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R. S. Leemhuis

W. Soedel

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KINEMATICS OF WANKEL COMPRESSORS (OR ENGINES)
BY WAY OF VECTOR LOOPS

R. S. Leemhuis, Graduate Teaching Assistant
W. Soedel, Professor of Mechanical Engineering
School of Mechanical Engineering
Purdue University
West Lafayette, Indiana 47907

INTRODUCTION

The vector loop equation approach is a simple and accurate method for analyzing the kinematics of the Wankel system. In particular, equations describing the shape of the cylinder, displacement of the piston, velocity of the apexes, and volume of the chambers, are easily derived and are all in closed form. In addition, these equations are very general, and apply to any variations of the Wankel system, regardless of the number of lobes. The derivation of each of the equations described above as well as how they are to be applied will be presented in this paper.

During the preparation of this paper, a short survey was made of the state of the art to determine what methods are being used presently to analyze the kinematics of the Wankel system. Although this survey was by no means extensive, it did reveal that those books which are available on the subject approach it totally by means of classical geometry. This method yields perfectly satisfactory results, at least for the classical two lobe design; however, the derivation of the equations is not as simple to understand nor are the resulting equations as easy to use.

In summary, this paper will:

1. Define what is meant by the vector loop equation approach and show how it can be applied to the Wankel system.
2. Show the derivation for the displacement of the piston as a function of the crank angle which will also define the shape of the cylinder.
3. Show the effect certain parameters have on the geometry of the piston and cylinder.
4. Derive an equation for the velocity of an apex of the piston as a function of crank speed and will provide examples.

5. Derive an equation for the volume of any chamber of the Wankel as a function of crank angle and will provide examples.

VECTOR LOOP EQUATION APPROACH

The method used in the analysis of the Wankel involved writing a vector loop equation for the mechanism. First, a definition of what is meant by the vector loop equation approach will be given.

If some point P is located on a link which is rotating about a fixed origin as in Figure 1, the position of the point can be defined by a vector \bar{r} in the complex plane. The real component of the vector is in the X direction and the imaginary component is in the Y direction. Thus:

$$\bar{r} = X + iY \quad (1)$$

where

$$X = r \cos \theta \quad (2a)$$

$$Y = r \sin \theta \quad (2b)$$

Equation (1) can also be written as:

$$\bar{r} = r e^{i\theta} \quad (3)$$

where

$$e^{i\theta} = \cos \theta + i \sin \theta \quad (4)$$

By connecting a series of vectors in a loop, it is possible to write an equation which will describe the position of one vector relative to another provided, of course, that the vector loop is chosen such that the number of equations is equal to the number of unknowns. If the vectors describe the position of two or more links, the relative position between the links can be determined. Furthermore, by differentiating the vector loop equation with respect to time, the relative velocity of the links can be found. This was the method used in the analysis of the Wankel system.

In Figure (2) the vector loop used in the analysis of the Wankel system is shown for

the two lobe case. However, the analysis is also valid for any number of lobes. Several additional equations were found using geometry as will be seen later in this paper. The following features should be noted from Figure (2):

1. The position of the input crank is given by θ_2 and the position of an apex of the piston is given by r_7 and θ_7 .
2. When $\theta_2 = 0$, point A on the stationary gear and point A' on the piston gear coincide. As the input crank rotates counterclockwise, the piston gear rolls off the stationary gear as shown.
3. The angle α is measured between an apex of the piston and point A' about the center of the piston.
4. Point B is always at the point of contact between the pitch circles of the stationary gear and piston gear.

The first step in the analysis is to determine expressions for r_7 and θ_7 as functions of the input crank angle θ_2 . These equations will be developed in the next section.

DEVELOPMENT OF WANKEL SYSTEM DISPLACEMENT EQUATIONS

Referring to Figure (2), it is desired to derive a set of equations which locate point C as a function of the crank angle θ_2 .

Point C is the point of contact between a piston apex and the cylinder and therefore traces the shape of the cylinder as the piston is rotated. The derivation of the equations for r_7 and θ_7 are as follows.

Since the piston gear is rotating about the stationary gear, the arc length \overline{AB} must equal the arc length $\overline{A'B}$. Thus:

$$r_5 \beta_2 = r_3 \beta_1 \quad (5)$$

From geometry it can be seen that:

$$\beta_1 = \theta_2 + \pi - \theta_3 \quad \text{and} \quad (6)$$

$$\beta_2 = \theta_2 = \theta_5 \quad (7)$$

Using these three equations, the following equation for θ_3 in terms of θ_2 can be written:

$$\theta_3 = \theta_2 \left(1 - \frac{r_5}{r_3}\right) + \pi \quad (8)$$

It can be seen from Figure (2) that:

$$\theta_6 = \theta_3 - \alpha \quad (9)$$

Thus:

$$\theta_6 = \theta_2 \left(1 - \frac{r_5}{r_3}\right) + \pi - \alpha \quad (10)$$

The vector loop equation which describes the position of the piston apex (point C) is:

$$\overline{r}_2 + \overline{r}_6 - \overline{r}_7 = 0 \quad (11)$$

or

$$r_2 e^{i\theta_2} + r_6 e^{i\theta_6} - r_7 e^{i\theta_7} = 0 \quad (12)$$

Substituting Euler's formula (equation (4)) into equation (12) and equating the real and imaginary parts to zero:

$$r_2 \cos \theta_2 + r_6 \cos \theta_6 - r_7 \cos \theta_7 = 0 \quad (13a)$$

$$r_2 \sin \theta_2 + r_6 \sin \theta_6 - r_7 \sin \theta_7 = 0 \quad (13b)$$

Solving these two equations for r_7 and θ_7 yields the final equations:

$$\theta_7 = \tan^{-1} \left[\frac{r_2 \sin \theta_2 + r_6 \sin(\pi + \theta_2 (1 - \frac{r_5}{r_3}) - \alpha)}{r_2 \cos \theta_2 + r_6 \cos(\pi + \theta_2 (1 - \frac{r_5}{r_3}) - \alpha)} \right] \quad (15)$$

$$r_7 = [r_2^2 + r_6^2 + 2r_2 r_6 \cos(\alpha - \pi + \theta_2 \frac{r_5}{r_3})]^{1/2} \quad (16)$$

Since these equations are functions of only the variable θ_2 , they give the position of an apex of the piston as a function of the crank angle θ_2 . As stated previously, these equations also define the shape of the cylinder. The effect the parameters have in the above two equations on the geometry of the cylinder will be discussed in the next section.

CYLINDER GEOMETRY

In designing a Wankel compressor or engine, one must choose the parameters r_2 , r_3 , r_5 , r_6 , and α . The ratio of r_5 to r_3 determines the number of cylinder lobes. This ratio must be: $1/2, 2/3, 3/4, 4/5, \dots, M/M+1$ (where M is a whole number). Ratios other than these will result in a cylinder that does not close on itself in one revolution of the piston. From Figure (2) it can be seen that:

$$r_2 = r_3 - r_5 \quad (17)$$

Thus, for a particular number of lobes, r_2 , r_3 , and r_5 are all proportional. Some

examples of Wankels with one to six lobes are shown in Figure (3).

