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ELLPACK '78 User's Guide-Preliminary Version

John R. Rice
Purdue University, jrr@cs.purdue.edu

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ABSTRACT

This report outlines the additional features in ELLPACK 78 compared to ELLPACK 77. It is assumed that the reader is familiar with ELLPACK 77 (CSD-TR 289). ELLPACK 78 provides facilities for general 2-dimensional domains and the internal representations of the geometry information is described in Appendix A.

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   5-POINT STAR
   2 DEPEP
ELLPACK 78 USER'S GUIDE
John R. Rice

1. INTRODUCTION.

The ELLPACK 78 version of ELLPACK adds the capability to handle general geometry in two dimensions. Both versions are in the same system and ELLPACK 77 programs will continue to operate unchanged. The principal changes for ELLPACK 78 are:

(a) Definition and processing of the geometry information
(b) Discretization of the operator in general domains.

The first changes affect the ELLPACK system and they have been carried out. The second affect contributed modules for the DISCRETIZATION segment. Two such modules are already installed (5 POINT STAR and 2DEPEP) and two more (HODIE ACF and P3C1 COLLOCATION) are in progress. The INDEXING and SOLUTION modules should, by and large, be unaffected by the geometry. There are additions to the OUTPUT segment as well as changes in the operation of some OUTPUT statements. Note that the same ELLPACK statements can invoke different routines in ELLPACK 77 or ELLPACK 78, e.g. 5-POINT STAR and PLOT are implemented by entirely different programs in the two cases.

Figure 1 shows a simple ELLPACK 77 program and its equivalent in ELLPACK 78. The principal change is that the BOUNDARY segment is split into two segments (DOMAIN and BOUNDARY). In ELLPACK 78, the DOMAIN specifies the geometry and BOUNDARY specifies the boundary conditions. The connection is made between the domain definition and the boundary conditions by numbering the boundary pieces and giving conditions according to this numbering.
Figure 1. Top - a simple ELLPACK 77 program. Bottom - an equivalent ELLPACK 78 program.

The keyword RECTANGULAR may be used with the BOUNDARY segment to indicate that ELLPACK 77 is being used but it is not necessary; it does help provide protection from errors.

The remainder of this report gives specific details of the four segments affected: DOMAIN, BOUNDARY, HOLE and OUTPUT.

2. THE DOMAIN SEGMENT.

The domain of the PDE is defined by its boundary which is given in piecewise, parametric form. That is, each piece is defined by

\[ x = x(t), \quad y = y(t) \] for \( t \in [a, b] \)

which appears in ELLPACK as (if this is the 4th piece)

4. \( x = x(T), \quad y = y(T) \) \( T \in [4.1, 6] \)
The functions $X(T)$ and $Y(T)$ must be ordinary Fortran expressions; complicated
definitions are made by defining Fortran functions which are included in the
Fortran segment. The limits on the parameter $T$ must be constant expressions
e.g. $\exp(1.)$ is allowable to express $e$, $\text{ATAN}(1.)$ for $\pi/4$ and must be increasing

The pieces must be given in order and close to define the domain. The
domain is in the interior of the boundary curve and the user must specify the
orientation of the boundary: clockwise or counter-clockwise. The keywords
CLOCKWISE and COUNTER-CLOCKWISE make this specification with CLOCKWISE the
default value.

There is a special simple form for straight line pieces which can be
specified just by giving their end points. A sequence of straight lines can
be specified by giving the sequence of end points. The keyword LINE indicates
this special form and the general syntax for $K$ pieces (starting with the $N$th one)
is

\[
N. \text{ TO } (N+K). \ \text{LINE} \ A_1, B_1 \text{ TO } A_2, B_2 \text{ TO } --- \text{ TO } A_{K+1}, B_{K+1}
\]

Thus pieces 4, 5 and 6 could be specified by

4. to 6. LINE 0,1. TO 0, 2. TO 1,3. TO 4.2368,4.

Four complete examples of domains and their specifications are given in
Figures 2 and 3.

