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1. Introduction. We describe modifications of the Purdue University version of ELLPACK77 (see Rice [1977]) to solve the Laplace equation subject to mixed boundary conditions on the boundary of a rectangule and Neumann conditions on straight line segments in the interior of the rectangle.

In Section 2 a brief description is given of the physical origin of the problem, namely the analysis of the performance of a magnetohydrodynamic electric generator. Section 3 contains a mathematical model of the generator. Section 4 contains a new result: we show that one can determine the efficiency of the model for all values of one of its parameters by solving a single pair of boundary value problems. Section 5 very briefly describes how the efficiency of the generator can be increased by inserting insulating vanes in the fluid; in the mathematical model, these are modeled by slits in the interior of the rectangular domain on which Neumann boundary conditions are specified. Section 6 contains a brief outline of the ELLPACK system and its application to this problem. Section 7 gives the complete system of finite difference equations whose solution is taken as an approximation to the electric potential in the generator.

The ELLPACK module 5-POINT STAR, which was written by Ronald F. Boisvert of Purdue University, was modified to solve this problem. We thank him for his help during our modification of his routines.

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2. <u>Physical phenomenon</u>. In this section we describe the idealized physical situation which we consider.

An inviscid incompressible fluid flows with constant speed v in the x-direction in a channel between a pair of planes at $y = -h$ and at $y = h$. All variation in the z-direction is neglected. The fluid is electrically conductive with uniform conductivity σ.

Except between $x = -x_e$ and $x = x_e$, the channel walls are electric insulators. Between $-x_e$ and x_e they are electrodes with infinite electric conductivity.

A magnetic field of intensity $\overline{B}$ is imposed in the z-direction. It is uniform across the channel and its magnitude depends only on x. Its magnitude is symmetric with respect to x, it is constant for x between $-x_e$ and x_e, and it tends to zero as x tends to infinity; a special case is $\overline{B}(x) = 0$ for $|x| > 0$.

The Lorentz force which acts on the electric charges in the fluid causes charge separation and charges of different sign collect near the walls of the channel. This creates an electric field in the fluid with potential ϕ. We use $2\phi_0$ to denote the resulting potential difference of the electrodes and take ϕ_0 to be the potential of the electrode on the wall at $x = h$. The channel height is $2h$ and it follows from Maxwell's equations that $2\phi_0$ is given by

$$2\phi_0 = 2hvB_0$$

The device becomes an electric generator when wires are attached to the electrodes and ends of an electric resistance; this produces an electric current in the wires and resistance. The potential difference between the electrodes is reduced to a fraction η of its open curcuit value $2\phi_0$; the fraction depends on the load and η is called the load factor.

The current density $\bar{j}$ in the fluid is

$$\bar{j} = \sigma(\bar{E} + \bar{v}\times\bar{B}) = \sigma(-\text{grad } \phi + \bar{v}\times\bar{B}).$$

We write the magnitude of the magnetic field as $B_0 b(x)$ with $b(x)$ equal to unity for x between $-x_e$ and x_e. The z-component j_z of the current density is zero and its other two components are

$$j_x(x,y) = \sigma(-\partial\phi(x,y)/\partial x),$$

$$j_y(x,y) = \sigma[-\partial\phi(x,y)/\partial x + vB_0 b(x)].$$

We call the term in j_y which involves ϕ the electrostatic contribution and the term which involves the magnetic field the magnetic contribution.

When there is an external current, then j_y is positive in the vicinity of the electodes because there the magnetic contribution dominates the electrostatic one. The magnetic field decreases rapidly as x increases from x_e and for large x, j_y is negative because there the electrostatic contribution dominates the magnetic one and this give rise to a counter-current. The net amount of current which can be drawn from the system is diminished by this counter-current. It is part of our research to determine the effect on the performance predicted by the mathematical model of thin insulators (vanes) placed in the fluid; as expected, these vanes decrease the counter-current.

To determine the net current which can be drawn from the system per unit channel width (in the z-direction), one integrates j_y along a line parallel to the x-axis and by symmetry, one can integrate $2j_y$ from $x = 0$ to $x = \infty$. The value obtained is independent of y for y between $y = -h$ and $y = h$. The total electric power P per unit channel width is the product of the current and $2\eta\phi_0 = 2\eta v B_0 h$. One obtains

$$P = 4\eta v B_0 h\sigma \int_0^\infty [-\partial\phi(x,y)/\partial y + vB_0 b(x)]\, dx, \quad P \text{ independent of } y.$$

The total power W per unit channel width needed to maintain the flow is the integral of vj_yB_0b over the whole channel. This is given by

$$W = 4v\sigma \int_0^\infty dy \int_0^\infty dx\, [-\partial\phi(x,y)/\partial y + vB_0 b(x)]\, B_0 b(x).$$

Considering only these two powers, the efficiency E of the device is taken to be

$$E = P/W.$$

[We remark that one of the things which is neglected in this model is the effect of viscosity which gives rise to friction effects and makes the fluid speed nonuniform across the channel. A second thing which is neglected is the variation of the magnetic field across the channel and between $-x_e$ and x_e. A third thing which is neglected is the finite width of the channel.]