The ratio of r_2 to r_6 determines the general shape of the cylinder. Examples for the two and four lobe case are shown in Figure (4). Note that the smaller the ratio of r_2 to r_6 , the more circular the cylinder becomes. The magnitude of r_6 is arbitrary provided it is large enough such that the piston gear will fit within the piston. Note that the pitch circles of the piston gear and stationary gear have been drawn to scale in Figure (4). It should be apparent that the shape of the piston between adjacent apexes has some bearing on the permissible ratio of r_2 to r_6 . This will be discussed in the next section.

The value of α effects the torque characteristics of the Wankel system. For the examples given at the end of this paper, α was chosen as π/N where N is the number of apexes; however, α is totally arbitrary.

Some useful relationships between the parameters are given in the following table.

TABLE 1

# of Lobes	# of Piston Apexes	$\frac{r_5}{r_3}$	$\frac{r_5}{r_2}$	$\frac{\text{Rev. of Crank}}{\text{Rev. of Piston}}$
1	2	1/2	1	2
2	3	2/3	2	3
3	4	3/4	3	4
M*	M+1	M/M+1	M	M+1

*(where M is a whole number)

PISTON GEOMETRY

The magnitude of the vector r_6 determines the size of the piston by defining the location of the apexes from the center of the piston. However, by itself it gives no information as to the geometry of each face of the piston between adjacent apexes. The designer must therefore choose the shape of the piston face within certain limitations. First, the piston must be large enough to contain the piston gear. Secondly, each face of the piston must clear the cylinder wall for a complete revolution. The following derivation models the piston face as an arc of a circle and determines the limiting

value for the radius of curvature of that arc.

Referring to Figure (5), it is desired to derive an equation for the clearance (C). Note that this derivation is valid for a Wankel system with any number of lobes. Let:

1. N = number of apexes
2. W = width of a face of the piston
3. C = clearance between a flat faced piston and the innermost surface of the cylinder wall.
4. R = radius of curvature of the piston face.

Referring to Figure (5a)

$$L = r_6 - r_2 \quad (18)$$

$$d = r_6 \cos\left(\frac{\pi}{N}\right) \quad (19)$$

Thus:

$$C = L - (d+r_2) \quad (20)$$

or

$$C = r_6 \left(1 - \cos\left(\frac{\pi}{N}\right)\right) - 2r_2 \quad (21)$$

$$W = 2r_6 \sin\left(\frac{\pi}{N}\right) \quad (22)$$

Referring to Figure (5b), it is now desired to find the radius of the arc (R) described above.

$$\cos \gamma = \frac{R-C}{R} \quad (23)$$

$$\sin \gamma = \frac{W}{2R} \quad (24)$$

Squaring and adding these two equations yields the final equation for the radius.

$$R = \frac{4C^2 + W^2}{8C} \quad (25)$$

Note that the radius (R) can be either positive or negative depending upon the sign of C. Also note that this gives the limiting value of the radius and its magnitude must be increased if clearance is desired between the piston and cylinder.

The total volume of the piston (V_T) can be found by geometry once the radius (R) has been determined.

$$V_T = t r_6 N \left[\frac{r_6}{2} \sin\left(\frac{2\pi}{N}\right) + (C-R) \sin\left(\frac{\pi}{N}\right) \right] + t R^2 N \sin^{-1} \left[\frac{r_6}{R} \sin\left(\frac{\pi}{N}\right) \right] \quad (26)$$

This volume is usually modified because of thermodynamic considerations by relief

slots. In the following, V_T will mean the actual piston volume, not necessarily the theoretical one of Equation (26).

PISTON APEX VELOCITIES

The piston apex velocities are very important from a design aspect since they have a great effect upon the wear of the apex seal. The velocity of an apex can be found by differentiating equations (15) and (16) with respect to time and then calculating the velocity component parallel to the cylinder wall. Thus:

$$\dot{\theta}_7 = \theta_2 \left[\frac{r_2^2 + r_6^2 \left(1 - \frac{r_5}{r_3}\right) + r_2 r_6 \left[2 - \frac{r_5}{r_3}\right] \cos\left(\alpha - \pi + \theta_2 \frac{r_5}{r_3}\right)}{r_2^2 + r_6^2 + 2r_2 r_6 \cos\left(\alpha - \pi + \theta_2 \frac{r_5}{r_3}\right)} \right] \quad (27)$$

$$\dot{r}_7 = \theta_2 \frac{4 r_2 r_5 \frac{r_6}{r_3} \sin\left(\alpha - \pi + \theta_2 \frac{r_5}{r_3}\right)}{\sqrt{r_2^2 + r_6^2 + 2r_2 r_6 \cos\left(\alpha - \pi + \theta_2 \frac{r_5}{r_3}\right)}} \quad (28)$$

The velocity of the apex located at angle θ_7 is:

$$V_A = \sqrt{\dot{r}_7^2 + (r_7 \dot{\theta}_7)^2} \quad (29)$$

and is always tangential to the cylinder wall.

Equation (29) is a function of θ_2 and $\dot{\theta}_2$ only. Thus, the velocity of an apex can be found for any position and speed of the crank. Examples are given in Figures (6d) and (7d) for the two and five lobe case respectively.

VOLUME ANALYSIS OF THE WANKEL SYSTEM

Using the vector loop described previously, it is possible to derive a closed form expression for the volume of any chamber of the Wankel compressor or engine. This volume will be expressed in terms of the input crank angle θ_2 . First, it is necessary to define the terms used.

1. N = number of apexes
2. t = thickness of the cylinder
3. V_T = volume of the piston including the volume of the hollow center. (See Figure (8a))
4. V_S = volume that \bar{r}_7 sweeps as it moves from C to D. (See Figure (8b))

5. V_P = volume of the piston that \bar{r}_7 sweeps across as it moves from C to D. (See Figure (8c))

6. V_C = volume of one of the Wankel chambers. (See Figure (8d))

7. θ_2^* = the input crank angle at which the Wankel chamber volumes are to be found.

Before proceeding with the volume analysis, it is necessary to elaborate on an important feature of the Wankel system. Referring to Figure 2, assume points C and D to be fixed to the cylinder. The vector r_7 will sweep from point C to point D in exactly one revolution of the input crank. Thus when deriving an expression for V_S , the limits of integration are as shown.

$$V_S = \frac{t}{2} \int_{\theta_2}^{\theta_2 + 2\pi} r_7^2 \frac{d\theta_7}{d\theta_2} d\theta_2 \quad (30)$$

Performing this integration yields:

$$V_S = \pi t \left\{ r_2^2 + r_6^2 \left(1 - \frac{r_5}{r_3}\right) \right\} + t r_2 r_6 \left[\frac{r_3}{r_5} - \frac{1}{2} \right] \cdot \left\{ \sin\left(\alpha - \pi + (\theta_2 + 2\pi) \frac{r_5}{r_3}\right) - \sin\left(\alpha - \pi + \theta_2 \frac{r_5}{r_3}\right) \right\} \quad (31)$$

The volume (V_P) is found by geometry and is given by the equation:

$$V_P = \frac{t}{2} r_7(\theta_2) r_7(\theta_2 + 2\pi) \left(\sin(\theta_7(\theta_2 + 2\pi)) - \sin(\theta_7(\theta_2)) \right) + \frac{V_T}{N} - \frac{t}{2} r_6^2 \sin\left(\frac{2\pi}{N}\right) \quad (32)$$

Note that the subscript in parenthesis in the above equations means "evaluated at" (i.e. $r_7(\theta_2 + 2\pi)$ indicates r_7 evaluated at

$\theta_2 + 2\pi$). The other subscripts have similar meanings. Figure (9) shows what volume each of the terms in equation (32) include. Thus, the volume of one chamber located between $\theta_7(\theta_2)$ and $\theta_7(\theta_2 + 2\pi)$ is given by:

$$V_C = V_S - V_P \quad (33)$$

Typical volumes for the two and five lobe case are found in Figures (6C) and (7C) respectively.