********** WARNING - WARNING **********

The domain must be specified in a reasonable way or the ELLPACK 78 system
will not be able to identify all the intersections of the boundary with the
pieces. We list certain requirements on the specifications of the domain:

**ESSENTIAL REQUIREMENTS**

1. *Fine Grid* The grid must be fine enough that the boundary is well behaved
   within any particular grid square. The boundary does not intersect any
   side of a grid square more than twice (and most of them only once).
2. **Accurate Boundary with Ordinary Size.** The boundary must be given accurately and in units of ordinary size. There is a fixed constant EPSGRD in ELLPACK that is a tolerance for the pieces joining or intersecting the grid lines. (EPSGRD = 10^{-8} for a 14 decimal digit machine, 10^{-6} for 10 digits, 10^{-5} for 8 digits and for 7 digits 5*10^{-5}. If the units of the x,y coordinate system lead to values of more than 1000 or all less than .001, then there is considerable risk that the system will fail to process the boundary properly.

3. **Nice Parameters.** The parameters of the pieces must be of ordinary size and should not vary erratically. It is easy to parameterize a simple curve so that ELLPACK does not follow it properly; just make a variations 1 in T correspond to 1 inch on the curve at the start and 10^8 inches at the end.

**WARNING**

4. **Protection.** The functions defining the pieces should be protected from illegal or nonsense values. A semi-circular piece can be specified by

6. \( X = T \quad Y = \sqrt{1-T^2} \quad T = -1 \text{ TO } 1. \)

This can cause trouble near \( X = -1 \) and +1. As initial phases of the numerical method may attempt to use values of \( X^2 \) greater than 1, which causes the square root of a negative number to be computed. This will terminate the computation on most systems.
ELLPACK 78 USER GUIDE EXAMPLE 1 - MAY 1979

EQUATION. UXXS + UYY = 2.·EXP(X·Y)

CIRCULAR DISK

DOMAIN. 1. X = 1.·COS(3.141592654·THETA)
       Y = 1.·SIN(3.141592654·THETA) $\theta$ THETA = 0. TO 2.

ELLPACK 78 USER GUIDE EXAMPLE 2 - MAY 1979

REGION FROM HALL, LUCZAK, SERDY - TOMS 2(1976) 257-274

LAPLACE $ UXXS + UYY = 0 $ SELF-ADJOINT

DOMAIN. 1. TO 2. LINE 0.0 TO 10.23.
       3. X = XD0M3(P) $ Y = YD0M3(P) $ P = 0 TO 15.
       4. LINE 34.13 TO 34.7
       5. X = XD0M5(P) $ Y = YD0M5(P) $ P = 0 TO 20.
       6. X = 12.·P $ Y = 0.0 $ P = 0 TO 12.

FUNCTION XD0M3(T)
XD0M3 = 15. + T
IF( T .GE. 5. ) RETURN
P1730 = AMAX1(AMIN1(T,1000.),-2.)
XD0M3 = 20.·T·COS(3.141592654·P1730·.1)
RETURN
END

FUNCTION YD0M3(T)
YD0M3 = 13.
IF( T .GE. 5. ) RETURN
P1730 = AMAX1(AMIN1(T,1000.),-2.)
YD0M3 = 23.·T·COS(3.141592654·P1730·.1)
RETURN
END

FUNCTION XD0M5(T)
XD0M5 = 34. - T
IF( T .LE. 15. ) RETURN
P1755 = AMAX1(AMIN1(T,75.),-25.)
XD0M5 = 19.·T·COS(3.141592654·P1755-15.)*.1)
RETURN
END

FUNCTION YD0M5(T)
YD0M5 = 7.
IF( T .LE. 15. ) RETURN
P1755 = AMAX1(AMIN1(T,75.),-85.)
YD0M5 = 7.·COS(3.141592654·P1755-15.)*.1)
RETURN
END

FUNCTION TRUE(X,Y)
TRUE = X·Y - Y - 2.
RETURN
END

Figure 2. Two domains with their specifications in ELLPACK.
ELLPACK 78 USER GUIDE EXAMPLE 3 - MAY 1979

EQUATION.  \( \nabla \psi + \psi = -1. \)

DOMAIN.  COUNTER-CLOCKWISE

1. TO 3. LINE 4.,4. TO 1.,4. TO 1.,.5 TO 4.,.5
4. \( X = 4. + .1*P*(P-4.5) \) \( Y = .5 + P \) \( P = 0. \) TO 4.5

ELLPACK 78 USER GUIDE EXAMPLE 4 - MAY 1979

EQUATION.  \( GR(X,Y) \nabla \psi + GR(X,Y) \psi = 0.0 \)

