular for \( a \leq 3 \)

Boundary condition: Dirichlet, homogeneous

Parameter: \( 1 < a \leq 5 \) adjusts singularity strength

**PROB 7** Artificial [6]

\[ u_{xx} + u_{yy} = 1 \]

**DOMAIN** unit square

**BC** \[ u = 0 \]

**TRUE** Approximate series solution gives \( 10^{-9} \) accuracy

Operator: Laplace

Right side: Constant

Boundary condition: Dirichlet, homogeneous

Solution: Has logarithmic singularities at corners in second derivatives; approximate solution is a polynomial.

Parameter: None
PROB 8  Artificial [13]
\[ u_{xx} + u_{yy} = f \]

**DOMAIN**  unit square  
**BC**  \( u = g \)  
**TRUE**  \( \psi(x)\psi(y) \)  where  \( \psi(x) = 1 \) for  \( x \leq 0.5 - \alpha, = 0 \) for  \( x > 0.5 + \alpha \) and  \( \psi(x) \) is a quintic polynomial for  \( 0.5 - \alpha \leq x \leq 0.5 + \alpha \) so  \( \psi \) has two continuous derivatives.

Operator: Laplace  
Right side: Just continuous with a right angle ridge.  
Boundary conditions: Dirichlet  
Solution: Wave front along a right angle joining two regions where it is constant.  
Parameter:  \( \alpha \) adjusts width and sharpness of wave front.

PROB 9  Artificial [13]
\[ u_{xx} + u_{yy} - 100u = 0.5(\alpha^2 - 100)cosh(\alpha y)/cosh \alpha \]

**DOMAIN**  unit square  
**BC**  \( u = g \)  
**TRUE**  \( 0.5(cosh 10x/cosh 10 + cosh \alpha y/cosh \alpha) \)

Operator: Helmholtz, constant coefficients, somewhat singular.  
Right side: Entire but nearly singular for  \( \alpha \neq 10 \).  
Boundary conditions: Dirichlet  
Solution: Boundary layer, nearly singular.  
Parameter:  \( \alpha \) adjusts strength of y-side boundary layer.

PROB 10  Artificial [13]
\[ u_{xx} + u_{yy} = f \]

**DOMAIN**  unit square  
**BC**  \( u = 0 \)  
**TRUE**  \( e^{-\alpha((x-.5)^2+(y-\beta)^2)}(x^2-x)(y^2-y) \)

Operator: Laplace  
Right side: Strongly peaked if  \( \alpha \) large, but entire.  
Boundary condition: Dirichlet, homogeneous  
Solution: Strongly peaked for large  \( \alpha \).  
Parameters:  \( \alpha \) adjusts strength of the peak,  \( \beta \) moves it in the y-direction.

PROB 11  Artificial
\[ u_{xx} + u_{yy} = f \]

**DOMAIN**  unit square  
**BC**  \( u = g \)  
**TRUE**  \( \sin[\alpha(x - y + 2)^5/(1 + (x - y + 2)^4)] \)

Operator: Laplace  
Right side: Oscillatory, analytic  
Boundary conditions: Dirichlet  
Solution: Oscillatory  
Parameter:  \( \alpha \) adjusts frequency of oscillations.
PROB 13  Artificial

\[
\left(1 + (x - .4)\right)^2 + \frac{1}{x}u_x + u_{yy} = f
\]

**DOMAIN**  unit square
**BC**  \(u = g\)
**TRUE**  \(\min\left[x + .3, 7.5(x - .4) + (x - .4)^2/(1 + x^2)\right] (1 + (y - 1)^2)e^{-y}\)

**Operator:** Self-adjoint, discontinuous coefficients.
**Right side:** Line of singularities along \(x = 0.4\)
**Boundary conditions:** Dirichlet
**Solution:** Derivative in \(x\) is singular.
**Parameter:** None

---

**PROB 15**  Artificial

\[
u_{xx} + u_{yy} + \alpha/(y + y)u_y = f
\]

**DOMAIN**  unit square
**BC**  \(u = g\)
**TRUE**  \(y^6 + \cos(xy^2) - 1)x^2(x - 1)^2\)

**Operator:** Laplace plus nearly singular derivative term
**Right side:** Singularity in \(\beta - 1\) \(y\)-derivative, nearly singular for small \(\gamma\).
**Boundary conditions:** Dirichlet
**Solution:** Boundary layer at \(y = 0\), derivative singular.
**Parameters:** \(\alpha, \gamma\) adjust operator singularity, \(\beta\) adjusts unrelated derivative singularity in solution.