3. Mathematical model. We approximate the infinite channel with a finite one with end points $\pm x_{max}$ where x_{max} is much larger than either x_e or h. The matematical model of the idealized situation described in Section 2 is taken to be

$$\text{(3-1a)} \qquad \partial^2\phi(x,y)/\partial x^2 + \partial^2\phi(x,y)/\partial y^2 = 0, \qquad 0 < x < x_{max}, \quad 0 < y < h,$$

$$\text{(3-1b)} \qquad \phi(x,h) = \eta h v B_0, \qquad 0 \leqq x \leqq x_e,$$

$$\text{(3-1c)} \qquad \partial\phi(x,h)/\partial y = vB_0\, b(x), \qquad x_e < x < x_{max},$$

$$\text{(3-1d)} \qquad \phi(x_{max},y) = 0, \qquad 0 \leqq y \leqq h,$$

$$\text{(3-1e)} \qquad \phi(x,0) = 0, \qquad 0 \leqq x \leqq x_{max},$$

$$\text{(3-1f)} \qquad \partial\phi(0,y)/\partial x = 0, \qquad 0 < y < h.$$

The function b which specifies the magnetic field is equal to unity for x between 0 and x_e and in some of our experiments it was equal to zero for x larger than x_e; in others it was

$$\text{(3-2)} \qquad b(x) = \begin{cases} 1 & 0 \leqq x \leqq x_e \\ \exp(-[x-x_e]/\text{efold}), & x_e < x \end{cases}$$

where efold is a constant which specifies the rate of decay of the magnetic field beyond the end of the conductor.

The channel half-width h is taken as the unit of length and we set

$$x = hX, \quad y = hY, \quad \phi(x,y) = \phi(hX,hY) = \Phi'(X,Y),$$

$$x_e = hX_e, \quad x_{max} = hX_{max}, \quad b(x) = b(hX) = B(X), \quad \text{efold} = h\,\text{EFOLD}$$

$$\partial\phi(x,y)/\partial x = (1/h)\partial\Phi'(X,Y)/\partial X, \quad \partial\phi(x,y)/\partial y = (1/h)\partial\Phi'(X,Y)/\partial Y.$$

With the exception of (3-1c), the equations (3-1) and (3-2) are transformed by making the changes:

$$x \to X, \quad y \to Y, \quad \phi \to \Phi', \quad b \to B, \quad h \to 1, \quad \text{efold} \to \text{EFOLD}.$$

Equation (3-1c) becomes

$$\partial\Phi'(X,1)/\partial Y = hvB_0 B(X), \qquad X_e < X < X_{max}.$$

Now set $\Phi' = hvB_0\Phi$ to get the system

(3-3a) $$\partial^2\Phi(X,Y)/\partial X^2 + \partial^2\Phi(X,Y)/\partial Y^2 = 0, \qquad 0 < X < X_{max},\ 0 < Y < 1,$$

(3-3b) $$\Phi(X,1) = \eta, \qquad 0 \leqq X \leqq X_e,$$

(3-3c) $$\partial\Phi(X,1)/\partial Y = B(X), \qquad X_e < X < X_{max},$$

(3-3d) $$\Phi(X_{max},Y) = 0, \qquad 0 \leqq Y \leqq 1,$$

(3-3e) $$\Phi(X,0) = 0, \qquad 0 \leqq X \leqq X_{max},$$

(3-3f) $$\partial\Phi(0,Y)/\partial X = 0, \qquad 0 < Y < 1,$$

(3-4) $$B(X) = \begin{cases} 1, & 0 \leqq X \leqq X_e, \\ \exp(-[X-X_e]/\text{EFOLD}), & X_e < X. \end{cases}$$

The power P and the rate of work W then become

(3-5a) $$P = \eta 4v^2B_0^2\sigma \int_0^{X_{max}} [-\partial\Phi(X,Y)/\partial Y + B(X)]\, dX$$

(3-5b) $$W = 4v^2B_0^2\sigma \int_0^1 dY \int_0^{X_{max}} dX\, [-\partial\Phi(X,Y)/\partial Y + B(X)]\, B(X)$$

Thus, in this model, the efficiency $E = P/W$ does not depend on the value of $v^2B_0^2\sigma$.

An analysis of this mathematical model by conformal mapping is given by Sutton, Hurwitz, and Poritsky [1961].