The equations derived for position, velocity, and volume have been limited to one apex or chamber, given the input crank angle. To find these relationships for the other apexes or chambers let:

$$\begin{aligned} \theta_2 &= \theta_2^* \\ \theta_2 &= \theta_2^* + 2\pi \\ \theta_2 &= \theta_2^* + 4\pi \\ &\vdots \\ \theta_2 &= \theta_2^* + 2\pi(K-1) \quad K = 1, 2, 3, \dots, N \quad (34) \end{aligned}$$

This procedure will now be summarized. Assume the volumes of a two lobe Wankel compressor or engine are to be found given the input crank angle θ_2^* . ($N = 3$)

1. Let $\theta_2 = \theta_2^*$
2. Find V_T either by tests or equation (26) noting the assumptions made in the derivation of this equation.
3. Evaluate θ_7 at θ_2 and at $\theta_2 + 2\pi$ (Eq. 15)
4. Evaluate r_7 at θ_2 and at $\theta_2 + 2\pi$ (Eq. 16)
5. Solve for V_S at θ_2 (Eq. 31)
6. Solve for V_P (Eq. 32) using the results found in steps (3) and (4) above.
7. Solve for V_C (Eq. 33)
8. Repeat steps 2 through 7 for the values of θ_2 given in (Eq. 34). This will result in three values of V_C for the crank position θ_2^* .

CONCLUSION

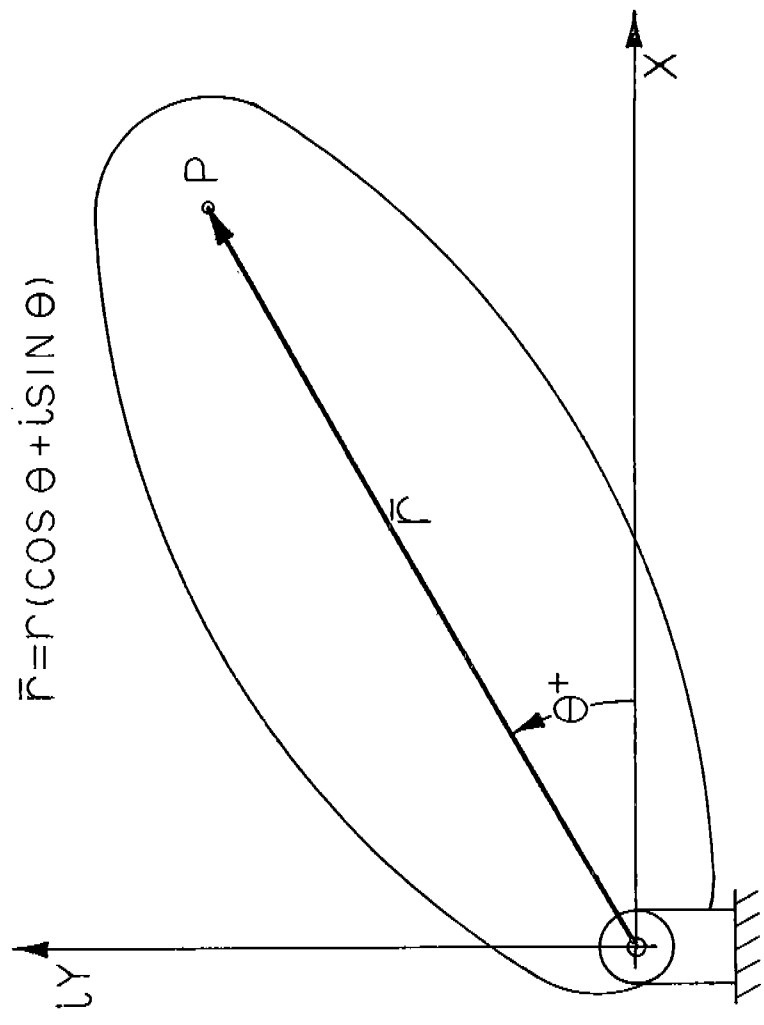
The purpose of this paper has been to show how the vector loop approach can be applied to the Wankel system. It should be apparent that this method provides a relatively simple way of arriving at important kinematic relationships. It should be noted that all relationships are general and apply to all variations, regardless of the number of lobes. Also, since the results are in

closed form, they should provide the designer with a convenient tool to understand the interdependence of the various variables.

Equations for the inversions of the basic Wankel system can be derived by the same approach.