---

**PROB 17**  Artificial

\[
u_{xx} + u_{yy} = f
\]

**DOMAIN**  unit square
**BC**  \(u = g\)
**TRUE**  \(e^{y^2/\left(1 + (\beta x)^3\right)^2} + \sin(x - y + .5)\)

**Operator:** Laplace
**Right side:** Large values for \(x\) near \(0.15\)
**Boundary conditions:** Dirichlet
**Solution:** Sharp wave front near \(x = 0.15\), entire.
**Parameters:** \(\alpha, \beta\) adjust the strength and shape of the wave front.

---

**PROB 20**  From \(u_{xx} + u_{yy} = e^u\) [1]

\[
u_{xx} + u_{yy} = wu = f, w = e^{\varphi(x)}
\]

**DOMAIN**  \([0, 0.5] \times [0, 0.75]\)
**BC**  \(u = g\)
**TRUE**  \((10\varphi(x)\varphi(y) + \alpha)\) where \(\varphi(x) = e^{-100(x-.5)^2(x^2-x)}\)

**Operator:** Helmholtz type, approximates nonlinear \(\varphi\) operator.
**Right side:** Sharp, large values near \(x = y = 0.5\).
**Boundary conditions:** Dirichlet, homogeneous.
**Solution:** \(T\) has a peak at \(x = y = 0.5\).
**Parameter:** \(\alpha\) adjusts singularity of operator.
PROB 28 Artificial

\[(w_{u_x})_x + (w_{u_y})_y = 1 \text{ where } w = a \text{ if } 0 \leq x, y \leq 0.5 \]
\[= 1 \text{ otherwise} \]

**DOMAIN** \([-1,1) \times [-1,1)\]
**BC** \(u = 0\)
**TRUE** unknown

Operator: Self-adjoint, discontinuous coefficients.
Right side: Constant
Boundary conditions: Dirichlet, homogeneous
Solution: Approximate solutions given for \(a = 1, 10, 100\). Strong wave fronts for \(a \gg 1\).
Parameter: \(a\) adjusts size of discontinuity in operator coefficients which introduces large, sharp jumps in solution.

---

PROB 30 Artificial

\[\frac{1}{(2+(y-l)\text{e}^x+1\text{+lu}+y[x(x-l)+(y-.3)(6-.7)]u}} \]

**DOMAIN** unit square
**BC** \(u = g\)
**TRUE** \(\frac{x+y^2}{1+(2x)}e^{-ay} + \frac{(y-2)(1+x)e^{-ay}}{1+(2x)} + \frac{\gamma(x+y)\cos(xy)}{1+(2x)}\)

Operator: Coefficients may be widely varying, singular.
Right side: Complicated behavior
Boundary conditions: Dirichlet
Solution: Complicated behavior, with wave fronts, etc.
Parameters: \(a, \beta, \gamma\) adjust the contribution of 3 independent complexities of the problem.

---

PROB 34 From infinite region problem [5]

\[u_{xx} + u_{yy} = -1\]

**DOMAIN** \([-1,1) \times [-1,1)\]
**BC** \(u = g\)
**TRUE** \(\frac{1}{2}95776 - \frac{(x^2+y^2)^2}{2} - 14476(x^4-6x^2y^2+y^4)/319424\)
\[+ 429(x^8-28x^6y+70x^4y^4-28x^2y^6+y^8)/319424\]

Operator: Laplace
Right side: Constant
Boundary conditions: Dirichlet
Solution: Harmonic polynomial expansion for homogeneous boundary conditions.
Parameter: None

---

PROB 35 Torsion for a beam [5]

\[u_{xx} + u_{yy} = 0\]

**DOMAIN** \([-1,1) \times [-1,1)\]
**BC** \(u = g\) for \(y = \pm 1\), \((1+a)u + u_N = g\) for \(x = \pm 1\)
**TRUE** \(1.1786 - 0.1801p + (0.006)q\)
\[p(x,y) = x^6-2x^2y^4, q(x,y) = x^6-28x^6y+70x^4y^4-28x^2y^6+y^8\]