4. <u>Efficiency as a function of load factor</u>. In this section we present a result which does not seem to be in the literature which significantly reduces the amount of computation to determine the efficiency $E = P/W$ as a function of load factor η. Namely, for a fixed geometry and magnetic field, the efficiency can be determined by solving a <u>single pair</u> of boundary value problems.

Suppose that the geometry and the function B are fixed. Then the solution of (3-3) depends only on the value of η. Let Φ_1 denote the solution when η is equal to some given value η_1. Consider the solution Φ for an arbitrary value of η. We can write

$$\Phi = \Phi_1 + (\eta - \eta_1)\,\Psi$$

where Ψ satisfies

(4-1a) $$\partial^2\Psi(X,Y)/\partial X^2 + \partial^2\Psi(X,Y)/\partial Y^2 = 0, \quad 0 < X < X_{max},\ 0 < Y < 1,$$

(4-1b) $$\Psi(X,1) = 1, \quad 0 \leq X \leq X_e,$$

(4-1c) $$\partial\Psi(X,1)/\partial Y = 0 \quad X_e < X < X_{max},$$

(4-1d) $$\Psi(X_{max},Y) = 0, \quad 0 \leq Y \leq 1,$$

(4-1e) $$\Psi(X,0) = 0, \quad 0 \leq X \leq X_{max},$$

(4-1f) $$\partial\Psi(0,Y)/\partial X = 0, \quad 0 < Y < 1,$$

The efficiency $E(\eta)$ for a given value of load factor η is then given by

(4-2a) $$E(\eta) = \eta[\alpha + (\eta - \eta_1)\beta]/[\gamma + (\eta - \eta_1)\delta]$$

where the constants $\alpha,\beta,\gamma,\delta$ are given by

(4-2b) $$\alpha = \int_0^{X_{max}} [-\partial\Phi_1(X,Y)/\partial Y + B(X)]\, dX$$

(4-2c) $$\beta = \int_0^{X_{max}} \Psi(X,Y)\, dX$$

(4-2d) $$\gamma = \int_0^1 dY \int_0^{X_{max}} [-\partial\Phi_1(X,Y)/\partial Y + B(X)]\, B(X)\, dX$$

(4-2e) $$\delta = \int_0^1 dY \int_0^{X_{max}} [-\partial\Psi(X,Y)/\partial Y]\, B(X)\, dX$$

5. Reduction of the counter-current. The efficiency of the device is limited by the amount of counter-current mentioned in Section 2. This counter-current can be reduced by inserting insulating vanes parallel to the X-axis in the fluid. These vanes are symmetric with respect to $X = 0$ and to $Y = 0$ and a pair might be on the X-axis. We denote the left (L) and right (R) endpoints of these vanes in the first quadrant by

$$(X^{(L,k)},Y^{(k)}) \quad \text{and} \quad (X^{(R,k)},Y^{(k)}), \quad k = 1,2,\ldots$$

The mathematical models (3-3) and (4-1) are augmented by the boundary conditions

(5-1a) $$\partial\Phi(X,Y^{(k)})/\partial Y = B(X), \quad X^{(L,k)} < X < X^{(R,k)}, \quad k = 1,2,\ldots$$

(5-1b) $$\partial\Psi(X,Y^{(k)})/\partial Y = 0, \quad X^{(L,k)} < X < X^{(R,k)}, \quad k = 1,2,\ldots$$

If one of these vanes, say for $k = 1$, is on the X-axis, then (3-3e) and (4-1e) are replaced with

(5-1c) $$\Phi(X,0) = \Psi(X,0) = 0, \quad 0 \leqq X < X^{(L,k)}, \quad X^{(R,k)} < X \leqq X_{max}$$

together with (5-1a) and (5-1b) with $k = 1$.

The thickness of the vanes is neglected in the mathematical model and thus the rectangle $[0,X_{max}]\times[0,h]$ contains slits. Although the Y-derivative of the potential is continuous across these slits (because of (5-1a), the potential itself is discontinuous at the slits.

6. General operator of ELLPACK. The ELLPACK system is a research tool for testing the performance of the various components which make up a program designed to obtain approximations to solutions of elliptic partial differential equations. ELLPACK is made up a a number of modules, for example, the DISCRETIZATION module constructs a set of linear algebraic equations whose solution gives an approximation to the solution of the user specified partial differential equation. Each module has several different versions, for example there are several different discretizations one can use which vary from the standard divided central difference approximation to the most recently developed methods of approximating elliptic partial differential equations. There have been numerous contributors to the components which make up ELLPACK and these include people at several different universities. The current version of ELLPACK, called ELLPACK77, treats problems with the domain of the partial differential equation the interior of a rectangle or a cube. A new version of ELLPACK is being prepared which will treat general domains.

An ELLPACK progrm consists of (a) a series of statements written in the ELLPACK user oriented language grouped to form ELLPACK segments and (b) a set of user supplied Fortran FUNCTION and SUBROUTINE subprograms.