DOMAIN.  HALF OF SQUARE WITH CIRCULAR HOLE

1. \( X = P \) \( Y = .4 \) \( P = 0 \) TO .8
2. \( X = .8 - TAU \) \( Y = 0.0 \) \( TAU = 0.0 \) TO 1.0
3. \( X = .8 + \cos(3.141592*XI) \) \( Y = .2*\sin(3.141592*XI) \) \( XI = 0.0 \) TO .4
4. \( X = .2 - P12345 \) \( Y = 0.0 \) \( P12345 = 0.00000 \) TO 0.20000
5. \( X = 0.0 \) \( Y = P \) \( P = 0.0 \) TO 0.4

FORTRAN:

```
FUNCTION GR(X,Y)
  GR = -EXP(-X+Y)
RETURN
END

FUNCTION TRUE(X,Y)
  TRUE = EXP(X+Y)
RETURN
END

FUNCTION D3EST(X,Y)
  D3EST = 1.0
RETURN
END

END.
```

Figure 3. Two domains with their specifications in ELLPACK.
The protection is to use

\[ \text{SQRT(ABS(1.-T**2))} \]

instead of the natural \[ \text{SQRT(1.-T**2)} \] expression.

There are many ways that error conditions can be generated inadvertently; the main thing is to be aware of the possibilities and to use protective devices whenever a potential trouble is identified. The April, 1980 version of ELLPACK has no protection against the parameter being outside the designated range.

3. THE BOUNDARY SEGMENT.

The boundary conditions are specified in the BOUNDARY segment as a list of conditions associated with the list of boundary pieces. This list must be given in order. Thus

6. \( U = X+1 \).

specifies that on the sixth piece the given Dirichlet condition is to be satisfied. The form for giving Dirichlet, Neumann and Mixed boundary conditions are exactly the same as in ELLPACK 77 and the same keywords (DIRICHLET, NEUMANN, MIXED and HOMOGENEOUS) are used. Figure 4 shows example boundary condition specifications for the four domains given in Figures 2 and 3.
BOUNDARY.  1. U = TRUE(X,Y)  
GRID.  UNIFORM X = 4, 0.0 TO 2.0 $ UNIFORM Y = 5, 0.0 TO 2.0

BOUND.  1. U = TRUE(X,Y)  
2. U = TRUE(X,Y)  
3. U = TRUE(X,Y)  
4. U = TRUE(X,Y)  
5. U = TRUE(X,Y)  
6. U = TRUE(X,Y)

GRID.  UNIFORM X = 4, 0.0 TO 2.0  
UNIFORM Y = 6, 0.0 TO 2.0

BOUND.  1. U = EXP(1.0 + X) $ U = EXP(1.0 + Y)  
2. U = EXP(X) $ U = TRUE(X,Y)  
3. U = EXP(Y) $ U = EXP(Y)

GRID.  UNIFORM Y = 6, 0.0 TO 0.4  
UNIFORM X = 7, 0.0 TO 0.8

BOUNDARY.  1. U = 4.0 - X  $ U = 4.0  
2. U = 4.0 - X  $ U = 4.0

GRID.  UNIFORM X = 12, 0.0 TO 5.5  
UNIFORM Y = 11, -1.0 TO 4.0

Figure 4. The specifications of boundary conditions and grids for the domains of Figures 2 and 3.

4. THE GRID SEGMENT.

The GRID segment of ELLPACK 77 is changed in the way a uniform grid is specified. In ELLPACK 77 we have UNIFORM X=11 meaning that 11 grid lines are placed in the x-range of the domain. In ELLPACK 78 the x-range is not known (or is not well defined) so the range in which the uniform grid is placed must be specified as in:

UNIFORM X=11, 0.0 TO 5.0

The comma and the keyword TO separate the information about the number (11) of grid lines, the starting line (0.0) and the ending line (5.0). The same convention applies to the y coordinate and non-uniform grids are specified as in ELLPACK 77.
The ELLPACK system checks for the boundary going out of the grid defined and this is a fatal error.