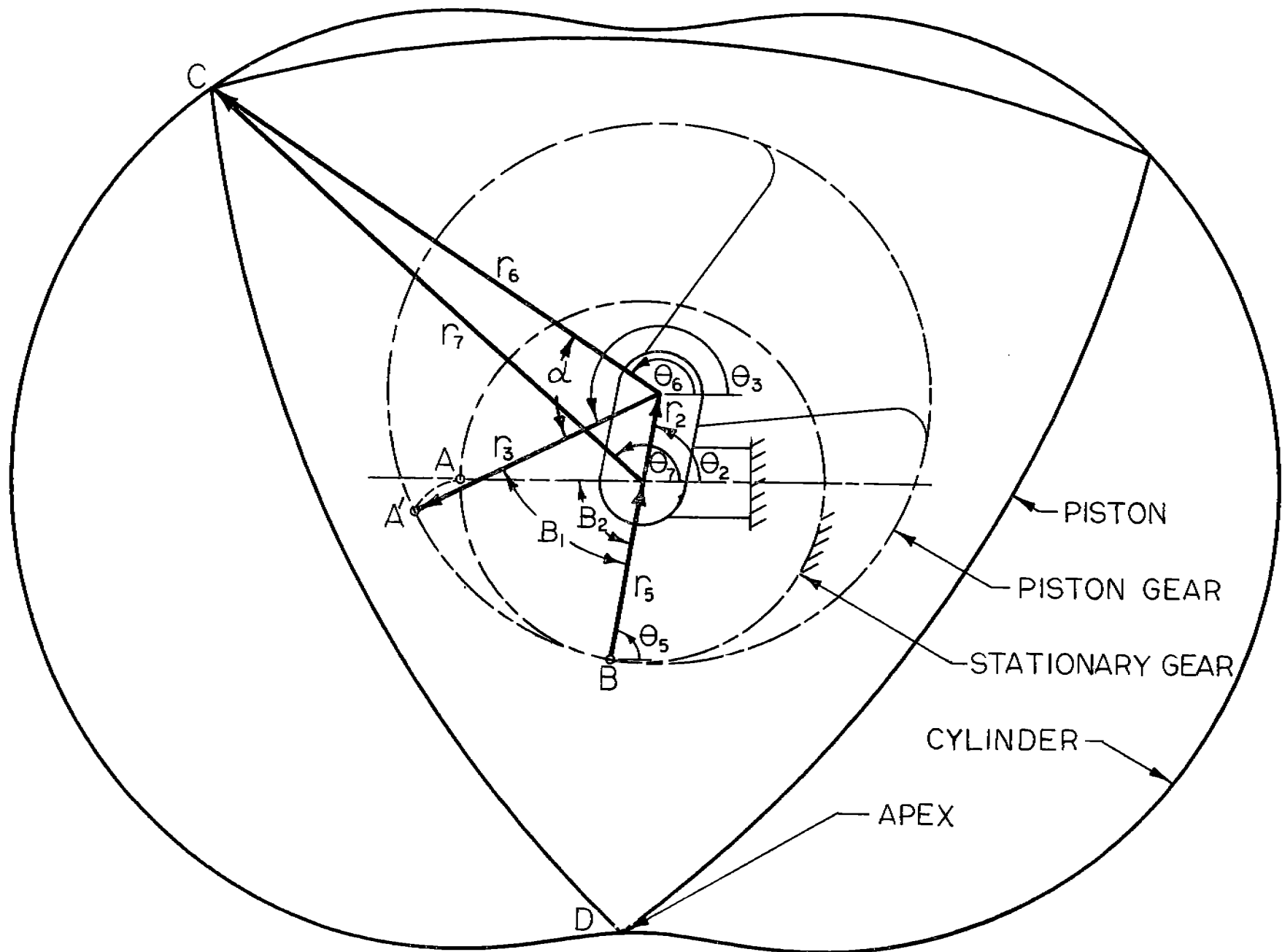
NOMENCLATURE

- r_2 = magnitude of input crank [mm]
- r_3 = pitch circle radius of the piston gear [mm]
- r_5 = pitch circle radius of the stationary gear [mm]
- r_6 = distance between the center of the piston to any one of the apexes [mm]
- r_7 = location of an apex as well as the shape of the cylinder relative to the center of the crankshaft [mm]
- α = angle measured from the center of the piston between an apex and an arbitrary reference point on the piston [RADIANS]
- θ_2 = input crank angle [RADIANS]
- C = the smallest distance between the center of a flat faced piston and the innermost portion of the cylinder [mm]
- W = width of a face of the piston measured between apexes [mm]
- R = radius of curvature of a face of the piston [mm]
- t = thickness of the cylinder or thickness of the piston plus seals [mm]
- V_T = total volume of the piston assuming it has a solid center [mm³]
- V_A = velocity of an apex [mm/sec]
- N = number of apexes
- V_S = the volume \bar{r}_7 sweeps if it were to move across one face of the piston [mm³]
- V_P = volume of the piston \bar{r}_7 sweeps if it were to move across one face of the piston [mm³]
- V_C = volume of one of the Wankel chambers [mm³]
- θ_2^* = input crank angle at which the volume of the Wankel or the velocity of one of the apexes is desired [RADIANS]