The ELLPACK77 segments are:

(i) EQUATION. which specifies the elliptic partial differential equation to be solved;

(ii) BOUNDARY. which specifies the rectangular domain and the conditions on the solution of the partial differential equation at the boundaries.

(iii) GRID. which specifies the number of vertical and horizontal mesh lines, a rectangular mesh is generated;

(iv) OPTIONS. which specifies, for example, the amount of ELLPACK generated information about the execution of the program;

(v) DISCRETIZATION. which specifies which of the discretization method is to be used;

(vi) INDEX. which specifies the indexing of the algebraic equations and the unknowns;

(vii) SOLUTION. which specifies which of several linear algebraic equation solvers is to be used;

(viii) OUTPUT. which specifies the ELLPACK generated output, for example: print a table of the values of the approximation at mesh points, construct a contour plot of the approximation;

(ix) SEQUENCE. which specifies the order of the exectuion of the ELLPACK segments and the number of times the sequence is to be executed;

(x) FORTRAN. which specified that the lines of the program which follow are user supplied Fortran FUNCTION and SUBROUTINE subprograms, these are for the boundary conditions and other needed routines as well as a SUBROUTINE called TEST which is described below.

The current version, ELLPACK77, of ELLPACK can be used to generate an approximation U to the potential Φ or Ψ, the solutions of (3-1) and (4-1). Paul Gherson of Purdue used ELLPACK77 to find estimates of Φ

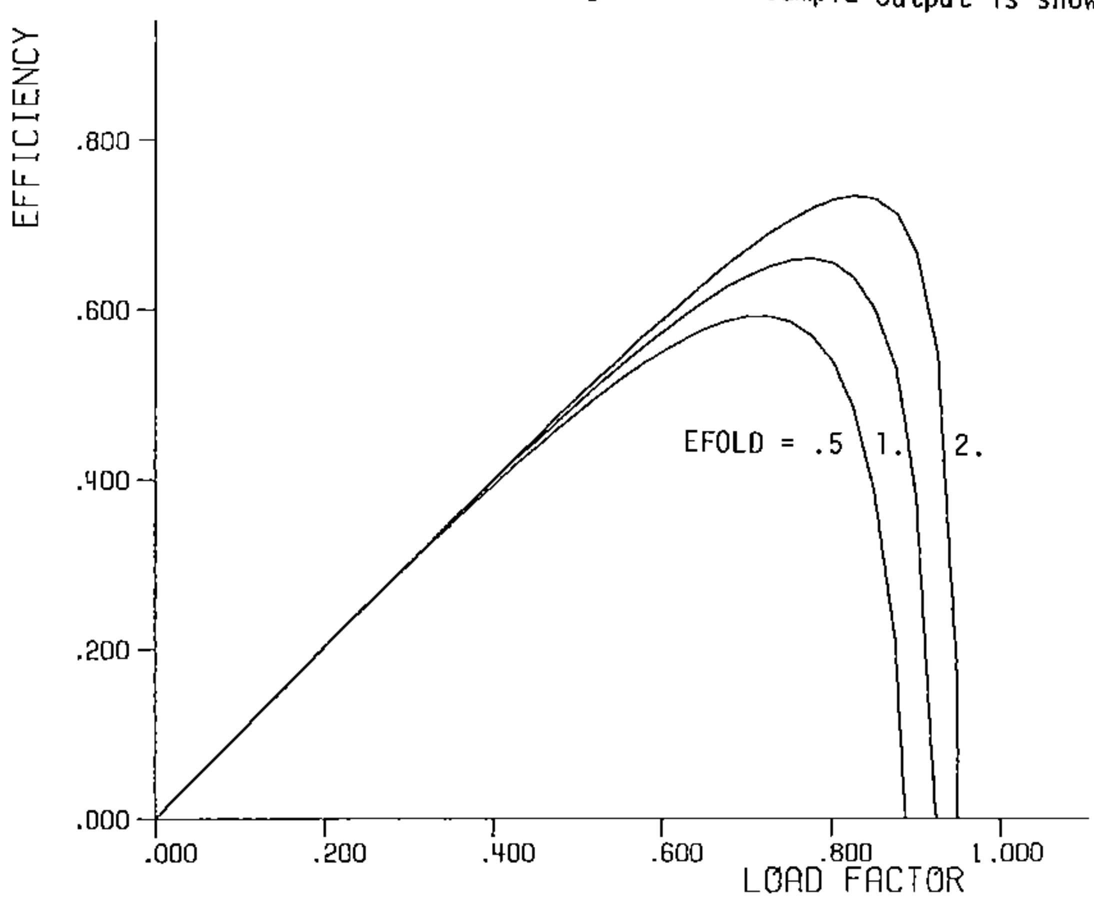
for various values of the electrode endpoint X_-, the endpoint of the rectangle X_{max}, different magnetic fields B, different load factors η, and for various conditions on $Y = 0$, namely the boundary condition (3-3e) and the condition (5-1) for $k = 1$ and $Y^{(1)} = 0$. His results are given in the report by Gherson and Lykoudis [1978].

