A Finite Difference Approach for Predicting Acoustic Behavior of the Poro-Elastic Particle Stacks

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A Finite Difference Approach for Predicting Acoustic Behavior of the Poro-Elastic Particle Stacks

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Motivation

At low input levels, the particle stack behaves like a solid
- Many absorption peaks due to vibrational modes of edge-constrained cylindrical solid

At high input levels, the particle stack behaves like a fluid
- Simple plane wave behavior
- A numerical model that can explain both types of behavior is needed

Glass bubbles: 3M K20
Bubble radius: 30 μm
Density: 200 kg/m³
Introduction

• Acoustic behavior of particles has been of interests recently - hybrid materials, particle-nonwoven fiber composite [Mo et al., 2021. No. 2021-01-1127. SAE Technical Paper, 2021].

• Biot theory has been used to model such behavior, for example, Tsuruha et al. [(2020). J. Acoust. Soc. Am. 147(5), 3418–3428] modeled glass bubble with poro-elastic model.

Part of figure 10 in (Tsuruha et al., 2020). On the left side is absorption coefficient calculation based on Biot theory, compared with measurements of cased glass bubbles on the right side.

To better accommodate testing conditions, different boundary conditions, and better explain observations in measurements, a finite difference model based on Biot theory is developed.
Introduction – Biot Poro-Elastic Theory

• Advantage of Biot poro-elastic theory

The poro-elastic model can capture features that are not captured by rigid model, by considering all three waves propagating in one direction (two compressional waves and one shear wave).

Part of figure 2 in [Mo et al., 2021. Inter-Noise and Noise-Con Congress and Conference Proceedings, 263(3), 3523-3529]. The solid phase resonance peak of a 30-mm-thick activated carbon particle stack at about 200 Hz is modeled by the poro-elastic model, but not the rigid model.

• Why finite difference?

Geometry, boundary conditions, varying stiffness.

• Geometry: 2D axisymmetric model representing the material tested in a cylindrical standing wave tube.
• Boundary conditions: finite difference scheme allows implementing various boundary conditions at tube wall – slip B.C. and fully constrained B.C.
• Varying stiffness: finite difference scheme allows us to assign different stiffnesses to material at different locations.
Governing Equations – Stress-Strain

According to Biot theory stress-strain equation,

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_x \\
\tau_y \\
\tau_z \\
S
\end{bmatrix} = \begin{bmatrix}
2N + A \\
2N + A \\
2N + A
\end{bmatrix}
\]

Normal stresses
Solid phase
Shear stresses
Solid phase
Pressure
Fluid phase

\[P = \frac{4}{3}N + K_b + (1 - \phi)^2 K_{eq}\]

\[Q = \phi (1 - \phi) K_{eq}\]

\[R = \phi^2 K_{eq}\]

\[e_x = \frac{\partial u_x}{\partial x}\]
\[\gamma_z = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\]
The derivation starts from equations of motion,

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_z}{\partial y} + \frac{\partial \tau_y}{\partial z} = \rho_{11} \ddot{u}_x + \rho_{12} \ddot{u}_x
\]

If we take stiffness variation into consideration,

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_z}{\partial y} + \frac{\partial \tau_y}{\partial z} = 2 \frac{\partial N}{\partial x} e_x + \frac{\partial A}{\partial x} + \frac{\partial N}{\partial y} \gamma_z + \frac{\partial N}{\partial z} \gamma_y + 2N \frac{\partial e_x}{\partial x} + A \frac{\partial e}{\partial x} + N \frac{\partial \gamma_z}{\partial y} + N \frac{\partial \gamma_y}{\partial z} + Q \frac{\partial \epsilon}{\partial x} = \rho_{11} \ddot{u}_x + \rho_{12} \ddot{U}_x
\]

\[
Q \frac{\partial e}{\partial x} + R \frac{\partial \epsilon}{\partial x} = \rho_{12} \ddot{u}_x + \rho_{22} \ddot{U}_x
\]

Fluid phase displacement
Vector noted as \( \textbf{U} \)

Solid phase displacement
Vector noted as \( \textbf{u} \)
Governing Equations – Cylindrical Coordinates

\[
(A + N) \nabla e + N \nabla^2 \mathbf{u} + 2 \mathbf{e} \cdot \nabla N + \nabla A \cdot \mathbf{e} + Q \nabla \epsilon = \rho_{11} \ddot{\mathbf{u}} + \rho_{12} \ddot{\mathbf{U}}
\]
\[
Q \nabla e + R \nabla \epsilon = \rho_{12} \ddot{\mathbf{u}} + \rho_{22} \ddot{\mathbf{U}}
\]

\[
\epsilon = \begin{bmatrix}
    u_{x,x} & \frac{1}{2} (u_{r,x} + u_{x,r}) & \frac{1}{2} \left( \frac{u_{x,\theta}}{r} + u_{\theta,x} \right) \\
    \frac{1}{2} (u_{r,x} + u_{x,r}) & u_{r,r} & \frac{1}{2} \left( \frac{u_{r,\theta} - u_\theta}{r} + u_{\theta,r} \right) \\
    \frac{1}{2} \left( \frac{u_{x,\theta}}{r} + u_{\theta,x} \right) & \frac{1}{2} \left( \frac{u_{r,\theta} - u_\theta}{r} + u_{\theta,r} \right) & u_r + u_{\theta,\theta}
\end{bmatrix}
\]
Finite Difference – Discretization

For non-boundary locations,

\[
\frac{\partial f}{\partial x}_{m,n} = \frac{f_{m+1,n} - f_{m-1,n}}{2\Delta x}
\]

\[
\frac{\partial f}{\partial r}_{m,n} = \frac{f_{m,n+1} - f_{m,n-1}}{2\Delta r}
\]

\[
\frac{\partial^2 f}{\partial x^2}_{m,n} = \frac{f_{m+1,n} - 2f_{m,n} + f_{m-1,n}}{\Delta x^2}
\]

\[
\frac{\partial^2 f}{\partial r^2}_{m,n} = \frac{f_{m,n+1} - 2f_{m,n} + f_{m,n-1}}{\Delta r^2}
\]

\[
\frac{\partial^2 f}{\partial x \partial r}_{m,n} = \frac{f_{m+1,n+1} - f_{m-1,n+1} - f_{m+1,n-1} + f_{m-1,n-1}}{4\Delta x \Delta r}
\]
Absorption coefficient can be obtained by decomposing the incident and reflected waves.