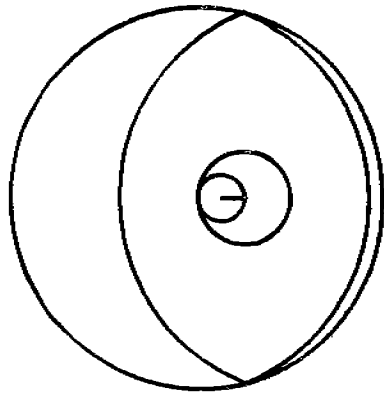


DEFINITION OF A VECTOR IN THE COMPLEX PLANE

FIGURE 1

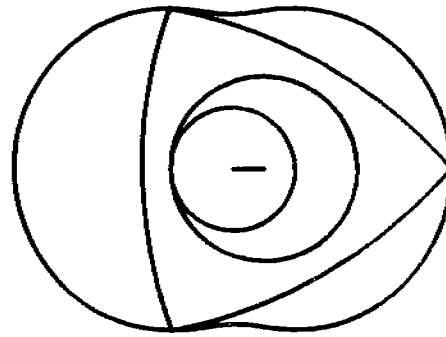


VECTOR LOOP USED IN WANKEL SYSTEM ANALYSIS
FIGURE 2



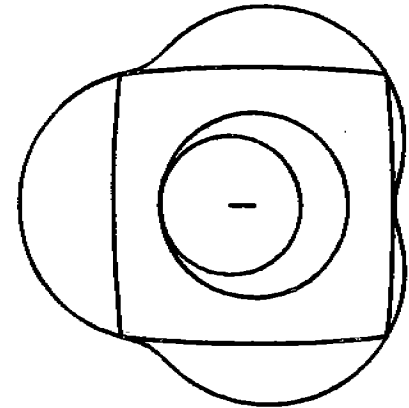
$$\frac{r_5}{r_3} = \frac{1}{2}$$

1 LOBE



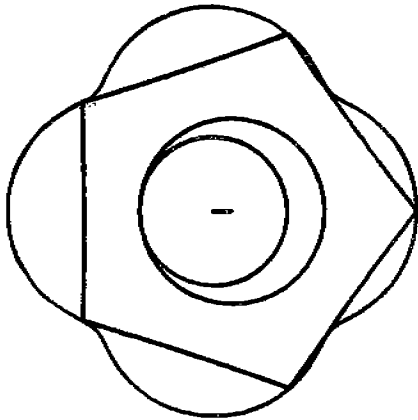
$$\frac{r_5}{r_3} = \frac{2}{3}$$

2 LOBES



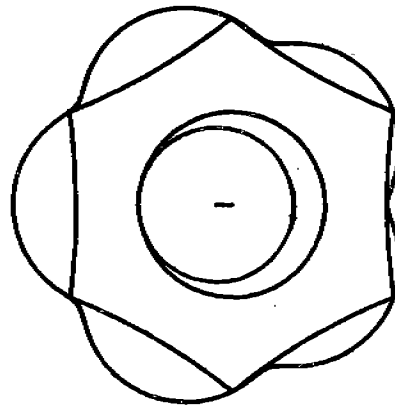
$$\frac{r_5}{r_3} = \frac{3}{4}$$

3 LOBES



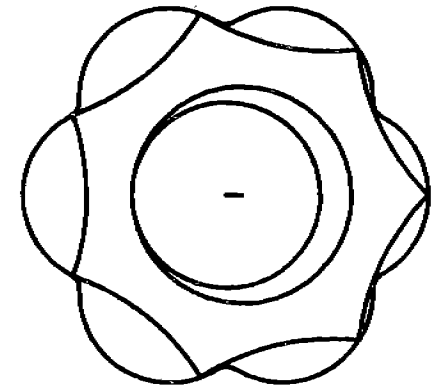
$$\frac{r_5}{r_3} = \frac{4}{5}$$

4 LOBES



$$\frac{r_5}{r_3} = \frac{5}{6}$$

5 LOBES

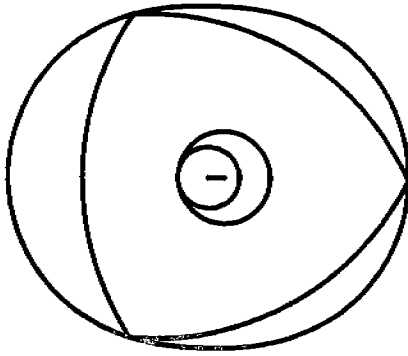


$$\frac{r_5}{r_3} = \frac{6}{7}$$

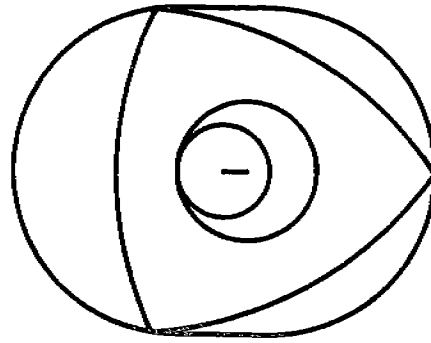
6 LOBES

EFFECT OF THE RATIO $\frac{r_5}{r_3}$ ON THE