One of the discretization modules 5-POINT STAR which was written by Ronald F. Boisvert was modified by Robert E. Lynch to handle the case of insulator vanes in the interior of the rectangular domain. In addition to modifying the routines used by 5-POINT STAR, Gherson's BOUNDARY. segment was changed slightly and an additional DISCRETIZATION module was specified in order to increase the sizes of certain of the ELLPACK generated arrays so that the modifed 5-POINT STAR would operator properly. Also, Gherson's FORTRAN. seqment was completely rewritten in order to handle all the various cases to be treated; the new FORTRAN. segment has about three times the number of lines of Gherson's. The new segment has about 900 lines with numerous comments.

In some of our recent test cases, the SEQUENCE. segment was:

```
SEQUENCE.   LOOP = 6
            DISCRETIZATION
            INDEXING
            SOLUTION
            OUTPUT
            TEST
```

The statement LOOP = 6 instructs the ELLPACK system to execute the SEQUENCE six times. As with Gherson's SUBROUTINE TEST, ours set initial values of parameters with DATA statements; some of these specify values such as the location of the endpoints of the interior insulating strips, the value of EFOLD, and so on; others control the operation and they are "switches", for example the LOGICAL variable MAGFON (MAgnetic Field ON) is set to .TRUE. when B is to be nonzero and it is set to .FALSE. when B is to be zero. During the execution of the SEQUENCE the odd numbers of times (1,3,5), MAGFON is set to .TRUE. and the approximation U gives estimates of the potential Φ for a fixed value of load factor η_1; a different value of EFOLD is used so that Φ for η_1 is obtained for three different values of EFOLD. During the even number of times (2,4,6), MAGFON is set to .FALSE., the load factor is set to unity, and the approximation U gives estimates of the potential Ψ. The values of the integrals $\alpha,\beta,\gamma,\delta$ in (4-2) are estimated by the Trapezoid Rule, values of efficiency for $\eta = 0, 1/40, 2/40, \ldots$ is computed, and a graph of efficiency versus load factor is generated. Sample output is shown below.



7. <u>Discrete model</u>. The ELLPACK module 5-POINT STAR which we modified uses finite difference approximations to derivatives. There are three basic ones which we display below in terms of a given function u of z:

$$d^2u(0)/dz^2 = [u(-\Delta z) - 2u(0) + u(\Delta z)]/\Delta z^2 + (\Delta z^2/12)d^4u(0)/dz^4 + \ldots$$

$$du(0)/dz = [-3u(0) + 4u(\Delta z) - u(2\Delta z)]/2\Delta z - (\Delta z^2/3)d^3u(0)/dz^3 + \ldots$$

$$du(0)/dz = [u(-2\Delta z) - 4u(\Delta z) + 3u(0)]/2\Delta z - (\Delta z^2/3)d^3u(0)/dz^3 + \ldots$$

Use of the values of the divided differences as approximations to the values of the derivatives leads to local error which is the order of Δz^2. Provided that the u is smooth, then when one halves Δz one expects the error to be approximately quartered. This applies to finite difference approximations to ellptic partial differential equations for sufficiently small Δz and provided the solution has continuous fourth derivatives in the region and at the boundary. For the mathematical models decribed above for the potentials Φ and Ψ, the solutions have derivatives with singularities at the end of the electrode and at the ends of insulator vanes. Consequently, one expect that the error does not decrease as the square of the mesh spacing as this spacing tends to zero. The error probably decreases as the first power of the mesh spacing as it tends to zero.

We now describe the approximations used in 5-POINT STAR and their modifications which allow the slit-region problems to be treated. We use some of the variable names of ELLPACK and our routines, such as TEST so that this report will also serve as partial documentation for the programs involved.

The horizontal sides of the rectangle are $AX \leq X \leq BX$ and the vertical sides are $AY \leq Y \leq BY$. Numerical values for AX, BX, AY, and BY are user supplied in the BOUNDARY. segement of the ELLPACK program. In our case, AX = AY = 0., BX = X_{max}, and BY = 1.