Operator: Laplace, homogeneous
Right side: Zero
Boundary conditions: Mixed, Dirichlet for \(a = 0\).
Solution: Harmonic polynomial combination.
Parameter: \(a\) adjusts contribution of mixed boundary condition; \(a = 0\) is the physical problem.
PROB 36  Adapted from Problem 27

\[(1+\beta)u_{xx} + \frac{2}{x+a} u_x + \frac{1}{(x+a)^2} u_y + \cot \frac{y}{x+a} u_y = f\]

**DOMAIN**  unit square
**BC**  \( u = g \)
**TRUE**  

\[(1 - \beta)e^{x+y} + \beta \log_e(x + a)\]

Operator: Possibly singular coefficients for \( \alpha = 0 \).
Right side: Analytic except for \( \alpha = 0 \); then singular.
Boundary conditions: Dirichlet
Solution: Logarithmic singularity for \( \alpha = 0 \).
Parameters: \( \alpha \) adjusts distance of singularity from domain, \( \beta \) adjusts relative size of exponential and logarithmic terms in solution.

---

PROB 39  From nonlinear problem [4]

\[u_{xx} + u_{yy} + \left[1 - h(x)\right] \frac{w(x,y)^2}{\beta u} = 0\]

**DOMAIN**  unit square
**BC**  \( u = 1 \)
**TRUE**  unknown

Operator: Helmholtz type, homogeneous
Right side: Zero
Boundary conditions: Dirichlet, constant
Solution: Approximate solution \( w(x,y) \) calculated and tabulated for 5 cases.
Parameters: \( h(x) = 1/x \) for \( \beta = .5, 1 \) (Cases 1 and 2).
\( h(x) = e^x \) for \( \beta = .25, .5, 1 \) (Cases 3, 4 and 5).

---

PROB 44  From nonlinear problem [20]

\[u_{xx} + u_{yy} + wu = w\]

**DOMAIN**  unit square
**BC**  \( u = 0 \)
**TRUE**  unknown

Operator: Helmholtz type
Right side: Complicated
Boundary conditions: Dirichlet, homogeneous
Solution: Approximate solution given for \( r = r(x,y) \) tabulated from a solution to the nonlinear problem; \( r \) should be \( u \), \( w(x,y) = \frac{\alpha^2(1-r)\beta-1}{\gamma \delta r/(1+\gamma r)} \).
Parameters: \( \alpha, \beta, \gamma \) and \( \delta \) are physical parameters.

Four cases are given:
1. \( \alpha = 1.425 \)  \( \beta = 1 \)  \( \gamma = .5 \)  \( \delta = 2 \)
2. \( \alpha = 10 \)  \( \beta = 1 \)  \( \gamma = .5 \)  \( \delta = 2 \)
3. \( \alpha = 1.425 \)  \( \beta = 2 \)  \( \gamma = .04 \)  \( \delta = 25 \)
4. \( \alpha = 1.425 \)  \( \beta = 2 \)  \( \gamma = .5 \)  \( \delta = 2 \)
PROB 47  Artificial
\[ u_{xx} + u_{yy} = f \]
**DOMAIN** unit square
**BC** \( u = g \)
**TRUE** \( (xy)^{\alpha/2} \)
**Operator:** Laplace
**Right side:** Variable singularities
**Boundary conditions:** Dirichlet
**Solution:** Singularity of variable strength.
**Parameter:** \( \alpha \) adjusts singularity strength.

PROB 49  Nonlinear diffusion in catalysts [2]
\[ u_{xx} + u_{yy} + wu = f \]
**DOMAIN** unit square
**BC** \( u = 1 \)
**TRUE** Unknown
**Operator:** Helmholtz type
**Right side:** Complicated
**Boundary conditions:** Dirichlet
**Solution:** Approximate solution given for \( r = r(x,y) \)
Tabulated from a solution to the nonlinear problem; \( r \) should be \( u \),
\[ w(x,y) = -(1.425)^2 \left[ \left( \frac{1+\beta-r}{\beta} \right)^{\alpha-1} e^{\gamma} \right] \]
**Parameters:** \( (\alpha,\beta,\gamma) \) are physical parameters. Four cases are given: \( (1,5,2), (1.5,25), (2,0.4,2) \) and \( (2,5,2) \).