5. Effects on OUTPUT.

There are now output commands for the internal information defining the boundary and domain. See Appendix A for a brief discussion of this information and for four examples. The new commands are

TABLE-DOMAIN
TABLE-BOUNDARY

The TABLE-DOMAIN command produces the GTYPE table which indicates the nature of all grid point (interior, exterior, boundary and next-to-boundary). Various information is encoded in the boundary and next-to-boundary entries.

The TABLE-BOUNDARY command produces the values of variables that define the boundary (see Appendix A for definitions). These include NBOUND (No. of boundary points), NGRID X and NGRID Y (No. of grid lines). There are seven arrays (XBOUND, YBOUND, PARAM, PIECE, BPTYPE, BGRID and BNEIGH) tabulated whose entries contain information pertinent to each boundary point.

The PLOT command applies in ELLPACK 78 for contour plots, but they might not be as accurate as desired. The reason is that it is a lengthy calculation to determine the intersection of contour lines with a curved boundary. Only an approximate calculation is made which may lead to contour lines stopping short of the boundary. Note that contours are plotted based on a 40 x 40 grid and their accuracy is further limited by this. See Figure 5 for an example contour plot.

6. THE HOLE SEGMENT

The HOLE segment allows for multiply connected domains; the form of the statement is exactly the same as DOMAIN except the keyword HOLE is used. Thus, for a square with a semi-circular hole, we have
Figure 5. An example contour plot for a curved domain. Notice the lack of precision of the contour lines next to the boundary.

This facility does not have any effect on the ELLPACK modules, but it does require the user to be careful to make the grid fine enough. There should be at least two interior grid points separating the boundaries of the domain and the hole; even this is unlikely to give much accuracy in the solution.

The HOLE segment may appear several times to define more complex regions.
APPENDIX A: INTERNAL REPRESENTATION OF THE GEOMETRY INFORMATION

The basic geometrical situation in two dimensions is shown in Figure A-1. The relationship with a rectangular grid is shown in Figure A-2. Each piece is represented parametrically in the form \( x = x(p), y = y(p) \) for \( p \in [a,b] \).

3.3 Interface 2: Domain Representation:

The objectives of the domain processing module are:

a. locate the domain boundary with respect to the grid.

b. relate all grid points to the domain (e.g. interior or exterior)

c. provide this information in various forms so that later processing of the domain requires a minimum of geometric analysis.

The information generated is thus of two types. One is associated with the boundary intersections with the grid and "follows the boundary around the domain". The other is various information encoded in GTYPE associated with the two dimensional array of grid points. The specific information in GTYPE is

\[
GTYPE(IX,IV) \text{ FOR } IX = 1 \text{ TO } NGRIDX, IV = 1 \text{ TO } NGRIDY
\]

- = 0 GRD POINT OUTSIDE DOMAIN, AWAY FROM BOUNDARY.
- = INSIDE GRID POINT INSIDE DOMAIN, AWAY FROM BOUNDARY.
- = INSIDE = CONSTANT INTEGER, E.G. 999
- = INTEGER LESS THAN INSIDE, GRID PT = BOUNDARY PT OF INDEX GTYPE
- = INTEGER GREATER THAN INSIDE+1 IN ABSOLUTE VALUE

GRID POINT IS NEXT TO THE BOUNDARY. RELATION TO THE BOUNDARY IS ENCODED BY

\[ GTYPE = INDEX + IPACKB*J \]

WHERE

- INDEX = SMALLEST INDEX OF NEIGHBORING BOUNDARY PT
- IPACKB = CONSTANT FOR PACKING = INSIDE + 1
- J = FOUR BITS FOR DIRECTION TO BOUNDARY PTS.
  - = 1 FOR NOON
  - = 2 FOR 3 O CLOCK
  - = 4 FOR 6 O CLOCK
  - = 8 FOR 9 O CLOCK
  - = POSITIVE FOR GRID POINT IN THE DOMAIN OR ON BOUNDARY
  - = NEGATIVE OR 0 FOR POINT OUTSIDE THE DOMAIN
Figure A-1. A typical two-dimensional rectangular domain. The pieces of the boundary are numbered clockwise and described parametrically.