The number of vertical and horizontal mesh lines are NGRIDX and NGRIDY, respectively. Numerical values of these are specified in the GRID. segment of the ELLPACK program. Uniform mesh spacing is used by 5-POINT STAR and the spacing are HX and HY defined by

$$HX = (-AX+BX)/(NGRIDX-1),$$

$$HY = (-AY+BY)/(NGRIDY-1).$$

We denote the mesh points by (X_I, Y_J) and the values are stored in the ELLPACK arrays GRIDX and GRIDY. The values are

$$X_I = AX + (I-1)HX = GRIDX(I), \quad I = 1,\ldots,NGRIDX,$$

$$Y_J = AY + (J-1)HY = GRIDY(J), \quad J = 1,\ldots,NGRIDY.$$

The approximation to the solution of the partial differential equation is denoted by U and this can be considered as a two-dimensional array and, depending on which problem is solved,

$$U(I,J) \simeq \Phi(X_I, Y_J) \quad \text{or} \quad U(I,J) \simeq \Psi(X_I, Y_J).$$

At each interior mesh point, the Laplace equation is approximated with its usual divided central difference approximation (this is modified at certain points as explained below):

$$[U(I-1,J) - 2U(I,J) + U(I+1,J)]/HX^2$$

(6-1a) $$+ [U(I,J-1) - 2U(I,H) + U(I,J+1)]/HY^2 = 0, \quad I = 2,\ldots,NGRIDX-1, \quad J = 2,\ldots,NGRIDX-1.$$

The vertical mesh line number of the endpoint of the electrode (conductor) is denoted by ICONDR and its value is set in a DATA statement in SUBROUTINE TEST. Since AX = 0, the X-coordinate of the endpoint of the conductor is

$$X_e = (ICONDR-1)HX = GRIDX(ICONDR).$$

The LOGICAL variable HAVANE (HAve VANE) is set to .TRUE. or .FALSE. in a DATA statement in TEST. When it is .TRUE., then there is an insulator strip (vane) on the X-axis, and in this case the horizontal mesh line numbers must also be set by a DATA statement in TEST. The endpoints are denoted by IVANEL (L for "left") and IVANER (R for "right"). The coordinates of the endpoints are then determined by the modified 5-POINT STAR to be at $(X^{(L,1)},0)$ and $(X^{(R,1)},0)$ where

$$X^{(L,1)} = (IVANEL-1)HX = GRIDX(IVANEL),$$

$$X^{(R,1)} = (IVANER-1)HX = GRIDX(IVANER).$$

The boundary conditions on the boundary of the rectangle are given below in the same order as in (3-3) [see also (5-1)]:

(6-1b) $$U(I,NGRIDY) = LOADFA, \quad I = 1,\ldots,ICONDR,$$

(LOADFA for LOAD FActor; this is declared REAL and set in a DATA statment in SUBROUTINE TEST),

$$[U(I,NGRIDY-2) - 4U(I,NGRIDY-1) + 3U(I,NGRIDY)]/2HY$$
(6-1c)
$$= B(GRIDX(I)), \quad I = ICONDR+1,...,NGRIDX-1,$$

(6-1d) $$U(NGRIDX,J) = 0, \quad J = 1,...,NGRIDY,$$

$$U(I,1) = 0, \quad I = 1,...,IVANEL-1 \text{ and}$$
(6-1e)
$$I = IVANER+1,...,NGRIDX-1,$$

$$[-3U(I,1) + 4U(I,2) - U(I,3)]/2HY$$
(6-1e')
$$= B(GRIDX(I)), \quad I = IVANEL,...,IVANER,$$

$$[-3U(1,J) + 4U(2,J) - U(3,J)]/2HY$$
(6-1f)
$$= 0, \quad J = 2,...,NGRIDY-1.$$

The total number of algebraic equations in (6-1) is equal to the total number of mesh points NGRIDX*NGRIDY, hence there is one equation for each mesh point and thus for each unknown U(I,J).

The equations (6-1) give the set of algebraic equations generated by 5-POINT STAR and which are used in the case that there are no insulator vanes in the <u>interior</u> of the rectangle. We now consider ther case in which there are insulator vanes in the interior; we allow for one or two such vanes or <u>slits</u>.

The horizontal mesh line number of a slit is set in a DATA statment in TEST. The variable names are JYSL1D and JYSL2D (D for "down"). If one or both of these are nonzero, then this informs the system that there is one or two interior slits and in this case the corresponding vertical mesh line numbers for the endpoints must be set in a DATA statement in TEST. The endpoints are denoted by

IXSL1L, IXSL1R for JYSL1D and

IXSL2L, IXSL2R for JYSL2D,

where "L" and "R" stand for "left" and "right". The boundary condition at the slit is $\partial\Phi/\partial Y = B(X)$ and at the downward side of the slit, the difference equation approximation is

$$[U(I,JYSLkD-2) - 4U(I,JYSLkD-1) + 3U(I,JSLkD)]/2HY$$

$$= B(GRIDX(I)), \quad I = IXSLkL,\ldots,IXSLkR \quad \text{for } k = 1 \text{ and/or } k = 2. \tag{6-1g}$$