Undetermined fluid phase displacements governed by curl relation,

\[ \nabla \times \mathbf{U} = -\frac{\rho_{12}}{\rho_{22}} \nabla \times \mathbf{u} \]

At axis
\[ \frac{\partial p}{\partial r} \bigg|_{r=0} = 0 \]
\[ u^x \bigg|_{r=0} = 0 \]
\[ U^r \bigg|_{r=0} = 0 \]

At wall
\[ \frac{\partial p}{\partial r} \bigg|_{r=R} = 0 \]
\[ u^x \bigg|_{r=R} = 0 \] (2D)
\[ \frac{\partial u^x}{\partial r} \bigg|_{r=R} = 0 \] (1D)
\[ u^r \bigg|_{r=R} = 0 \]
\[ U^r \bigg|_{r=R} = 0 \]

At bottom
\[ u^x \bigg|_{x=L} = 0 \]
\[ U^x \bigg|_{x=L} = 0 \]
\[ u^r \bigg|_{x=L} = 0 \]
Finite Difference – 1D Validation
- Solid phase can slip at tube wall – planar motion

- 1D validation of glass bubble stack: \( \frac{\partial u^x}{\partial r} \bigg|_{r=R} = 0 \)
- Compared with layered system approach [Dazel et al., 2013. J. Appl. Phys. 113(8), 083506]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tube radius</td>
<td>5 cm</td>
</tr>
<tr>
<td>Stack thickness</td>
<td>2 cm</td>
</tr>
<tr>
<td>Particle radius</td>
<td>30 μm</td>
</tr>
<tr>
<td>Porosity</td>
<td>0.373</td>
</tr>
<tr>
<td>Skeleton density</td>
<td>200 kg/m³</td>
</tr>
<tr>
<td>Frame modulus</td>
<td>(2 \times 10^5) Pa</td>
</tr>
<tr>
<td>Loss factor</td>
<td>0.02</td>
</tr>
</tbody>
</table>
Finite Difference – 1D Validation
- Stiffness variation with depth – due to gravity, friction, and loads

Stiffness variation is considered in calculation by (Tsuruha et al., 2020), also according to (Matchett and Yanagida, 2003; Duran, 2000, Springer), the stiffness of particle stack is dependent on the depth,

\[ E = E_0 \sigma^{1/3} = E_0 \left[ \frac{\rho g}{\beta} (1 - e^{-\beta x}) + P_L e^{-\beta x} \right]^{1/3} \]

And thus, the derivatives of moduli should also be considered,

\[ \frac{\partial E}{\partial x} = \frac{1}{3} E_0 \sigma^{-2/3} \frac{\partial \sigma}{\partial x} \]

\[ \times \frac{1}{3} E_0 \left[ \frac{\rho g}{\beta} (1 - e^{-\beta x}) + P_L e^{-\beta x} \right]^{-2/3} \]

\[ \times (\rho g - \beta P_L) e^{-\beta x} \]

The same stack of glass bubbles with \( \beta = 0.4 \), \( E_0 = 2 \times 10^5 \text{ Pa}^{2/3} \) under 10 Pa input sound pressure is compared with layered system.
Solid phase constrained at wall
2cm-thick Glass bubble:
Tube radius: 5 cm, Bulk density: 125.4 kg/m³
Particle radius: 30 μm, Porosity: 0.373
Poisson’s ratio: 0.35, Loss factor: 0.01
Janssen coefficient $\beta$: 5
Proportional constant $E_0$: $2 \times 10^5$ Pa$^{2/3}$
Standing Wave Tube Test

20 mm glass bubble

- At low input level, particles “stick” to the wall – modal response of solid phase
- At high input level, particles “slip” at wall – planar response of solid phase

20 mm glass bubbles under 68 dB input

20 mm glass bubbles under 122 dB input
Standing Wave Tube Test

40 mm glass bubble

- At low input level, particles “stick” to the wall – modal response of solid phase
- At high input level, particles “slip” at wall – planar response of solid phase

**40 mm glass bubbles under 68 dB input**

- **B.C.: solid phase constrained at tube wall**

**40 mm glass bubbles under 121 dB input**

- **B.C.: solid phase allowed to slip at tube wall**
Conclusions

When solid phase constrained, absorption peaks correspond to radial modes of solid phase.

- At high input level, it appears that solid particles “slip” at tube wall.
- At low input level, it appears that solid particles “stick” at tube wall.
- Modal response at low input levels offers additional sound absorption possibilities compared to a fibrous layer.
Conclusions

- A finite difference method was built based on Biot theory. This approach allows implementation of varying stiffness and different boundary conditions to accommodate the measurement conditions.

- The finite difference simulation provides displacement/velocity profile of the particle stacks, which relates the sound absorption to vibration modes of the particle stack.

- The finite difference simulation results show similar patterns to testing results, both with low-level and high-level input, which indicates that the change of boundary condition may be the reason for different behavior of particle stack under those different inputs.
Acknowledgment

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References

References