NUMBER OF CYLINDER LOBES

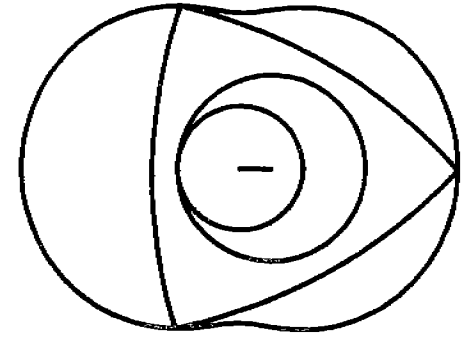
FIGURE 3



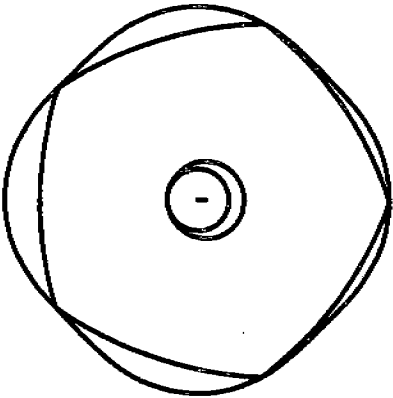
$$\frac{r_2}{r_6} = \frac{1}{12}$$



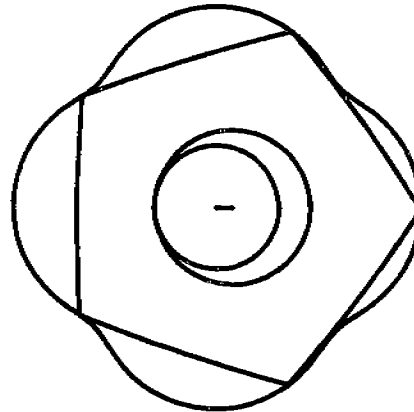
$$\frac{r_2}{r_6} = \frac{1}{8}$$



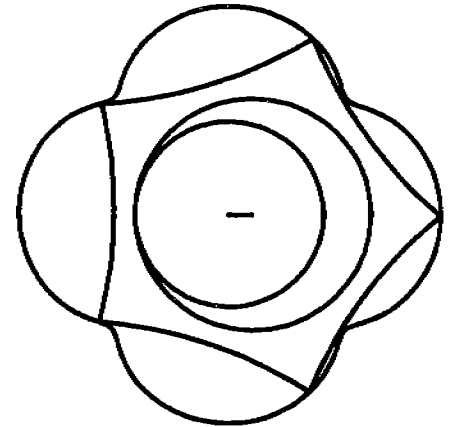
$$\frac{r_2}{r_6} = \frac{1}{6}$$



$$\frac{r_2}{r_6} = \frac{1}{24}$$

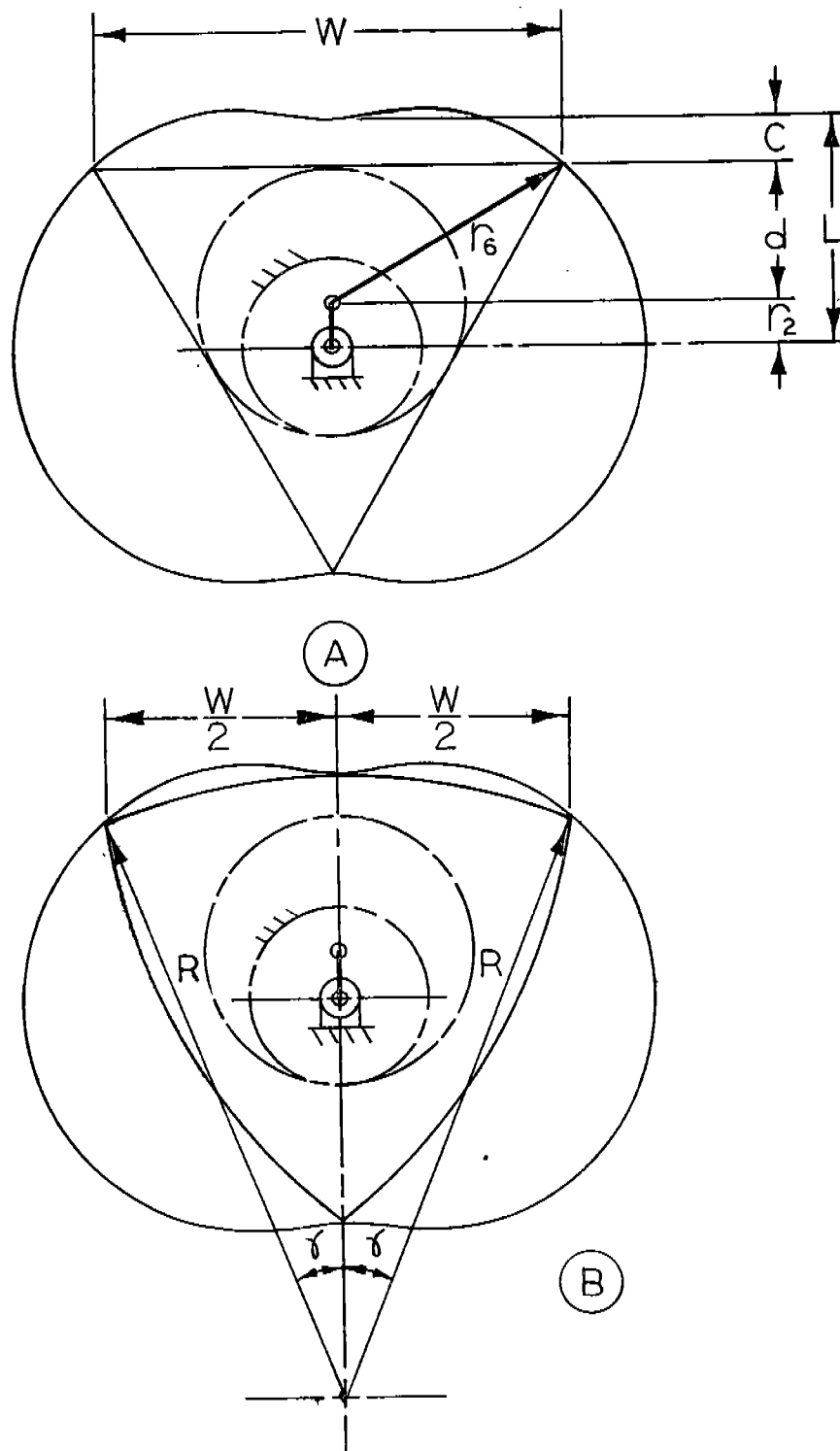


$$\frac{r_2}{r_6} = \frac{1}{12}$$



$$\frac{r_2}{r_6} = \frac{1}{8}$$

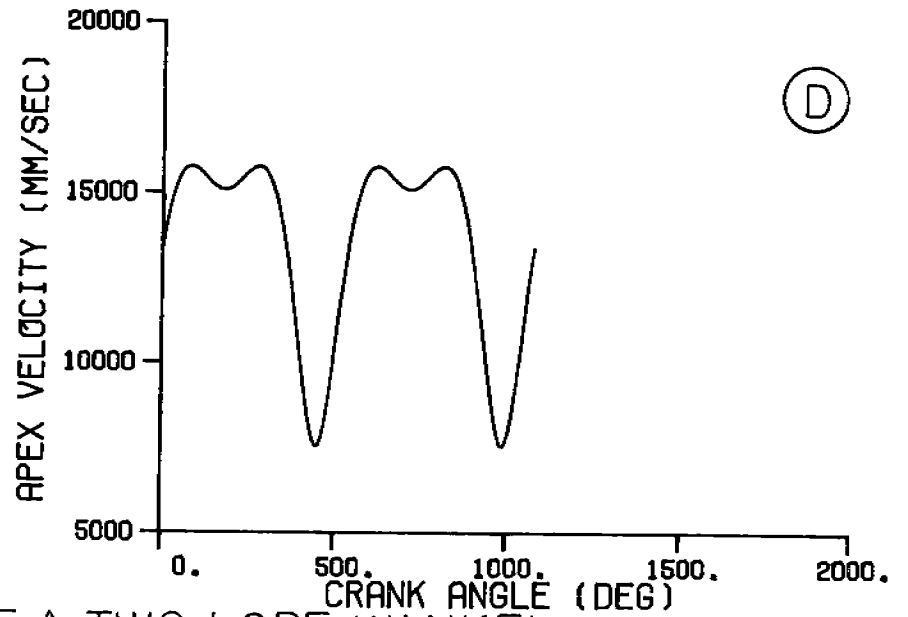
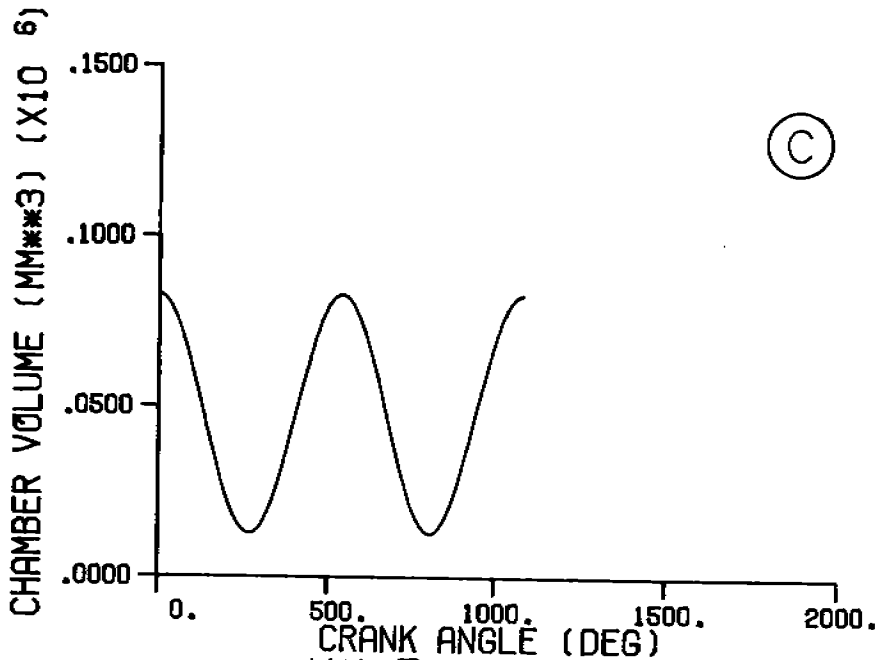
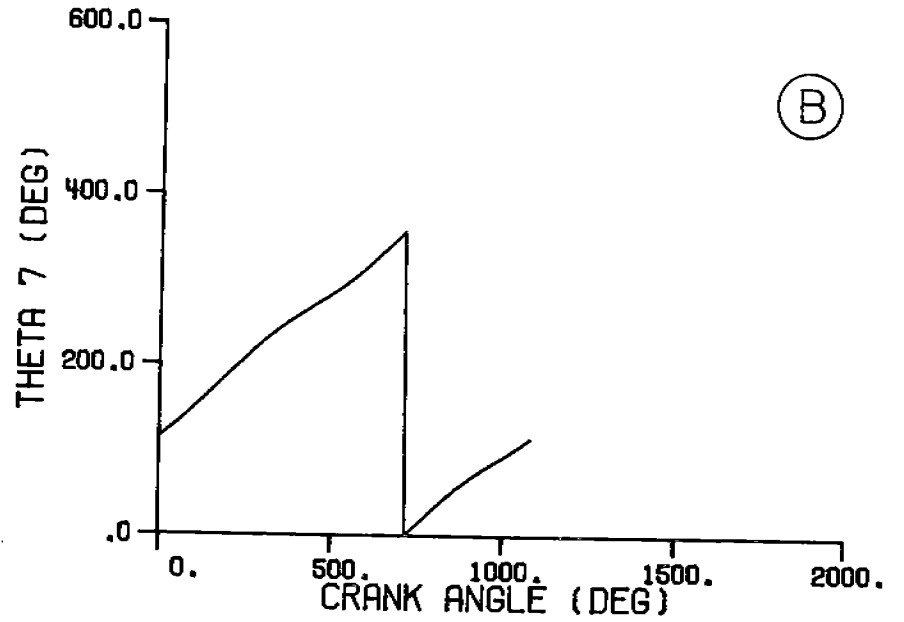
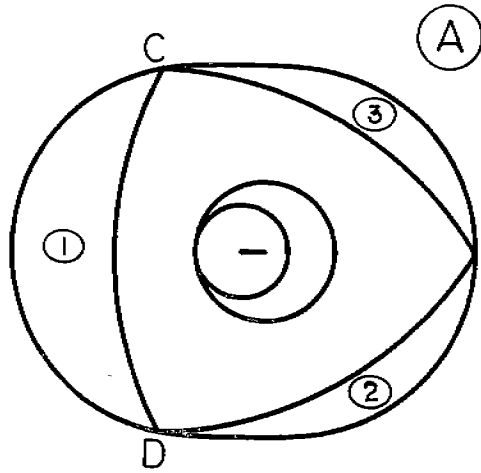
EFFECT OF THE RATIO $\frac{r_2}{r_6}$ ON THE SHAPE OF THE
CYLINDER FOR THE TWO AND FOUR LOBE WANKEL
FIGURE 4