The value of $\partial\Phi/\partial Y$ is continuous across a slit, but the value of Φ is discontinuous; that is

$$\lim_{\varepsilon\downarrow 0} [-\Phi(X,Y^{(k)}-\varepsilon) + \Phi(X,Y^{(k)}+\varepsilon)] \neq 0 \quad \text{for} \quad X^{(L,k)} \leq X \leq X^{(R,k)}.$$

Since the value of U at both the bottom and the top of the slit are unknowns, we need to add another equation:

$$[-3U(I,JYSLkD+1) + 4U(I,JYSLkD+2) - U(I,JYSLkD+3)]/2HY$$

$$= B(GRIDX(I)), \quad I = IXSLkL,\ldots,IXSLkR \quad \text{for } k = 1 \text{ and/or } k = 2. \tag{6-1h}$$

The mesh points (I,JYSLkD) and (I,JYSLkD+1) correspond to the same point $(X_I, Y^{(k)}_{JYSLkD})$ in the region. Consequently, in order to obtain an approximation to the potential in a rectangular region with interior slits, we double the number of horizontal mesh points along each interior slit.

This is done by a modification of 5-POINT STAR. When the module is called to be executed, it calls a subroutine which determines whether or not there are interior slits and modifies the horizontal mesh line coordinates. We give three examples. Suppose that one wants estimates of Φ on a mesh with horizontal mesh spacing HY = .1 and vertical mesh spacing HX = 1. where $X_{max} = 20$.

Example 1: no interior slits.

In TEST one includes the statement

DATA JYSL1D, JYSL2D / 0, 0 /

The GRID. segment of the ELLPACK program is

GRID. UNIFORM X = 21 $ UNIFORM Y = 11

Then NGRIDX = 21, NGRIDY = 11, HX = 20/20 = 1, HY = 1/10 = .1

Example 2: one interior slit with endpoints (4,.3) and (10,.3)

In TEST one includes the statement

DATA JYSL1D, JYSL2D, IXSL1L, IXSL1R / 4, 0, 5, 11 /

The GRID. segment of the ELLPACK program is

GRID. UNIFORM X = 21 $ UNIFORM Y = 12

Then NGRIDX = 21, NGRIDY = 12, and the ELLPACK system set HY = 20/20 = 1, HY = 1/11 = .0909090... It also generates values for GRIDX and GRIDY with spacing 1. and 1/11, respecitively. The modified 5-POINT STAR determines that JYSL1D is nonzero so that there is an interior slit. It then knows that there is a pair of horizontal mesh lines which correspond to the same value of Y along the slit. It changes HY from 1/11 to 1/10 = .1 and it resets the

the values of the Y coordinates of the mesh lines:

$$Y_J = (J-1)HY = GRIDY(J), \qquad J = 1,\ldots,JYSL1D$$

$$Y_{JYSL1D+1} = Y_{JYSL1D} + HY*10^{-5} = GRIDY(JYSL1D+1)$$

$$Y_J = (J-2)HY = GRIDY(J), \qquad J = JYSL1D+2,\ldots,NGRIDY.$$

The values in GRIDY are then:

$$J = 1,\ 2,\ 3,\ 4,\ 5,\ \ 6,\ 7,\ 8,\ 9,10,11,12$$
$$GRIDY(J) = 0.,.1,.2,.3,.3+10^{-6},.4,.5,.6,.7,.8,.9,1.$$

Example 3: two interior slits with endpoints (4,.3),(10,.3) and (3,.7),(5,.7):

In TEST one includes the statements

```
DATA JYSL1D, IXSL1L, IXSL1R / 4, 5, 11 /
DATA JYSL2D, IXSL2L, IXSL2R / 9, 4,  6 /
```

The GRID. segment of the ELLPACK program is

```
GRID. UNIFORM X = 21 $ UNIFORM Y = 13
```

The ELLPACK system sets HX = 20/20 = 1., HY = 1/12 = .08333... and generates values of GRIDX and GRIDY with these spacing. The modified 5-POINT STAR finds that both JYSL1D and JYSL2D are nonzero so that there are two interior slits. It then knows that there are two pairs of horizontal mesh lines which correspond to the two values of Y along the slits. It changes the value HY from 1/12 to 1/10 = .1 and it resets the values of the Y coordinates of the mesh lines:

$$Y_J = (J-1)HY = GRIDY(J), \qquad J = 1,\ldots,JYSL1D$$

$$Y_{JYSL1D+1} = Y_{JYSL1D} + HY*10^{-5} = GRIDY(JYSL1D+1)$$

$$Y_J = (J-2)HY = GRIDY(J), \qquad J = JYSL1D+2,\ldots,JYSL2D$$

$$Y_{JYSL2D+1} = Y_{JYSL2D} + HY*10^{-5} = GRIDY(JYSL2D+1)$$

$$Y_J = (J-3)HY = GRIDY(J), \qquad J = JYSL2D+2,\ldots,NGRIDY$$

The values in GRIDY are then:

$$\begin{array}{l} J \ = 1,\ 2,\ 3,\ 4,\ 5,\quad 6,\ 7,\ 8,\ 9,\ 10,\quad 11,12,13 \\ GRIDY(J) = 0.,.1,.2,.3,.3+10^{-6},.4,.5,.6,.7,.7+10^{-6},.8,.9,1. \end{array}$$