Figure A-2. The domain of Figure A-1 with a rectangular grid overlay. Here we have NGRIDX = 9 and NGRIDY = 7. In practice one would expect the horizontal and vertical sides to be grid lines.
The information about the intersection of the boundary with the grid lines is contained in a set of 7 linear arrays of length NBNDPT. These arrays all have problem dependent dimensions MAXBND and are used and defined as shown below:

\[
\begin{align*}
\text{INTEGER PIECE(MAXBND), BPTYPE(MAXBND), BNEIGH(MAXBND), BGRID(MAXBND)} \\
\text{REAL XBOUND(MAXBND), YBOUND(MAXBND), BPARAM(MAXBND)} \\
\end{align*}
\]

- **XBOUND(I), YBOUND(I)** = Coordinates of I-th boundary point.
- **BPARAM(I)** = Parameter value P of I-th boundary point.
- **PIECE(I)** = Index of boundary piece to which point belongs.
- **BPTYPE(I)** = The type of the boundary point:
  - **HORZ** if point is on a Y grid line.
  - **VERT** if point is on a X grid line.
  - **BOTH** if point is also a grid point.
  - **CORN** if point is a corner point on a grid line.
  - **INTE** if point is not on a grid line.
- **BNEIGH(I)** = \( \pm 100*1\) if point is a corner point on a grid line.
  - \( \pm 1000*1\) if point is not on the grid.

- **BGRID(I)** = \( \pm 1000*1\) where the I-th boundary point is in the grid square \( x,y \).
- **GRIDX(x)**, \( \text{I}, \text{LT}, \text{GRIDX(x+1)} \)
- **GRIDY(y)**, \( \text{I}, \text{LT}, \text{GRIDY(y+1)} \)

Figures A-3 through A-6 give the information for representing the four domains of Figures 2 and 3.
THE 10 BOUNDARY POINTS IN THE 4 BY 5 GRID

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>PARAM</th>
<th>PIECE</th>
<th>GRID_TYPE</th>
<th>NEIGH</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1.00000E+00</td>
<td>0</td>
<td>0</td>
<td>CORN</td>
<td>3001</td>
</tr>
<tr>
<td>1</td>
<td>1.33333E+00</td>
<td>0.66667E+00</td>
<td>1</td>
<td>0.66667E-01</td>
<td>0001</td>
<td>4001</td>
</tr>
<tr>
<td>2</td>
<td>2.00000E+00</td>
<td>0.66667E+00</td>
<td>1</td>
<td>0.66667E-01</td>
<td>0001</td>
<td>4001</td>
</tr>
<tr>
<td>3</td>
<td>2.00000E+00</td>
<td>1.33333E+00</td>
<td>1</td>
<td>0.66667E-01</td>
<td>0001</td>
<td>4001</td>
</tr>
<tr>
<td>4</td>
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<td>1.33333E+00</td>
<td>1</td>
<td>0.66667E-01</td>
<td>0001</td>
<td>4001</td>
</tr>
<tr>
<td>5</td>
<td>1.33333E+00</td>
<td>0.66667E+00</td>
<td>1</td>
<td>0.66667E-01</td>
<td>0001</td>
<td>4001</td>
</tr>
<tr>
<td>6</td>
<td>1.33333E+00</td>
<td>0.66667E+00</td>
<td>1</td>
<td>0.66667E-01</td>
<td>0001</td>
<td>4001</td>
</tr>
<tr>
<td>7</td>
<td>1.33333E+00</td>
<td>0.66667E+00</td>
<td>1</td>
<td>0.66667E-01</td>
<td>0001</td>
<td>4001</td>
</tr>
<tr>
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<td>0.66667E+00</td>
<td>1</td>
<td>0.66667E-01</td>
<td>0001</td>
<td>4001</td>
</tr>
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<td>0.66667E+00</td>
<td>1</td>
<td>0.66667E-01</td>
<td>0001</td>
<td>4001</td>
</tr>
<tr>
<td>10</td>
<td>1.33333E+00</td>
<td>0.66667E+00</td>
<td>1</td>
<td>0.66667E-01</td>
<td>0001</td>
<td>4001</td>
</tr>
</tbody>
</table>

THE GRID POINT TYPES WITH (XGRID(I),YGRID(I)) AT LOWER LEFT

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>PARAM</th>
<th>PIECE</th>
<th>GRID_TYPE</th>
<th>NEIGH</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-4003</td>
<td>0</td>
<td>-4004</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-6001</td>
<td>8002</td>
<td>0</td>
<td>3004</td>
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</tr>
<tr>
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<td>2008</td>
<td>6</td>
<td></td>
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</tr>
<tr>
<td>3</td>
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<td>12005</td>
<td>6007</td>
<td>-5066</td>
<td></td>
<td>0</td>
</tr>
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<td>-1008</td>
<td>-1008</td>
<td>0</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