WANKEL SYSTEM SHOWING THE TERMS USED TO FIND AN ALLOWABLE RADIUS OF CURVATURE (R)

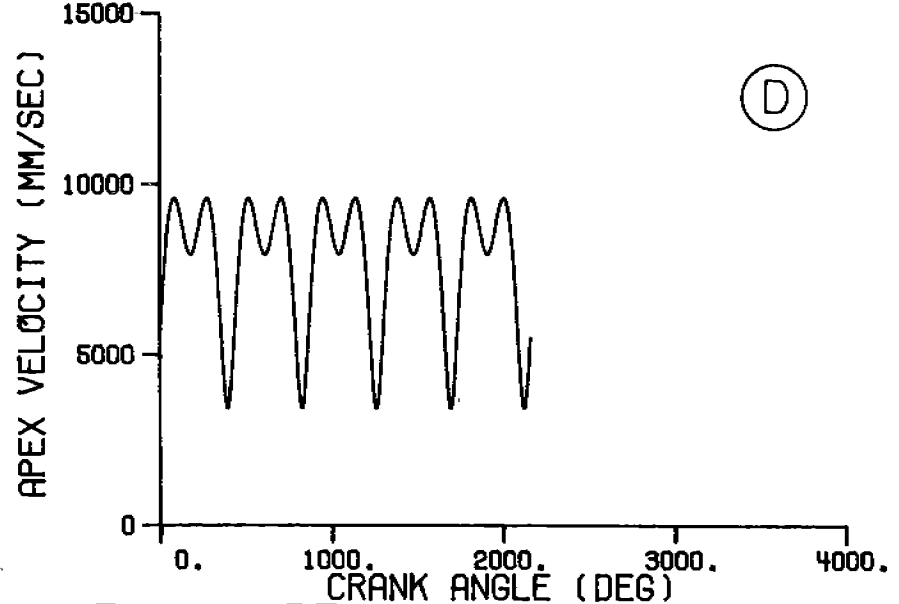
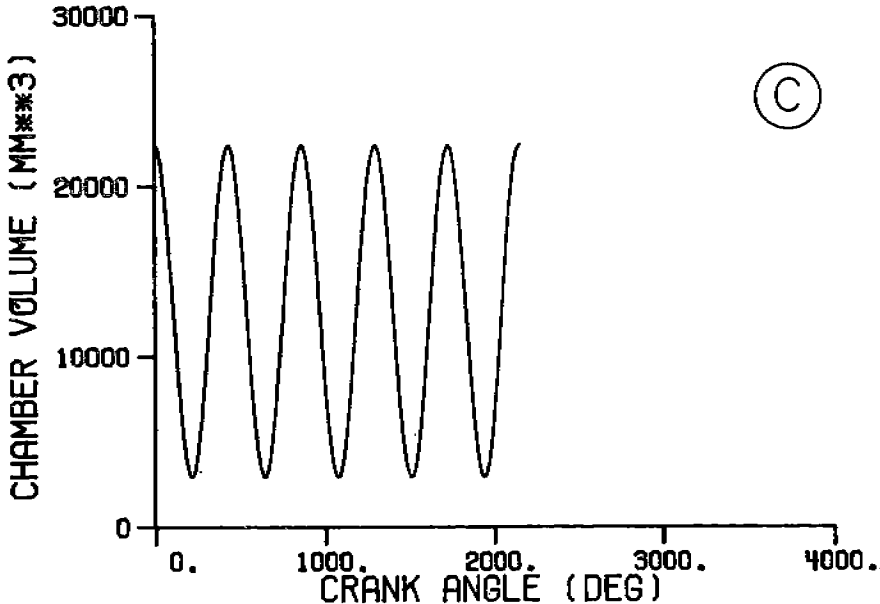
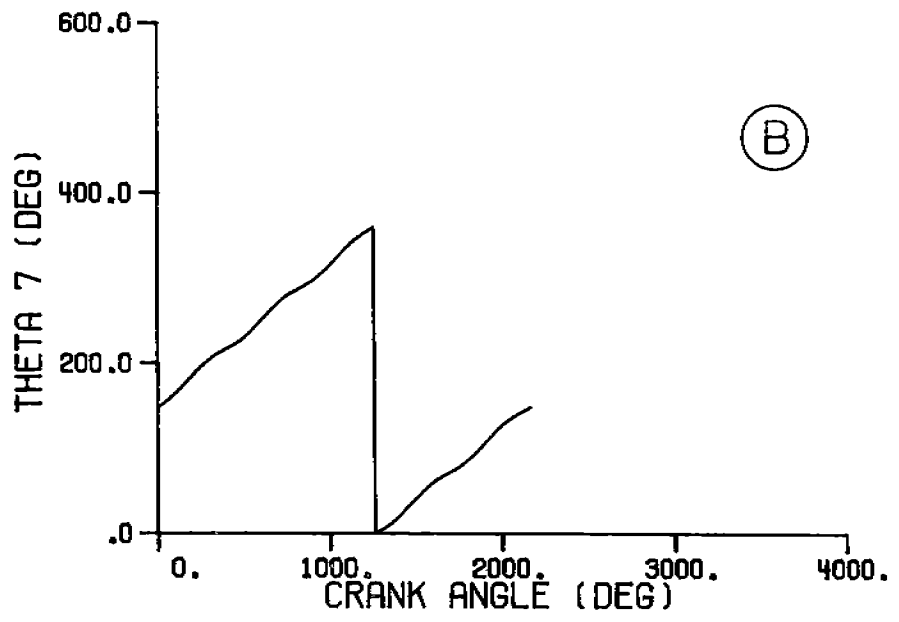
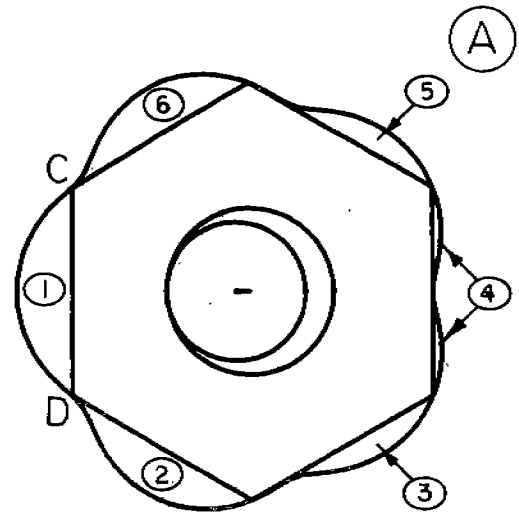
FIGURE 5

$r_2 = 10$ [mm]
 $r_3 = 30$ [mm]
 $r_4 = 20$ [mm]
 $r_5 = 90$ [mm]
 $r_6 = 15$ [mm]
 $\theta_2 = 60$ [deg]
 $\omega_2 = 3600$ [rpm]