By introducing these lines of "double" mesh points, we have added additional unknowns, namely the values of U(I,JYSLkD) and U(I,JYSLkD+1), I = 1,...,NGRIDY. Equations corresponding to some of these are given in (6-1g) and (6-1h), specifically for I = IXSLkL,...,IXSLkR. At points off of the slits, the potential is continuous, so we add the equations

$$U(I,JYSLkD) - U(I,JYSLkD+1) = 0, \quad I = 1,\ldots,IXSLkL-1 \text{ and } I = IXSLkR+1,\ldots,NGRIDX \qquad (6\text{-}1i)$$

In equation (6-1a) we give the approximation for the Laplace equation; but now that does not apply for (I,J) on these lines of double mesh points. In (6-1a) the range of the subscripts changes from

$$I = 2,\ldots,NGRIDX-1, \quad J = 2,\ldots,NGRIDX-1$$

to

(6-1a') I = 2,...,NGRIDX-1, J = 2,...,JYSL2D-2
J = JYSL1D+2,...,JYSL2D-2, J = JYSL2D+2,...,NGRIDY-1

The approximation to the Laplace equation for a mesh point with JYSLkD and at a distance at least 2HX from the end of a slit is taken as [note the "+2" in the second pair of square brackets]

$$[U(I-1,JYSLkD) - 2U(I,JYSLkD) + U(I+1,JYSLkD)]/HX^2$$

$$(6\text{-}1j) \quad + [U(I,JYSLkD-1) - 2U(I,JYSLkD) + U(I,JYSLkD+2)]/HY^2 = 0,$$

I = 2,...,IXSLkL-2, and I = IXLSkR+2,...,NGRIDX.

For mesh points immediately to the right and to the left of the end points of the slits, we incorporate the average value of U at the top and bottom of the end of the slit and use [again, note the "+2"]

$$[U(I-1,JYSLkD) - 2U(I,JYSLkD) + U(I+1,JYSLkD)/2 + U(I+1,JYSLkD+1)/2]/HX^2$$

$$(6\text{-}1k) \quad + [U(I,JYSLkD-1) - 2U(I,JYSLkD) + U(I,JYSLkD+2)]/HY^2 = 0,$$

for I = IXSLkL-1

and

$$[U(I-1,JYSLkD)/2 + U(I-1,JYSLkD+1)/2 - 2U(I,JYSLkD) + U(I+1,JYSLkD)]/HX^2$$

$$(6\text{-}1\ell) \quad + [U(I,JYSLkD-1) - sU(I,JYSLkD) + U(I,JYSLkD+2)]/HY^2 = 0$$

for I = IXSLkR+1

These last two equations involve a linear combination of six of the unknowns; the usual approximation to Laplace's equation involves five as in (6-1a). The ELLPACK module 5-POINT STAR constructs a set of difference equations which has at most five of the unknowns in them; consequently, the ELLPACK system does not allocate a sufficient amount of storage when the DISCRETIZATION. segement of the ELLPACK program is

DISCRETIZATION. 5-POINT STAR

In order to obtain the necessary storage, we include a second discretization module--this is not subsequently used, its purpose is merely to obtain extra storage. Hence in an ELLPACK program which estimates the potential of a region with slits, one uses

DISCRETIZATION(1) 5-POINT STAR

DISCRETIZATION(2) HOLR 9-POINT (IORDER=40)

and in the SEQUENCE. segment, DISCRETIZATION is replaced with DISCRETIZATION(1).

Caution: The locations of the end of the conductor, and the endpoints of the slits and vane are given in terms of mesh line numbers. Consequently, if the number of grid lines is changed in the ELLPACK GRID. then the integer values ICONDR, JYSL1D, and so on, must be changed in order to obtain the same geometry.

Caution: Because a three term approximation is used for the normal derivative at an insultor, for accuracy one must have at least two interior (non-slit) mesh lines between any pair of insulators. In Example 3 above, with two slits, note that there are two interior mesh lines between the lower slit and Y = 0 (J = 2,3) and two between the upper slit and Y = 1 (J = 11,12) and these are the minimal number. There are three between the pair of interior slits (J = 6,7,8) and this is one more than the required minimum number.

Caution: The derivatives of the potential are singular at the end point of the conductor and at the end of the insulator slits and vane. One should run a few test cases to determine the mesh spacing which yieldssufficient accuracy. In particular, the vertical mesh line spacing should be smaller than the length of the conductor.

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