******* 1  2  3  4

Figure A-3. The boundary/grid information for Example 1.
THE 28 BOUNDARY POINTS IN THE 9 BY 6 GRID

\[ \begin{array}{ccccccc}
1 & X & Y & \text{PARAM} & \text{PIECE} & \text{TYPE} & \text{GRID} & \text{NEIGH} \\
0 & 0 & 0 & 0 & 1 & \text{CORN} & 1001 & 0 \\
2 & 0 & 4.80000E+00 & 2.86656E-01 & 1 & \text{BOTH} & 2001 & 2002 \\
3 & 0 & 9.60000E+00 & 4.17816E-01 & 1 & \text{BOTH} & 3001 & 3002 \\
4 & 0 & 1.44000E+01 & 5.25987E-01 & 1 & \text{BOTH} & 4001 & 4002 \\
5 & 0 & 1.92000E+01 & 6.34783E-01 & 1 & \text{BOTH} & 5001 & 5002 \\
6 & 0 & 2.39000E+01 & 1.00000E+00 & 1 & \text{CORN} & 6001 & 0 \\
7 & 4.25000E+00 & 2.39000E+01 & 2.45200E-01 & 1 & \text{CORN} & 4002 & 5002 \\
8 & 9.50000E+00 & 2.39000E+01 & 6.50000E+00 & 2 & \text{VERT} & 5003 & 5002 \\
9 & 1.38000E+01 & 2.39000E+01 & 1.00000E+00 & 2 & \text{INT} & 5003 & 0 \\
10 & 1.77000E+01 & 1.92000E+01 & 1.24076E+00 & 3 & \text{HORIZ} & 3003 & 5003 \\
11 & 1.77000E+01 & 1.61152E+01 & 2.41846E+00 & 3 & \text{VERT} & 4004 & 4003 \\
12 & 1.48171E+01 & 1.44000E+01 & 3.28537E+00 & 3 & \text{HORIZ} & 4004 & 4004 \\
13 & 1.70000E+01 & 1.34656E+01 & 4.03138E+00 & 3 & \text{VERT} & 3005 & 3005 \\
14 & 8.02500E+01 & 1.30000E+01 & 6.25500E+00 & 3 & \text{VERT} & 3006 & 3005 \\
15 & 2.02500E+01 & 1.30000E+01 & 1.65909E+01 & 3 & \text{VERT} & 3007 & 3007 \\
16 & 2.57500E+01 & 1.30000E+01 & 1.47500E+01 & 3 & \text{VERT} & 3008 & 3008 \\
17 & 3.40000E+01 & 1.30000E+01 & 1.80000E+01 & 3 & \text{CORN} & 3009 & 0 \\
18 & 3.40000E+01 & 9.82000E+00 & 5.85587E+01 & 4 & \text{VERT} & 3009 & 3008 \\
19 & 3.40000E+01 & 7.40000E+00 & 1.00000E+00 & 4 & \text{CORN} & 2005 & 0 \\
20 & 2.57500E+01 & 7.40000E+00 & 4.59090E+00 & 5 & \text{VERT} & 2006 & 3008 \\
21 & 2.55000E+01 & 7.00000E+00 & 8.00000E+00 & 5 & \text{VERT} & 2007 & 3007 \\
22 & 8.12500E+01 & 7.00000E+00 & 1.27500E+01 & 5 & \text{VERT} & 2008 & 3006 \\
23 & 8.12500E+01 & 6.70000E+00 & 1.93823E+01 & 5 & \text{VERT} & 2005 & 3005 \\
24 & 1.39458E+01 & 4.80000E+00 & 1.75949E+01 & 5 & \text{HORIZ} & 2004 & 2004 \\
25 & 1.37500E+01 & 3.10000E+00 & 1.05130E+01 & 5 & \text{VERT} & 1004 & 2004 \\
26 & 1.25000E+01 & 0 & 2.00000E+01 & 5 & \text{CORN} & 1003 & 0 \\
27 & 6.50000E+00 & 0 & 3.00000E+00 & 6 & \text{BOTH} & 1003 & 2003 \\
28 & 4.25000E+00 & 0 & 7.75900E+00 & 6 & \text{BOTH} & 1002 & 2002 \\
\end{array} \]