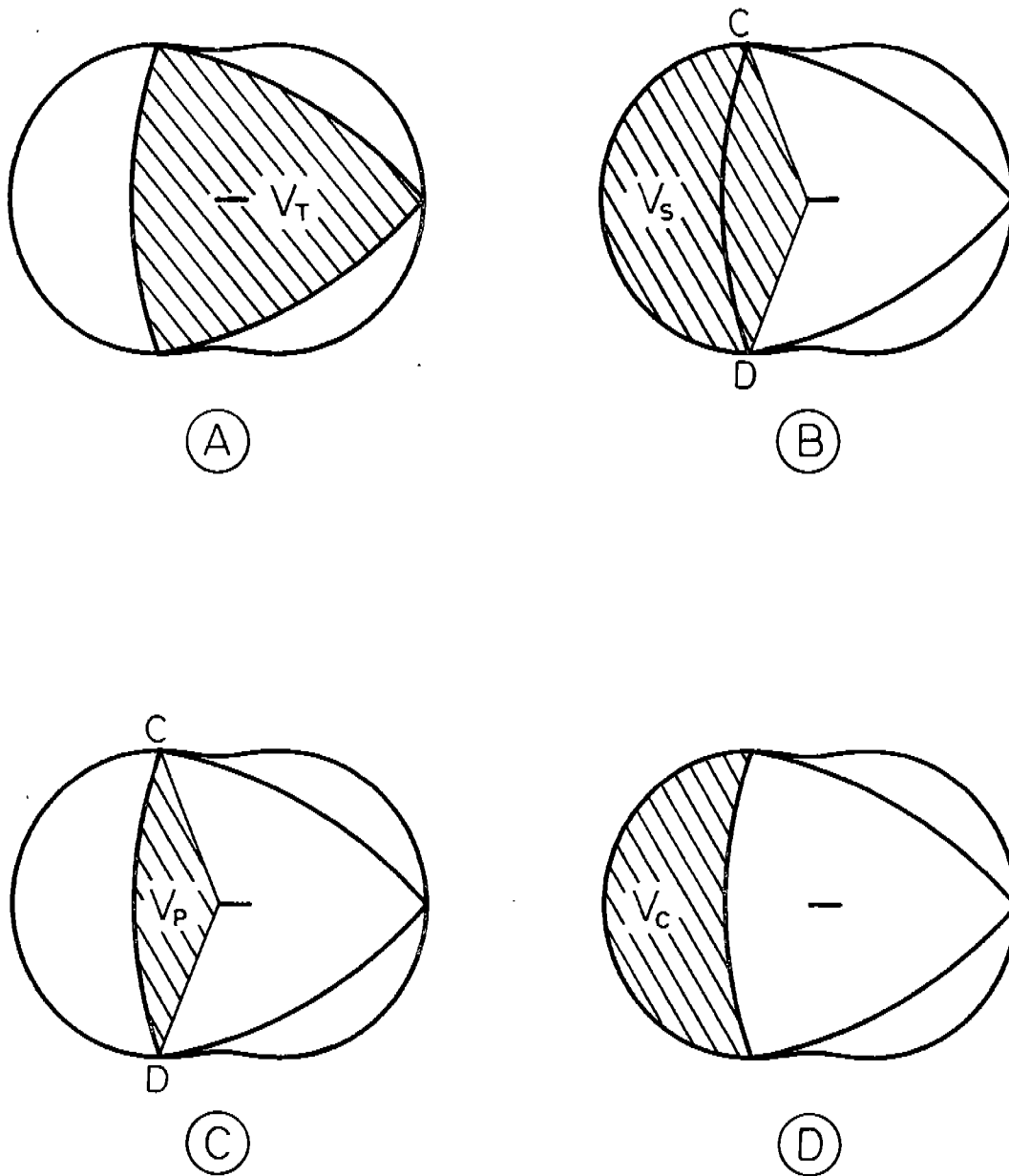


KINEMATIC ANALYSIS OF A TWO LOBE WANKEL
 (A) PISTON AND CYLINDER (B) θ_7 VERSUS CRANK ANGLE θ_2
 (C) CHAMBER VOLUME $\textcircled{1}$ VERSES θ_2 (D) APEX VELOCITY C VERSES θ_2
 FIGURE 6

$r_2 = 6$ [mm]
 $r_3 = 36$ [mm]
 $r_5 = 30$ [mm]
 $r_6 = 90$ [mm]
 $t = 15$ [mm]
 $\rho = 30$ [deg]
 $\theta_2 = 3600$ [rpm]

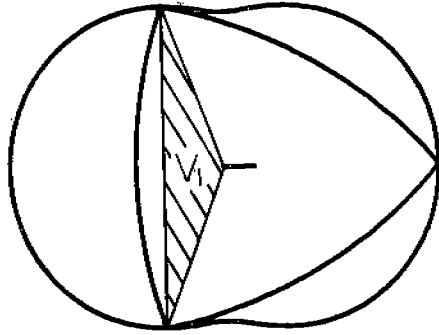


KINEMATIC ANALYSIS OF A FIVE LOBE WANKEL
 (A) PISTON AND CYLINDER (B) θ_7 VERSUS CRANK ANGLE θ_2
 (C) CHAMBER VOLUME C VERSES θ_2 (D) APEX VELOCITY C VERSES θ_2
 FIGURE 7

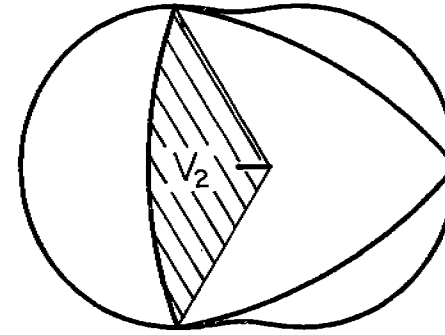


WANKEL SYSTEM PROFILES DEFINING VARIOUS VARIABLES USED IN THE VOLUME ANALYSIS

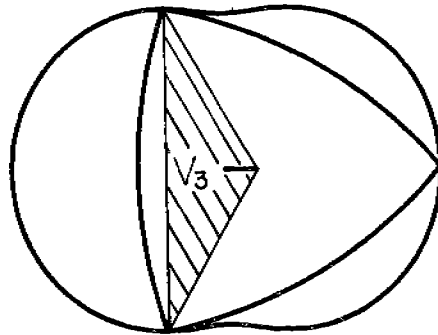
FIGURE 8



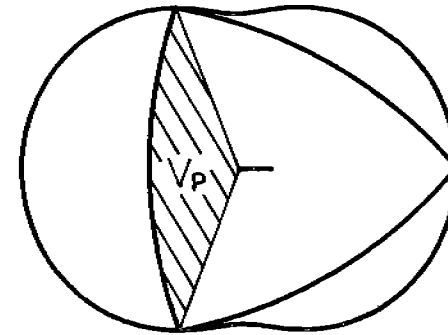
$$V_1 = \frac{t}{2} r_7(\theta_2) r_7(\theta_2 + 2\pi) \sin(\theta_7(\theta_2 + 2\pi) - \theta_7(\theta_2))$$



$$V_2 = \frac{V_T}{N}$$



$$V_3 = \frac{t}{2} r_6^2 \sin\left\{\frac{2\pi}{N}\right\}$$



$$V_p = V_1 + V_2 - V_3$$

WANKEL SYSTEM PROFILES DEFINING THE TERMS USED IN EQUATION (32)

FIGURE 9