PROB 51  Fluid flow [18]
\[ u_{xx} + \frac{1}{x} u_{x} + \frac{1}{x^2} u_{yy} = -10 \]
**DOMAIN** unit square
**BC** \( u = 0 \) for \( x = 1 \); \( u_{y} = 0 \) for \( x,y = 0; A_{u} + u_{y} = 0 \) for \( y = 1 \)
**TRUE** Unknown
**Operator:** Singular coefficients
**Right side:** Constant
**Boundary conditions:** Mixed
\[ A(x) = \begin{cases} 0 & x > \alpha \\ 1 & x \leq \alpha \end{cases} \]
\[ B(x) = \begin{cases} 1 & x > \alpha \\ 0 & x \leq \alpha \end{cases} \]
**Solution:** Has singularity, unusual behavior.
**Parameters:** \( \alpha \) adjusts position of change in boundary condition for \( y = 1 \).

PROB 54  Artificial
\[ (1+x^2)u_{xx} + (1+4)^2 u_{yy} + 12xu_x + 16y u_y -(1+8y-x-4)^2 u = f \]
**DOMAIN** unit square
**BC** \( u = g \)
**TRUE** \( B = \max \{0,(3-x/A(y))^3\} \), \( C = \max \{0,x-A(y)\} \)
\[ D = 0 \text{ if } C < .02, \quad D = e^{-B/C} \text{ if } C \geq .02 \]
\[ u(x,y) = 2.25 (x-A(y))^2(1-D)/(4A(y)^3)+1/(1+(8y-x-4)^2) \]
**Operator:** Expanded form of self-adjoint operator.
**Analytic.**
**Right side:** Complicated with possible wild behavior.
**Boundary conditions:** Dirichlet
**Solution:** Wildly behaving for \( \alpha \) possible, has singularities for \( x - 4y^2 = \alpha \) or \( 4y^2 = -\alpha \).
APPENDIX TWO: THE COMPARISONS MADE AND SUBJECTIVE RANKINGS

Table 6 presents the information about the comparisons made. The programs are abbreviated as follows:

- $S$ = 5-POINT STAR
- $C$ = P3-C1 COLLOCATION
- $D$ = DYAKANOV CG
- $D_4$ = DYAKANOV CG-4
- $F_2$ = FFT9(IORDER=2)
- $F_4$ = FFT9(IORDER=4)
- $F_6$ = FFT9(IORDER=6)
- $H$ = HODIE ACF

The columns correspond to all the pairs of evaluations made with the two programs indicated by the abbreviations. An asterisk shows the comparison was used in the study. P shows that the comparison was not made because one of the methods was doing so poorly that the comparison values would not be computed reliably from the raw data. I shows that the comparison was not made because of problems with the raw data, but it is inconclusive as to the performance situation. The blanks shows that the comparison has not been made for reasons such as (i) one of the methods is inapplicable, (ii) the program has a limitation that makes it inapplicable (iii) chance behavior ruins the comparison (e.g. symmetry produces the exact answer).

The number $N$ of PDEs in each comparison is given at the bottom of the table.

The raw data for this experiment is saved in the ELLPACK data base of elliptic PDE software performance data and may be obtained by persons interested in giving it serious study.

Table 7 presents the rankings made by a subjective examination of the plots of error versus computer time. The comparisons presented in Table 5 are based on this data. A ranking of 1 corresponds to the best performance.
| PDE | S | S | S | S | S | C | C | C | C | D | D | D | D | D | D | D | D | H | H | H |
| 3 - 1 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 3 - 2 | I | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 3 - 1 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 8 - 2 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 9 - 1 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 9 - 2 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 10 - 2 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 10 - 3 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 10 - 4 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 10 - 7 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 11 - 2 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 11 - 3 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 11 - 4 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 13 - 1 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 15 - 1 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 15 - 2 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 17 - 1 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 17 - 2 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 20 - 1 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 20 - 2 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 28 - 2 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 30 - 4 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 30 - 8 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 34 - 1 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 35 - 1 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 36 - 2 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 39 - 4 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 44 - 2 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 44 - 3 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 47 - 2 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 49 - 3 | I | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 51 - 1 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| 54 - 1 | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |

Table 6: PDEs IN THE COMPARISONS

Method Pair

N 28 24 24 16 14 20 20 15 18 15 12 27 19 22 19 16 14 16 18 18 15 16
Table 7: SUBJECTIVE RANKINGS

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