THE GRID POINT TYPES WITH (XGRIDU, YGRIDU) AT LOWER LEFT

\[ \begin{array}{ccccccccccc}
6 & -4006 & -4007 & -4008 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 4 & 8004 & 899 & 3011 & -12012 & -4014 & -4015 & -4016 & -4017 \\
3 & 3 & 8003 & 899 & 899 & 3013 & 3014 & 3015 & 7015 & 18 \\
2 & 2 & 12992 & 4027 & 5004 & -9023 & -1022 & -1021 & -1020 & -1019 \\
1 & 1 & 28 & 27 & -9225 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \]

Figure A-4. The boundary/grid information for Example 2.
Figure A-5. The boundary/grid information for Example 3.
Figure A-6. The boundary/grid information for Example 5.
APPENDIX B: ELLPACK 78 DISCRETIZATION MODULES.

MODULE NAME: 5-POINT STAR

AUTHOR/DATE: Ron Boisvert, June 1977 (revised Nov. 1978)

INITIAL/FINAL INTERFACES: EQUATION FORMATION - EQUATION INDEXING

MODULE FUNCTIONS: Discretizes a general linear elliptic PDE with boundary conditions of the form $p_x u_x + q_y u_y + r = g$.

RESTRICTIONS ON USE: Two dimensions, no $u_x$ term, uniform grid, non-self-adjoint form. Caution: this is an experimental version of the 5-POINT STAR code; no claims are made about its correctness.

METHOD DESCRIPTION: The usual 5-POINT finite difference discretization is used in the interior of the domain. Adjacent to the boundary a lower order non-symmetric divided difference approximation is used. The derivatives in boundary conditions are approximated by three point one-sided differences, with values at non-grid

PARAMETERS: None

KEYWORDS THAT AFFECT MODULE: CONSTANT COEFFICIENTS

STORAGE AND TIMING ESTIMATES: Approximately $N\text{GRID}X \times N\text{GRID}Y$ equations are generated with up to 7 unknowns per equation. A workspace of $25 + N\text{GRID}X + N\text{GRID}Y + \text{number of boundary-grid intersections}$ is used.
MODULE NAME: 2DEPEP

AUTHOR/DATE: G. Sewell, 10/16/78

INITIAL/FINAL INTERFACE: EQUATION FORMATION-OUTPUT

RESTRICTIONS: Two-dimensions, self-adjoint

MODULE DESCRIPTION: Galerkin's method with 6-node quadratic triangular elements, user-controlled grading of the triangular mesh, and the frontal method to organize out-of-core storage of the matrix when necessary. This is a small, specialized subset of the program TWODEPEP, a commercial product of IMSL, Inc. For further details see G. Sewell, A finite element program with automatic user controlled mesh grading B-2, in Advances in Computer Methods for Partial Differential Equations III (R. Vishnevetsky, ed.) Rutgers Univ., New Brunswick, NJ.

PARAMETERS: NTRI- number of triangles desired in final triangulation
MEM- workspace storage = 71*NTRI if external storage to be used
46*NTRI + 15*NTRI**1.5 otherwise

The grid is used to define an initial triangulation. This triangulation will have about 4 triangles for each grid square which intersects the region, so NTRI must be larger than this number. The closure of the intersection of any grid square with the region must be convex or nearly so. Thus it is necessary, in general, that any region corner with exterior angle less than 180° be cut by a grid line which divides the exterior angle into two parts. (In the case of a 90° exterior angle with edges parallel to the axes, however, it is sufficient to put a grid point at that corner.)

FORTRAN: Function D3EST(X,Y) must be user-supplied.
The program grades the initial triangulation so that the final triangulation is most dense where D3EST is largest. In particular, it attempts to distribute D3EST(X,Y)*H**3 uniformly, where (X,Y) is the center of triangle j and H_j is its diameter. If D3EST is an estimate of the function
\[
\max_{1+j=3} \left| \frac{\partial^2 u}{\partial x \partial y} \right|
\]
It is possible to obtain optimal order convergence to the solution of some singular problems.